The Aharonov-Bohm solenoids in a constant magnetic field

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Abstract: We study the spectral properties of two-dimensional magnetic Schrödinger operator $H_N = (\frac{1}{i}\nabla + \boldsymbol{a}_N)^2$. The magnetic field is given by $\operatorname{rot} \boldsymbol{a}_N = B + \sum_{j=1}^N 2\pi\alpha_j\delta(z - z_j)$, where $B > 0$ is a constant, $1 \le N \le \infty$, $0 < \alpha_j < 1$ ($j = 1, \ldots, N$) and the points $\{z_j\}_{j=1}^N$ are uniformly separated. We give an upper bound for the number of eigenvalues of H_N between two Landau levels or below the lowest Landau level, when N is finite. We prove the spectral localization of H_N near the spectrum of the single solenoid operator, when $\{z_j\}_{j=1}^N$ are far from each other, all the values $\{\alpha_j\}_{j=1}^N$ are the same and the boundary conditions at all z_j are the same. We give a characterization of self-adjoint extensions of the minimal operator.

1 Introduction

After the paper of Aharonov and Bohm ([Ah-Bo]), a lot of physicists and mathematicians have developed the spectral and scattering theory for Schrödinger operators on a plane with δ-like magnetic fields, as it has an importance in gauge theories. In the present paper, we consider the spectral problem for the Schrödinger operators in a plane with a non-zero uniform magnetic field in addition to δ-like magnetic fields. The operator of this type is studied by Nambu ([Nam]) and Exner, Šťovíček and Vytřas ([Ex-St-Vy]).

Let $N = 1, 2, 3, \ldots$ or $N = \infty$. Let $\{z_j\}_{j=1}^N$ be points in $\boldsymbol{R}^2$ and put $S_N = \cup_{j=1}^N \{z_j\}$. We assume that

$$R = \inf_{j \neq k} |z_j - z_k| > 0. \tag{1.1}$$

This assumption is satisfied if N is finite. Define a differential operator $\mathcal{L}_N$ on $\boldsymbol{R}^2 \setminus S_N$ by

$$\mathcal{L}_N = \mathbf{p}_N^2, \quad \mathbf{p}_N = \frac{1}{i}\nabla + \boldsymbol{a}_N,$$

where $i = \sqrt{-1}$ and $\nabla = (\partial_x, \partial_y)$ is the gradient vector with respect to the coordinate $z = (x, y) \in \boldsymbol{R}^2$. We assume that the magnetic vector potential $\boldsymbol{a}_N = (a_{N,x}, a_{N,y})$ belongs to $C^\infty(\boldsymbol{R}^2 \setminus S_N; \boldsymbol{R}^2) \cap L^1_{loc}(\boldsymbol{R}^2; \boldsymbol{R}^2)$. The function $\operatorname{rot} \boldsymbol{a}_N(z) = (\partial_x a_{N,y} - \partial_y a_{N,x})(z)$ represents the intensity of the magnetic field perpendicular to the plane. We assume that

$$\operatorname{rot} \boldsymbol{a}_N(z) = B + \sum_{j=1}^N 2\pi\alpha_j\delta(z - z_j) \tag{1.2}$$

in $\mathcal{D}'(\boldsymbol{R}^2)$ (the Schwartz distribution space), where B, α_j are constants satisfying $B > 0$ and

$$0 < \alpha_j < 1 \text{ for every } j = 1, \ldots, N. \tag{1.3}$$

The constant B represents the intensity of a uniform magnetic field. The constant $2\pi\alpha_j$ represents the magnetic flux of an infinitesimally thin solenoid placed at z_j. We will show that the difference of integer magnetic fluxes can be gauged out by a suitable unitary gauge transform (see Proposition 2.3 below). Since we consider only the spectral problem, the assumption (1.3) loses no generality. We find a proof of the existence of the vector potential with δ-like singularities in Arai's paper (see [Ar1] and [Ar2]; see also Proposition 2.1 in section 2 below). When $N = 1$ and $\alpha_1 = \alpha$, we always assume that $z_1 = 0$ and take the circular gauge, that is,

$$\boldsymbol{a}_1(z) = \left(-\frac{B}{2}y - \frac{\alpha}{|z|^2}y, \frac{B}{2}x + \frac{\alpha}{|z|^2}x \right). \tag{1.4}$$

When we need to indicate the value α explicitly, we denote $\mathcal{L}_1^\alpha$ for $\mathcal{L}_1$ (this notation is used for the operator L_1 defined below).

Define a linear operator L_N on $L^2(\boldsymbol{R}^2)$ by

$$L_N u = \mathcal{L}_N u, \ u \in D(L_N) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N),$$

where $D(L)$ is the operator domain of a linear operator L and $C_0^\infty(U)$ is the space of compactly supported smooth functions in an open set U. The operator L_N is symmetric, positive and has the deficiency indices $(2N, 2N)$ (see Proposition 4.5 below). Thus the operator L_N has self-adjoint extensions parameterized by $(2N \times 2N)$-unitary matrices (see [Re-Si, Theorem X.2]). We denote one of self-adjoint extensions of L_N by H_N. In particular, we denote the Friedrichs extension of L_N (the self-adjoint operator associated to the form closure of $D(L_N)$, see [Re-Si, Theorem X.23]) by H_N^{AB}, which is called the standard Aharonov-Bohm Hamiltonian (this name is used in [Ex-St-Vy], when $N = 1$).

The Schrödinger operator with constant magnetic field is given by

$$\mathcal{L}_0 = \left(\frac{1}{i}\nabla + \boldsymbol{a}_0 \right)^2, \ \boldsymbol{a}_0 = \left(-\frac{B}{2}y, \frac{B}{2}x \right). \tag{1.5}$$

It is well-known that the linear operator defined by

$$L_0 u = \mathcal{L}_0 u, \ D(L_0) = C_0^\infty(\boldsymbol{R}^2) \tag{1.6}$$

is essentially self-adjoint and the spectrum of the unique self-adjoint extension H_0 of L_0 satisfies

$$\sigma(H_0) = \{(2n-1)B; n = 1, 2, \ldots\},$$

which is called the Landau levels.

When solenoids exist, the spectrum in a gap of the Landau levels appears. Our aim is to give an estimate for the number of eigenvalues between two Landau levels or below the lowest Landau level.

We recall known results in the case $N = 1$. Nambu ([Nam]) treats the standard Aharonov-Bohm Hamiltonian H_1^{AB} and gives an explicit representation of all eigenvalues and eigenfunctions using complex integration (he treats also the case $B = 0$). Exner, Šťovíček and Vytřas ([Ex-St-Vy]) give a detailed analysis for every self-adjoint extension H_1. In the polar coordinate $x + iy = re^{i\theta}$, the operator H_1 takes the form

$$H_1 = -\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\left(\frac{1}{i}\partial_\theta + \alpha + \frac{B}{2}r^2\right)^2 \tag{1.7}$$

if one takes the circular gauge (1.4). Exner, Šťovíček and Vytřas solve the eigenequation explicitly by using Fourier expansion with respect to the θ-variable and represent the eigenvalue as a solution of an equation including Gamma functions. We summarize a part of their results as follows.

Theorem 1.1 (Nambu, Exner-Šťovíček-Vytřas) *(i) The spectrum of the standard Aharonov-Bohm Hamiltonian H_1^{AB} is given by*

$$\sigma(H_1^{AB}) = \{(2n-1)B; n = 1, 2, \ldots, \} \cup \{(2n + 2\alpha - 1)B; n = 1, 2, \ldots\}.$$

The multiplicity of each eigenvalue is given by

$$\begin{aligned} \mathrm{mult}((2n-1)B; H_1^{AB}) &= \infty, \ n = 1, 2, \ldots, \\ \mathrm{mult}((2n + 2\alpha - 1)B; H_1^{AB}) &= n, \ n = 1, 2, \ldots, \end{aligned}$$

where $\mathrm{mult}(\lambda; H)$ *is the multiplicity of an eigenvalue λ of a self-adjoint operator H.*

(ii) $L^2(\boldsymbol{R}^2)$ is decomposed into the direct sum of two closed subspaces $\mathcal{H}_s$ and $\mathcal{H}_c$, called the stable subspace and critical subspace, respectively. The spaces $\mathcal{H}_s$ and $\mathcal{H}_c$ are invariant subspaces for any self-adjoint extension H_1 of L_1. The restricted operator $H_1|_{\mathcal{H}_s}$ is independent of the choice of H_1 and the spectrum of $H_1|_{\mathcal{H}_s}$ is given by

$$\sigma\left(H_1|_{\mathcal{H}_s}\right) = \{(2n-1)B; n = 1, 2, \ldots\} \cup \{(2n + 2\alpha - 1)B; n = 2, 3, \ldots\}.$$

The multiplicity of each eigenvalue is given by

$$\begin{aligned}\mathrm{mult}((2n-1)B; H_1|_{\mathcal{H}_s}) &= \infty,\ n=1,2,\ldots,\\ \mathrm{mult}((2n+2\alpha-1)B; H_1|_{\mathcal{H}_s}) &= n-1,\ n=2,3,\ldots.\end{aligned}$$

The restricted operator $H_1|_{\mathcal{H}_c}$ *depends on the choice of self-adjoint extension* H_1*. However, the following estimates hold independently of the choice of* H_1*.*

$$\begin{aligned}\dim\mathrm{Ran}\, P_{(-\infty,(2\alpha-1)B)}(H_1|_{\mathcal{H}_c}) &\le 2,\\ \dim\mathrm{Ran}\, P_{((2n+2\alpha-1)B,(2n+1)B)}(H_1|_{\mathcal{H}_c}) &\le 2,\ n=0,1,2,\ldots\\ \dim\mathrm{Ran}\, P_{((2n-1)B,(2n+2\alpha-1)B)}(H_1|_{\mathcal{H}_c}) &\le 2,\ n=1,2,\ldots\end{aligned}$$

where $P_I(H)$ *denotes the spectral projection of a self-adjoint operator* H *corresponding to an interval* I*. The left-hand side of each of three inequalities above takes the values* $0,1,2$ *if we take an appropriate self-adjoint extension* H_1*.*

From Theorem 1.1, it follows that

$$n-1 \le \dim\mathrm{Ran}\, P_{((2n-1)B,(2n+1)B)}(H_1) \le n+3 \tag{1.8}$$

for $n=1,2,\ldots$, if $(2n+2\alpha-1)B$ is not an eigenvalue of $H_1|_{\mathcal{H}_c}$ (this condition holds for generic self-adjoint extension H_1). Later we show that the upper bound can be sharpened (see (1.12) below).

According to (i) of Theorem 1.1, there are n eigenstates of the Hamiltonian H_1^{AB} with the energy between n th Landau level and the $(n+1)$ st Landau level. We shall try to give a physical interpretation of this phenomenon.

In classical mechanics, an electron in a uniform magnetic field moves along a circle (cyclotron motion). The energy of an electron is quantized by the condition that the phase variation of the electron wave in one cyclotron rotation is 2π times an integer. Thus the energy of an electron takes one of the values in the Landau levels.

If some solenoids are contained in the circle of the cyclotron motion, then the phase of the electron wave is shifted by $e/\hbar$ times the magnetic flux of solenoids in the circle (the Aharonov-Bohm phase shift). Thus the energy of the electron is obliged to change, to correct the phase shift caused by the magnetic flux of solenoids. Hence the spectrum between Landau levels appears.

For this reason, the number of eigenstates with an energy between n th and $(n+1)$ st Landau level is roughly estimated by the possible number of electrons with the n th Landau level energy, in the circle of the Larmor radius centered at the position of solenoid. This number is calculated as follows. If we normalize physical constants as the mass $m = 1/2$, the Planck constant (divided by 2π) $\hbar = 1$ and the charge of an electron $e = 1$, then the cyclotron radius r of an electron with n th Landau level energy $(2n-1)B$ equals to $\sqrt{(2n-1)/B}$. It is known that the density of states (the number of eigenstates per unit area) for each Landau level is $B/2\pi$ (see [Nak, Proposition 15]). Thus, the number of possible eigenstates in the circle is

$$\pi r^2 \times \frac{B}{2\pi} = n - \frac{1}{2}.$$

The difference between this estimate and the rigorous result ((i) of Theorem 1.1) is only 1/2.

When $N \geq 2$, Nambu ([Nam]) gives a representation of eigenfunctions for the Landau levels by the multiple integral in the complex plane. But no information about the eigenvalues between the Landau levels are known. However, the physical explanation above gives us a conjecture about the number of eigenvalues in a gap of Landau levels, when $N \geq 2$. This number is roughly estimated by the number of eigenstates with the n th Landau energy in the union set, with respect to $j = 1, \dots, N$, of the disks of Larmor radius centered at z_j. Each disk contains n eigenstates with n th Landau energy. These disks may intersect in general, but they are disjoint if solenoids are far from each other. Thus we reach the following conjecture.

Conjecture *(I) The number of eigenvalues between n th and $(n+1)$ st Landau levels is bounded by nN.*

(II) If solenoids are far from each other compared with the cyclotron radius, the number of eigenvalues between n th and $(n+1)$ st Landau levels equals to nN.

Our aim in this paper is to give an answer to these conjectures. Our answer to the conjecture (I) is the following.

Theorem 1.2 *Let $1 \leq N < \infty$. Then, the following holds.*

(i) For any self-adjoint extension H_N of L_N, we have that $(2n-1)B$ is an infinitely degenerated eigenvalue for every $n = 1, 2, 3, \dots$.

(ii) For the standard Aharonov-Bohm Hamiltonian H_N^{AB}, we have

$$\begin{aligned} \dim \mathrm{Ran}\, P_{(-\infty, B)}(H_N^{AB}) &= 0, \\ \dim \mathrm{Ran}\, P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) &\leq nN, \text{ for } n = 1, 2, 3, \dots. \end{aligned} \tag{1.9}$$

(iii) For any self-adjoint extension H_N of L_N, we have

$$\dim \mathrm{Ran}\, P_{(-\infty, B)}(H_N) \leq 2N, \tag{1.10}$$

$$\dim \mathrm{Ran}\, P_{((2n-1)B,(2n+1)B)}(H_N) \leq (n+1)N, \text{ for } n = 1, 2, 3, \dots. \tag{1.11}$$

In the case of $N = 1$, our result and (1.8) imply that

$$n - 1 \leq \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_1) \leq n + 1 \tag{1.12}$$

for $n = 1, 2, \ldots$. The upper bound of (1.12) is sharper than that of (1.8) (however, [Ex-St-Vy, Fig 1,2] seems to indicate that there are at most two eigenvalues of $H_1|_{\mathcal{H}_c}$ in each gap of Landau levels).

Next, we shall exhibit our answer to the conjecture (II). In this paper, we shall consider the special case where the physical situations around every z_j are the same. To represent this situation rigorously, we shall prepare an operator which intertwines two magnetic Schrödinger operators.

Definition 1.1 *Let $w \in \boldsymbol{R}^2$. Let U be a simply connected open set, and $V = U + w = \{z + w; z \in U\}$. Let S be an at most countable subset of U with no accumulation points in U and $T = S + w$. Let $\boldsymbol{a} \in C^\infty(U \setminus S; \boldsymbol{R}^2) \cap L^1_{loc}(U; \boldsymbol{R}^2)$ and $\boldsymbol{b} \in C^\infty(V \setminus T; \boldsymbol{R}^2) \cap L^1_{loc}(V; \boldsymbol{R}^2)$ be two vector potentials satisfying*

$$\operatorname{rot} \boldsymbol{a}(z) = \operatorname{rot} \boldsymbol{b}(z + w)$$

in $\mathcal{D}'(U)$. Then, there exists an operator t_{-w} from $\mathcal{D}'(V \setminus T)$ to $\mathcal{D}'(U \setminus S)$ satisfying the following (i) and (ii):

(i) There exists a complex-valued smooth function $\Phi(z) \in C^\infty(U \setminus S)$ with $|\Phi(z)| = 1$ for every $z \in U \setminus S$, such that

$$t_{-w}v(z) = \Phi(z)v(z + w), \; v \in \mathcal{D}'(V \setminus T).$$

(ii) The following distributional equality holds:

$$\mathbf{p}(\boldsymbol{a})t_{-w}v = t_{-w}\mathbf{p}(\boldsymbol{b})v, \; \mathcal{L}(\boldsymbol{a})t_{-w}v = t_{-w}\mathcal{L}(\boldsymbol{b})v \tag{1.13}$$

for $v \in \mathcal{D}'(V \setminus T)$, where

$$\mathbf{p}(\boldsymbol{a}) = \frac{1}{i}\nabla + \boldsymbol{a}, \qquad \mathbf{p}(\boldsymbol{b}) = \frac{1}{i}\nabla + \boldsymbol{b},$$
$$\mathcal{L}(\boldsymbol{a}) = \mathbf{p}(\boldsymbol{a})^2, \qquad \mathcal{L}(\boldsymbol{b}) = \mathbf{p}(\boldsymbol{a})^2.$$

We call the operator t_{-w} the magnetic translation operator from V to U intertwining $\mathcal{L}(\boldsymbol{b})$ with $\mathcal{L}(\boldsymbol{a})$. We denote the inverse operator of t_{-w} by t_w, that is,

$$t_w u(z) = \Phi(z - w)u(z - w)$$

for $u \in \mathcal{D}'(U \setminus S)$.

We call the equality (1.13) the intertwining property of t_{-w}. The existence of the function Φ will be proved in Proposition 2.4 below.

Definition 1.2 *Let H_N be a self-adjoint extension of L_N. We say the operator H_N has the same boundary condition at every z_j, if the following two conditions hold:*

(i) There exists a constant α with $0 < \alpha < 1$ such that $\alpha_j = \alpha$ for every $j = 1, \ldots, N$.

(ii) Let t_{-z_j} be the magnetic translation operator from $\{|z - z_j| < \frac{R}{2}\}$ to $\{|z| < \frac{R}{2}\}$ intertwining $\mathcal{L}_N$ with $\mathcal{L}_1^{\alpha_j}$. Let $\chi \in C_0^\infty(\boldsymbol{R}^2)$ be a function satisfying $0 \le \chi \le 1$ on $\boldsymbol{R}^2$, $\chi = 0$ in $|z| > \frac{R}{2}$ and $\chi = 1$ in $|z| < \frac{R}{3}$. Put $\chi_j(z) = \chi(z - z_j)$. There exists a self-adjoint extension H_1 of L_1 independent of j such that

$$D(H_N) = \left\{u \in D(L_N^*); t_{-z_j}(\chi_j u) \in D(H_1) \text{ for every } j = 1, \ldots, N\right\}. \tag{1.14}$$

Here, L_N^* is the adjoint operator of L_N.

Remark 1. The right hand side of (1.14) is independent of the choice of the function χ (this fact follows from (ii) of Proposition 4.2 below).
Remark 2. There exists a self-adjoint extension of L_N satisfying (1.14) for any given self-adjoint extension H_1 of L_1 (see Lemma 8.1).

Our answer to the conjecture (II) is the following.

Theorem 1.3 *Let $1 \le N < \infty$ or $N = \infty$. Let H_N be a self-adjoint extension of L_N which has the same boundary condition at every z_j. Let $I = [c, d]$ be a closed interval satisfying that $I \cap \{(2n-1)B; n = 1, 2, \ldots\} = \emptyset$, that $c, d \notin \sigma(H_1)$ and that $\sigma(H_1) \cap I = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \neq \emptyset$.*

Then, there exist constants $u > 0$ and $R_0 > 0$ dependent on B, α, I, H_1 (independent of N, R) satisfying the following:

(i) If $R \ge R_0$, we have

$$\sigma(H_N) \cap I \subset \bigcup_{l=1}^{k} [\lambda_l - \delta, \lambda_l + \delta],$$

where $\delta = e^{-uR^2}$.

(ii) If $R \ge R_0$, we have

$$\dim \operatorname{Ran} P_I(H_N) = N \dim \operatorname{Ran} P_I(H_1).$$

Note that $\sigma(H_1) \cap I$ is a finite set by (iii) of Theorem 1.2.

Combining (ii) of Theorem 1.3 with Theorem 1.1, we have the following corollary.

Corollary 1.4 *Let* $1 \leq N < \infty$ *and let* $\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha$*. Then, for every* $n_0 = 1, 2, \ldots$*, there exists a constant* $R_0 > 0$ *dependent on* B*,* α*,* n_0 *(independent of* N*,* R*) satisfying the following: If* $R \geq R_0$*, then there exist self-adjoint extensions* $H_N^0, H_N^1, \ldots, H_N^{n_0}$ *of* L_N *such that*

$$\begin{aligned}
\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) &= nN, \\
\dim \operatorname{Ran} P_{(-\infty,B)}(H_N^0) &= 2N, \\
\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^n) &= (n+1)N
\end{aligned}$$

for $n = 1, 2, \ldots, n_0$*.*

We shall shortly exhibit the outline of the proof of our theorems.

In the proof of Theorem 1.2, the canonical commutation relation (CCR) of the annihilation operator A_N and the creation operator $A_N^\dagger$ plays a crucial role (for the definition of these operators, see section 6 below). It is well-known that the spectrum of the Schrödinger operators with constant magnetic fields are completely determined by CCR. In our case, CCR holds with a perturbation by δ-like magnetic fields. This perturbation makes two self-adjoint operators $(A_N^\dagger)^* \overline{A_N^\dagger} - B$ and $\overline{A_N^\dagger}(A_N^\dagger)^* + B$ different (the overline denotes the operator closure; notice that the note * denotes the operator adjoint, while the note † denotes only the formal adjoint). Comparing the spectrum of these two operators, we can reach the conclusion of Theorem 1.2. Note that Iwatsuka ([Iw]) uses the argument of this type, to determine the essential spectrum of the Schrödinger operators on $\boldsymbol{R}^2$ with the magnetic fields converging to a non-zero constant at infinity.

Theorem 1.3 is an analogy of the result of Cornean and Nenciu ([Co-Ne, Theorem III.1, Corollary III.1]). They treat the case

$$\begin{aligned}
\operatorname{rot} \boldsymbol{a}_N(z) &= B + \sum_{j=1}^{N} \operatorname{rot} \boldsymbol{a}_0(z - z_j), \ \ \boldsymbol{a}_0 \in C_0^\infty(\{|z| < 1\}; \boldsymbol{R}^2), \\
V_N(z) &= \sum_{j=1}^{N} V_0(z - z_j), \ \ V_0 \in C_0^\infty(\{|z| < 1\}; \boldsymbol{R}),
\end{aligned}$$

and obtain the same conclusion as that of Theorem 1.3, for the operator $(\frac{1}{i}\nabla - \boldsymbol{a}_N)^2 + V_N$. The proof of Theorem 1.3 is similar to that of their result. The main difference is that our operators H_N $(N < \infty)$ and H_∞ do not have the same core in general, while $C_0^\infty(\boldsymbol{R}^2)$ is the common core for the Schrödinger operators with smooth vector potentials (see [Ik-Ka, Theorem 1] or [Le-Si, Theorem 2]). Thus we do not use the approximating argument

$N \to \infty$, which is used in their paper. We prove the statement of Theorem 1.3 directly even when $N = \infty$, using the argument of approximating eigenfunctions.

