THE HARDY INEQUALITY AND NONLINEAR PARABOLIC EQUATIONS ON CARNOT GROUPS

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Abstract. In this paper we shall investigate the nonexistence of positive solutions for the following nonlinear parabolic partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{\mathbb{G},p} u + V(x)u^{p-1} & \text{in} \quad \Omega \times (0,T), \quad 1 < p < 2, \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \end{cases}$$

where $\Delta_{\mathbb{G},p}$ is the p-sub-Laplacian on Carnot group $\mathbb{G}$ and $V \in L^1_{\text{loc}}(\Omega)$.

1. Introduction

In this paper we study the nonexistence of positive solutions for the following nonlinear degenerate parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{\mathbb{G},p}\, u + V(x)u^{p-1} & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Here Ω is a bounded domain with smooth boundary in $\mathbb{G}$, $V \in L^1_{\text{loc}}(\Omega)$ and $1 < p < 2$. The nonlinear operator

$$\Delta_{\mathbb{G},p}\, u = \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u) \tag{1.2}$$

is the sub-p-Laplacian on Carnot group $\mathbb{G}$.

It is well known that Euclidean space $\mathbb{R}^n$ with its usual Abelian group structure is a trivial Carnot group and the problem (1.1) has been studied by Goldstein and Kombe [20] on $\mathbb{R}^n$. In this case we should also mention that the problem (1.1) with the potential $V(x) = \frac{c}{|x|^p}$ has been studied in great detail by Garcia and Peral [16], and by Aguilar and Peral [1]. Recently, Goldstein and Kombe [21] extended the results in [20] to the Heisenberg group $\mathbb{H}^n$ which is the simplest non-commutative Carnot group.

It turns out that nonexistence of positive solutions that kind of problems largely depends on the size of bottom of the normalized p-energy form

$$\sigma^p_{\text{inf}}(V) := \inf_{0 \not\equiv \phi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla_{\mathbb{G}} \phi|^p dx - \int_\Omega V |\phi|^p dx}{\int_\Omega |\phi|^p dx} \tag{1.3}$$

and the range of p. This was discovered by Cabré and Martel [8] for $p = 2$ in Euclidean space $\mathbb{R}^n$ which was a nice generalization of Baras and Goldstein result [4]. The observation of Cabré and Martel [8] used to get additional results on nonexistence of positive solutions

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Date: April 3, 2004.

Key words and phrases. nonlinear parabolic equations, positive solutions, Hardy-type inequality.

AMS Subject Classifications: 35K65,35R05, 22E30, 26D10 .

for wide class of linear and nonlinear parabolic problems on Euclidean space $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$ (see [18], [19], [20], [21], [23], [26], and [27]).

The purpose of this paper is to investigate nonexistence of positive solutions of the problem(1.1) on more general Carnot group $\mathbb{G}$ and our results recover the previous results [20], [21]. We considered singular potentials $V(x) = c/|x|^p$ and $V(z,l) = c|z|^p/(|z|^4 + l^2)^{p/2}$ in [20] and [21], respectively. These potentials appear in the Hardy's inequality on Euclidean space $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$, respectively. It is natural to investigate such a singular potential for the problem (1.1) on more general Carnot groups and we shall first prove a Hardy type inequality on polarizable Carnot groups.

The plan of the paper is as follows. In section 2 we recall the basic properties of Carnot group $\mathbb{G}$ and some well known results that will be used in the sequel. In section 3 we prove L^p- Hardy inequality on polarizable Carnot groups. Finally, in Section 4 we study problem (1.1).

2. Carnot group

A Carnot group (see [2], [3], [13], [14], [15], [31] and [34]) is a connected, simply connected, nilpotent Lie group $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is, there exist linear subspaces $V_1, ..., V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \quad i = 1, 2, ..., k-1 \quad \text{and} \quad [V_1, V_k] = 0 \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This defines a k-step Carnot group and integer $k \geq 1$ which is called the step of $\mathbb{G}$.

Via the exponential map, it is possible to induce on $\mathbb{G}$ a family of automorphisms of the group, called dilations, $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n (\lambda > 0)$ such that

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \leq ... \leq \alpha_n$ are integers and $m = \dim(V_1)$.

The group law can be written in the following form

$$x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

where $P : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has polynomial components and $P_1 = ... = P_m = 0$. Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, ... - x_n)$.