We have organized the present paper as follows. In section 2, we recall the construction of the magnetic vector potentials and magnetic translations by Arai ([Ar1], [Ar2]). In section 3, we review the result of Exner, Šťovíček and Vytřas ([Ex-St-Vy]) for single solenoid operators and define some operators and basis functions useful for later calculus. In section 4, we determine the deficiency indices of the operator L_N. In section 5, we characterize all self-adjoint extensions of L_N. In section 6, we define the annihilation operator A_N and the creation operator $A_N^\dagger$ and determine the domain of the linear operator $\overline{A_N^\dagger(A_N^\dagger)^*} + B$ and that of $\overline{(A_N^\dagger)^* A_N^\dagger} - B$. In section 7, we prove Theorem 1.2. In section 8, we prove Theorem 1.3.

Remark. In the sequel, we sometimes identify the 2-fold vector $z = (x, y) \in \boldsymbol{R}^2$ with the complex number $z = x + iy \in \boldsymbol{C}$. This identification simplifies our calculus in many cases.

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2 Vector potentials

In this section, shall construct vector potentials whose rotation has δ-like singularities. Most of these are the result of Arai ([Ar1], [Ar2]); we shall give a proof for the completeness of our paper.

Proposition 2.1 *Let U be an open set in $\boldsymbol{R}^2$. Let $N = 1, 2, \ldots$ or $N = \infty$ and $\{z_j\}_{j=1}^N$ be (at most) countable points in U with no accumulation points in U. Put $S_N = \cup_{j=1}^N \{z_j\}$. Let $\{\alpha_j\}_{j=1}^N$ be an arbitrary sequence of real numbers. Then, there exists a vector potential $\boldsymbol{a} \in C^\infty(U \setminus S_N; \boldsymbol{R}^2) \cap L^1_{loc}(U; \boldsymbol{R}^2)$ satisfying the equality*

$$\operatorname{rot} \boldsymbol{a}(z) = \sum_{j=1}^N 2\pi\alpha_j \delta(z - z_j) \text{ in } \mathcal{D}'(U). \tag{2.1}$$

Thus we can construct a vector potential $\boldsymbol{a}_N$ satisfying (1.2) by putting $\boldsymbol{a}_N = \boldsymbol{a}_0 + \boldsymbol{a}$, where $\boldsymbol{a}_0$ is the vector potential given by (1.5) and $\boldsymbol{a}$ is the vector potential given in Proposition 2.1.

Proof. By the Mittag-Leffler theorem (see [Ru1, Theorem 13.10]), there exists a meromorphic function $\phi(z)$ whose principal part at each z_j is $\frac{\alpha_j}{z-z_j}$ and which has no other poles in $U \setminus S_N$. Put

$$\boldsymbol{a}(z) = (\operatorname{Im}\phi(z), \operatorname{Re}\phi(z)).$$

If $z \in U \setminus S_N$, we have $\operatorname{rot}\boldsymbol{a} = \partial_x \operatorname{Re}\phi - \partial_y \operatorname{Im}\phi = 0$ by the Cauchy-Riemann relation. If z is in a small neighborhood U_j of z_j, we have

$$\phi(z) = \frac{\alpha_j}{z - z_j} + \phi_1(z),$$

where ϕ_1 is a holomorphic function in U_j. Since $\operatorname{rot}(\operatorname{Im}\phi_1, \operatorname{Re}\phi_1) = 0$ by the Cauchy-Riemann relation, we have

$$\begin{aligned}
\operatorname{rot}\boldsymbol{a}(z) &= \partial_x\left(\operatorname{Re}\frac{\alpha_j}{z-z_j}\right) - \partial_y\left(\operatorname{Im}\frac{\alpha_j}{z-z_j}\right) \\
&= \alpha_j(\partial_x^2 \log|z-z_j| + \partial_y^2 \log|z-z_j|) \\
&= 2\pi\alpha_j\delta(z-z_j)
\end{aligned}$$

in $\mathcal{D}'(U_j)$. Here we used the distributional equality $\Delta \log|z| = 2\pi\delta(z)$ in the last equality. □

The meromorphic function $\phi(z)$ in the proof above can be constructed explicitly in some cases. When N is finite, one can take

$$\phi(z) = \sum_{j=1}^{N} \frac{\alpha_j}{z - z_j}.$$

When $N = \infty$ and the condition

$$z_1 = 0,\ R = \inf_{j \neq k}|z_j - z_k| > 0,\ 0 < \alpha_j < 1 \ for\ j = 1, 2, \ldots$$

holds, one can take

$$\phi(z) = \frac{\alpha_1}{z} + \sum_{j=2}^{\infty}\alpha_j\left(\frac{1}{z-z_j} + \frac{1}{z_j} + \frac{z}{z_j^2}\right) = \frac{\alpha_1}{z} + \sum_{j=2}^{\infty}\alpha_j\frac{z^2}{(z-z_j)z_j^2}.$$

The sum converges absolutely, uniformly in a compact set K in $\boldsymbol{R}^2 \setminus S_\infty$, since the absolute value of the summand is $O(|z_j|^{-3})$ uniformly with respect to $z \in K$. In particular, when S_∞ is a lattice and every α_j takes the same value α, the function ϕ equals to α times the Weierstrass ζ function.

In the previous lemma, we take a special gauge for given magnetic fields. The following lemma suggests that the choice of vector potentials does not affect our spectral problem.

Lemma 2.2 *Let U be a simply connected open set in $\boldsymbol{R}^2$. Let $N = 0, 1, 2, \dots$ or $N = \infty$. Let $\{z_j\}_{j=1}^N$ are at most countable points in U with no accumulation points in U. Put $S_N = \cup_{j=1}^N \{z_j\}$. Let $\boldsymbol{a}, \boldsymbol{b} \in C^\infty(U \setminus S_N; \boldsymbol{R}^2) \cap L^1_{loc}(U; \boldsymbol{R}^2)$ and assume that*

$$\mathrm{rot}\, \boldsymbol{a} = \mathrm{rot}\, \boldsymbol{b} \text{ in } \mathcal{D}'(U).$$

Then, there exists a multiplication operator by a complex-valued smooth function $\Phi \in C^\infty(\boldsymbol{R}^2 \setminus S_N)$ with $|\Phi(z)| = 1$ for every $z \in \boldsymbol{R}^2 \setminus S_N$, such that the equality

$$\Phi^{-1}(\nabla + i\boldsymbol{b})\Phi = \nabla + i\boldsymbol{a} \tag{2.2}$$

holds as an operator from $\mathcal{D}'(U \setminus S_N)$ to $(\mathcal{D}'(U \setminus S_N))^2$.

Remark that we cannot apply the result by Leinfelder ([Le]), since $\boldsymbol{a}$ does not belong to $L^2_{loc}(U)$ in our situation.

Proof. Replacing $\boldsymbol{a}$ by $\boldsymbol{a} - \boldsymbol{b}$ if necessary, we may assume $\boldsymbol{b} = 0$. Then $\boldsymbol{a} = (a_x, a_y)$ satisfies

$$\mathrm{rot}\, \boldsymbol{a} = \partial_x a_y - \partial_y a_x = 0 \text{ in } \mathcal{D}'(U). \tag{2.3}$$

Fix a point $z_0 \in U \setminus S_N$. For $z \in U \setminus S_N$, define

$$\phi(z) = \int_{\gamma_{z_0,z}} (a_x dx + a_y dy),$$

where $\gamma_{z_0,z}$ is a piecewise linear path from z_0 to z not crossing the points in S_N.

We shall see that $\phi(z)$ is independent of the choice of paths $\gamma_{z_0,z}$. Take another piecewise linear path $\tilde{\gamma}_{z_0,z}$ from z_0 to z. We may assume that two paths cross only at z_0 and z (if not, divide paths by their crossing points and consider each divided paths). Since U is simply connected, the union of two paths is the boundary of an open set Ω contained in U. If Ω does not contain points in S_N, then rot $\boldsymbol{a}$ is smooth in Ω and we have

$$\left(\int_{\gamma_{z_0,z}} - \int_{\tilde{\gamma}_{z_0,z}} \right) (a_x dx + a_y dy) = \pm \int_\Omega \mathrm{rot}\, \boldsymbol{a} dx dy, \tag{2.4}$$

by the Stokes theorem (the sign ± is determined by the direction of paths $\gamma_{z_0,z} \cup (-\tilde{\gamma}_{z_0,z})$). By (2.3), we have

$$\int_{\gamma_{z_0,z}} (a_x dx + a_y dy) = \int_{\tilde{\gamma}_{z_0,z}} (a_x dx + a_y dy). \tag{2.5}$$

We shall show that (2.5) holds even if Ω contains points in S_N. To see this, take $\chi \in C_0^\infty(\{|z| < 1\})$ such that $\int \chi = 1$ and put

$$\chi_\epsilon(z) = \chi\left(\frac{z}{\epsilon}\right), \ \boldsymbol{a}_\epsilon(z) = \int_{\Omega_\epsilon} \chi_\epsilon(z - \tilde{z}) \boldsymbol{a}(\tilde{z}) d\tilde{x} d\tilde{y}$$

for sufficiently small $\epsilon > 0$, where $\Omega_\epsilon = \{\tilde{z} = (\tilde{x}, \tilde{y}); \mathrm{dist}(\tilde{z}, \Omega) < \epsilon\}$. Since $\boldsymbol{a} \in L^1_{loc}(U; \boldsymbol{R}^2)$, we see that the integral above converges for $z \in U$, that $\boldsymbol{a}_\epsilon \in C_0^\infty(\Omega_{2\epsilon})$ and that $\boldsymbol{a}_\epsilon(z) \to \boldsymbol{a}(z)$ uniformly on any compact subset in $\overline{\Omega} \setminus S_N$ as $\epsilon \to 0$. Moreover,

$$\begin{aligned} & \mathrm{rot}\, \boldsymbol{a}_\epsilon(z) \\ = \ & \int_{\Omega_\epsilon} \left(\partial_x \left(\chi_\epsilon(z - \tilde{z})\right) a_y(\tilde{z}) - \partial_y \left(\chi_\epsilon(z - \tilde{z})\right) a_x(\tilde{z})\right) d\tilde{x} d\tilde{y} \\ = \ & - \int_U \chi_\epsilon(z - \tilde{z})\, \mathrm{rot}\, \boldsymbol{a}(\tilde{z}) d\tilde{x} d\tilde{y} = 0 \end{aligned}$$

for $z \in \overline{\Omega}$, where we used (2.3) and the fact $\chi_\epsilon(z - \cdot) \in C_0^\infty(U)$ for $z \in \overline{\Omega}$ and sufficiently small $\epsilon > 0$. Thus (2.4) and (2.5) holds if $\boldsymbol{a}$ is replaced by $\boldsymbol{a}_\epsilon = (a_{\epsilon,x}, a_{\epsilon,y})$. Taking the limit $\epsilon \to 0$, we have (2.5) for any two paths $\gamma_{z_0,z}$ and $\tilde{\gamma}_{z_0,z}$. Thus $\phi(z)$ is well-defined.

It is easy to check $\nabla \phi = \boldsymbol{a}$ in $U \setminus S_N$. Put $\Phi(z) = e^{i\phi(z)}$, then we have (2.2) for $\boldsymbol{b} = 0$. This completes the proof. □

When $N = 1$, it is known that the integral difference of α_1 can be gauged out by an appropriate gauge transform. This fact holds true when $1 \leq N \leq \infty$.

Lemma 2.3 *In addition to the assumption of Proposition 2.1, assume that all α_j's are integers. Let $\boldsymbol{a}$ be the vector potential constructed in Proposition 2.1. Then, there exists a multiplication operator by a complex-valued smooth function $\Phi(z) \in C^\infty(\Omega \setminus S_N)$ with $|\Phi(z)| = 1$, satisfying the equality*

$$\Phi(z)^{-1} \nabla \Phi(z) = \nabla + i\boldsymbol{a} \tag{2.6}$$

as an operator from $\mathcal{D}'(U \setminus S_N)$ to $(\mathcal{D}'(U \setminus S_N))^2$.

Proof. Assume all α_j's are integers. Put $\psi(z) = \int^z \phi(z)$ and $\Phi(z) = \exp(i \operatorname{Im} \psi(z))$, where ϕ is a meromorphic function appeared in the proof of Proposition 2.1. Then $\Phi(z)$ is a single-valued smooth function on $U \setminus S_N$ with $|\Phi(z)| = 1$, since

$$\oint_{C_j} \phi(z) = 2\pi i \alpha_j \in 2\pi i \boldsymbol{Z}$$

for any small circular path C_j going around z_j counterclockwise. Then we have

$$\begin{aligned} & \Phi(z)^{-1} \nabla \Phi(z) \\ = \ & \nabla + i(\partial_x \operatorname{Im} \psi(z), \partial_y \operatorname{Im} \psi(z)) \\ = \ & \nabla + i(\operatorname{Im} \partial_x \psi(z), \operatorname{Re} \partial_x \psi(z)) \\ = \ & \nabla + i(\operatorname{Im} \phi(z), \operatorname{Re} \phi(z)) = \nabla + i\boldsymbol{a}(z), \end{aligned}$$

where we used the Cauchy-Riemann relation. □

The existence of the magnetic translation operator defined in Definition 1.1 is guaranteed by the following more general statement.

Proposition 2.4 *Let U and V are open sets in $\boldsymbol{R}^2$ satisfying $V = U + w = \{z + w; z \in U\}$ for some constant vector $w \in \boldsymbol{R}^2$. Let $N = 0, 1, 2, \ldots$ or $N = \infty$, $\{z_j\}_{j=1}^N$ are (at most) countable points in U with no accumulation points in U, $S_N = \cup_{j=1}^N \{z_j\}$ and $T_N = S_N + w$ ($S_N = T_N = \emptyset$ when $N = 0$). Let $\boldsymbol{a} \in C^\infty(U \setminus S_N; \boldsymbol{R}^2) \cap L^1_{loc}(U; \boldsymbol{R}^2)$ and $\boldsymbol{b} \in C^\infty(V \setminus T_N; \boldsymbol{R}^2) \cap L^1_{loc}(V; \boldsymbol{R}^2)$. Assume that*

$$\operatorname{rot} \boldsymbol{b}(z + w) = \operatorname{rot} \boldsymbol{a}(z) + \sum_{j=1}^N 2\pi \alpha_j \delta(z - z_j) \text{ in } \mathcal{D}'(U) \tag{2.7}$$

and that all α_j's are integers. Then there exists an operator t_{-w} from $\mathcal{D}'(V \setminus T_N)$ to $\mathcal{D}'(U \setminus S_N)$ satisfying the following conditions:

(i) There exists $\Phi \in C^\infty(U \setminus S_N)$ with $|\Phi(z)| = 1$ for every $z \in U \setminus S_N$ such that

$$t_{-w} u(z) = \Phi(z) u(z + w)$$

for every $u \in \mathcal{D}'(V \setminus T_N)$.

(ii) The intertwining property

$$\left(\frac{1}{i}\nabla + \boldsymbol{a}\right)^2 t_{-w} u = t_{-w} \left(\frac{1}{i}\nabla + \boldsymbol{b}\right)^2 u \text{ in } \mathcal{D}'(U \setminus S_N) \tag{2.8}$$

holds for any $u \in D'(V \setminus T_N)$.

Proof. Combining Lemma 2.2 with Lemma 2.3, we can prove that there exists $\Phi \in C^\infty(U \setminus S_N)$ with $|\Phi| = 1$ satisfying

$$\Phi(z)^{-1}\nabla\Phi(z) = \nabla + i\boldsymbol{b}(z+w) - i\boldsymbol{a}(z) \text{ in } \mathcal{D}'(U \setminus S_N).$$

Thus we obtain

$$\begin{aligned} \Phi(z)^{-1}(\nabla + i\boldsymbol{a}(z))\Phi(z) &= \nabla + i\boldsymbol{b}(z+w), \\ \Phi(z)^{-1}(\nabla + i\boldsymbol{a}(z))^2\Phi(z) &= (\nabla + i\boldsymbol{b}(z+w))^2. \end{aligned} \tag{2.9}$$

Put $t_{-w}v(z) = \Phi(z)v(z+w)$ for $v \in \mathcal{D}'(V \setminus T_N)$. One can easily verify that (2.8) holds by a short computation using (2.9). $\square$

3 Single solenoid operator

In this section, we shall give some properties of the single solenoid operator H_1, some of which are given in [Ex-St-Vy].

We shall introduce some operators and functions convenient for later calculus.

Proposition 3.1 *(i) Define differential operators* $\Pi_x, \Pi_y, \mathcal{A}_1, \mathcal{A}_1^\dagger$ *from* $\mathcal{D}'(\boldsymbol{R}^2 \setminus \{0\})$ *to* $\mathcal{D}'(\boldsymbol{R}^2 \setminus \{0\})$ *by*

$$\begin{aligned} \Pi_x &= \frac{1}{i}\partial_x - \frac{B}{2}y - \frac{\alpha}{|z|^2}y, \\ \Pi_y &= \frac{1}{i}\partial_y + \frac{B}{2}x + \frac{\alpha}{|z|^2}x, \\ \mathcal{A}_1 &= i\Pi_x + \Pi_y = 2\partial_z + \frac{B}{2}\bar{z} + \frac{\alpha}{z} \\ \mathcal{A}_1^\dagger &= -i\Pi_x + \Pi_y = -2\partial_{\bar{z}} + \frac{B}{2}z + \frac{\alpha}{\bar{z}}, \end{aligned}$$

where

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Then, the following relation holds:

$$\begin{aligned} &\mathcal{A}_1^\dagger\mathcal{A}_1 = \mathcal{L}_1 - B, && \mathcal{A}_1\mathcal{A}_1^\dagger = \mathcal{L}_1 + B, \\ &(\mathcal{A}_1 u, v) = (u, \mathcal{A}_1^\dagger v) && \text{for any } u, v \in C_0^\infty(\boldsymbol{R}^2 \setminus \{0\}). \end{aligned}$$

(ii) Define smooth functions $\{\phi_{m,n}\}$, $\{\psi_{m,n}\}$ $(m, n \in \boldsymbol{Z})$ *on* $\boldsymbol{R}^2 \setminus \{0\}$ *by*

$$\begin{aligned} \phi_{m,n}(z) &= |z|^{\alpha} z^m \bar{z}^n e^{-\frac{B}{4}|z|^2}, \\ \psi_{m,n}(z) &= |z|^{-\alpha} z^m \bar{z}^n e^{-\frac{B}{4}|z|^2}. \end{aligned}$$

Then the following relation holds for $m, n \in \boldsymbol{Z}$*:*

$$\begin{aligned} \mathcal{A}_1 \phi_{m,n} &= 2(\alpha + m)\phi_{m-1,n}, \\ \mathcal{A}_1^{\dagger} \phi_{m,n} &= -2n\phi_{m,n-1} + B\phi_{m+1,n}, \\ \mathcal{A}_1 \psi_{m,n} &= 2m\psi_{m-1,n}, \\ \mathcal{A}_1^{\dagger} \psi_{m,n} &= 2(\alpha - n)\psi_{m,n-1} + B\psi_{m+1,n}, \\ \mathcal{L}_1 \phi_{m,0} &= (2\alpha + 2m + 1)B\phi_{m,0}, \\ \mathcal{L}_1 \psi_{0,n} &= B\psi_{0,n}. \end{aligned}$$

In particular, the four functions

$$\phi_{-1} = \phi_{-1,0},\ \psi_1 = \psi_{0,1},\ \phi_0 = \phi_{0,0},\ \psi_0 = \psi_{0,0} \tag{3.1}$$

satisfy

$$\begin{aligned} \mathcal{L}_1 \phi_{-1} = (2\alpha - 1)B\phi_{-1}, &\qquad \mathcal{L}_1 \psi_1 = B\psi_1, \\ \mathcal{L}_1 \phi_0 = (2\alpha + 1)B\phi_0, &\qquad \mathcal{L}_1 \psi_0 = B\psi_0. \end{aligned} \tag{3.2}$$

Proof. We can prove statements above by the following formulas.

$$\begin{aligned} [\Pi_x, \Pi_y] = \frac{B}{i}, &\qquad \mathcal{L}_1 = \Pi_x^2 + \Pi_y^2 \\ \partial_z z^m = \frac{m}{z} z^m,\ \partial_{\bar{z}} z^m = 0, &\qquad \partial_z \bar{z}^n = 0,\ \partial_{\bar{z}} \bar{z}^n = \frac{n}{\bar{z}} \bar{z}^n, \\ \partial_z (|z|^{\pm\alpha}) = \frac{\pm\alpha}{2z} |z|^{\pm\alpha}, &\qquad \partial_{\bar{z}} (|z|^{\pm\alpha}) = \frac{\pm\alpha}{2\bar{z}} |z|^{\pm\alpha}, \\ \partial_z e^{-\frac{B}{4}|z|^2} = -\frac{B}{4} \bar{z} e^{-\frac{B}{4}|z|^2}, &\qquad \partial_{\bar{z}} e^{-\frac{B}{4}|z|^2} = -\frac{B}{4} z e^{-\frac{B}{4}|z|^2}. \end{aligned}$$

□

Remark 1. In the polar coordinate $z = re^{i\theta}$,

$$\begin{aligned} \phi_{m,n}(z) = r^{\alpha+m+n} e^{i(m-n)\theta} e^{-\frac{B}{4}r^2}, \\ \psi_{m,n}(z) = r^{-\alpha+m+n} e^{i(m-n)\theta} e^{-\frac{B}{4}r^2}. \end{aligned} \tag{3.3}$$

This expressions imply that the four functions ϕ_{-1}, ψ_1, ϕ_0, ψ_0 belong to $L^2(\boldsymbol{R}^2)$.
Remark 2. When we need to indicate the value α explicitly, we denote ϕ^α_{-1}, ψ^α_1, ϕ^α_0, ψ^α_0 for ϕ_{-1}, ψ_1, ϕ_0, ψ_0.

We quote a part of the result by Exner, Šťovíček and Vytřas ([Ex-St-Vy]) for later use. Notice that some notes used in [Ex-St-Vy] are changed.

Proposition 3.2 (Exner-Šťovíček-Vytřas) *(i) The domain of the adjoint operator* L_1^* *of* L_1 *is*

$$D(L_1^*) = \left\{u \in L^2(\boldsymbol{R}^2) \cap H^2_{loc}(\boldsymbol{R}^2 \setminus \{0\}); \mathcal{L}_1 u \in L^2(\boldsymbol{R}^2)\right\}.$$

(ii) The deficiency indices of L_1 *are* $(2, 2)$.

(iii) The four linear functionals

$$\begin{aligned}
\Phi_{-1}(u) &= \lim_{r\to+0} r^{1-\alpha}\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\boldsymbol{e}^{i\theta}d\theta, \\
\Psi_1(u) &= \lim_{r\to+0} r^{-1+\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\boldsymbol{e}^{i\theta}d\theta - \Phi_{-1}(u)r^{-1+\alpha}\right), \\
\Psi_0(u) &= \lim_{r\to+0} r^{\alpha}\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta, \\
\Phi_0(u) &= \lim_{r\to+0} r^{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta - \Psi_0(u)r^{-\alpha}\right)
\end{aligned}$$

are well-defined and finite for $u \in D(L_1^*)$.

(iv) Every $u \in D(L_1^*)$ *is uniquely decomposed as*

$$u = \Phi_{-1}(u)\phi_{-1} + \Psi_1(u)\psi_1 + \Phi_0(u)\phi_0 + \Psi_0(u)\psi_0 + \xi, \tag{3.4}$$

where $\xi \in D(\overline{L_1})$. *In particular, the element* $u \in D(L_1^*)$ *belongs to* $D(\overline{L_1})$ *if and only if*

$$\Phi_{-1}(u) = \Psi_1(u) = \Phi_0(u) = \Psi_0(u) = 0.$$

Remark 1. The four functions ϕ_{-1}, ψ_1, ϕ_0, ψ_0 are eigenfunctions of $\mathcal{L}_1$ by (3.2). Thus these functions belong to $D(L_1^*)$ by (i) of this proposition.
Remark 2. The decomposition (3.4) is a paraphrase of [Ex-St-Vy, page2158, line8], since the four functions have the asymptotics

$$\phi_{-1} \sim r^{\alpha-1}e^{-i\theta},\ \psi_1 \sim r^{1-\alpha}e^{-i\theta}, \phi_0 \sim r^{\alpha},\ \psi_0 \sim r^{-\alpha}$$

as $r \to 0$.

We shall prepare a lemma for the future use. In the sequel, the note $||u||$ is the L^2-norm $||u||^2 = \int |u|^2 dxdy$. We regard $D(L_1^*)$ as a Hilbert space equipped with the graph norm

$$||u||_1^2 = ||\mathcal{L}_1 u||^2 + ||u||^2,\ u \in D(L_1^*).$$

Lemma 3.3 *(i) The following equalities hold.*

$$\begin{aligned}
||\phi^\alpha_{-1}||_1^2 &= \pi\left\{(2\alpha-1)^2B^2+1\right\}\left(\frac{2}{B}\right)^{\alpha}\Gamma(\alpha),\\
||\psi^\alpha_{1}||_1^2 &= \pi(B^2+1)\left(\frac{2}{B}\right)^{2-\alpha}\Gamma(2-\alpha),\\
||\phi^\alpha_{0}||_1^2 &= \pi\left\{(2\alpha+1)^2B^2+1\right\}\left(\frac{2}{B}\right)^{1+\alpha}\Gamma(1+\alpha),\\
||\psi^\alpha_{0}||_1^2 &= \pi(B^2+1)\left(\frac{2}{B}\right)^{1-\alpha}\Gamma(1-\alpha),
\end{aligned}$$

where $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ *is the Gamma function.*

(ii) For $u, v \in D((L^\alpha_1)^*)$*, define*

$$[u,v]_1 = (\mathcal{L}^\alpha_1 u, v) - (u, \mathcal{L}^\alpha_1 v).$$

Then, the following equalities hold.

$$\begin{aligned}
&[\phi^\alpha_{-1}, \psi^\alpha_1]_1 = 4\pi(\alpha-1), [\phi^\alpha_0, \psi^\alpha_0]_1 = 4\pi\alpha,\\
&[\phi^\alpha_{-1}, \phi^\alpha_0]_1 = [\phi^\alpha_{-1}, \psi^\alpha_0]_1 = [\psi^\alpha_1, \phi^\alpha_0]_1 = [\psi^\alpha_1, \psi^\alpha_0]_1 = 0\\
&[\phi^\alpha_l, \phi^\alpha_l]_1 = [\psi^\alpha_n, \psi^\alpha_n]_1 = 0 \quad \text{for } l=-1,0,\ n=0,1.
\end{aligned}$$

(iii) The operators Φ^α_{-1}*,* Ψ^α_1*,* Φ^α_0*,* Ψ^α_0 *are bounded linear functionals on* $D(L^\alpha_1)^*$ *whose operator norm are estimated by*

$$\begin{aligned}
&||\Phi^\alpha_{-1}|| \le \frac{1}{4\pi(1-\alpha)}||\psi^\alpha_1||_1, \quad && ||\Psi^\alpha_1|| \le \frac{1}{4\pi(1-\alpha)}||\phi^\alpha_{-1}||_1,\\
&||\Phi^\alpha_0|| \le \frac{1}{4\pi\alpha}||\psi^\alpha_0||_1, && ||\Psi^\alpha_0|| \le \frac{1}{4\pi\alpha}||\phi^\alpha_0||_1.
\end{aligned}$$

Proof. (i) (ii) One can prove these equalities by a short calculation using (3.2) and (3.3).
(iii) Notice that

$$[u,v]_1 = (L_1^* u, v) - (u, \overline{L_1} v) = 0$$

for any $u \in D(L_1^*)$ and $v \in D(\overline{L_1})$. From this equality, (ii) of this lemma and (3.4), we have

$$\begin{aligned}
&\Phi^\alpha_{-1}(u) = \frac{1}{4\pi(1-\alpha)}[\psi^\alpha_1, u]_1, \quad && \Psi^\alpha_1(u) = \frac{1}{4\pi(\alpha-1)}[\phi^\alpha_{-1}, u]_1,\\
&\Phi^\alpha_0(u) = \frac{1}{-4\pi\alpha}[\psi^\alpha_0, u]_1, && \Psi^\alpha_0(u) = \frac{1}{4\pi\alpha}[\phi^\alpha_0, u]_1.
\end{aligned} \tag{3.5}$$

Moreover, we have by the Schwarz inequality

$$|[u,v]_1| \leq ||\mathcal{L}_1 u||\ ||v|| + ||u||\ ||\mathcal{L}_1 v|| \leq ||u||_1 ||v||_1 \tag{3.6}$$

for $u, v \in D(L_1)^*$. By (3.5) and (3.6), the conclusion holds. $\square$

4 Deficiency indices of L_N

In this section, we shall determine the deficiency indices of L_N. We shall introduce some differential operators convenient for later calculus.

Lemma 4.1 *Define differential operators* $\Pi_{N,x}, \Pi_{N,y}, \mathcal{A}_N, \mathcal{A}_N^\dagger$ *from* $\mathcal{D}'(\boldsymbol{R}^2 \backslash S_N)$ *to* $\mathcal{D}'(\boldsymbol{R}^2 \backslash S_N)$ *by*

$$\begin{aligned}
\Pi_{N,x} &= \frac{1}{i}\partial_x + a_{N,x}, \\
\Pi_{N,y} &= \frac{1}{i}\partial_y + a_{N,y}, \\
\mathcal{A}_N &= i\Pi_{N,x} + \Pi_{N,y} = 2\partial_z + \phi(z), \\
\mathcal{A}_N^\dagger &= -i\Pi_{N,x} + \Pi_{N,y} = -2\partial_{\bar{z}} + \overline{\phi(z)},
\end{aligned}$$

where $\phi(z) = ia_{N,x}(z) + a_{N,y}(z)$. *Let* t_{-z_j} *be the magnetic translation from* $\{|z - z_j| < \frac{R}{2}\}$ *to* $\{|z| < \frac{R}{2}\}$ *intertwining* $\mathcal{L}_N$ *with* $\mathcal{L}_1$. *Then the following equalities holds:*

$$\mathcal{A}_N^\dagger \mathcal{A}_N = \mathcal{L}_N - B,\ \mathcal{A}_N \mathcal{A}_N^\dagger = \mathcal{L}_N + B, \tag{4.1}$$

as operators from $\mathcal{D}'(\boldsymbol{R}^2 \setminus S_N)$ *to* $\mathcal{D}'(\boldsymbol{R}^2 \setminus S_N)$.

$$(\mathcal{A}_N \phi, \psi) = (\phi, \mathcal{A}_N^\dagger \psi) \tag{4.2}$$

for every $\phi, \psi \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$.

$$\mathcal{A}_1^{\alpha_j} t_{-z_j} = t_{-z_j} \mathcal{A}_N,\ \mathcal{A}_1^{\dagger,\alpha_j} t_{-z_j} = t_{-z_j} \mathcal{A}_N^\dagger \tag{4.3}$$

as operators from $\mathcal{D}'(\{|z - z_j| < \frac{R}{2}\})$ *to* $\mathcal{D}'(\{|z| < \frac{R}{2}\})$.

Proof. One can prove (4.1) by a short computation using the formula

$$[\Pi_{N,x}, \Pi_{N,y}] = \frac{B}{i},\ \mathcal{L}_N = \Pi_{N,x}^2 + \Pi_{N,y}^2.$$

The equality (4.2) follows from an integration by parts. Notice that the intertwining property

$$\left(\frac{1}{i}\nabla + \boldsymbol{a}_1^{\alpha_j}\right) t_{-z_j} = t_{-z_j} \left(\frac{1}{i}\nabla + \boldsymbol{a}_N\right)$$

implies

$$\Pi_x^{\alpha_j} t_{-z_j} = t_{-z_j} \Pi_{N,x},\ \Pi_y^{\alpha_j} t_{-z_j} = t_{-z_j} \Pi_{N,y}.$$

The equality (4.3) follows from these equalities. □

In the sequel, we denote

$$U(r) = \bigcup_{j=1}^{N} \{z \in \boldsymbol{R}^2; |z - z_j| < r\}.$$

Proposition 4.2 *Let $N = 1, 2, \ldots,$ or $N = \infty$. Then the following holds:*
(i) The operator domain of $D(L_N^*)$ is given by

$$D(L_N^*) = \{u \in L^2(\boldsymbol{R}^2) \cap H^2_{loc}(\boldsymbol{R}^2 \setminus S_N);\ \mathcal{L}_N u \in L^2(\boldsymbol{R}^2)\}.$$

(ii) Let $u \in D(L_N^*)$. Suppose that there exist a constant R_1 with $0 < R_1 < R$ such that $\operatorname{supp} u \subset \boldsymbol{R}^2 \setminus U(R_1)$. *Then, $u \in D(\overline{L_N})$.*

Proof. (i) By definition, $u \in D(L_N^*)$ if and only if there exists some $v \in L^2(\boldsymbol{R}^2)$ such that $(v, \phi) = (u, -(\nabla + i\boldsymbol{a}_N)^2\phi)$ for every $\phi \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$. By definition, this statement is equivalent to $v = -(\nabla + i\boldsymbol{a}_N)^2 u$ in $\mathcal{D}'(\boldsymbol{R}^2 \setminus S_N)$. Then we have $u \in H^2_{loc}(\boldsymbol{R}^2 \setminus S_N)$ by an elliptic estimate (see [Ag]).

(ii) Take $\eta \in C^\infty(\boldsymbol{R}^2)$ satisfying $\eta = 1$ on $\boldsymbol{R}^2 \setminus U(\frac{1}{3}R_1)$, $\eta = 0$ on $U(\frac{1}{4}R_1)$ and η, $\nabla\eta$ and $\Delta\eta$ are bounded functions. Put $\widetilde{\boldsymbol{a}_N} = \eta \boldsymbol{a}_N$. Then the operator

$$\widetilde{L_N} = (\frac{1}{i}\nabla + \widetilde{\boldsymbol{a}_N})^2,\ D(\widetilde{L_N}) = C_0^\infty(\boldsymbol{R}^2)$$

is essentially self-adjoint since $\widetilde{\boldsymbol{a}_N} \in C^\infty(\boldsymbol{R}^2)$ (see [Ik-Ka]). Thus we have

$$D(\overline{\widetilde{L_N}}) = \{u \in L^2(\boldsymbol{R}^2); (\nabla + i\widetilde{\boldsymbol{a}_N})^2 u \in L^2(\boldsymbol{R}^2)\}, \tag{4.4}$$

because the domain of the maximal operator coincides with that of the minimal operator. Since $\widetilde{L_N} u = \mathcal{L}_N u \in L^2(\boldsymbol{R}^2)$, we have $u \in D(\overline{\widetilde{L_N}})$ by (4.4).

By the essential self-adjointness of $\widetilde{L_N}$, we can take a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\boldsymbol{R}^2)$ which approximates u with respect to the graph norm of $\overline{\widetilde{L_N}}$. Take $\xi \in C^{\infty}(\boldsymbol{R}^2)$, satisfying that $\xi = 1$ on $\boldsymbol{R}^2 \setminus U(\frac{R_1}{2})$, that $\xi = 0$ on $U(\frac{R_1}{3})$ and that ξ, $\nabla\xi$, $\Delta\xi$ are bounded. Then we have $\{\xi u_n\}_{n=1}^{\infty} \subset D(L_N)$. It is sufficient to show that the sequence $\{\xi u_n\}$ approximates u with respect to the graph norm of L_N^*. Since

$$\overline{\widetilde{L_N}}(\xi\phi) = \xi\overline{\widetilde{L_N}}\phi - 2(\nabla\xi)(\nabla + i\widetilde{\boldsymbol{a}_N})\phi - (\Delta\xi)\phi$$

for any $\phi \in D(\overline{\widetilde{L_N}})$, we have

$$\begin{aligned} & ||L_N^*(\xi u_n - u)|| = ||\overline{\widetilde{L_N}}(\xi(u_n - u))|| \\ \leq \; & ||\xi\overline{\widetilde{L_N}}(u_n - u)|| + 2||(\nabla\xi)(\nabla + i\widetilde{\boldsymbol{a}_N})(u_n - u)|| + ||(\Delta\xi)(u_n - u)||. \end{aligned}$$

One can show that the last expression converges to 0 as $n \to \infty$ by using the inequality (which can be proved by an integration by parts)

$$||(\nabla + i\widetilde{\boldsymbol{a}_N})v||^2 \leq \frac{1}{2}\left(||(\nabla + i\widetilde{\boldsymbol{a}_N})^2 v||^2 + ||v||^2\right)$$

for $v \in D(\overline{\widetilde{L_N}})$ and the uniform boundedness of $|\nabla\xi|$, $|\Delta\xi|$. □

We shall prove the continuity of the cutting-off map. In the sequel, we regard $D(L_N^*)$ as a Hilbert space equipped with the graph norm

$$||u||_N^2 = ||\mathcal{L}_N u||^2 + ||u||^2, \; u \in D(L_N^*).$$

The bracket $[X, Y]$ denotes the commutator of two operators X, Y, that is, $[X, Y] = XY - YX$.

Lemma 4.3 *Let* $1 \leq N < \infty$ *or* $N = \infty$. *Let* $\eta \in C^{\infty}(\boldsymbol{R}^2)$ *and suppose* $\operatorname{supp}|\nabla\eta|$ *is a compact set in* $\boldsymbol{R}^2 \setminus S_N$. *Then, for any* $u \in D(L_N^*)$, *we have* $\eta u \in D(L_N^*)$ *and*

$$||[\mathcal{L}_N, \eta]u||^2 \leq C_0 \int_{\operatorname{supp}|\nabla\eta|} (|\mathcal{L}_N u|^2 + |u|^2)dxdy, \tag{4.5}$$

where $C_0 = 10\sup(|\nabla\eta|^2 + |\nabla(\partial_x\eta)|^2 + |\nabla(\partial_y\eta)|^2)$. *Moreover, we have*

$$||\eta u||_N^2 \leq C_1 \int_{\operatorname{supp}\eta} (|\mathcal{L}_N u|^2 + |u|^2)dxdy, \tag{4.6}$$

where $C_1 = 2(\sup|\eta|^2 + C_0)$.

Proof. Take $u \in D(L_N^*)$. By (i) of Proposition 4.2, we have $\mathcal{L}_N u \in L^2(\boldsymbol{R}^2)$ and $u \in H^2_{loc}(\boldsymbol{R}^2 \setminus S_N)$. Then $\eta u \in L^2(\boldsymbol{R}^2)$ and

$$\mathcal{L}_N(\eta u) = \eta \mathcal{L}_N u - 2(\nabla\eta)(\nabla + i\boldsymbol{a}_N)u - (\Delta\eta)u \in L^2(\boldsymbol{R}^2),$$

since $\operatorname{supp}\nabla\eta$ is compact. Thus $\eta u \in D(L_N^*)$ by (i) of Proposition 4.2.

Let $\xi \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N; \boldsymbol{R})$. The following equality is a little modified version of the equality given in [Co-Ne, Lemma II.2]:

$$||(\nabla + i\boldsymbol{a}_N)(\xi u)||^2 = -\operatorname{Re}(u, \xi^2(\nabla + i\boldsymbol{a}_N)^2 u) + (u, |\nabla\xi|^2 u). \tag{4.7}$$

We shall check that (4.7) holds in our case. By the Leibniz rule, we have

$$\begin{aligned} & \text{(l.h.s. of (4.7))} \\ = \ & ||\xi(\nabla + i\boldsymbol{a}_N)u||^2 + 2\operatorname{Re}((\nabla\xi)u, \xi(\nabla + i\boldsymbol{a}_N)u) + ||(\nabla\xi)u||^2. \end{aligned} \tag{4.8}$$

Since

$$\begin{aligned} \xi^2(\nabla + i\boldsymbol{a}_N)^2 &= [\xi^2, (\nabla + i\boldsymbol{a}_N)](\nabla + i\boldsymbol{a}_N) + (\nabla + i\boldsymbol{a}_N)\xi^2(\nabla + i\boldsymbol{a}_N) \\ &= -2\xi(\nabla\xi)(\nabla + i\boldsymbol{a}_N) + (\nabla + i\boldsymbol{a}_N)\xi^2(\nabla + i\boldsymbol{a}_N), \end{aligned}$$

we have

$$\begin{aligned} & \text{(r.h.s. of (4.7))} \\ = \ & 2\operatorname{Re}(u, \xi(\nabla\xi)(\nabla + i\boldsymbol{a}_N)u) - \operatorname{Re}(u, (\nabla + i\boldsymbol{a}_N)\xi^2(\nabla + i\boldsymbol{a}_N)u) + (u, |\nabla\xi|^2 u) \\ = \ & 2\operatorname{Re}(u, \xi(\nabla\xi)(\nabla + i\boldsymbol{a}_N)u) + \operatorname{Re}((\nabla + i\boldsymbol{a}_N)u, \xi^2(\nabla + i\boldsymbol{a}_N)u) + ||(\nabla\xi)u||^2, \end{aligned}$$

by partial integration (since $\xi \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$, the boundary term vanishes). The last expression coincides with (4.8). Thus (4.7) holds.

By (4.7) and the Schwarz inequality, we have

$$||(\nabla + i\boldsymbol{a}_N)(\xi u)||^2 \le \sup(|\xi|^2 + |\nabla\xi|^2) \int_{\operatorname{supp}\xi} (|\mathcal{L}_N u|^2 + |u|^2) dxdy$$

for any $\xi \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$. Applying this inequality to $\xi = \partial_x\eta, \partial_y\eta$, we have

$$||(\nabla + i\boldsymbol{a}_N)\{(\nabla\eta)u\}||^2 \le C \int_{\operatorname{supp}|\nabla\eta|} (|\mathcal{L}_N u|^2 + |u|^2) dxdy, \tag{4.9}$$

where $C = \sup(|\nabla\eta|^2 + |\nabla(\partial_x\eta)|^2 + |\nabla(\partial_y\eta)|^2)$. Since

$$[\mathcal{L}_N, \eta]u = -2(\nabla + i\boldsymbol{a}_N)\,((\nabla\eta)u) - (\Delta\eta)u,$$

we have by (4.9)

$$\begin{aligned} & ||[\mathcal{L}_N, \eta]u||^2 \\ \leq\ & 8||(\nabla + i\boldsymbol{a}_N)\,((\nabla\eta)u)\,||^2 + 2||(\Delta\eta)u||^2 \\ \leq\ & C_0 \int_{\mathrm{supp}\,|\nabla\eta|} (|\mathcal{L}_N u|^2 + |u|^2)dxdy. \end{aligned}$$

Thus (4.5) holds. Moreover, by this inequality we have

$$\begin{aligned} & ||\eta u||_N^2 = ||\mathcal{L}_N(\eta u)||^2 + ||\eta u||^2 \\ \leq\ & 2||[\mathcal{L}_N, \eta]u||^2 + 2||\eta\mathcal{L}_N u||^2 + ||\eta u||^2 \\ \leq\ & C_1 \int_{\mathrm{supp}\,\eta} (|\mathcal{L}_N u|^2 + |u|^2)dxdy. \end{aligned}$$

Thus (4.6) holds. □

The following lemma is useful to determine the deficiency indices of L_N. In the sequel, we denote $\overset{N}{\underset{j=1}{\oplus}} \mathcal{H}_j$ the orthogonal direct sum of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$ ($N = 1, 2, \ldots,$ or $N = \infty$). The norm of $\overset{N}{\underset{j=1}{\oplus}} \mathcal{H}_j$ is defined by

$$||(u_1, \ldots, u_N)||^2_{\oplus\mathcal{H}_j} = \sum_{j=1}^{N} ||u_j||^2_{\mathcal{H}_j},$$

where $u_j \in \mathcal{H}_j$ for $j = 1, \cdots, N$. In particular, when $\{\mathcal{H}_j\}_{j=1}^N$ are mutually orthogonal closed subspaces of a Hilbert space $\mathcal{H}$, we identify $\overset{N}{\underset{j=1}{\oplus}} \mathcal{H}_j$ with the closed subspace of $\mathcal{H}$ generated by $\{\mathcal{H}_j\}_{j=1}^N$.
Remark. When $N = \infty$, the note '$j = 1, \cdots, N$' above implies 'for every positive integer j'. We often use the notation of this type in the sequel, to simplify the statements; for example, the note '$u_1 + \cdots + u_N$' implies '$\sum_{j=1}^{\infty} u_j$', when $N = \infty$.

Lemma 4.4 *Let* $1 \leq N < \infty$ *or* $N = \infty$. *Let* $\chi \in C_0^\infty(\{|z| < \frac{R}{3}\})$ *with* $0 \leq \chi \leq 1$ *on* $\boldsymbol{R}^2$, $\chi = 1$ *on* $\{|z| < \frac{R}{4}\}$ *and put* $\chi_j(z) = \chi(z - z_j)$. *Let* t_{-z_j} *be the magnetic translation from* $\{|z - z_j| < \frac{R}{2}\}$ *to* $\{|z| < \frac{R}{2}\}$ *intertwining* $\mathcal{L}_N$ *with* $\mathcal{L}_1^{\alpha_j}$. *Then, the following holds.*

(i) The map

$$T_j : D(L_N^*) \ni u \mapsto t_{-z_j}(\chi_j u) \in D((L_1^{\alpha_j})^*) \tag{4.10}$$

is well-defined and continuous for each $j = 1, \dots, N$. Moreover, the map

$$T : D(L_N^*) \ni u \mapsto (T_1 u, \dots, T_N u) \in \bigoplus_{j=1}^{N} D((L_1^{\alpha_j})^*) \tag{4.11}$$

is well-defined and continuous, where $\oplus_{j=1}^N D((L_1^{\alpha_j})^*)$ is the orthogonal direct sum of Hilbert spaces $\{D((L_1^{\alpha_j})^*)\}_{j=1}^N$.