Let $X_1, ..., X_m$ be a family of left invariant vector fields that is an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, ..., X_m(0) = \partial_{x_m}$. The vector fields X_j have polynomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x) \partial_i, \quad X_j(0) = \partial_j, j = 1, ..., m,$$

where each polynomial a_{ij} is homogeneous with respect to the dilations of the group, that is $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$. The horizontal gradient on $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, ..., X_m)$$

where $X_1, ..., X_m$ are the generators of $\mathbb{G}$.

The nonlinear operator

$$\Delta_{\mathbb{G},p} u = \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u)$$

is the sub-p-Laplacian on Carnot group $\mathbb{G}$. If $p = 2$ then we have sub-Laplacian

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2$$

which is a second-order partial differential operator on $\mathbb{G}$. The fundamental solution u for $\Delta_{\mathbb{G}}$ is defined to be a weak solution to the equation

$$-\Delta_{\mathbb{G}} u = \delta$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. In [13], Folland proved that in any Carnot group $\mathbb{G}$, there exists a homogeneous norm N such that

$$u = N^{2-Q}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$ (see also [6]).

We now set

$$N(x) := \begin{cases} u^{\frac{1}{2-Q}} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.3}$$

We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $N : \mathbb{G} \longrightarrow [0, \infty)$ smooth away from the origin which satisfies the conditions : $N(\delta_\lambda(x)) = \lambda N(x)$, $N(x^{-1}) = N(x)$ and $N(x) = 0$ iff $x = 0$.

The curve $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{G}$ is called horizontal if its tangents lie in V_1, i.e, $\gamma'(t) \in span\{X_1, ..., X_m\}$ for all t. Then, the Carnot-Caréthedory distance $d_{CC}(x, y)$ between two points $x, y \in \mathbb{G}$ is defined to be the infimum of all horizontal lengths $\int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2} dt$ over all horizontal curves $\gamma : [a, b] \longrightarrow \mathbb{G}$ such that $\gamma(a) = x$ and $\gamma(y) = b$. Notice that d_{CC} is a homogeneous norm and satisfies the invariance property

$$d_{CC}(z \cdot x, z \cdot y) = d_{CC}(x, y), \quad \forall\, x, y, z \in \mathbb{G},$$

and is homogeneous of degree one with respect to the dilation δ_λ, i.e.

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x, y), \quad \forall\, x, y, z \in \mathbb{G}, \forall\, \lambda > 0.$$

The Carnot-Carethédory balls are defined by $B(x, R) = \{y \in \mathbb{G} | d_{CC}(x, y) < R\}$. The n-dimensional Lebesgue measure $\mathcal{L}^n$, is the Haar measure of group $\mathbb{G}$. This means that if $E \subset \mathbb{R}^n$ is a measurable, then $\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E)$ for all $x \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$. Clearly

$$\mathcal{L}^n\big(B(x, R)\big) = R^Q \mathcal{L}^n\big(B(x, 1)\big) = R^Q \mathcal{L}^n\big(B(0, 1)\big)$$

where

$$Q = \sum_{j=1}^{k} j(\mathrm{dim} V_j)$$

is the homogeneous dimension of $\mathbb{G}$.

It is well known that Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. The following Sobolev inequality holds on $\mathbb{G}$

$$\Big(\int_{\mathbb{G}} |\phi(x)|^q dx\Big)^{1/q} \leq C_{p,q} \Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} \phi(x)|^p dx\Big)^{1/p} \tag{2.4}$$

where $1 \leq p < Q$, $q = \frac{Qp}{Q-p}$ (see [13], [35]) and plays a crucial role in this paper.

Example 1. The simplest non trivial example of Carnot group is given by the Heisenberg group $\mathbb{H}^n$. Denoting points in $\mathbb{H}^n$ by (z,t) with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$ we have the group law given as

$$(z,t) \circ (z', t') = (z + z', t + t' + 2\sum_{j=1}^{n} Im(z_j \bar{z}'_j))$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

The one dimensional center V_2 is spanned by the vector field $T = \frac{\partial}{\partial t}$. We have the commutator relations $[X_j, Y_j] = -4T$, and all other brackets of $\{X_1, Y_1, ..., X_n, Y_n\}$ are zero. The sub-Laplacian associated with the basis $\{X_1, Y_1, ..., X_n, Y_n\}$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