(ii) We have $T_j D(\overline{L_N}) \subset D(\overline{L_1^{\alpha_j}})$, for every $j = 1, \dots, N$.

(iii) The map

$$S : \bigoplus_{j=1}^{N} D((L_1^{\alpha_j})^*) \ni (u_1, \dots, u_N) \mapsto t_{z_1}(\chi u_1) + \ldots + t_{z_N}(\chi u_N) \in D(L_N^*) \tag{4.12}$$

is well-defined and continuous.

(iv) We have $S(\oplus_{j=1}^N D(\overline{L_1^{\alpha_j}})) \subset D(\overline{L_N})$.

(v) An element $u \in D(L_N^*)$ belongs to $D(\overline{L_N})$ if and only if

$$T_j u \in D(\overline{L_1^{\alpha_j}}) \text{ for every } j = 1, \dots, N. \tag{4.13}$$

(vi) For every $(u_1, \dots, u_N) \in \oplus_{j=1}^N D((L_1^{\alpha_j})^*)$, we have

$$(TS - Id)(u_1, \dots, u_N) \in \bigoplus_{j=1}^{N} D(\overline{L_1^{\alpha_j}}).$$

Proof. (i) Let $u \in D(L_N^*)$. Then $\chi_j u \in D(L_N^*)$ by Lemma 4.3. By the intertwining property and unitarity of t_{-z_j}, we have

$$\begin{aligned} ||\chi_j u||_N^2 &= ||\mathcal{L}_N(\chi_j u)||^2 + ||\chi_j u||^2 \\ &= ||\mathcal{L}_1^{\alpha_j} t_{-z_j}\chi_j u||^2 + ||t_{-z_j}\chi_j u||^2 = ||t_{-z_j}\chi_j u||_{1,\alpha_j}^2, \end{aligned} \tag{4.14}$$

where $||v||_{1,\alpha_j}^2 = ||\mathcal{L}_1^{\alpha_j} v||^2 + ||v||^2$ for $v \in D((L_1^{\alpha_j})^*)$. Thus $t_{-z_j}\chi_j u \in D((L_1^{\alpha_j})^*)$ and the map T_j is well-defined. The continuity of T_j follows from (4.14) and (4.6). Moreover, we have by (4.14) and (4.6)

$$\sum_{j=1}^{N} ||T_j u||_1^2 \le C_1 \sum_{j=1}^{N} \int_{\{|z-z_j|<\frac{R}{2}\}} (|\mathcal{L}_N u|^2 + |u|^2) dxdy \le C_1 ||u||_N^2 < \infty, \tag{4.15}$$

where C_1 is a constant dependent only on χ. We used that the balls $\left\{\{|z-z_j|<\frac{R}{2}\}\right\}_{j=1}^{N}$ are disjoint by the assumption (1.1). Thus the map T is well-defined and continuous.

(ii) Since T_j maps $D(L_N)=C_0^\infty(\boldsymbol{R}^2\setminus S_N)$ into $D(L_1^{\alpha_j})=C_0^\infty(\boldsymbol{R}^2\setminus\{0\})$, the assertion holds from the continuity of the map T_j.

(iii) Take $(u_1,\dots,u_N)\in\oplus_{j=1}^N D((L_1^{\alpha_j})^*)$. By the unitarity and the intertwining property of t_{z_j}, we have $t_{z_j}(\chi u_j)\in L^2(\boldsymbol{R}^2)$, $\mathcal{L}_N t_{z_j}(\chi u_j)=t_{z_j}\mathcal{L}_1(\chi u_j)\in L^2(\boldsymbol{R}^2)$ and

$$||t_{z_j}(\chi u_j)||_N^2=||\chi u_j||_{1,\alpha_j}^2. \tag{4.16}$$

Thus we have $t_{z_j}(\chi u_j)\in D(L_N^*)$ by (i) of Proposition 4.2. When N is finite, the continuity of the map follows from (4.16).

Consider the case $N=\infty$. Since $\left\{\mathrm{supp}(t_{z_j}(\chi u_j))\right\}_{j=1}^\infty$ are disjoint, we have that $\{t_{z_j}(\chi u_j)\}_{j=1}^\infty$ is a mutually orthogonal sequence in $D(L_N^*)$. Thus we have by (4.16) and (4.6)

$$\begin{aligned} ||\sum_{j=N_1}^{N_2} t_{z_j}(\chi u_j)||_N^2 &= \sum_{j=N_1}^{N_2}||t_{z_j}(\chi u_j)||_N^2 \\ = \sum_{j=N_1}^{N_2}||\chi u_j||_{1,\alpha_j}^2 &\le C_1\sum_{j=N_1}^{N_2}||u_j||_{1,\alpha_j}^2 \end{aligned} \tag{4.17}$$

for $1\le N_1<N_2<\infty$, where C_1 is a constant dependent only on χ. The last expression of (4.17) converges to 0 as N_1 and N_2 goes to infinity, since $(u_j)_{j=1}^\infty\in\oplus_{j=1}^\infty D((L_1^{\alpha_j})^*)$. Thus the convergence of the sum follows from the Cauchy criterion. Putting $N_1=1$ and taking the limit $N_2\to\infty$ in (4.17), we obtain the continuity statement.

(iv) The set

$$\begin{aligned} D_0 \;=\; &\{(u_1,\dots,u_n,0,0,\dots,0);\ n\le N,\ n<\infty, \\ &u_1,\dots,u_n\in C_0^\infty(\boldsymbol{R}^2\setminus\{0\})\} \end{aligned}$$

is dense in $\oplus_{j=1}^N D(\overline{L_1^{\alpha_j}})$ and $SD_0\subset C_0^\infty(\boldsymbol{R}^2\setminus S_N)=D(L_N)$. Thus the assertion follows from the continuity of S.

(v) By (ii), it is sufficient to show that $u\in D(L_N^*)$ and (4.13) implies $u\in D(\overline{L_N})$. Suppose an element u in $D(L_N^*)$ satisfies (4.13). By (iv), we have

$$STu=S(T_1u,\dots,T_Nu)\in D(\overline{L_N}).$$

Thus it is sufficient to show

$$v=u-STu\in D(\overline{L_N}). \tag{4.18}$$

By (i) and (iii), we have $v \in D(L_N^*)$. Moreover, we have

$$\operatorname{supp} v \subset \boldsymbol{R}^2 \setminus U\left(\frac{R}{4}\right)$$

by definition of T_j and S. Thus (4.18) follows from (ii) of Proposition 4.2.

(vi) By definition,

$$\operatorname{supp}\{(T_j S)(u_1, \dots, u_N) - u_j\} = \operatorname{supp}(\chi^2 - 1)u_j \subset \{|z| \geq \frac{R}{4}\}.$$

Thus the conclusion follows from (ii) of Proposition 4.2. □

We shall determine the deficiency indices of L_N.

Proposition 4.5 *Let $1 \leq N < \infty$ or $N = \infty$. Then, the deficiency indices of L_N are $(2N, 2N)$.*

Proof. Since the operator L_N is symmetric and positive, the deficiency indices $m_\pm = \dim \operatorname{Ker}(L_N^* \mp i)$ are equal (see [Re-Si, Corollary of Theorem X.1]). Since $D(L_N^*) = D(\overline{L_N}) \oplus \operatorname{Ker}(L_N^* - i) \oplus \operatorname{Ker}(L_N^* + i)$ (see [Re-Si, (b) of Lemma in page 138]), it is sufficient to show that

$$\dim D(L_N^*)/D(\overline{L_N}) = 4N, \tag{4.19}$$

where $D(L_N^*)/D(\overline{L_N})$ is the quotient Banach space equipped with the norm

$$|||[u]|||_N^2 = \inf\{||u + v||_N^2;\ v \in D(\overline{L_N})\}$$

($[u]$ is the equivalence class of u). Since $D(L_N^*)$ is a Hilbert space, we have

$$|||[u]|||_N = ||P_N u||_N$$

for $u \in D(L_N^*)$, where P_N denotes the orthogonal projection onto $D(\overline{L_N})^\perp = \operatorname{Ker}(L_N^* - i) \oplus \operatorname{Ker}(L_N^* + i)$.

Define a map $\tilde{T}$ from $D(L_N^*)/D(\overline{L_N})$ to $\oplus_{j=1}^N D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$ by

$$\tilde{T}([u]) = \bigoplus_{j=1}^{N} [T_j u]$$

for $u \in D(L_N^*)$, where T_j is the operator defined in Lemma 4.4. We see that the vectors $\{[T_j u]\}_{j=1}^N$ are independent of the choice of a representative u in a equivalence class $[u]$, by

(ii) of Lemma 4.4. Since $u - P_N u \in D(\overline{L_N})$, we have $[u] = [P_N u]$ and thus $[T_j u] = [T_j P_N u]$. Thus we have by (4.15)

$$\sum_{j=1}^{N} |||[T_j u]|||_{1,\alpha_j}^2 = \sum_{j=1}^{N} |||[T_j P_N u]|||_{1,\alpha_j}^2 \leq C_1 ||P_N u||_N^2 = C_1 |||[u]|||_N^2.$$

Thus $\widetilde{T}$ is well-defined and continuous. Moreover, we have that $\widetilde{T}$ is injective by (v) of Lemma 4.4.

We shall show that $\widetilde{T}$ is surjective. Take $([u_1], \ldots, [u_N]) \in \oplus_{j=1}^N D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$. Since $[u_j] = [P_1^{\alpha_j} u_j]$ and $|||[u_j]|||_1 = ||P_1^{\alpha_j} u_j||_1$ ($P_1^{\alpha_j}$ is the orthogonal projection onto $D(\overline{L_1^{\alpha_j}})^\perp$), we may assume $u_j \in D(\overline{L_1^{\alpha_j}})^\perp$ and $|||[u_j]|||_1 = ||u_j||_1$. Then we have

$$\sum_{j=1}^{N} ||u_j||_{1,\alpha_j}^2 = \sum_{j=1}^{N} |||[u_j]|||_{1,\alpha_j}^2 < \infty$$

and thus $(u_1, \ldots, u_N) \in \oplus_{j=1}^N D((L_1^{\alpha_j})^*)$. By (vi) of Lemma 4.4, we have $([u_1], \ldots, [u_N]) = \widetilde{T} S(u_1, \ldots, u_N) \in \operatorname{Ran} \widetilde{T}$.

Therefore we have the map $\widetilde{T}$ is a bijective continuous map from a Hilbert space to a Hilbert space. This implies $\widetilde{T}$ is isomorphism, by the open mapping theorem (see [Ru2, Theorem 2.11]). Thus we have

$$\dim D(L_N^*)/D(\overline{L_N}) = N \dim D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}}) = 4N,$$

since $\dim D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}}) = 4$ by (ii) of Proposition 3.2. Thus (4.19) holds. $\square$

5 Characterization of self-adjoint extensions

In this section, we shall give a characterization of self-adjoint extensions of L_N. For this purpose, we shall introduce some basis functions.

Lemma 5.1 *(i) Let $N = 1, 2, \ldots$ or $N = \infty$. Put*

$$\begin{aligned} \phi_{-1}^{(j)} &= t_{z_j}(\chi \phi_{-1}^{\alpha_j}),\ \psi_1^{(j)} = t_{z_j}(\chi \psi_1^{\alpha_j}), \\ \phi_0^{(j)} &= t_{z_j}(\chi \phi_0^{\alpha_j}),\ \psi_0^{(j)} = t_{z_j}(\chi \psi_0^{\alpha_j}) \end{aligned}$$

for $j=1,\ldots,N$*, where* χ *is a function given in Lemma 4.4 and* t_{z_j} *is the magnetic translation operator from* $\{|z|<\frac{R}{2}\}$ *to* $\{|z-z_j|<\frac{R}{2}\}$ *intertwining* $\mathcal{L}_1$ *with* $\mathcal{L}_N$*. For* $u,v\in D(L_N^*)$*, define*

$$[u,v]_N=(\mathcal{L}_N u,v)-(u,\mathcal{L}_N v).$$

Then, the following equalities hold:

$$[\phi_l^{(j)},\phi_{l'}^{(j')}]_N=[\phi_l^{(j)},\psi_n^{(j')}]_N=[\psi_n^{(j)},\psi_{n'}^{(j')}]_N=0 \tag{5.1}$$

for $j\neq j'$*,* $l,l'=-1,0$*,* $n,n'=0,1$*.*

$$\begin{aligned}&[\phi_{-1}^{(j)},\psi_1^{(j)}]_N=4\pi(\alpha_j-1),[\phi_0^{(j)},\psi_0^{(j)}]_N=4\pi\alpha_j,\\&[\phi_{-1}^{(j)},\phi_0^{(j)}]_N=[\phi_{-1}^{(j)},\psi_0^{(j)}]_N=[\psi_1^{(j)},\phi_0^{(j)}]_N=[\psi_1^{(j)},\psi_0^{(j)}]_N=0,\\&[\phi_l^{(j)},\phi_l^{(j)}]_N=[\psi_n^{(j)},\psi_n^{(j)}]_N=0\quad\text{for } l=-1,0,\ n=0,1\end{aligned}\tag{5.2}$$

for $j=1,\ldots,N$*.*

(ii) Define linear functionals on $D(L_N^*)$ *by*

$$\begin{aligned}\Phi_{-1}^{(j)}(u)=\Phi_{-1}^{\alpha_j}(T_ju),\quad \Psi_1^{(j)}(u)=\Psi_1^{\alpha_j}(T_ju),\\\Phi_0^{(j)}(u)=\Phi_0^{\alpha_j}(T_ju),\quad \Psi_0^{(j)}(u)=\Psi_0^{\alpha_j}(T_ju),\end{aligned}\tag{5.3}$$

for $j=1,\ldots,N$ *and* $u\in D(L_N^*)$*, where* T_j *is the operator defined by (*4.10*). Then the following equalities hold:*

$$\Phi_l^{(j)}(u_0)=\Psi_n^{(j)}(u_0)=0 \tag{5.4}$$

for $j=1,\ldots,N$*,* $l=-1,0$*,* $n=0,1$ *and* $u_0\in D(\overline{L_N})$*.*

$$\Phi_l^{(j)}(\phi_{l'}^{(j')})=\Psi_n^{(j)}(\psi_{n'}^{(j')})=\Phi_l^{(j)}(\psi_{l'}^{(j')})=\Psi_n^{(j)}(\phi_{n'}^{(j')})=0, \tag{5.5}$$

for $j\neq j'$*,* $l,l'=-1,0$ *and* $n,n'=0,1$*.*

$$\Phi_l^{(j)}(\phi_{l'}^{(j)})=\delta_{l,l'},\ \Psi_n^{(j)}(\psi_{n'}^{(j)})=\delta_{n,n'},\Phi_l^{(j)}(\psi_{l'}^{(j)})=0,\ \Psi_n^{(j)}(\phi_{n'}^{(j)})=0, \tag{5.6}$$

for $j=1,\ldots,N$*,* $l,l'=-1,0$ *and* $n,n'=0,1$*, where* δ *is the Kronecker delta, that is,*

$$\delta_{m,n}=\begin{cases}1 & (m=n)\\0 & (m\neq n\ )\end{cases}.$$

Proof. (i) The equalities (5.1) clearly hold since the intersection of the support of two functions in each bracket is empty. The first equality of (5.2) is proved as follows.

$$\begin{aligned}
& [\phi_{-1}^{(j)}, \psi_1^{(j)}]_N \\
= & \left(\mathcal{L}_N(t_{z_j}\chi\phi_{-1}^{\alpha_j}), t_{z_j}\chi\psi_1^{\alpha_j}\right) - \left(t_{z_j}\chi\phi_{-1}^{\alpha_j}, \mathcal{L}_N(t_{z_j}\chi\psi_1^{\alpha_j})\right) \\
= & \left(\mathcal{L}_1^{\alpha_j}(\chi\phi_{-1}^{\alpha_j}), \chi\psi_1^{\alpha_j}\right) - \left(\chi\phi_{-1}^{\alpha_j}, \mathcal{L}_1^{\alpha_j}(\chi\psi_1^{\alpha_j})\right) \\
= & (\mathcal{L}_1^{\alpha_j}\phi_{-1}^{\alpha_j}, \psi_1^{\alpha_j}) - (\phi_{-1}^{\alpha_j}, \mathcal{L}_1^{\alpha_j}\psi_1^{\alpha_j}) \\
= & 4\pi(\alpha_j - 1),
\end{aligned}$$

where we used the intertwining property and unitarity of t_{z_j} in the second equality, the fact $(1-\chi)\phi_{-1}^{\alpha_j} \in D(\overline{L_1})$ and $(1-\chi)\psi_1^{\alpha_j} \in D(\overline{L_1})$ in the third equality and (ii) of Lemma 3.3 in the fourth equality. The remaining equalities can be proved similarly.

(ii) (5.4) follows from (ii) of Lemma 4.4 and (iv) of Proposition 3.2. Since $\operatorname{supp}\chi_j \cap \operatorname{supp}\chi_{j'} = \emptyset$ if $j \neq j'$, we have $T_j\phi_l^{(j')} = T_j\psi_{n'}^{(j')} = T_j\psi_{l'}^{(j')} = T_j\phi_{n'}^{(j')} = 0$ for $j \neq j'$. Thus (5.5) holds. By definition, we have

$$\Phi_l^{(j)}(\phi_{l'}^{(j)}) = \Phi_l(t_{-z_j}(\chi_j t_{z_j}\phi_{l'})) = \Phi_l(\chi\phi_{l'}) = \delta_{l,l'}.$$

Thus the first equality of (5.6) holds. The remaining equalities can be proved similarly. □

We shall determine the structure of $D(L_N^*)$ under the following assumption.

Assumption 1 *There exist constants α_-, α_+ such that*

$$0 < \alpha_- \leq \alpha_j \leq \alpha_+ < 1 \text{ for every } j = 1, \ldots, N.$$

Remark 1. This assumption is satisfied when N is finite.
Remark 2. It is known that the deficiency indices of $L_1^0 = -\Delta|_{C_0^\infty(\mathbf{R}^2\setminus\{0\})}$ are $(1,1)$ (see [Al-Ge-Ho-Ho]). Since the operator L_1^1 is unitarily equivalent to L_1^0, the deficiency indices of L_1^1 are also $(1,1)$. Thus the deficiency indices of L_1^α change discontinuously at $\alpha = 0, 1$. Hence some of our estimates have singularities at $\alpha = 0, 1$ (for example, see Lemma 3.3). We impose Assumption 1 to avoid these singularities.

In the sequel, $\boldsymbol{C}^{4N}$ denotes the Hilbert space of the sequence of $4N$ complex numbers (infinite complex numbers, when $N = \infty$) equipped with the norm

$$||\mathbf{c}||_{\mathbf{C}^{4N}}^2 = \sum_{j=1}^{4N} |c_j|^2, \ \mathbf{c} = (c_1, \ldots, c_{4N}).$$

Lemma 5.2 *Let $N = 1, 2, \dots$ or $N = \infty$. Suppose Assumption 1 holds. Then, the following holds.*

(i) For $u \in D(L_N^*)$, the sequence

$$\begin{aligned} \Xi u &= (\Phi_{-1}^{(1)}(u), \Psi_1^{(1)}(u), \Phi_0^{(1)}(u), \Psi_0^{(1)}(u), \dots \\ &\quad \dots, \Phi_{-1}^{(N)}(u), \Psi_1^{(N)}(u), \Phi_0^{(N)}(u), \Psi_0^{(N)}(u)) \end{aligned} \tag{5.7}$$

belongs to $\boldsymbol{C}^{4N}$ for any $u \in D(L_N^*)$ (when $N = \infty$, Ξu is an infinite sequence). The map

$$D(L_N^*) \ni u \mapsto \Xi u \in \boldsymbol{C}^{4N}$$

is continuous. Moreover, there exist a constant C_2 dependent on B, R, α_-, α_+ (independent of N) such that

$$||\Xi u||_{\mathbf{C}^{4N}}^2 \le C_2 ||u||_N^2 \tag{5.8}$$

for any $u \in D(L_N^*)$.

(ii) For $\mathbf{c} = (c_1, \dots, c_{4N}) \in \boldsymbol{C}^{4N}$, the function

$$\begin{aligned} E(\mathbf{c}) &= c_1\phi_{-1}^{(1)} + c_2\psi_1^{(1)} + c_3\phi_0^{(1)} + c_4\psi_0^{(1)} + \dots \\ &+ c_{4N-3}\phi_{-1}^{(N)} + c_{4N-2}\psi_1^{(N)} + c_{4N-1}\phi_0^{(N)} + c_{4N}\psi_0^{(N)} \end{aligned} \tag{5.9}$$

belongs to $D(L_N^*)$ (when $N = \infty$, the right hand side of (5.9) is a convergent sequence in $D(L_\infty^*)$). Moreover, there exist a constant C_3 dependent on B, R, α_-, α_+ (independent of N) such that

$$||E(\mathbf{c})||_N^2 \le C_3 ||\mathbf{c}||_{\mathbf{C}^{4N}}^2 \tag{5.10}$$

for any $\mathbf{c} \in \boldsymbol{C}^{4N}$.

(iii) $u - E(\Xi u) \in D(\overline{L_N})$ for every $u \in D(L_N^*)$.

(iv) $\Xi E(\mathbf{c}) = \mathbf{c}$ for every $\mathbf{c} \in \boldsymbol{C}^{4N}$.

(v) $\Xi u = 0$ for every $u \in D(\overline{L_N})$.