A homogeneous norm on $\mathbb{H}^n$ is given by

$$\rho = |(z,t)| = (|z|^4 + t^2)^{1/4}$$

and the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

A remarkable analogy between sub-Laplacian and the classical Laplace operator has been obtained by Folland [12]. He found that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole zero is given by

$$\Psi(z,t) = \frac{c_Q}{\rho(z,t)^{Q-2}} \quad \text{where} \quad c_Q = \frac{2^{(Q-2)/2}\Gamma((Q-2)/4)^2}{\pi^{Q/2}}.$$

Example 2. Another important model of Carnot groups are the H-type (Heisenberg type) groups which were introduced by Kaplan [25] as direct generalizations of the Heisenberg group $\mathbb{H}^n$. An H-type group is a Carnot group with a two-step Lie algebra $\mathcal{G} = V_1 \oplus V_2$ and an inner product $\langle , \rangle$ in $\mathcal{G}$ such that the linear map

$$J : V_2 \longrightarrow \text{End}V_1,$$

defined by the condition

$$\langle J_z(u), v \rangle = \langle z, [u,v] \rangle, \quad u, v \in V_1, z \in V_2$$

satisfies

$$J_z^2 = -||z||^2 \mathbf{Id}$$

for all $z \in V_2$, where $||z||^2 = \langle z, z \rangle$.

Sub-Laplacian is defined in terms of a fixed basis $X_1, ..., X_m$ for V_1:

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2. \tag{2.5}$$

The exponential mapping of a simply connected Lie group is an analytic diffeomorphism. One can then define analytic mappings $v : \mathbb{G} \longrightarrow V_1$ and $z : \mathbb{G} \longrightarrow V_2$ by

$$x = \exp(v(x) + z(x))$$

for every $x \in \mathbb{G}$. In [25], Kaplan proved that there exists a constant $c > 0$ such that the function

$$\Phi(x) = c\Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{2-Q}{4}}$$

is a fundamental solution for the operator $-\Delta_{\mathbb{G}}$. We note that

$$K(x) = \Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{1}{4}} \tag{2.6}$$

defines a homogeneous norm and $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$ where $m = \mathrm{dim} V_1$ and $k = \mathrm{dim} V_2$. This result generalized Folland's fundamental solution for the Heisenberg group $\mathbb{H}^n$ [12]. It is remarkable that the homogeneous norm $K(x)$ (2.6) involves also in the expression of fundamental solution of the following $p-$sub-Laplace operator

$$\mathcal{L}_p u = \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u), \quad 1 < p < \infty. \tag{2.7}$$

More precisely, Capogna, Danielli and Garofalo proved that for every $1 < p < \infty$ there exists $c_p > 0$ and $\tilde{c}_p < 0$ such that the function

$$\Gamma_p(x) = \begin{cases} c_p K^{(p-Q)/(p-1)} & \text{when}\, p \neq Q, \\ \tilde{c}_p \log K & \text{when}\, p = Q, \end{cases} \tag{2.8}$$

is a fundamental solution for the operator $-\mathcal{L}_p$ [9]. This nice result immediately motivate the following question: Is it possible to find explicit formulas for the fundamental solutions to the sub-$p-$Laplacian on more general Carnot groups? An interesting result in this direction has been obtained by Balogh and Tyson [2]. They found explicit formulas for the fundamental solutions to the sub-$p-$Laplacian on polarizable Carnot goups.

3. Poloraziable Carnot groups and Hardy type inequality

Polarizable Carnot group. A Carnot group $\mathbb{G}$ is called polarizable if the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, satisfies the following ∞- sub-Laplace equation,

$$\Delta_{\mathbb{G},\infty} N := \frac{1}{2}\langle \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} N|^2), \nabla_{\mathbb{G}} N\rangle = 0, \qquad \text{in} \quad \mathbb{G} \setminus \{0\}. \tag{3.1}$$

In [2], Balogh and Tyson proved that the fundamental solutions of the sub-p-Laplacian

$$\Delta_{\mathbb{G},p} f = \mathrm{div}(|\nabla_{\mathbb{G}} f|^{p-2} \nabla_{\mathbb{G}} f), \qquad f \in C^2(\mathbb{G}), \tag{3.2}$$

is given by

$$u_p = \begin{cases} N^{\frac{p-Q}{p-1}}, & \text{if} \quad p \neq Q \\ -\mathrm{log} N, & \text{if} \quad p = Q. \end{cases} \tag{3.3}$$

Moreover, for each $1 < p < \infty$ there exists a constants $c_p > 0$ so that the function $c_p u_p$ solves the equation $-\Delta_p(c_p u_p) = \delta$ in the sense of distributions. This class of groups is the largest class for which the fundamental solution of the sub-p-Laplacian (3.2) is given by (3.3) [2].

It is known that Euclidean space, Heisenberg group and the Kaplan's H-type group are polarizable Carnot group. Further information about polarizable Carnot groups can be found in [2].

Hardy-type inequalities. To motivate our results, let us recall the following L^p- Hardy inequality in Euclidean space $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |\nabla\phi(x)|^p dx \geq |\frac{n-p}{p}|^p \int_{\mathbb{R}^n} \frac{|\phi(x)|^p}{|x|^p} dx \tag{3.4}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $|\frac{n-p}{p}|^p$ is the best constant. The inequality was discovered by Hardy [24] in the one-dimensional case. Later on it has been extended to higher dimensions and generalized involving various kinds of distance functions in Euclidean space (see [5], [32] and the references therein)

In recent years there has been considerable progress in the study of Hardy-type inequalities on sub-Rimeannian spaces. The following inequality is the analogue of the inequality (3.4) on the Heisenberg group $\mathbb{H}^n$

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi(z,l)|^p dzdl \geq (\frac{Q-p}{p})^p \int_{\mathbb{H}^n} \frac{|z|^p}{(|z|^4+l^2)^{\frac{p}{2}}} |\phi(z,l)|^p dzdl \tag{3.5}$$

where $Q = 2n+2$, $1 < p < Q$ and $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The inequality (3.5) was first proved by Garofalo and Lanconelli [17] for $p = 2$ and later by Niu, Zhang and Wang [30] for any p. (See also [10], [22]).

These inequalities play an important role in the study of linear and nonlinear partial differential equations on Euclidean space $\mathbb{R}^n$ and on the Heisenberg group $\mathbb{H}^n$ (see [1], [4], [7], [8], [11], [16], [19], [20], [21], [22], [23], [26], [27], [33]).

In this paper we prove L^p- Hardy inequality on polarizable Carnot groups and it is an open problem for more general Carnot groups. Recently, Kombe has proved Hardy inequality with a sharp constant for $p = 2$ on general Carnot groups [28]. The following theorem is the main result of this section.

Theorem 3.1. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $Q \geq 3$ and $1 < p < Q$. Then the following inequality is valid*

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^p dx \geq (\frac{Q-p}{p})^p \int_{\mathbb{G}} \frac{|\nabla_{\mathbb{G}}N|^p}{N^p} |\phi|^p dx. \tag{3.6}$$

Furthermore, the constant $(\frac{Q-p}{p})^p$ is sharp.

Proof. Let $\phi = N^\gamma\psi$ where $\psi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and $\gamma \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}(N^\gamma\psi)| = |\gamma N^{\gamma-1}\psi\nabla_{\mathbb{G}}N + N^\gamma\nabla_{\mathbb{G}}\psi|. \tag{3.7}$$

We now use the following inequality which is valid for any $a, b \in \mathbb{R}^n$ and $1 < p < 2$,

$$|a+b|^p - |a|^p \geq c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a \cdot b \tag{3.8}$$

where $c(p) > 0$ (see [5], [29]). In view of (3.8) we have that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p \int_{\mathbb{G}} N^{\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx + |\gamma|^{p-2}\gamma \int_{\mathbb{G}} N^{\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}}(|\psi|^p) dx$$
$$+ c(p)\int_{\mathbb{G}} \frac{N^{2\gamma}|\nabla_{\mathbb{G}}\psi|^2}{(|\gamma N^{\gamma-1}\psi\nabla_{\mathbb{G}}N| + |N^{\gamma}\nabla_{\mathbb{G}}\psi|)^{2-p}} dx.$$

Clearly

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p \int_{\mathbb{G}} N^{\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx + |\gamma|^{p-2}\gamma \int_{\mathbb{G}} N^{\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}}(|\psi|^p) dx,$$

and the integration by parts gives

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p \int_{\mathbb{G}} N^{\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx - |\gamma|^{p-2}\gamma \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx.$$