Proof. (i) Since the Gamma function $\Gamma(z)$ is continuous on $0 < z < 1$, $\Gamma(\alpha)$ is uniformly bounded for $\alpha_- \le \alpha \le \alpha_+$. From this fact and (i) of Lemma 3.3, we have that there exist a constant C dependent only on B, α_-, α_+ such that

$$\sup_{\alpha_- \le \alpha \le \alpha_+} \max\{||\phi_{-1}^\alpha||_1^2, ||\psi_1^\alpha||_1^2, ||\phi_0^\alpha||_1^2, ||\phi_{-1}^\alpha||_1^2\} \le C. \tag{5.11}$$

By (iii) of Lemma 3.3, we have that there exist a constant C' dependent only on B, α_-, α_+ such that

$$\sup_{\alpha_- \le \alpha \le \alpha_+} \max\{||\Phi_{-1}^\alpha||^2, ||\Psi_1^\alpha||^2, ||\Phi_0^\alpha||^2, ||\Psi_0^\alpha||^2\} \le C'. \tag{5.12}$$

By (5.11), (5.12) and (4.15), we have

$$\begin{aligned}
&\sum_{j=1}^{N}(|\Phi_{-1}^{(j)}(u)|^2+|\Psi_{1}^{(j)}(u)|^2+|\Phi_{0}^{(j)}(u)|^2+|\Psi_{0}^{(j)}(u)|^2)\\
=\ &\sum_{j=1}^{N}(|\Phi_{-1}^{\alpha_j}(T_ju)|^2+|\Psi_{1}^{\alpha_j}(T_ju)|^2+|\Phi_{0}^{\alpha_j}(T_ju)|^2+|\Psi_{0}^{\alpha_j}(T_ju)|^2)\\
\le\ &4C'\sum_{j=1}^{N}||T_ju||_1^2\\
\le\ &4C'C_1||u||_N^2.
\end{aligned}$$

Thus $\Xi u\in \boldsymbol{C}^{4N}$ and (5.8) holds for $C_2=4C'C_1$.

(ii) Notice that two basis functions are mutually orthogonal if the upper index j are different. From this fact, (5.11) and (4.6), we have for $\mathbf{c}\in\boldsymbol{C}^{4N}$

$$\begin{aligned}
&||\sum_{j=N_1}^{N_2}c_{4j-3}\phi_{-1}^{(j)}+c_{4j-2}\psi_{1}^{(j)}+c_{4j-1}\phi_{0}^{(j)}+c_{4j}\psi_{0}^{(j)}||_N^2\\
\le\ &\sum_{j=N_1}^{N_2}||c_{4j-3}\phi_{-1}^{(j)}+c_{4j-2}\psi_{1}^{(j)}+c_{4j-1}\phi_{0}^{(j)}+c_{4j}\psi_{0}^{(j)}||_N^2\\
\le\ &4CC_1\sum_{j=N_1}^{N_2}(|c_{4j-3}|^2+|c_{4j-2}|^2+|c_{4j-1}|^2+|c_{4j}|^2),
\end{aligned}\tag{5.13}$$

for any N_1, N_2 such that $1\le N_1\le N_2\le N$ and $N_2<\infty$. When $N=\infty$, this inequality guarantees the convergence of the infinite sum (5.9), since the right hand side of (5.13) converges to 0 as $N_1,N_2\to\infty$. Moreover, taking $N_1=1$ and $N_2=N$ (or $N_2\to\infty$) we have that (5.10) holds with $C_3=4CC_1$.

(iii) By (v) of Lemma 4.4 and (iv) of Proposition 3.2, it suffices to show $u_0=u-E(\Xi u)$ satisfies

$$\Phi_{-1}^{(j)}(u_0)=\Psi_{1}^{(j)}(u_0)=\Phi_{0}^{(j)}(u_0)=\Psi_{0}^{(j)}(u_0)=0$$

for every $j=1,\ldots,N$. These equalities follows from the definition of E, Ξ and (ii) of Lemma 5.1.

(iv)(v) These assertions follow from the definition of E, Ξ and (ii) of Lemma 5.1. □

In the sequel, we denote the algebraic direct sum of two subspace W_1 and W_2 of a vector space V by $W_1\oplus_{alg}W_2$, that is,

$$W_1\oplus_{alg}W_2=\{w_1+w_2\in V\ ;\ w_1\in W_1,\ w_2\in W_2\}.$$

Proposition 5.3 *Let* $1 \le N < \infty$ *or* $N = \infty$. *Suppose that Assumption 1 holds. Then, the operator domain* $D(L_N^*)$ *is decomposed into a direct sum*

$$D(L_N^*) = D(\overline{L_N}) \oplus_{alg} \overset{N}{\underset{j=1}{\oplus}} \mathrm{L.h.}\{\phi_{-1}^{(j)}, \psi_1^{(j)}, \phi_0^{(j)}, \psi_0^{(j)}\},$$

where $\oplus_{alg}$ *denotes the algebraic direct sum and* $\oplus_{j=1}^N$ *denotes the orthogonal direct sum of mutually orthogonal closed subspaces. More precisely, every* $u \in D(L_N^*)$ *is decomposed as*

$$\begin{aligned} u &= u_0 + E(\Xi u) \\ &= u_0 + \sum_{j=1}^N \Big(\Phi_{-1}^{(j)}(u)\phi_{-1}^{(j)} + \Psi_1^{(j)}(u)\psi_1^{(j)} + \Phi_0^{(j)}(u)\phi_0^{(j)} + \Psi_0^{(j)}(u)\psi_0^{(j)} \Big), \end{aligned} \tag{5.14}$$

where $u_0 \in D(\overline{L_N})$ *(when* $N = \infty$*, the right hand side of (5.14) is a convergent sequence in* $D(L_N^*)$*). This decomposition is unique in the sense that if* $u_0 + u_1 = u_0' + u_1'$*, and if* u_0*,* $u_0' \in D(\overline{L_N})$ *and* u_1*,* $u_1' \in \operatorname{Ran} E$*, then we have* $u_0 = u_0'$ *and* $u_1 = u_1'$*. The map*

$$\tilde{\Xi} : D(L_N^*) \ni u \mapsto (u_0, \Xi u) \in D(\overline{L_N}) \oplus \boldsymbol{C}^{4N} \tag{5.15}$$

is a Banach space isomorphism, where $D(\overline{L_N}) \oplus \boldsymbol{C}^{4N}$ *is the orthogonal direct sum of Hilbert spaces. The inverse map of* $\tilde{\Xi}$ *is given by*

$$\tilde{\Xi}^{-1} : D(\overline{L_N}) \oplus \boldsymbol{C}^{4N} \ni (u_0, \mathbf{c}) \mapsto u_0 + E\mathbf{c}. \tag{5.16}$$

Moreover, there exist a constant $C_4 > 1$ *dependent on* B*,* R*,* α_-*,* α_+ *(independent of* N*) such that*

$$C_4^{-1}||u||_N^2 \le ||u_0||_N^2 + ||\Xi u||_{\mathbf{C}^{4N}}^2 \le C_4||u||_N^2 \tag{5.17}$$

for every $u \in D(L_N^*)$*.*

Proof. We have already shown that the sum in the right hand side of (5.14) is well-defined, since it equals to $E(\Xi u)$ and E and Ξ is well-defined by Lemma 5.2. Put $u_0 = u - E\Xi u$. By (iii) of Lemma 5.2, we have $u_0 \in D(\overline{L_N})$. If $u_0 + E(\mathbf{c}) = u_0' + E(\mathbf{c}')$, u_0, $u_0' \in D(\overline{L_N})$ and $\mathbf{c}$, $\mathbf{c}' \in \boldsymbol{C}^{4N}$, then we have

$$\mathbf{c} - \mathbf{c}' = \Xi E(\mathbf{c} - \mathbf{c}') = \Xi(u_0' - u_0) = 0$$

by (iv) and (v) of Lemma 5.2. Thus we have $\mathbf{c} = \mathbf{c}'$ and $u_0 = u_0'$, so the decomposition is unique and the map $\tilde{\Xi}$ is well-defined. Since $\tilde{\Xi}u = 0$ implies $u = u_0 + E(\Xi u) = 0$, we have that $\tilde{\Xi}$ is injective. Moreover, for any $(u_0, \mathbf{c}) \in D(\overline{L_N}) \oplus \boldsymbol{C}^{4N}$, we have

$$\begin{aligned} \tilde{\Xi}(u_0 + E(\mathbf{c})) &= (u_0 + E(\mathbf{c}) - E(\Xi(u_0 + E(\mathbf{c}))), \Xi(u_0 + E(\mathbf{c}))) \\ &= (u_0, \mathbf{c}) \end{aligned}$$

by (iv) and (v) of Lemma 5.2. Thus $\tilde{\Xi}$ is bijective and the inverse map is given by (5.16). By (5.8) and (5.10), we have

$$\begin{aligned} ||u_0||_N^2 + ||\Xi u||_{\mathbf{C}^{4N}}^2 &\leq ||u - E(\Xi u)||_N^2 + ||\Xi u||_{\mathbf{C}^{4N}}^2 \\ &\leq \{2(1 + C_2 C_3) + C_2\} ||u||_N^2, \\ ||u||_N^2 = ||u_0 + E(\Xi u)||_N^2 &\leq 2(||u_0||_N^2 + C_3 ||\Xi u||_{\mathbf{C}^{4N}}^2). \end{aligned}$$

Thus (5.17) holds for $C_4 = \max(2(1 + C_2 C_3) + C_2, 2(1 + C_3))$. This implies $\tilde{\Xi}$ is a Banach space isomorphism. □

We shall give a characterization of self-adjoint extensions of L_N.

Proposition 5.4 *Let $N = 1, 2, \ldots$ or $N = \infty$. Suppose that Assumption 1 holds. Then the following holds:*

(i) Let M be a bounded operator on $\boldsymbol{C}^{4N}$ satisfying

$$\operatorname{Ran} M : \text{closed}, \quad \operatorname{Ker} M^* J = \operatorname{Ran} M, \tag{5.18}$$

where M^* is the adjoint operator of M and J is a bounded operator on $\boldsymbol{C}^{4N}$ given by the matrix

$$J = \begin{pmatrix} J_{\alpha_1} & O & \ldots & O \\ O & J_{\alpha_2} & \ldots & O \\ \ldots & \ldots & \ldots & \ldots \\ O & O & \ldots & J_{\alpha_N} \end{pmatrix}, \quad J_\alpha = 4\pi \begin{pmatrix} 0 & \alpha - 1 & 0 & 0 \\ 1 - \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}.$$

Define an operator H_N^M by

$$\begin{aligned} D(H_N^M) &= D(\overline{L_N}) \oplus_{alg} \operatorname{Ran} EM, \\ H_N^M u &= \mathcal{L}_N u, \quad u \in D(H_N^M). \end{aligned} \tag{5.19}$$

Then, H_N^M is a self-adjoint extension of L_N.

(ii) Let H_N be any self-adjoint extension of L_N. Then, there exists a bounded operator M on $\mathbf{C}^{4N}$ satisfying (5.18) and $H_N^M = H_N$ in the operator sense.

(iii) For two bounded operators M and M' satisfying (5.18), we have that the condition $H_N^M = H_N^{M'}$ is equivalent to the condition $\operatorname{Ran} M = \operatorname{Ran} M'$.

(iv) An element u in $D(L_N^*)$ belongs to $D(H_N^M)$ if and only if $\Xi u \in$ $\operatorname{Ran} M$.

Remark. Proposition 5.4 is a paraphrase of the proposition in [Ex-St-Vy, page 2159], when $N = 1$.

Proof. First we prove (iv). This follows from the equality

$$\tilde{\Xi}\left(D(\overline{L_N})) \oplus_{alg} \operatorname{Ran} EM\right) = D(\overline{L_N}) \oplus \operatorname{Ran} M$$

and the fact $\tilde{\Xi}$ is a Banach space isomorphism (see Proposition 5.3).

(i) Since $D(\overline{L_N}) \oplus_{alg} \operatorname{Ran} EM = \tilde{\Xi}^{-1}(D(\overline{L_N}) \oplus \operatorname{Ran} M)$ and $\operatorname{Ran} M$ is closed, we have that $D(H_N^M)$ is a closed subspace in $D(L_N^*)$. Thus H_N^M is a closed extension of L_N.

Notice that

$$[E(\mathbf{c}), E(\mathbf{c}')]_N = (\mathbf{c}, J\mathbf{c}')_{\mathbf{C}^{4N}} \tag{5.20}$$

for any $\mathbf{c}, \mathbf{c}' \in \boldsymbol{C}^{4N}$, since $\mathbf{c} = \delta_k, \mathbf{c}' = \delta_{k'}$ satisfies (5.20) for any $k, k' = 1, \dots, 4N$ by (i) of Lemma 5.1 (δ_k is the element of $\boldsymbol{C}^{4N}$ satisfying $\delta_k(n) = \delta_{kn}$). For an element u in $D(L_N^*)$, we have an equivalence relation (the note '$\Leftrightarrow$' means the equivalence)

$$\begin{aligned} u \in D((H_N^M)^*) \quad &\Leftrightarrow \quad [u, v]_N = 0 \text{ for every } v \in D(H_N^M) \\ &\Leftrightarrow \quad [u, EMc]_N = 0 \text{ for every } \mathbf{c} \in \boldsymbol{C}^{4N} \\ &\Leftrightarrow \quad (\Xi u, JMc)_{\mathbf{C}^{4N}} = 0 \text{ for every } \mathbf{c} \in \boldsymbol{C}^{4N} \\ &\Leftrightarrow \quad \Xi u \in (\operatorname{Ran} JM)^{\perp} \\ &\Leftrightarrow \quad \Xi u \in \ker M^* J, \end{aligned} \tag{5.21}$$

where we used the fact $[u, v]_N = 0$ for every $v \in D(\overline{L_N})$ in the second equivalence, (5.20) in the third. By (iv) of this proposition and the assumption (5.18), the last statement of (5.21) is equivalent to the condition $u \in D(H_N^M)$. Thus we have $D(H_N^M) = D((H_N^M)^*)$. Moreover, since $\overline{L_N} \subset H_N^M \subset L_N^*$, we have $\overline{L_N} \subset (H_N^M)^* \subset L_N^*$. Thus

$$(H_N^M)^* u = L_N^* u = \boldsymbol{\mathcal{L}}_N u = H_N^M u$$

for every $u \in D(H_N^M)$. Hence H_N^M is self-adjoint.

(ii) Let H_N be any self-adjoint extension of L_N. Then $D(H_N)$ is a closed subspace of $D(L_N^*)$ containing $D(\overline{L_N})$.

We show that $V = \Xi D(H_N)$ is a closed subspace of $\boldsymbol{C}^{4N}$. Suppose that $\{u_n\} \subset D(H_N)$ and that $\Xi u_n \to \mathbf{c} \in C^{4N}$. Then $E(\Xi u_n) \to E(\mathbf{c})$ in $D(L_N^*)$. Since $u_n - E\Xi u_n \in D(\overline{L_N})$ by (iii) of Lemma 5.2, we have $E(\Xi u_n) = u_n - (u_n - E(\Xi u_n)) \in D(H_N)$. Since $D(H_N)$ is a closed subspace in $D(L_N^*)$, we have $E(\mathbf{c}) \in D(H_N)$. By (iv) of Lemma 5.2, we have $\mathbf{c} = \Xi E(\mathbf{c}) \in V$. Hence V is closed.

Thus the following decomposition holds:

$$\tilde{\Xi} D(H_N) = D(\overline{L_N}) \oplus V.$$

Let M be the orthogonal projection from $\boldsymbol{C}^{4N}$ onto V. Then $\operatorname{Ran} M = V$ is closed and

$$D(H_N^M) = \tilde{\Xi}^{-1}\left(D(\overline{L_N}) \oplus V\right) = D(H_N)$$

by definition (5.19) and the form of the inverse map (5.16). Thus $H_N = H_N^M$. Moreover, by (iv), (5.21) and the self-adjointness of H_N, we have

$$\operatorname{Ran} M = \Xi D(H_N^M) = \Xi D((H_N^M)^*) = \operatorname{Ker} M^* J.$$

Thus (5.18) holds.

(iii) By definition,

$$\begin{aligned}
& \operatorname{Ran} M = \operatorname{Ran} M' \\
\Leftrightarrow\ & D(\overline{L_N}) \oplus \operatorname{Ran} M = D(\overline{L_N}) \oplus \operatorname{Ran} M' \\
\Leftrightarrow\ & \tilde{\Xi}^{-1}\left(D(\overline{L_N}) \oplus \operatorname{Ran} M\right) = \tilde{\Xi}^{-1}\left(D(\overline{L_N}) \oplus \operatorname{Ran} M'\right) \\
\Leftrightarrow\ & D(H_N^M) = D(H_N^{M'}),
\end{aligned}$$

where we used $\tilde{\Xi}$ is an isomorphism. □

By the proof of the previous proposition, we have the following corollary.

Corollary 5.5 *Suppose that Assumption 1 holds. Then, the map* $M \mapsto H_N^M$ *is an one-to-one, onto correspondence from the set of all orthogonal projections* M *on* $\boldsymbol{C}^{4N}$ *satisfying*

$$\operatorname{Ker} MJ = \operatorname{Ran} M \tag{5.22}$$

to the set of all self-adjoint extensions H_N *of* L_N.

Proof. When M is an orthogonal projection, $\operatorname{Ran} M$ is closed and $M^* = M$. Thus (5.22) is equivalent to (5.18). Hence the map is well-defined. For two orthogonal projections M and M', the condition $M = M'$ is equivalent to the condition $\operatorname{Ran} M = \operatorname{Ran} M'$. Thus the injectivity of the map follows from (iii) of Proposition 5.4. The surjectivity follows from the proof of (ii) of Proposition 5.4. □

When the operator M has a simple form, the condition (5.18) is simplified as follows:

Lemma 5.6 *(i) Let* $N < \infty$. *Then, the condition (5.18) is equivalent to*

$$\operatorname{rank} M = 2N,\ M^* J M = O, \tag{5.23}$$

where $\mathrm{rank}M = \dim \mathrm{Ran}\, M$.

(ii) Let $N = 1, 2, \ldots$ *or* $N = \infty$. *Suppose that Assumption 1 holds. Suppose that the operator* M *is the (finite or infinite) direct sum of* 4×4 *orthogonal projection matrices, i.e.*

$$M = \begin{pmatrix} M_1 & O & \ldots & O \\ O & M_2 & \ldots & O \\ \multicolumn{4}{c}{\dotfill} \\ O & O & \ldots & M_N \end{pmatrix}, \quad M_j : 4 \times 4,\ M_j^2 = M_j,\ M_j^* = M_j.$$

Then, the condition (5.18) is equivalent to

$$\mathrm{rank}M_j = 2,\ M_j J_{\alpha_j} M_j = O,\ \text{for every } j = 1, \ldots, N. \tag{5.24}$$

Proof. (i) When M is finite, $\mathrm{Ran}\, M$ is a closed subspace. The equality $M^* J M = O$ is equivalent to

$$\mathrm{Ran}\, M \subset \mathrm{Ker}\, M^* J. \tag{5.25}$$

Since J is a regular matrix,

$$\mathrm{rank}M^* J = \mathrm{rank}M.$$

Thus

$$\dim \mathrm{Ker}\, M^* J = 4N - \mathrm{rank}M.$$

Hence we have that the condition $\mathrm{Ran}\, M = \mathrm{Ker}\, M^* J$ is equivalent to the condition $\mathrm{rank}M = 2N$, under the condition (5.25).

(ii) Consider the following decomposition

$$\boldsymbol{C}^{4N} \ni (c_1, \ldots, c_{4N}) \mapsto ((c_{4j-3}, c_{4j-2}, c_{4j-1}, c_{4j}))_{j=1}^{N} \in \mathop{\oplus}_{j=1}^{N} \boldsymbol{C}^4.$$

Since M_j ($j = 1, 2, \ldots, N$) and M are orthogonal projections, we have that $\mathrm{Ran}\, M_j$ and $\mathrm{Ran}\, M$ are closed and that the following decomposition holds:

$$\mathrm{Ran}\, M = \mathop{\oplus}_{j=1}^{N} \mathrm{Ran}\, M_j,\ \ \ker M^* J = \mathop{\oplus}_{j=1}^{N} \ker (M_j)^* J_{\alpha_j}.$$

Thus the condition $\mathrm{Ran}\, M = \mathrm{Ker}\, M^* J$ is equivalent to

$$\mathrm{Ran}\, M_j = \mathrm{Ker} (M_j)^* J_{\alpha_j} \text{ for every } j = 1, \ldots, N.$$

This is equivalent to (5.23), by (i) and $M_j^* = M_j$. □

We shall examine some properties of the standard Aharonov-Bohm Hamiltonian H_N^{AB}.

Proposition 5.7 *Let* $1 \leq N < \infty$ *or* $N = \infty$. *Let* H_N^{AB} *be the Friedrichs extension of* L_N. *Suppose that Assumption 1 holds. Then, the following holds.*

(i) The operator domain $D(H_N^{AB})$ *is decomposed into the direct sum*

$$D(H_N^{AB}) = D(\overline{L_N}) \oplus_{alg} \overset{N}{\underset{j=1}{\oplus}} \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\},$$

where $\phi_0^{(j)}, \psi_1^{(j)}$ *are the functions defined in Lemma 5.1.*

(ii) $D(H_N^{AB}) = \{u \in D(L_N^*); (\nabla + i\boldsymbol{a}_N)u \in (L^2(\boldsymbol{R}^2))^2\}$.

(iii) For any $u \in D(H_N^{AB})$,

$$\begin{aligned} ||(\nabla + i\boldsymbol{a}_N)u||^2 &\leq ||H_N^{AB}u||\,||u||, &(5.26)\\ (H_N^{AB}u, u) &\geq B||u||^2. &(5.27)\end{aligned}$$

(iv) Let $u \in D(H_N^{AB})$. *Let* $\chi \in C^\infty(\boldsymbol{R}^2)$ *and assume that* χ, $\nabla\chi$, $\Delta\chi$ *are bounded functions on* $\boldsymbol{R}^2$. *Then* $\chi u \in D(H_N^{AB})$ *and*

$$\begin{aligned} ||H_N^{AB}(\chi u)|| \leq\; & (\sup|\chi|)\,||H_N^{AB}u|| + 2\,(\sup|\nabla\chi|)\,||(\nabla + i\boldsymbol{a}_N)u|| \\ & + (\sup|\Delta\chi|)\,||u||. \end{aligned} \tag{5.28}$$

Proof. (i) (ii) According to the construction of the Friedrichs extension, the form domain $Q(H_N^{AB})$ of H_N^{AB} is a closure of $D(L_N) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$ with respect to the norm

$$(\mathcal{L}_N u, u) = (-(\nabla + i\boldsymbol{a}_N)^2 u, u) = ||(\nabla + i\boldsymbol{a}_N)u||^2, \; u \in C_0^\infty(\boldsymbol{R}^2 \setminus S_N), \tag{5.29}$$

where we used an integration by parts. This norm is positive definite since

$$(\mathcal{L}_N u, u) = ((\mathcal{A}_N^\dagger \mathcal{A}_N + B)u, u) = ||\mathcal{A}_N u||^2 + B||u||^2 \geq B||u||^2 \tag{5.30}$$

by Lemma 4.1. Thus $D(H_N^{AB})$ is contained in the Hilbert space

$$Q_N = \{u \in L^2(\boldsymbol{R}^2);\; (\nabla + i\boldsymbol{a}_N)u \in (L^2(\boldsymbol{R}^2))^2\} \tag{5.31}$$

equipped with the norm $||u||_{Q_N}^2 = ||(\nabla + i\boldsymbol{a}_N)u||^2 + ||u||^2$. Since H_N^{AB} is a self-adjoint extension of L_N, we have $D(H_N^{AB}) \subset D(L_N^*)$. Thus we have

$$D(H_N^{AB}) \subset D(L_N^*) \cap Q_N = \{u \in D(L_N^*);\; (\nabla + i\boldsymbol{a}_N)u \in (L^2(\boldsymbol{R}^2))^2\}. \tag{5.32}$$

We shall show that

$$D(L_N^*) \cap Q_N = D(\overline{L_N}) \oplus_{alg} \overset{N}{\underset{j=1}{\oplus}} \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\}. \tag{5.33}$$

Since $D(\overline{L_N}) \subset D(H_N^{AB}) \subset D(L_N^*) \cap Q_N$, it suffices to show $\mathrm{Ran}\, E \cap (D(L_N^*) \cap Q_N) = \oplus_{j=1}^N \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\}$, by Proposition 5.3.