We now choose $\gamma = \frac{p-Q}{p}$ then we get

$$\begin{aligned} \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx &= \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{1-Q}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx \\ &= \int_{\mathbb{G}} (\Delta_{\mathbb{G},p}u_p)|\psi|^p dx. \end{aligned} \tag{3.9}$$

Since u_p is the fundamental solution of sub-p-Laplacian $-\Delta_{\mathbb{G},p}$, then we have

$$\begin{aligned} \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{1-Q}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx &= \int_{\mathbb{G}} (\Delta_{\mathbb{G},p}u_p)|\psi|^p dx \\ &= -|\phi(0)|^p N^{(Q-p)}(0) \\ &= 0. \end{aligned} \tag{3.10}$$

Therefore,

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^p dx \geq \Big(\frac{Q-p}{p}\Big)^p \int_{\mathbb{G}} \frac{|\nabla_{\mathbb{G}}N|^p}{N^p}|\phi|^p dx. \tag{3.11}$$

The theorem (3.1) also holds for $p > 2$ and in this case we use the following inequality

$$|a+b|^p - |a|^p \geq c(p)|b|^p + p|a|^{p-2}a \cdot b \tag{3.12}$$

where $a \in \mathbb{R}^n, b \in \mathbb{R}^n$ and $c(p) > 0$ (see [5], [29]). If $p = 2$ then the inequality (3.11) holds in any Carnot group without the hypothesis of polarizability [28].

To show that the constant $\left(\frac{Q-p}{p}\right)^p$ is sharp, we use the following family of functions

$$\phi_\epsilon(x) = \begin{cases} 1 & \text{if} \quad N(x) \in [0,1], \\ N^{-(\frac{Q-p}{p}+\epsilon)} & \text{if} \quad N(x) > 1, \end{cases} \tag{3.13}$$

and pass to the limit as $\epsilon \longrightarrow 0$. Here we notice that $|\nabla_{\mathbb{G}}N|$ is uniformly bounded and polar coordinate integration formula holds on $\mathbb{G}$ [2], [14].

4. NONEXISTENCE RESULTS

We now turn our attention to problem (1.1). Throughout this section, Ω is a bounded domain in $\mathbb{G}$ with smooth boundary and $V \in L^1_{\text{loc}}(\Omega \setminus \mathcal{K})$ for some closed Lebesque null set $\mathcal{K}$. We define the positive solutions in the following sense.

Definition 4.1. *By a positive local solution continuous off of $\mathcal{K}$, we mean*

(i) *$\mathcal{K}$ is a closed Lebesgue null subset of Ω,*
(ii) *$u : [0,T) \longrightarrow L^1(\Omega)$ is continuous for some $T > 0$,*
(iii) *$(x,t) \longrightarrow u(x,t) \in C((\Omega \setminus \mathcal{K}) \times (0,T))$,*
(iv) *$u(x,t) > 0$ on $(\Omega \setminus \mathcal{K}) \times (0,T)$,*
(v) *$\lim_{t\to 0} u(.,t) = u_0$ in the sense of distributions,*
(vi) *$\nabla_{\mathbb{G}} u \in L^p_{loc}(\Omega \setminus \mathcal{K})$ and u is a solution in the sense of distributions of the PDE.*

Remark. *If $0 < a < b < T$ and $\mathcal{K}_o$ is a compact subset of $\Omega \setminus \mathcal{K}$, then $u(x,t) \geq \epsilon_1 > 0$ for $(x,t) \in \mathcal{K}_o \times [a,b]$ for some $\epsilon_1 > 0$. We can weaken (iii), (iv) to be*

(iii)' *$u(x,t)$ is positive and locally bounded on $(\Omega \setminus \mathcal{K}) \times (0,T)$,*
(iv)' *$\frac{1}{u(x,t)}$ is locally bounded on $(\Omega \setminus \mathcal{K}) \times (0,T)$.*

If a solution satisfies (i), (ii), (iii)', (iv)', (v), and (vi) then we call it a " general positive local solution off of $\mathcal{K}$ ". This is more general than a positive local solution continuous off of $\mathcal{K}$. If $\mathcal{K} = \emptyset$, we simply call u "general positive local solution".

Theorem 4.1. *Let $\frac{2Q}{Q+1} \leq p < 2$ and $V \in L^1_{loc}(\Omega \setminus \mathcal{K})$ where $\mathcal{K}$ is a Lebesgue null subset of Ω. If*

$$\sigma^p_{\inf}((1-\epsilon)V) := \inf_{0 \neq \phi \in C_0^\infty(\Omega \setminus \mathcal{K})} \frac{\int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx - \int_\Omega (1-\epsilon)V|\phi|^p dx}{\int_\Omega |\phi|^p dx} = -\infty \tag{4.1}$$

for some $\epsilon > 0$, then (1.1) has no general positive local solution off of $\mathcal{K}$.

Proof. We argue by contradiction. Given any $T > 0$, let $u : [0,T) \longrightarrow L^1(\Omega)$ be a general positive local solution to (1.1) in $(\Omega \setminus \mathcal{K}) \times (0,T)$ with $u_0 \geq 0$ but not identically zero. Multiply both sides of (1.1) by the test function $|\phi|^p/u^{p-1}$ where $\phi \in C_0^\infty(\Omega \setminus \mathcal{K})$, and integrate over Ω, to get

$$\frac{1}{2-p}\frac{\partial}{\partial t}\int_\Omega u^{2-p}|\phi|^p dx - \int_\Omega \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u)(\frac{|\phi|^p}{u^{p-1}})dx = \int_\Omega V|\phi|^p dx. \tag{4.2}$$

It follows from the integration by parts that

$$L = \int_\Omega \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u)(\frac{|\phi|^p}{u^{p-1}})dx = -\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}}(\frac{|\phi|^p}{u^{p-1}})dx. \tag{4.3}$$

Since

$$|\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}}(\frac{|\phi|^p}{u^{p-1}}) = p|\nabla_{\mathbb{G}} u|^{p-2}\frac{|\phi|^{p-1}}{u^{p-1}}\nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}}|\phi| - (p-1)\frac{|\phi|^p}{u^p}|\nabla_{\mathbb{G}} u|^p,$$

$$L = (p-1)\int_\Omega |\nabla_{\mathbb{G}} u|^p \frac{\phi^p}{u^p}dx - \text{sign}_0(\phi)p\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2}\frac{\phi^{p-1}}{u^{p-1}}\nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}}|\phi|dx, \tag{4.4}$$

where

$$\operatorname{sign}_0(\phi) = \begin{cases} \frac{\phi}{|\phi|} & \text{if} \quad \phi \neq 0, \\ 0 & \text{if} \quad \phi = 0. \end{cases}$$

Therefore,

$$L \geq (p-1)\int_\Omega |\nabla_{\mathbb{G}} u|^p \frac{|\phi|^p}{u^p} dx - p\int_\Omega |\nabla_{\mathbb{G}} u|^{p-1} |\nabla_{\mathbb{G}} \phi| \frac{\phi^{p-1}}{u^{p-1}} dx. \tag{4.5}$$

Here we can use the following elementary inequality: Let $p > 1$ and $w_1 \neq w_2$ be two positive real numbers. Then

$$w_1^p - w_2^p - pw_2^{p-1}(w_1 - w_2) > 0;$$

it follows that

$$(p-1)w_2^p - pw_2^{p-1}w_1 > -w_1^p.$$

We can take $w_2 = |\frac{\phi}{u}\nabla_{\mathbb{G}} u|$, $w_1 = |\nabla_{\mathbb{G}}\phi|$; then we have

$$(p-1)\int_\Omega |\nabla_{\mathbb{G}} u|^p \frac{|\phi|^p}{u^p} dx - p\int_\Omega |\nabla_{\mathbb{G}} u|^{p-1} |\nabla_{\mathbb{G}} \phi| \frac{|\phi|^{p-1}}{u^{p-1}} dx \geq -\int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx. \tag{4.6}$$

Therefore

$$L = \int_\Omega \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u)(\frac{|\phi|^p}{u^{p-1}}) dx \geq -\int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx. \tag{4.7}$$

Substituting (4.7) into (4.2) and integrating from t_1 to t_2, where $0 < t_1 < t_2 < T$, we obtain

$$\int_\Omega V(z)|\phi|^p dx - \int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx \leq \frac{1}{(2-p)(t_2-t_1)} \int_\Omega (u^{2-p}(x,t_2) - u^{2-p}(x,t_1))|\phi|^p dx. \tag{4.8}$$