Take $u \in \mathrm{Ran}\, E$. u is written as

$$u = \sum_{j=1}^N \left(c_{4j-3}\phi_{-1}^{(j)} + c_{4j-2}\psi_1^{(j)} + c_{4j-1}\phi_0^{(j)} + c_{4j}\psi_0^{(j)} \right), \tag{5.34}$$

where $\mathbf{c} = (c_1, \ldots, c_{4N}) \in \boldsymbol{C}^{4N}$. Since $(\nabla + i\boldsymbol{a}_N)u = (\Pi_{N,x}u, \Pi_{N,y}u)$, we see that the condition $(\nabla + i\boldsymbol{a}_N)u \in (L^2(\boldsymbol{R}^2))^2$ is equivalent to the condition $\mathcal{A}_N u \in L^2(\boldsymbol{R}^2)$ and $\mathcal{A}_N^\dagger u \in L^2(\boldsymbol{R}^2)$. In the open set $\{|z - z_j| < \frac{R}{4}\}$, we have

$$\begin{aligned}
\mathcal{A}_N u &= t_{z_j} \mathcal{A}_1^{\alpha_j} \left(c_{4j-3}\phi_{-1,0}^{\alpha_j} + c_{4j-2}\psi_{0,1}^{\alpha_j} + c_{4j-1}\phi_{0,0}^{\alpha_j} + c_{4j}\psi_{0,0}^{\alpha_j} \right) \\
&= t_{z_j} \left\{ 2c_{4j-3}(\alpha_j - 1)\phi_{-2,0}^{\alpha_j} + 2c_{4j-1}\alpha_j\phi_{-1,0}^{\alpha_j} \right\}, \qquad (5.35) \\
\mathcal{A}_N^\dagger u &= t_{z_j} \mathcal{A}_1^{\dagger,\alpha_j} \left(c_{4j-3}\phi_{-1,0}^{\alpha_j} + c_{4j-2}\psi_{0,1}^{\alpha_j} + c_{4j-1}\phi_{0,0}^{\alpha_j} + c_{4j}\psi_{0,0}^{\alpha_j} \right) \\
&= t_{z_j} \Big\{ c_{4j-3}B\phi_{0,0}^{\alpha_j} + c_{4j-2}(2(\alpha_j - 1)\psi_{0,0}^{\alpha_j} + B\psi_{1,1}^{\alpha_j}) + c_{4j-1}B\phi_{1,0}^{\alpha_j} \\
&\qquad + c_{4j}(2\alpha_j\psi_{0,-1}^{\alpha_j} + B\psi_{1,0}^{\alpha_j}). \Big\} \qquad (5.36)
\end{aligned}$$

by (ii) of Proposition 3.1. Since $2(\alpha_j - 1)\phi_{-2,0}^{\alpha_j} \notin L^2(\{|z| < \frac{R}{4}\})$ and $\phi_{-1,0}^{\alpha_j} \in L^2(\{|z| < \frac{R}{4}\})$, we see that $\mathcal{A}_N u \in L^2(\boldsymbol{R}^2)$ implies $c_{4j-3} = 0$. Since $2\alpha_j\psi_{0,-1} \notin L^2(\{|z| < \frac{R}{4}\})$ and $\phi_{0,0}^{\alpha_j}$, $\psi_{0,0}^{\alpha_j}$, $\psi_{1,1}^{\alpha_j}$, $\phi_{1,0}^{\alpha_j}$, $\psi_{1,0}^{\alpha_j} \in L^2(\{|z| < \frac{R}{4}\})$, we see that $\mathcal{A}_N^\dagger u \in L^2(\boldsymbol{R}^2)$ implies $c_{4j} = 0$. Thus we obtain

$$D(L_N^*) \cap Q_N \subset D(\overline{L_N}) \oplus_{alg} \overset{N}{\underset{j=1}{\oplus}} \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\}.$$

Conversely, take $u = \sum_{j=1}^N (c_{4j-2}\psi_1^{(j)} + c_{4j-1}\phi_0^{(j)}) \in \oplus_{j=1}^N \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\}$. By the above calculus, the Leibniz rule and (5.11), we have that there exist a constant $C > 0$ dependent only on B, R, χ, α_-, α_+ such that

$$\begin{aligned}
||\mathcal{A}_N(c_{4j-2}\psi_1^{(j)} + c_{4j-1}\phi_0^{(j)})||^2 &\leq C(|c_{4j-2}|^2 + |c_{4j-1}|^2), \qquad (5.37) \\
||\mathcal{A}_N^\dagger(c_{4j-2}\psi_1^{(j)} + c_{4j-1}\phi_0^{(j)})||^2 &\leq C(|c_{4j-2}|^2 + |c_{4j-1}|^2). \qquad (5.38)
\end{aligned}$$

Since $\{\mathrm{supp}(c_{4j-2}\psi_1^{(j)}+c_{4j-1}\phi_{0,0}^{(j)})\}_{j=1}^N$ are disjoint, we have

$$\begin{aligned} & ||\mathcal{A}_N\sum_{j=1}^N(c_{4j-2}\psi_1^{(j)}+c_{4j-1}\phi_0^{(j)})||^2 \\ = & \sum_{j=1}^N||\mathcal{A}_N(c_{4j-2}\psi_1^{(j)}+c_{4j-1}\phi_0^{(j)})||^2 \\ \leq & C\sum_{j=1}^N(|c_{4j-2}|^2+|c_{4j-1}|^2)<\infty \end{aligned}$$

by (5.37). Thus $\mathcal{A}_N u\in L^2(\boldsymbol{R}^2)$. Similarly, $\mathcal{A}_N^\dagger u\in L^2(\boldsymbol{R}^2)$ follows from (5.38). Therefore (5.33) holds.

Define a bounded linear operator M_N^{AB} on $\boldsymbol{C}^{4N}$ by the (finite or infinite) matrix

$$M_N^{AB}=\begin{pmatrix} M^{AB} & O & \dots & O \\ O & M^{AB} & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & M^{AB}\end{pmatrix},\quad M^{AB}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}.$$

Then we have that M^{AB} is a orthogonal projection, that $\mathrm{rank}M^{AB}=2$ and that $M^{AB}J_{\alpha_j}M^{AB}=O$. By (ii) of Lemma 5.6, M^{AB} satisfies (5.18). By (i) of Proposition 5.4, we have that $H_N^{M^{AB}}$ is a self-adjoint extension of L_N and

$$D(H_N^{M^{AB}})=D(\overline{L_N})\oplus_{alg}\bigoplus_{j=1}^N\mathrm{L.h.}\{\phi_0^{(j)},\psi_1^{(j)}\}. \tag{5.39}$$

This equality, (5.32) and (5.33) implies the operator inclusion $H_N^{AB}\subset H_N^{M^{AB}}$. This implies $H_N^{AB}=H_N^{M^{AB}}$, since both operators are self-adjoint. Thus (i) and (ii) follows from (5.32), (5.33) and (5.39).

(iii) These inequalities follows from (5.29), (5.30) and an approximation argument.

(iv) For $u\in D(H_N^{AB})$ and $\chi\in C^\infty(\boldsymbol{R}^2)$ with χ, $\nabla\chi$ and $\Delta\chi$ bounded, the distributional equality

$$(\nabla+i\boldsymbol{a}_N)^2(\chi u)=\chi(\nabla+i\boldsymbol{a}_N)^2u+2(\nabla\chi)(\nabla+i\boldsymbol{a}_N)u+(\Delta\chi)u \tag{5.40}$$

holds in $D'(\boldsymbol{R}^2\setminus S_N)$. By (ii) and the assumption, the right hand side belongs to $L^2(\boldsymbol{R}^2)$. Thus $\chi u\in D(L_N^*)$ by (i) of Proposition 4.2. One can show that the multiplication operator by χ is a bounded operator from Q_N to Q_N, by the Leibniz rule. Thus χu belongs to $D(L_N^*)\cap Q_N$, which equals to $D(H_N^{AB})$ by (ii) of this proposition. The inequality (5.28) follows from (5.40) and an approximating argument. $\square$

6 Perturbed canonical commutation relation

In this section, we define the annihilation operators and the creation operators, and investigate their commutation relations as linear operators on the Hilbert space $L^2(\boldsymbol{R}^2)$.

Lemma 6.1 *Define linear operators* A_N, $A_N^\dagger$ *by*

$$\begin{aligned} A_N u &:= \mathcal{A}_N u, \ D(A_N) := C_0^\infty(\boldsymbol{R}^2 \setminus S_N), \\ A_N^\dagger u &:= \mathcal{A}_N^\dagger u, \ D(A_N^\dagger) := C_0^\infty(\boldsymbol{R}^2 \setminus S_N). \end{aligned}$$

Then, the following holds.

(i) We have $A_N^\dagger A_N = L_N - B$, $A_N A_N^\dagger = L_N + B$ *in the operator sense.*

(ii) We have $A_N^\dagger \subset A_N{}^*$, $A_N \subset A_N^\dagger{}^*$ *in the operator sense. Moreover, we have*

$$\begin{aligned} D(A_N{}^*) &= \{u \in L^2(\boldsymbol{R}^2);\ \mathcal{A}_N^\dagger u \in L^2(\boldsymbol{R}^2)\}, \\ A_N^* u &= \mathcal{A}_N^\dagger u, \ u \in D(A_N{}^*), \\ D(A_N^\dagger{}^*) &= \{u \in L^2(\boldsymbol{R}^2);\ \mathcal{A}_N u \in L^2(\boldsymbol{R}^2)\}, \\ A_N^\dagger{}^* u &= \mathcal{A}_N u, \ u \in D(A_N^\dagger{}^*). \end{aligned}$$

(iii) The operators A_N *and* $A_N^\dagger$ *are closable.*

Remark 1. The product ST of two linear operators S and T is defined by

$$\begin{aligned} D(ST) &= \{u \in D(T);\ Tu \in D(S)\}, \\ STu &= S(Tu), \quad u \in D(ST). \end{aligned}$$

Remark 2. The operators $\overline{A_N}$ and $A_N^\dagger{}^*$ are different. The operators $\overline{A_N^\dagger}$ and $A_N{}^*$ are also different.

Proof. (i) Since

$$A_N C_0^\infty(\boldsymbol{R}^2 \setminus S_N) \subset C_0^\infty(\boldsymbol{R}^2 \setminus S_N), \ A_N^\dagger C_0^\infty(\boldsymbol{R}^2 \setminus S_N) \subset C_0^\infty(\boldsymbol{R}^2 \setminus S_N),$$

we have $D(A_N A_N^\dagger) = D(A_N^\dagger A_N) = D(L_N)$. Thus the assertion follows from (4.1).

(ii) This assertion follows from (4.2) and the definition of the distributional derivatives.

(iii) Since A_N^* and $(A_N^\dagger)^*$ are closed operators, this assertion follows from (ii). □

We review some properties of a pair of operators XX^* and X^*X.

Lemma 6.2 *Let X be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Then, the following holds.*

(i) The operators X^*X and XX^* are self-adjoint.

(ii) The operator $(XX^*)|_{(\mathrm{Ker}\, XX^*)^\perp}$ is unitarily equivalent to the operator $(X^*X)|_{(\mathrm{Ker}\, X^*X)^\perp}$.

Proof. (i) See [Re-Si, Theorem X.25].

(ii) See [De, Theorem 3]. □

Applying the above lemma to the operators $\overline{A_N}$, $\overline{A_N^\dagger}$, $A_N{}^*$, $A_N^\dagger{}^*$, we obtain four self-adjoint operators. We shall examine some properties of these operators.

Proposition 6.3 *(i) $A_N{}^*\overline{A_N} = H_N^{AB} - B$, $A_N^\dagger{}^*\overline{A^\dagger} = H_N^{AB} + B$, in the operator sense. Moreover,*

$$\begin{aligned} & D(\overline{A_N}) \cap D(L_N^*) = D(\overline{A_N^\dagger}) \cap D(L_N^*) \\ = \ & D(H_N^{AB}) = D(\overline{L_N}) \oplus_{alg} \bigoplus_{j=1}^{N} \mathrm{L.h.}\{\phi_0^{(j)}, \psi_1^{(j)}\}. \end{aligned} \tag{6.1}$$

(ii) There exists two self-adjoint extensions H_N^+ and H_N^- of L_N such that $\overline{A_N}A_N{}^* = H_N^+ + B$, $\overline{A_N^\dagger}A_N^\dagger{}^* = H_N^- - B$ in the operator sense.

(iii) The following decomposition holds.

$$D(H_N^+) = D(\overline{L_N}) \oplus_{alg} \oplus_{j=1}^{N} \mathrm{L.h.}\{\phi_{-1}^{(j)}, \phi_0^{(j)}\} \tag{6.2}$$

$$D(H_N^-) = D(\overline{L_N}) \oplus_{alg} \oplus_{j=1}^{N} \mathrm{L.h.}\{\psi_1^{(j)}, \psi_0^{(j)}\}. \tag{6.3}$$

(iv) $\dim D(H_N^+)/(D(H_N^{AB}) \cap D(H_N^+)) = \dim D(H_N^-)/(D(H_N^{AB}) \cap D(H_N^-)) = N$.

Proof. (i) By (i) and (ii) of Lemma 6.1, we have $A_N{}^*\overline{A_N} \supset A_N^\dagger A = L_N - B$ and $A_N^\dagger{}^*\overline{A_N^\dagger} \supset A_N A_N^\dagger = L_N + B$ in the operator sense. By Lemma 6.2, $A_N{}^*\overline{A_N}$ and $A_N^\dagger{}^*\overline{A_N^\dagger}$ are self-adjoint operators. Thus, there are self-adjoint extensions X and Y of L_N such that $A_N{}^*\overline{A_N} = X - B$ and $A_N^\dagger{}^*\overline{A_N^\dagger} = Y + B$.

We shall show $X = H_N^{AB}$. By (5.29) and (5.30), we see that the graph norm of A_N is equivalent to the norm of the form domain $Q(H_N^{AB})$. Moreover, $D(A_N) = D(L_N) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$ is a form core for $D(H_N^{AB})$, since H_N^{AB} is the Friedrichs extension of L_N. Taking closure, we see that $D(\overline{A_N}) = Q(H_N^{AB})$. According to the proof of [Re-Si, Theorem X.25], $D(\overline{A_N})$ equals to the form domain of $A_N{}^*\overline{A_N}$, which equals to the form domain of X. Thus H_N^{AB} and X have the same form domain with the form core $C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$. Moreover, we have $(Xu,u) = (\mathcal{L}_{\mathcal{N}}u,u) = (H_N^{AB}u,u)$ for any function u in the form core $C_0^\infty(\boldsymbol{R}^2\backslash S_N)$. Thus X and H_N^{AB} have the same quadratic form. This implies $X = H_N^{AB}$. Moreover,

$$D(\overline{A_N}) \cap D(L_N^*) = Q(H_N^{AB}) \cap D(L_N^*) = D(H_N^{AB})$$

since H_N^{AB} is the Friedrichs extension of L_N. Thus (6.1) follows from (i) of Proposition 5.7. The proof of $Y = H_N^{AB}$ and $D(\overline{A_N^\dagger}) \cap D(L_N^*) = D(H_N^{AB})$ is similar.

(ii) One can prove this assertion by using the argument similar to the first part of the proof of (i).

(iii) By Proposition 5.4, the operator domain of H_N^+ and H_N^- is decomposed as

$$\begin{aligned} D(H_N^+) &= D(\overline{L_N}) \oplus_{alg} \left(\operatorname{Ran} E \cap D(H_N^+)\right) \\ D(H_N^-) &= D(\overline{L_N}) \oplus_{alg} \left(\operatorname{Ran} E \cap D(H_N^-)\right). \end{aligned}$$

Take $u \in \operatorname{Ran} E$. u is written as (5.34). Then u satisfies the conditions $u \in D({A_N}^*)$ and ${A_N}^*u = \mathcal{A}_N^\dagger u \in D(\overline{A_N})$ if and only if

$$c_{4j-2} = c_{4j} = 0 \text{ for every } j = 1, \dots, N,$$

by (ii) of Lemma 6.1, (6.1), (5.36) and the argument following (5.36). Thus (6.2) holds. Similarly, u satisfies the conditions $u \in D({A_N^\dagger}^*)$ and ${A_N^\dagger}^*u = \mathcal{A}_N u \in D(\overline{A_N^\dagger})$ if and only if

$$c_{4j-3} = c_{4j-1} = 0 \text{ for every } j = 1, \dots, N,$$

by (ii) of Lemma 6.1, (6.1), (5.35) and the argument following (5.35). Thus (6.3) holds.

(iv) This assertion follows from (i) of Proposition 5.7 and (iii) of this proposition. □

Remark. With the notation of Proposition 5.4, H_N^+, H_N^- are the self-adjoint extensions corresponding to the orthogonal projection matrices M_N^+, M_N^-, where

$$\begin{aligned} M_N^+ &= \begin{pmatrix} M^+ & O & \dots & O \\ O & M^+ & \dots & O \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ O & O & \dots & M^+ \end{pmatrix}, \quad M^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ M_N^- &= \begin{pmatrix} M^- & O & \dots & O \\ O & M^- & \dots & O \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ O & O & \dots & M^- \end{pmatrix}, \quad M^- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The matrices $M^\pm$ satisfy (5.24).

7 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. The second assertion of the following lemma is an immediate corollary of [We, Exercise 8.8]. We shall give a proof for the completeness.

Lemma 7.1 *Let L be a symmetric operators on a Hilbert space $\mathcal{H}$. Suppose that the deficiency indices of L are (n, n) and n is finite. Let A and B are two self-adjoint extensions of L. Then, the following holds.*

(i) We have $\sigma_{ess}(A) = \sigma_{ess}(B)$.

(ii) For any interval $I = (c_1, c_2)$ satisfying $c_1 < c_2$ and $\dim \operatorname{Ran} P_I(A) < \infty$, we have $\dim \operatorname{Ran} P_I(B) < \infty$ and

$$|\dim \operatorname{Ran} P_I(A) - \dim \operatorname{Ran} P_I(B)| \leq d,$$

where

$$d = \dim D(A)/\left(D(A) \cap D(B)\right) = \dim D(B)/\left(D(A) \cap D(B)\right).$$

Remark: The inequality $d \leq n$ always holds, since A and B are n rank extension of $\overline{L}$,

Proof. (i) See [We, Theorem 8.17].

(ii) First we show

$$\dim \operatorname{Ran} P_I(B) \leq \dim \operatorname{Ran} P_I(A) + d. \tag{7.1}$$

Suppose

$$\dim \operatorname{Ran} P_I(B) > \dim \operatorname{Ran} P_I(A) + d$$

holds. Put $X = \operatorname{Ran} P_I(B) \cap (\operatorname{Ran} P_I(A))^\perp$. Then $X \subset D(B)$ and $\dim X > d$.

We shall show that $X \cap D(A) = \{0\}$. Suppose that there exists nonzero vector $u \in X \cap D(A)$. Since $u \in (\operatorname{Ran} P_I(A))^\perp \cap D(A)$, we have

$$||(A - c)u||^2 \geq \epsilon^2 ||u||^2,$$

where $c = \frac{c_1+c_2}{2}$, $\epsilon = \frac{c_2-c_1}{2}$. However, since $L \subset A \subset L^*$, $L \subset B \subset L^*$ and since $u \in \operatorname{Ran} P_I(B) \cap D(A)$, we have

$$||(A - c)u||^2 = ||(L^* - c)u||^2 = ||(B - c)u||^2 < \epsilon^2 ||u||^2$$

(one can show that the strict inequality holds by the spectral decomposition theorem). This contradicts the previous inequality.

Thus, the projection map

$$X \ni u \mapsto [u] \in D(B)/\left(D(A) \cap D(B)\right)$$

is injective. Hence we have

$$\dim D(B)/\left(D(A) \cap D(B)\right) \geq \dim X > d.$$

This contradicts the assumption.

Thus we have proved (7.1) and that $\dim \operatorname{Ran} P_I(B)$ is finite. The equality

$$\dim \operatorname{Ran} P_I(A) \leq \dim \operatorname{Ran} P_I(B) + d$$

is proved by exchanging A for B in the above proof. $\square$

Proof of Theorem 1.2. By (ii) of Lemma 6.2, we have

$$\left. {A_N^\dagger}^* \overline{A_N^\dagger} \right|_{(\operatorname{Ker} {A_N^\dagger}^* \overline{A_N^\dagger})^\perp} \simeq \left. \overline{A_N^\dagger} {A_N^\dagger}^* \right|_{(\operatorname{Ker} \overline{A_N^\dagger} {A_N^\dagger}^*)^\perp},$$

where $\simeq$ denotes the (partial) unitarily equivalence. By (ii) and (iii) of Proposition 6.3, we have the unitarily equivalence relation

$$(H_N^{AB} + B)|_{(\operatorname{Ker}(H_N^{AB}+B))^\perp} \simeq (H_N^- - B)|_{(\operatorname{Ker}(H_N^- - B))^\perp}. \tag{7.2}$$

Let H_N be any self-adjoint extension of L_N. By (i) of Lemma 7.1, we have

$$S = \sigma_{ess}(H_N) = \sigma_{ess}(H_N^{AB}) = \sigma_{ess}(H_N^-)$$

for some closed subset $S \subset \boldsymbol{R}$ independent of the choice of the self-adjoint extension H. In particular we have $S \subset [B, +\infty)$, since $H_N^{AB} \geq B$ in the form sense by (iii) of Proposition 5.7. Thus $\{-B\} \notin S$. By (7.2), we have

$$S \setminus \{B\} = S + 2B = \{x + 2B; x \in S\}. \tag{7.3}$$

It is easy to show the set satisfying (7.3) is $\{(2n-1)B; n = 1, 2, \ldots\}$ or the empty set. We show S is not empty. To see this, it is sufficient to construct a Weyl sequence for the spectrum B, that is, an orthonormal sequence $\{u_n\}_{n=1}^\infty$ such that $||(H-B)u_n|| \to 0$ as $n \to \infty$. Take countable disjoint balls $\{B(n; w_n)\}_{n=1}^\infty$, $B(n; w_n) = \{z; |z - w_n| < n\}$ contained in $\boldsymbol{R}^2 \setminus S_N$ and take $\chi \in C_0^\infty(\{|z| < 1\})$ with $\chi(z) = 1$ in $\{|z| < \frac{1}{2}\}$. Put

$$v_n(z) = t_{w_n} \chi(\frac{z}{n}) e^{-\frac{B}{4}|z|^2},$$

where t_{w_n} is the magnetic translation from $\{|z| < n\}$ to $\{|z - w_n| < n\}$ intertwining $\mathcal{L}_0$ with $\mathcal{L}_N$. It is easy to check that $\{v_n/||v_n||\}_{n=1}^{\infty}$ is the Weyl sequence by using the equality

$$\mathcal{L}_N t_{w_n} e^{-\frac{B}{4}|z|^2} = t_{w_n} \mathcal{L}_0 e^{-\frac{B}{4}|z|^2} = B t_{w_n} e^{-\frac{B}{4}|z|^2}$$

and the Leibniz rule. Thus we have

$$\sigma_{ess}(H_N) = \{(2n-1)B;\ n = 1, 2, \ldots\} \tag{7.4}$$

for any self-adjoint extension H_N of L_N.

Next, we shall prove (ii). The assertion

$$\dim \operatorname{Ran} P_{(-\infty,B)}(H_N^{AB}) = 0 \tag{7.5}$$

follows from (iii) of Proposition 5.7. By the unitarily equivalence relation (7.2), we have

$$\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) = \dim \operatorname{Ran} P_{((2n+1)B,(2n+3)B)}(H_N^{-}) \tag{7.6}$$

for every $n = 0, 1, 2, \ldots$. By (iv) of Proposition 6.3 and (ii) of Lemma 7.1, we have

$$\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{-}) \leq \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) + N \tag{7.7}$$

for every $n = 0, 1, 2, \ldots$. By (7.5), (7.6), (7.7) and an elementary induction argument, we have (1.9). Thus (ii) of Theorem 1.2 holds.

Since the deficiency indices of L_N are $(2N, 2N)$, we have

$$\dim \Big(D(H)/D(H) \cap D(H_N^{-}) \Big) \leq 2N \tag{7.8}$$

for any self-adjoint extension H_N of L_N. Thus (iii) of Theorem 1.2 follows from (1.9), (7.6), (7.8) and (ii) of Lemma 7.1.

To show (i) of Theorem 1.2, consider the following equality:

$$\begin{aligned} & \dim \operatorname{Ran} P_{\{(2n-1)B\}}(H_N) \\ = \ & \dim \operatorname{Ran} P_{((2n-1)B-\epsilon,(2n-1)B+\epsilon)}(H_N) \\ - \ & \dim \operatorname{Ran} P_{((2n-1)B-\epsilon,(2n-1)B)}(H_N) - \dim \operatorname{Ran} P_{((2n-1)B,(2n-1)B+\epsilon)}(H_N), \end{aligned}$$

The first term of the last expression is infinity by (7.4). The second and the third terms are finite numbers for small $\epsilon > 0$, by (iii) of Theorem 1.2. Therefore we have

$$\dim \operatorname{Ran} P_{\{(2n-1)B\}} = \infty,$$

which implies (i) of Theorem 1.2. □

Remark. If we apply the argument in proving (7.6) to the pair of operators $\overline{A_N}{A_N}^*$ and ${A_N}^*\overline{A_N}$, we have

$$\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) = \dim \operatorname{Ran} P_{((2n-3)B,(2n-1)B)}(H_N^{+})$$

for $n = 1, 2, \ldots$. When $N = 1$, this equality and (7.6) imply

$$\begin{aligned} \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_1^{+}) &= n+1 \\ \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_1^{-}) &= n-1 \end{aligned}$$

for $n = 1, 2, \ldots$. Thus the operator H_1^+ attains the maximum and the operator H_1^- attains the minimum in the inequality (1.12).

8 Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. Before the proof, we shall make sure that an operator satisfying the assumption of Theorem 1.3 exists.

Lemma 8.1 *Let $N = 1, 2, \ldots$ or $N = \infty$. Let $0 < \alpha < 1$ and suppose $\alpha_j = \alpha$ for every $j = 1, 2, \ldots, N$. Let H_1 be any self-adjoint extension of L_1^α. Then, there exists a unique self-adjoint extension H_N of L_N satisfying (1.14).*

Proof. By Corollary 5.5, there exists a 4×4 orthogonal projection matrix M_1 satisfying

$$\begin{aligned} &\operatorname{rank} M_1 = 2,\ M_1 J_\alpha M_1 = O, \\ &D(H_1) = \{u \in D((L_1^\alpha)^*);\ \Xi_1 u \in \operatorname{Ran} M_1\}, \end{aligned} \tag{8.1}$$

where Ξ_1 be the map Ξ associated to the operator L_1^α. Put

$$M_N = \begin{pmatrix} M_1 & O & \ldots & O \\ O & M_1 & \ldots & O \\ \ldots & \ldots & \ldots & \ldots \\ O & O & \ldots & M_1 \end{pmatrix}.$$

Then M_N satisfies (5.24), and thus satisfies (5.18) by Lemma 5.6. Let $H_N^{M_N}$ be the corresponding self-adjoint extension given in Proposition 5.4. Let Ξ_N be the map Ξ associated to the operator L_N. By the form of the matrix M_N, we have an equivalence relation

$$\begin{aligned} u \in D(H_N^{M_N}) &\Leftrightarrow \Xi_N u \in \operatorname{Ran} M_N \\ &\Leftrightarrow \Xi_1(T_j u) \in \operatorname{Ran} M_1 \text{ for every } j = 1, 2, \ldots, N \\ &\Leftrightarrow t_{-z_j}\chi_j u \in D(H_1) \text{ for every } j = 1, 2, \ldots, N. \end{aligned} \tag{8.2}$$

Thus $H_N^{M_N}$ satisfies (1.14). The equivalence (8.2) also implies the uniqueness. $\square$

In the sequel, H_N denotes the operator given in Lemma 8.1. We shall give some domain properties of H_N.

Lemma 8.2 *Let $\eta \in C_0^\infty(\{|z| < \frac{R}{2}\})$ with $\eta = 1$ on some open neighborhood of 0. Let $1 \leq j \leq N$ and put $\eta_j(z) = \eta(z - z_j)$. Let t_{-z_j} be the magnetic translation from $\{|z - z_j| < \frac{R}{2}\}$ to $\{|z| < \frac{R}{2}\}$ intertwining $\mathcal{L}_N$ with $\mathcal{L}_1$, and let t_{z_j} be the inverse operator. Then, the following holds.*

(i) For any $u \in D(H_N)$, we have $\eta_j u \in D(H_N)$, $t_{-z_j}(\eta_j u) \in D(H_1)$ and

$$H_N(\eta_j u) = t_{z_j} H_1 t_{-z_j}(\eta_j u). \tag{8.3}$$

(ii) For any $v \in D(H_1)$, we have $\eta v \in D(H_1)$, $t_{z_j}(\eta v) \in D(H_N)$ and

$$t_{-z_j} H_N t_{z_j}(\eta v) = H_1(\eta v). \tag{8.4}$$

Proof. Take $u \in D(H_N)$. We have $\eta_j u \in D(L_N^*)$ by Lemma 4.3. By (8.2), we have $\Xi_1(T_{j'}u) \in \operatorname{Ran} M_1$ for $j' = 1, \ldots, N$. Then,

$$\Xi_1(t_{-z_{j'}}(\eta_j u)) = \Xi_1(T_{j'}(\eta_j u)) = \delta_{j'j}\Xi_1(T_j u) \in \operatorname{Ran} M_1$$

for $j' = 1, \ldots, N$. Thus $\eta_j u \in D(H_N)$ by (8.2) and $t_{-z_j}\eta_j u \in D(H_1)$ by (8.1). The equality (8.3) follows from the intertwining property of t_{-z_j}.

Thus the assertion (i) holds. The assertion (ii) can be proved similarly. $\square$

Lemma 8.3 *Let U be an simply connected open set in $\boldsymbol{R}^2 \setminus S_N$, $m \in \boldsymbol{R}^2$ and $V = U - m = \{z - m; z \in U\}$. Let $\eta \in C_0^\infty(V)$ and put $\eta_m = \eta(z - m)$. Let t_{-m} be the magnetic translation from U to V intertwining $\mathcal{L}_N$ with $\mathcal{L}_0$ and t_m be the inverse operator. Then, the following holds.*

(i) For any $u \in D(H_N)$, we have $\eta_m u \in D(H_N)$, $t_{-m}(\eta_m u) \in D(H_0)$ and

$$H_N(\eta_m u) = t_m H_0 t_{-m}(\eta_m u). \tag{8.5}$$

(ii) For any $v \in D(H_0)$, we have $\eta v \in D(H_0)$, $t_m(\eta v) \in D(H_N)$ and

$$t_{-m} H_N t_m(\eta v) = H_0(\eta v). \tag{8.6}$$

Proof. Take $u \in D(H_N)$. Then $\eta_m u \in D(\overline{L_N})$ by (ii) of Proposition 4.2. Since $H_N \supset \overline{L_N}$, we have $\eta_m u \in D(H_N)$. Moreover,

$$\mathcal{L}_0(t_{-m}\eta_m u) = t_{-m}\mathcal{L}_N\eta_m u \in L^2(\boldsymbol{R}^2).$$

This implies $t_{-m}\eta_m u \in D(H_0)$ and (8.5), since H_0 coincides with the maximal operator (see [Ik-Ka]).

Thus the assertion (i) holds. The assertion (ii) can be proved similarly. □

Proof of Theorem 1.3. We shall introduce the notation used in [Co-Ne] and some additional notations. For $p = ((p)_x, (p)_y) \in \boldsymbol{Z}^2$ and $\delta > 0$, put

$$K(p, \delta) = \{z = (x, y) \in \boldsymbol{R}^2;\ |x - \frac{R}{100}(p)_x| \le \frac{\delta}{2},\ |y - \frac{R}{100}(p)_y| \le \frac{\delta}{2}\}$$

and let $m(p) = \frac{R}{100}p$. Since

$$\bigcup_{p \in \mathbf{Z}^2} K(p, \delta) = \boldsymbol{R}^2$$

for $\delta \ge \frac{R}{100}$, we can take $p_j \in \boldsymbol{Z}^2$ such that $z_j \in K(p_j, \frac{R}{100})$ for each $j = 1, \dots, N$. For $\delta > 0$, put

$$\begin{aligned} \mathcal{K}_j(\delta) :\ &= \bigcup_{\beta,\gamma \in \{-1,0,1\}} K\left(\left((p_j)_x + \beta, (p_j)_y + \gamma\right), \delta\right), \\ \mathcal{F}_N &= \bigcup_{j=1}^{N} \mathcal{K}_j(\frac{R}{100}) \end{aligned}$$

and let $m_j = m(p_j)$. Let

$$\begin{aligned} \Gamma_0 &= \{m(p) \in \boldsymbol{R}^2;\ p \in \boldsymbol{Z}^2,\ K(p, \frac{R}{100}) \not\subset \mathcal{F}_N\}, \\ \Gamma_1 &= \{m(p_j);\ j = 1, \dots, N\}, \\ \Gamma &= \Gamma_0 \cup \Gamma_1. \end{aligned}$$

Proof of (i) of Theorem 1.3 . Let $I = [c, d]$ be an interval on $\boldsymbol{R}$ satisfying $I \cap \{(2n-1)B; n = 1, 2, \dots\} = \emptyset$ and $c, d \notin \sigma(H_1)$. Take $E \in \sigma(H_N) \cap I$. For any $\epsilon > 0$, there is some $\psi_\epsilon \in D(H_N)$ such that

$$||\psi\epsilon|| = 1,\ ||\xi_\epsilon|| < \epsilon,\ \xi_\epsilon = (H_N - E)\psi_\epsilon. \tag{8.7}$$

Remark that when N is finite, $\sigma(H_N) \cap I$ is a finite set by (iii) of Theorem 1.2. In this case, we take ψ_ϵ as an eigenfunction corresponding to the eigenvalue E and ξ_ϵ as 0.

We shall prepare three lemmas.

Lemma 8.4 *Let* $\eta \in C^\infty(\boldsymbol{R}^2)$. *Suppose* $\operatorname{supp}|\nabla\eta|$ *is a compact set in* $\boldsymbol{R}^2 \setminus S_N$ *and*

$$\sup|\nabla\eta|^2 + \sup|\nabla\partial_x\eta|^2 + \sup|\nabla\partial_y\eta|^2 \le C$$

for some constant C. *Then,* $\eta\psi_\epsilon \in D(H_N)$ *and there exists a constant* $C_5 > 0$ *dependent only on* E, C *such that*

$$||[H_N,\eta]\psi_\epsilon||^2 \le C_5 \int_{\operatorname{supp}|\nabla\eta|} dxdy(|\psi_\epsilon(z)|^2 + |\xi_\epsilon(z)|^2). \tag{8.8}$$

Proof. First we show $\eta\psi_\epsilon \in D(H_N)$. Let χ_j be the function given in Lemma 4.4. Then $\chi_j\eta\psi_\epsilon \in D(H_N)$ by Lemma 8.2. Thus

$$ST\eta\psi_\epsilon = \sum_{j=1}^N \chi_j^2\eta\psi_\epsilon \in D(H_N),$$

where S, T are the maps defined in Lemma 4.4. Moreover, we have $(1-ST)\eta\psi_\epsilon \in D(\overline{L_N})$ by the same proof as that of (4.18). Since $\overline{L_N} \subset H_N$, we have $(1-ST)\eta\psi_\epsilon \in D(H_N)$. Thus $\eta\psi_\epsilon = (1-ST)\eta\psi_\epsilon + ST\eta\psi_\epsilon \in D(H_N)$.

By (4.5), (8.7) and the Schwarz inequality, we have

$$\begin{aligned} ||[H_N,\eta]\psi_\epsilon||^2 &\le 10C\int_{\operatorname{supp}|\nabla\eta|}(|H_N\psi_\epsilon|^2 + |\psi_\epsilon|^2)dxdy \\ &\le 10C\int_{\operatorname{supp}|\nabla\eta|}\{(2E^2+1)|\psi_\epsilon|^2 + 2|\xi_\epsilon|^2)dxdy. \end{aligned}$$

Thus the assertion holds with $C_5 = 20(E^2+1)C$. □

Lemma 8.5 *Let* $p \in \Gamma_0$. *Then, there exist constants* $C_6 > 0$ *and* $c > 0$ *dependent only on* B, E *such that*

$$\begin{aligned} &\int_{K(p,\frac{R}{100})} dxdy|\psi_\epsilon(z)|^2 \\ \le\ & C_6\left(e^{-cR^2}\int_{\overline{K(p,\frac{R}{98})\setminus K(p,\frac{R}{99})}} dxdy|\psi_\epsilon(z)|^2 + \int_{K(p,\frac{R}{98})} dxdy|\xi_\epsilon(z)|^2\right), \text{ if } R \ge 1. \end{aligned} \tag{8.9}$$

Proof. Take a smooth function η_0 satisfying $0 \le \eta_0 \le 1$ and

$$\eta_0(z) = \begin{cases} 1 & z \in K(0,\frac{R}{99}) \\ 0 & z \notin K(0,\frac{R}{98}) \end{cases},$$

$$\sup|\nabla\eta_0|^2 + \sup|\nabla\partial_x\eta_0|^2 + \sup|\nabla\partial_y\eta_0|^2 \le \frac{C}{R^2},$$

where C is a positive constant independent of R. Put $\eta_m(z) = \eta_0(z-m)$, where $m = m(p)$, $p \in \Gamma_0$. Applying Lemma 8.3 with $U = K(p, \frac{R}{98})$, we have that $\eta_m\psi_\epsilon \in D(H_N)$, that $t_{-m}\eta_m\psi_\epsilon \in D(H_0)$ and that

$$[H_N, \eta_m]\psi_\epsilon + \eta_m\xi_\epsilon = (H_N - E)(\eta_m\psi_\epsilon) = t_m(H_0 - E)t_{-m}(\eta_m\psi_\epsilon). \tag{8.10}$$

Since the support of both sides of (8.10) are concluded in $K(p, \frac{R}{98})$, we can operate $(H_0 - E)^{-1}t_{-m}$ to both sides of (8.10) and obtain

$$(H_0 - E)^{-1}t_{-m}\left([H_N, \eta_m]\psi_\epsilon + \eta_m\xi_\epsilon\right) = t_{-m}(\eta_m\psi_\epsilon). \tag{8.11}$$

Since the support of the right hand side of (8.11) is concluded in $K(0, \frac{R}{100})$, so is the left hand side. Thus we can operate t_m to both sides of (8.11) and obtain

$$t_m(H_0 - E)^{-1}t_{-m}\left([H_N, \eta_m]\psi_\epsilon + \eta_m\xi_\epsilon\right) = \eta_m\psi_\epsilon. \tag{8.12}$$

Since $\operatorname{supp} t_{-m}[H_N, \eta_m]\psi_\epsilon \subset \overline{K(0, \frac{R}{98}) \setminus K(0, \frac{R}{99})}$ and $\operatorname{dist}(K(0, \frac{R}{100}), \overline{K(0, \frac{R}{98} \setminus \frac{R}{99})}) = \frac{R}{19800}$, we have by [Co-Ne, Corollary II.1] that there exists a constant $c > 0$ dependent only on B, E such that

$$\int_{K(p,\frac{R}{100})} dxdy|t_m(H_0 - E)^{-1}t_{-m}[H_N, \eta_m]\psi_\epsilon|^2 \leq Ce^{-cR^2}||[H_N, \eta_m]\psi_\epsilon||^2 \tag{8.13}$$

for every R. Thus we have

$$\begin{aligned}
&\int_{K(p,\frac{R}{100})} |\psi_\epsilon|^2 dxdy = \int_{K(p,\frac{R}{100})} |\eta_m\psi_\epsilon|^2 dxdy \\
\leq\ & 2Ce^{-cR^2}||[H_N, \eta_m]\psi_\epsilon||^2 + 2||t_m(H_0 - E)^{-1}t_{-m}(\eta_m\xi_\epsilon)||^2 \\
\leq\ & 2CC_5e^{-cR^2}\int_{\operatorname{supp}|\nabla\eta_m|} dxdy(|\psi_\epsilon|^2 + |\xi_\epsilon|^2) \\
& +2\operatorname{dist}(\sigma(H_0), E)^{-1}\int_{\operatorname{supp}\eta_m} dxdy|\xi_\epsilon|^2 \\
\leq\ & C_6\left(e^{-cR^2}\int_{\operatorname{supp}|\nabla\eta_m|} dxdy|\psi_\epsilon(z)|^2 + \int_{\operatorname{supp}\eta_m} dxdy|\xi_\epsilon(z)|^2\right), \text{ if } R \geq 1,
\end{aligned}$$

where C_6 is a constant dependent only on B, E. We used (8.12) in the first inequality, (8.13), Lemma 8.4 and the equality $||(H_0 - E)^{-1}|| = dist(\sigma(H_0), E)^{-1}$ in the second inequality. Thus we have the conclusion. □

Lemma 8.6 *There exists constants* $C_7 > 0$ *dependent only on* B*,* E *such that*

$$\int_{\mathcal{F}_N^c} dxdy |\psi_\epsilon(z)|^2 \leq C_7 e^{-cR^2}, \text{ if } R \geq 1, \tag{8.14}$$

where $\epsilon = e^{-\frac{c}{2}R^2}$*,* c *is a constant given in Lemma 8.5 and* ψ_ϵ *satisfying (8.7).*

Proof. Summing up the inequality (8.9) for all $p \in \Gamma_0$, we have

$$\int_{\mathcal{F}_N^c} dxdy |\psi_\epsilon(z)|^2 \leq 4C_6(e^{-cR^2}||\psi_\epsilon||^2 + ||\xi_\epsilon||^2) \leq 8C_6 e^{-cR^2}, \text{ if } R \geq 1.$$

In the first inequality, we used the fact any point $z \in \boldsymbol{R}^2$ is contained in $K(p, \frac{R}{98})$ at most four p's. In the second inequality, we used (8.7) and $\epsilon = e^{-\frac{1}{2}cR^2}$. □

We shall continue the proof of (i) of Theorem 1.3. Take $\eta_0 \in C_0^\infty(\boldsymbol{R}^2)$ such that $0 \leq \eta_0 \leq 1$ and

$$\eta_0(z) = \begin{cases} 1 & z \in \mathcal{K}_0(\frac{R}{99}) \\ 0 & z \notin \mathcal{K}_0(\frac{R}{98}) \end{cases},$$

$$\sup |\nabla \eta_0|^2 + \sup |\nabla \partial_x \eta_0|^2 + \sup |\nabla \partial_y \eta_0|^2 \leq \frac{C}{R^2}, \tag{8.15}$$

where C is a positive constant independent of R and

$$\mathcal{K}_0(\delta) = \bigcup_{\beta,\delta \in \{-1,0,1\}} K((\beta, \gamma); \delta).$$

Put $\eta_{p_j}(z) = \eta_0(z - p_j)$ for $j = 1, \ldots, N$. By Lemma 8.2, we have $\eta_{p_j}\psi_\epsilon \in D(H_N)$, $t_{-z_j}\eta_{p_j}\psi_\epsilon \in D(H_1)$ and

$$\begin{aligned} (H_1 - E)t_{-z_j}\eta_{p_j}\psi_\epsilon &= t_{-z_j}(H_N - E)(\eta_{p_j}\psi_\epsilon) \\ &= t_{-z_j}([H_N, \eta_{p_j}]\psi_\epsilon + \eta_{p_j}\xi_\epsilon). \end{aligned} \tag{8.16}$$

Let c be a constant given in Lemma 8.5 and put $\epsilon = e^{-\frac{c}{2}R^2}$. Thus we have

$$\begin{aligned} & \sum_{j=1}^N ||(H_1 - E)t_{-z_j}\eta_{p_j}\psi_\epsilon||^2 \\ \leq \; & 2\sum_{j=1}^N \left(||[H_N, \eta_{p_j}]\psi_\epsilon||^2 + ||\eta_{p_j}\xi_\epsilon||^2\right) \end{aligned}$$

$$\begin{aligned}
&\leq 2(C_5+1)\sum_{j=1}^{N}\left(\int_{\overline{\mathcal{K}_j(\frac{R}{98})\setminus\mathcal{K}_j(\frac{R}{99})}} dxdy|\psi_\epsilon(z)|^2 + \int_{\mathcal{K}_j(\frac{R}{98})} dxdy|\xi_\epsilon|^2\right)\\
&\leq C\left(\int_{\mathcal{F}_N^c} dxdy|\psi_\epsilon(z)|^2+\epsilon^2\right) \leq Ce^{-cR^2},\ \text{if } R\geq 1, \qquad (8.17)
\end{aligned}$$

where C is a constant dependent only on B, E. We used (8.16) in the first inequality, (8.8) in the second, (8.7) in the third and (8.14) in the last. Moreover, we have

$$\begin{aligned}
&\sum_{j=1}^{N}||(H_1-E)t_{-z_j}\eta_{p_j}\psi_\epsilon||^2\\
\geq\ & \text{dist}\{E,\sigma(H_1)\}^2\sum_{j=1}^{N}||\eta_{p_j}\psi_\epsilon||^2\\
\geq\ & \text{dist}\{E,\sigma(H_1)\}^2\int_{\mathcal{F}_N} dxdy|\psi_\epsilon|^2\\
\geq\ & \text{dist}\{E,\sigma(H_1)\}^2(1-C_7e^{-cR^2}),\ \text{if } R\geq 1, \qquad (8.18)
\end{aligned}$$

where we used (8.14) in the last inequality. Take a large number $R_0>1$ such that $C_7e^{-cR_0^2}<\frac{1}{2}$. Then we have by (8.17) and (8.18)

$$\text{dist}\{E,\sigma(H_1)\}\leq\sqrt{2C}e^{-\frac{c}{2}R^2}, \text{if } R\geq R_0.$$

Thus (i) of Theorem 1.3 holds. $\square$

Proof of (ii) of Theorem 1.3. Let I be an interval satisfying the assumption of Theorem 1.3. By (i) of Theorem 1.3, there exists a large number $R_0>1$ and a small number $\delta_0>0$ such that

$$\begin{aligned}
&I\cap\sigma(H_N)\subset\bigcup_{E\in I\cap\sigma(H_1)}(E-\delta_0,E+\delta_0),\ \text{if } R\geq R_0,\\
&(E-\delta_0,E+\delta_0)\cap\sigma(H_1)=\{E\}\ \text{for every } E\in I\cap\sigma(H_1).
\end{aligned}$$

Thus it is sufficient to show that there exists $R_1>R_0$ such that

$$\dim\text{Ran}\,P_{(E-\delta_0,E+\delta_0)}(H_N)=Nk,\ \text{if } R\geq R_1,$$

where $k=\text{mult}(E;H_1)$.