Using Jensen's inequality for concave functions, we obtain

$$\int_\Omega \Big(u(x,t_i)\Big)^{\frac{(2-p)(Q)}{p}} dx \leq C(|\Omega|)\Big(\int_\Omega u(x,t_i)dx\Big)^{\frac{(2-p)(Q)}{p}} < \infty.$$

Therefore

$$u^{2-p}(x,t_i) \in L^{\frac{Q}{p}}(\Omega).$$

We now use the following a priori inequality which is a consequence of the Sobolev inequality (2.4). For every $\epsilon > 0$ there exists $C(\epsilon)$ such that

$$\begin{aligned} &\frac{1}{(2-p)(t_2-t_1)} \int_\Omega \Big(u^{2-p}(x,t_2) - u^{2-p}(x,t_1)\Big)|\phi|^p dx \\ &\leq \frac{\epsilon}{1-\epsilon}\int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx + C(\epsilon)\int_\Omega |\phi|^2 dx. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), we obtain

$$\int_\Omega V(z)|\phi|^p dx - \int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx \leq \frac{\epsilon}{1-\epsilon}\int_\Omega |\nabla_{\mathbb{G}}\phi|^p dx + C(\epsilon)\int_\Omega |\phi|^p dx. \tag{4.10}$$

Therefore

$$\inf_{0\neq\phi\in C_c^\infty(\Omega\setminus\mathcal{K})}\frac{\int_\Omega|\nabla_{\mathbb{G}}\phi|^p dx-\int_\Omega(1-\epsilon)V(x)|\phi|^p dx}{\int_\Omega|\phi|^p dx}\geq-(1-\epsilon)C(\epsilon)>-\infty. \tag{4.11}$$

This contradicts our assumption (4.1). The proof of Theorem 4.1 is now complete. □

Singular Potentials. We now focus our attention on some singular potentials. First we treat the following positive singular potential:

$$V(x)=\lambda\frac{|\nabla_{\mathbb{G}}N|^p}{N^p} \tag{4.12}$$

where $\lambda>0$.

As a sign changing potential, we consider the following highly singular, oscillating potential:

$$V(x)=\lambda\frac{|\nabla_{\mathbb{G}}N|^p}{N^p}+\beta\frac{|\nabla_{\mathbb{G}}N|^p}{N^p}\sin(\frac{1}{N^\alpha}) \tag{4.13}$$

where $\lambda>0$, $\beta\in\mathbb{R}\setminus\{0\}$ and $\alpha\in\mathbb{R}$.

Theorem 4.1 gives the following corollaries.

Corollary 4.1. *Let $0\in\Omega$ and $V(x)$ be defined by (4.12). Then the problem (1.1) has no general positive local solution off of $\mathcal{K}$ if $\frac{2Q}{Q+1}\leq p<2$ and $\lambda>(\frac{Q-p}{p})^p$.*

Corollary 4.2. *Let $0\in\Omega$ and $V(x)$ be defined by (4.13). Then the problem (1.1) has no general positive local solution off of $\mathcal{K}$ if $\frac{2Q}{Q+1}\leq p<2$ and $\lambda>(\frac{Q-p}{p})^p$.*

To prove Corollary 4.1 and Corollary 4.2, we need to construct a sequence $\{\phi_n\}$ of test functions which satisfies

$$\frac{\int_\Omega|\nabla_{\mathbb{G}}\phi_n|^p dx-\int_\Omega(1-\epsilon)V(x)|\phi_n|^p(x)dx}{\int_\Omega|\phi_n|^p dx}\longrightarrow-\infty\quad\text{as}\quad n\longrightarrow\infty. \tag{4.14}$$

We can easily construct radial functions $\phi_n=f(N(x))$ which satisfy 4.13. (see [20], [21]).

Notice that when $|\beta|>\lambda$, the potential in Corollary 4.2 has very large positive and negative parts, in particular, it oscillates wildly, but important cancellations occur between the positive and the negative parts in the quadratic form. Therefore, nonexistence of positive solutions only depends on the size of λ.

Remark. We would like to point out that all the new results in Section 4 hold on the whole space $\mathbb{G}$ which is a consequence of bounded domain case (see [23] or [26]).

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