First we show there exists $R_1 > R_0$ such that

$$\dim \operatorname{Ran} P_{(E-\delta_0,E+\delta_0)}(H_N) \geq Nk \tag{8.19}$$

for each eigenvalue E of H_1 contained in I, if $R > R_1$. Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis (as a subspace of $L^2(\boldsymbol{R}^2)$) of the eigenspace $\operatorname{Ker}(H_1 - E)$. Put

$$\tilde{e}_{j,l} = t_{z_j}(\eta_0 e_l), \ j = 1, \ldots, N, \ l = 1, \ldots, k,$$

where η_0 is the function satisfying (8.15). Put

$$V_j = \text{L.h.}\{\tilde{e}_{j,l}\}_{l=1,\ldots,k}.$$

The subspaces $\{V_j\}_{j=1}^N$ are mutually orthogonal since the support of the element of V_j and that of $V_{j'}$ do not intersect if $j \neq j'$. Put

$$V = \overset{N}{\underset{j=1}{\oplus}} V_j,$$

where $\oplus$ denotes the orthogonal direct sum of mutually orthogonal subspaces in $L^2(\boldsymbol{R}^2)$. By (ii) of Lemma 8.2, we have $V \subset D(H_N)$. Applying Lemma 8.6 to the case $N = 1$, we have

$$\int_{K(0,\frac{R}{100})^c} dxdy|e_l|^2 \leq C_7 e^{-cR^2}, \text{ if } R \geq 1, \tag{8.20}$$

where c and C_7 is a constant dependent only on B, E. By the equality

$$\begin{aligned}(\tilde{e}_{j,l}, \tilde{e}_{j,l'}) = (\eta_0 e_l, \eta_0 e_{l'}) &= (e_l, e_{l'}) - ((1-\eta_0)e_l, e_{l'}) - (\eta_0 e_l, (1-\eta_0)e_{l'}) \\ &= \delta_{l,l'} - ((1-\eta_0)e_l, e_{l'}) - (\eta_0 e_l, (1-\eta_0)e_{l'}) \end{aligned} \tag{8.21}$$

and (8.20), we have

$$|(\tilde{e}_{j,l}, \tilde{e}_{j,l'}) - \delta_{l,l'}| \leq C_8 e^{-cR^2}, \text{ if } R \geq 1, \tag{8.22}$$

where C_8 is a constant dependent only on B, E. Let $M = (m_{l,l'})_{l,l'=1,\ldots,k}$ be a $k \times k$-matrix given by $m_{l,l'} = (\tilde{e}_{j,l}, \tilde{e}_{j,l'})$. By (8.21), we see that M is a matrix independent of j. By (8.22), we have that there exists a constant $R_1 > R_0$ dependent only on B, E such that

$$|m_{l,l'}| = |(\tilde{e}_{j,l}, \tilde{e}_{j,l'})| \leq 2, \ \det M > \frac{1}{2}, \text{ if } R \geq R_1. \tag{8.23}$$

This inequality implies that M is invertible and the vectors $\{\tilde{e}_{j,l}\}_{l=1,\ldots,k}$ are linearly independent, if $R \geq R_1$. Thus

$$\dim V = \sum_{j=1}^{N} \dim V_j = Nk. \tag{8.24}$$

Let P_j be the orthogonal projection onto V_j and put

$$P_j u = \sum_{l=1}^{k} c_{j,l} \tilde{e}_{j,l}$$

for $u \in L^2(\boldsymbol{R}^2)$. The coefficient $c_{j,l}$ is written as

$$(c_{j,1}, \ldots, c_{j,k}) = (\ (P_j u, e_{j,1}), \ldots, (P_j u, e_{j,k})\ )\, M^{-1},$$

where the right hand side is a product of a column vector and a matrix. Thus we have by (8.23)

$$\sum_{l=1}^{k} |c_{j,l}|^2 \leq C_k ||P_j u||^2, \text{ if } R \geq R_1,$$

where C_k is a constant dependent only on k. Summing up this inequality for all $j = 1, \ldots, N$, we have

$$\sum_{j=1}^{N} \sum_{l=1}^{k} |c_{j,l}|^2 \leq C_k ||u||^2, \tag{8.25}$$

for every $u \in L^2(\boldsymbol{R}^2)$, if $R \geq R_1$.

Suppose that, for any positive number R_3, there exists a sequence $\{z_j\}_{j=1}^{N}$ such that $R = \inf_{j \neq j'} |z_j - z_{j'}| \geq R_3$ and $\dim \operatorname{Ran} P_{(E-\delta_0, E+\delta_0)}(H_N) \leq Nk - 1$ hold (when $N = \infty$, we replace the latter assumption by '$\dim \operatorname{Ran} P_{(E-\delta_0, E+\delta_0)}(H_\infty)$ is finite'). By (8.24), there exists $\tilde{e} = \sum_{j,l} c_{j,l} \tilde{e}_{j,l} \in V$ such that $||\tilde{e}|| = 1$ and $\tilde{e} \in \left(\operatorname{Ran} P_{(E-\delta_0, E+\delta_0)}(H_N)\right)^{\perp}$. Then we have

$$||(H_N - E)\tilde{e}|| \geq \delta_0. \tag{8.26}$$

But, we have by (ii) of Lemma 8.2

$$\begin{aligned} (H_N - E)\tilde{e} &= \sum_{j,l} c_{j,l} (H_N - E) \tilde{e}_{j,l} \\ = \quad & \sum_{j,l} c_{j,l} t_{z_j} (H_1 - E) \eta_0 e_l = \sum_{j,l} c_{j,l} t_{z_j} [H_1, \eta_0] e_l. \end{aligned}$$

The vectors $t_{z_j}[H_1,\eta_0]e_l$ and $t_{z_{j'}}[H_1,\eta_0]e_l$ are mutually orthogonal if $j \neq j'$. Applying Lemma 8.4 and Lemma 8.6 to the case $N=1$, $\psi_\epsilon = e_l$ and $\xi_\epsilon = 0$, we have

$$||t_{z_j}[H_1,\eta_0]e_l||^2 \leq C_9 e^{-cR^2}, \text{ if } R \geq 1,$$

where C_9 and c is a constant dependent only on B, E. By this inequality and (8.25), we have

$$\begin{aligned}
& ||(H_N - E)\tilde{e}||^2 \\
= & \sum_{j=1}^{N} ||\sum_{l=1}^{k} c_{j,l} t_{z_j}[H_1,\eta_0]e_l||^2 \\
\leq & \sum_{j=1}^{N} \left(\sum_{l=1}^{k} |c_{j,l}|^2\right)\left(\sum_{l=1}^{k} ||[H_1,\eta_0]e_l||^2\right) \\
\leq & \, kC_9 e^{-cR^2} \sum_{j=1}^{N}\sum_{l=1}^{k} |c_{j,l}|^2 \\
\leq & \, kC_9 C_k e^{-cR^2} ||\tilde{e}||^2 = Ce^{-cR^2}, \text{ if } R \geq R_1,
\end{aligned}$$

where C is a constant dependent only on B, E. Since $R \geq R_3$ and R_3 is arbitrary, this contradicts (8.26). Thus (8.19) holds.

Next, we assume N is finite (when $N=\infty$, the inequality (8.27) below is obvious) and prove there exists a constant $R_2 > R_0$ dependent only on B, E, δ_0 and α such that

$$\dim \operatorname{Ran} P_{(E-\delta_0, E+\delta_0)}(H_N) \leq Nk, \text{ if } R \geq R_2. \tag{8.27}$$

We prepare three lemmas.

Lemma 8.7 *Let P be an orthogonal projection and P' be a finite rank operator on a Hilbert space. Suppose that*

$$||P - P'|| < 1.$$

Then, $\dim \operatorname{Ran} P \leq \dim \operatorname{Ran} P'$.

Proof. See [Co-Ne, Proposition III.1]. □

Lemma 8.8 *There exist smooth functions* $\{\eta_m\}_{m\in\Gamma}$*, satisfying*

$$\begin{aligned} 0 \le \eta_m \le 1 \text{ for } m \in \Gamma, &\qquad \sum_{m\in\Gamma} \eta_m^2 = 1 \text{ on } \boldsymbol{R}^2, \\ \operatorname{supp}\eta_m \subset K(m, \frac{R}{99}) &\qquad \text{for } m \in \Gamma_0, \\ \operatorname{supp}\eta_{m_j} \subset \mathcal{K}_j(\frac{R}{99}) &\qquad \text{for } m_j \in \Gamma_1, \\ \sup(|\nabla\eta_m|^2 + |\nabla\partial_x\eta_m|^2 + |\nabla\partial_x\eta_m|^2 \le \frac{C}{R^2} &\qquad \text{for } m \in \Gamma, \text{ if } R \ge 1, \end{aligned} \tag{8.28}$$

where C *is a constant independent of* m*,* R*.*

Proof. Take $\zeta_0 \in C_0^\infty(K(0, \frac{1}{99}))$ and $\zeta_1 \in C_0^\infty(\mathcal{K}_0(\frac{1}{99}))$ satisfying

$$\begin{aligned} 0 \le \zeta_0 \le 1 \text{ on } \boldsymbol{R}^2, &\qquad \zeta_0 = 1 \text{ on } K(0, \frac{1}{100}) \\ 0 \le \zeta_1 \le 1 \text{ on } \boldsymbol{R}^2, &\qquad \zeta_1 = 1 \text{ on } \mathcal{K}_0(\frac{1}{100}) \end{aligned}$$

and put

$$\begin{aligned} \eta_m &= \frac{\zeta_0(\frac{z-m}{R})}{\sqrt{\sum_{m'\in\Gamma_0}\zeta_0(\frac{z-m'}{R})^2 + \sum_{m_{j'}\in\Gamma_1}\zeta_1(\frac{z-m_{j'}}{R})^2}} \quad (m \in \Gamma_0), \\ \eta_{m_j} &= \frac{\zeta_1(\frac{z-m_j}{R})}{\sqrt{\sum_{m'\in\Gamma_0}\zeta_0(\frac{z-m'}{R})^2 + \sum_{m_{j'}\in\Gamma_1}\zeta_1(\frac{z-m_{j'}}{R})^2}} \quad (m_j \in \Gamma_1). \end{aligned}$$

One can easily check that the functions above have desired properties. □

Lemma 8.9 *For* $m \in \Gamma$*, put* $\eta_0^{(m)}(z) = \eta_m(z+m)$*. Define the operator* $A_N(w)$ *by*

$$A_N(w) = \sum_{m_j\in\Gamma_1} t_{z_j}\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j} + \sum_{m\in\Gamma_0} t_m\eta_0^{(m)}(H_0 - w)^{-1}t_{-m}\eta_m$$

for $w \in \boldsymbol{C}$*,* $|w - E| = \delta_0$*. Then, the sums converge in the strong operator topology of the bounded operator from* $L^2(\boldsymbol{R}^2)$ *to* $D(H_N)$*. Moreover, there exists a constant* $C_{10} > 0$ *dependent on* B*,* E*,* δ_0*, and* α *such that*

$$||A_N(w)||_{B(L^2(\mathbf{R}^2);D(H_N))} \le C_{10}$$

for every $w \in \boldsymbol{C}$ *with* $|w - E| = \delta_0$*, if* $R \ge 1$*.*

Proof. It is sufficient to show that there exists a constant $C_{10} > 0$ dependent on B, E, δ_0, and α such that

$$||A_N(w)\phi||^2_{B(L^2(\mathbf{R}^2);D(H_N))} \leq C_{10}||\phi||^2 \tag{8.29}$$

for every $\phi \in C_0^\infty(\boldsymbol{R}^2)$ (notice that the sum in the definition of $A_N(w)\phi$ becomes a finite sum for $\phi \in C_0^\infty(\boldsymbol{R}^2)$; the strong convergence follows from (8.29) and an $\epsilon/3$-argument).

Take $\phi \in C_0^\infty(\boldsymbol{R}^2)$. Then we have $(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi \in D(H_1)$. Thus $t_{z_j}\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi \in D(H_N)$ by Lemma 8.2, for $m_j \in \Gamma_1$. Similarly we have $(H_0 - w)^{-1}t_{-m}\eta_m\phi \in D(H_0)$ and $t_m\eta_0^{(m)}(H_0 - w)^{-1}t_{-m}\eta_m\phi \in D(H_N)$ for $m \in \Gamma_0$, by Lemma 8.3.

Since the support of the functions

$$\{t_{z_j}\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi\}_{m_j\in\Gamma_1}$$

are disjoint, we see that these functions are mutually orthogonal in $D(H_N)$. Similarly, the functions

$$\{t_m\eta_0^{(m)}(H_0 - w)^{-1}t_{-m}\eta_m\phi\}_{m\in\Gamma_{0,\beta,\gamma}},$$
$$\Gamma_{0,\beta,\gamma} = \Gamma_0 \cap \{m(p);\ p \in 2\boldsymbol{Z}^2 + (\beta,\gamma)\}$$

are mutually orthogonal in $D(H_N)$, for each $\beta = 0, 1$ and $\gamma = 0, 1$.

We have

$$\begin{aligned}
& ||t_{z_j}\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi||_N^2 \\
= \ & ||\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi||_1^2 \\
\leq \ & C_1||(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi||_1^2 \\
\leq \ & C_1(1 + E + \delta_0)^2(1 + ||(H_1 - w)^{-1}||^2_{B(L^2(\mathbf{R}^2);L^2(\mathbf{R}^2))})||\eta_{m_j}\phi||^2
\end{aligned}$$

for $m_j \in \Gamma_1$ and $w \in \boldsymbol{C}$ with $|w - E| = \delta_0$, if $R \geq 1$. We used (4.6). Notice that we can take the constant C_1 in Lemma 4.3 uniformly in $R \geq 1$, by (8.28). Similarly, we have

$$\begin{aligned}
& ||t_m\eta_0^{(m)}(H_0 - w)^{-1}t_{-m}\eta_m\phi||_N^2 \\
\leq \ & C_1(1 + E + \delta_0)^2(1 + ||(H_0 - w)^{-1}||^2_{B(L^2(\mathbf{R}^2);L^2(\mathbf{R}^2))})||\eta_m\phi||^2
\end{aligned}$$

for $m \in \Gamma_0$ and $w \in \boldsymbol{C}$ with $|w - E| = \delta_0$, if $R \geq 1$. Thus we have

$$\begin{aligned}
||A_N(w)\phi||_N^2 \ \leq \ & \sum_{m_j\in\Gamma_1} ||t_{z_j}\eta_0^{(m_j)}(H_1 - w)^{-1}t_{-z_j}\eta_{m_j}\phi||_N^2 \\
& + \sum_{\beta,\gamma=0,1}\ \sum_{m\in\Gamma_{0,\beta,\gamma}} ||t_m\eta_0^{(m)}(H_0 - w)^{-1}t_{-m}\eta_m\phi||_N^2 \\
\leq \ & C_{10}||\phi||^2, \text{ if } R \geq 1,
\end{aligned}$$

where

$$C_{10} = 4C_1(1+E+\delta_0)^2 \left(1 + \sup_{|w-E|=\delta_0} \max_{n=0,1} \left(||(H_n - w)^{-1}||^2_{B(L^2(\mathbf{R}^2);L^2(\mathbf{R}^2))} \right) \right)$$

and we used the fact $\#\{m \in \Gamma; z \in \operatorname{supp} \eta_m\} \leq 4$ for any $z \in \boldsymbol{R}^2$. $\Box$

We have

$$\begin{aligned}
(H_N - w)A_N(w) &= \sum_{m_j \in \Gamma_1} (H_N - w) t_{z_j} \eta_0^{(m_j)} (H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \\
&\quad + \sum_{m \in \Gamma_0} (H_N - w) t_m \eta_0^{(m)} (H_0 - w)^{-1} t_{-m} \eta_m \\
&= \sum_{m_j \in \Gamma_1} (t_{z_j} [H_1, \eta_0^{(m_j)}] (H_1 - w)^{-1} t_{-z_j} \eta_{m_j} + \eta_{m_j}^2) \\
&\quad + \sum_{m \in \Gamma_0} (t_m [H_0, \eta_0^{(m)}] (H_0 - w)^{-1} t_{-m} \eta_m + \eta_m^2) \\
&= 1 + T_N(w),
\end{aligned} \tag{8.30}$$

where

$$\begin{aligned}
T_N(w) &= \sum_{m_j \in \Gamma_1} t_{z_j} [H_1, \eta_0^{(m_j)}] (H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \\
&\quad + \sum_{m \in \Gamma_0} t_m [H_0, \eta_0^{(m)}] (H_0 - w)^{-1} t_{-m} \eta_m.
\end{aligned} \tag{8.31}$$

We used the intertwining property of t_{-m} and the equality $\sum_{m \in \Gamma} \eta_m^2 = 1$. Notice that the operator $[H_1, \eta_0^{(m_j)}](H_1 - w)^{-1}$ is well-defined by Lemma 8.2, and the operator $[H_0, \eta_0^{(m)}](H_0 - w)^{-1}$ is well-defined by Lemma 8.3.

Lemma 8.10 *Two sums in the right hand side of (8.31) converge in the strong operator topology of the bounded operator from $L^2(\boldsymbol{R}^2)$ to $L^2(\boldsymbol{R}^2)$. Moreover, there is a constant $C_{11} > 0$ dependent only on B, E, δ_0 and α such that*

$$||T_N(w)||_{B(L^2(\mathbf{R}^2);L^2(\mathbf{R}^2))} \leq \frac{C_{11}}{R}$$

for every $w \in \boldsymbol{C}$ with $|w - E| = \delta_0$, if $R \geq 1$.

Proof. By the similar argument in the proof of Lemma 8.9, we see that it is sufficient to show that there is a constant $C' > 0$ dependent only on B, E, δ_0 and α such that

$$\begin{aligned} \|t_{z_j}[H_1, \eta_0^{(m_j)}](H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \phi\|^2 &\leq \frac{C'}{R^2}\|\phi\|^2 \quad m_j \in \Gamma_1 & (8.32) \\ \|t_m[H_0, \eta_0^{(m)}](H_N - w)^{-1} t_{-m} \eta_m \phi\|^2 &\leq \frac{C'}{R^2}\|\phi\|^2 \quad m \in \Gamma_0 & (8.33) \end{aligned}$$

for every $\phi \in C_0^\infty(\boldsymbol{R}^2)$ and every $w \in \boldsymbol{C}$ with $|w - E| = \delta_0$, if $R \geq 1$. By (4.5) and (8.28), we have

$$\begin{aligned} & \|t_{z_j}[H_1, \eta_0^{(m_j)}](H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \phi\|^2 \\ = \; & \|[H_1, \eta_0^{(m_j)}](H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \phi\|^2 \\ \leq \; & C_0 \|(H_1 - w)^{-1} t_{-z_j} \eta_{m_j} \phi\|_1^2 \\ \leq \; & \frac{10C}{R^2}(1 + E + \delta_0)^2 (1 + \|(H_1 - w)^{-1}\|)_{B(L^2(\mathbf{R}^2); L^2(\mathbf{R}^2))} \|\phi\|^2, \text{ if } R \geq 1, \end{aligned}$$

where $C_0 = 10 \sup(|\nabla \eta_{m_j}|^2 + |\nabla(\partial_x \eta_{m_j})|^2 + |\nabla(\partial_y \eta_{m_j})|^2)$. Thus (8.32) holds. We can prove (8.33) similarly. □

Take a large number $R' > R_0$ such that $\frac{C_{11}}{R'} < \frac{1}{2}$. By Lemma 8.10, we have $\|T_N(w)\| \leq \frac{1}{2}$ and thus $1 + T_N(w)$ is invertible for $w \in C$ with $|w - E| = \delta_0$, if $R \geq R'$. Then $H_N - w$ is also invertible and its inverse is given by

$$(H_N - w)^{-1} = A_N(w) - A_N(w) T_N(w)(1 + T_N(w))^{-1}. \tag{8.34}$$

Integrating (8.34) on $\{w; |w - E| = \delta_0\}$, we have

$$P_{(E-\delta_0, E+\delta_0)}(H_N) = \sum_{m_j \in \Gamma_1} t_{z_j} \eta_0^{(m_j)} P_{(E-\delta_0, E+\delta_0)}(H_1) t_{-z_j} \eta_{m_j} + R_N, \tag{8.35}$$

where

$$R_N = -\oint_{|w-E|=\delta_0} A_N(w) T_N(w)(1 + T_N(w))^{-1}.$$

By Lemma 8.9 and Lemma 8.10, there exist a constant $R_2 > R'$ dependent only on B, E, δ_0 and α such that

$$\|R_N\| < 1, \text{ if } R \geq R_2. \tag{8.36}$$

Since the first term of the right hand side of (8.35) is a linear operator of at most rank Nk, we have (8.27) by Lemma 8.7 and (8.36).□

Thus the proof of Theorem 1.3 is completed. □

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