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Inverse Scattering on the Line with Incomplete Scattering Data

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ABSTRACT. The Schrödinger equation is considered on the line when the potential is real valued, compactly supported, and square integrable. The nonuniqueness is analyzed in the recovery of such a potential from the data consisting of the ratio of a corresponding reflection coefficient to the transmission coefficient. It is shown that there are a discrete number of potentials corresponding to the data and that their L^2-norms are related to each other in a simple manner. All those potentials are identified, and it is shown how an additional estimate on the L^2-norm in the data can uniquely identify the corresponding potential. The recovery is illustrated with some explicit examples.

1. Introduction

In this paper we analyze the recovery of the potential in the Schrödinger equation on the line from a set of scattering data containing no information on bound states. Our work is motivated by the following question of Paul Sacks: Consider two potentials in the Schrödinger equation where one potential is obtained from the other by adding a bound state. Can we compare the L^2-norms of these two potentials, and can we conclude that the potential with fewer bound states has a smaller L^2-norm? By using (2.13) and (2.17) these questions can be answered as follows: Take a square-integrable potential and add a bound state with bound-state energy $-\kappa^2$ and any bound-state norming constant. The new potential will have a larger L^2-norm differing from the previous L^2-norm by the exact value of $16\kappa^3/3$. Note that such a difference is independent of the value of the norming constant used, and hence L^2-norms of square-integrable potentials are affected only by bound-state energies and not by norming constants.

Our work is also motivated by the work of Rundell and Sacks [1], where it was shown that a bounded, real-valued, compactly-supported potential with a sufficiently small L^2-norm is uniquely determined by the corresponding ratio of a

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reflection coefficient to the transmission coefficient. With the help of the results in [2], our work here quantifies the smallness of the L^2-norm in the result of [1]. In Section 3 we present the exact least upper bound for that L^2-norm, below which we are assured the unique determination of a real-valued, compactly-supported, square-integrable potential in terms of the ratio of a reflection coefficient to the transmission coefficient; we do not require the potential to be bounded.

Let us now establish our notation. We consider the Schrödinger equation

$$\psi''(k,x)+k^2\,\psi(k,x)=V(x)\,\psi(k,x),\qquad x\in\mathbf{R},\tag{1.1}$$

where the potential V belongs to the Faddeev class, i.e. it is real valued, measurable, and in $L^1_1(\mathbf{R})$, the class of measurable functions on the real axis $\mathbf{R}$ such that $\int_{-\infty}^{\infty} dx\,(1+|x|)\,|V(x)|$ is finite. The prime is used for the derivative with respect to the spatial coordinate x. The Jost solutions $f_{\rm l}$ and $f_{\rm r}$, from the left and right, respectively, satisfy the respective boundary conditions

$$e^{-ikx}f_{\rm l}(k,x)=1+o(1),\quad e^{-ikx}f_{\rm l}'(k,x)=ik+o(1),\qquad x\to+\infty,$$

$$e^{ikx}f_{\rm r}(k,x)=1+o(1),\quad e^{ikx}f_{\rm r}'(k,x)=-ik+o(1),\qquad x\to-\infty,$$

and the transmission coefficient T, and the reflection coefficients L and R, from the left and right, respectively, are obtained from the spatial asymptotics

$$f_{\rm l}(k,x)=\frac{e^{ikx}}{T(k)}+\frac{L(k)\,e^{-ikx}}{T(k)}+o(1),\qquad x\to-\infty,$$

$$f_{\rm r}(k,x)=\frac{e^{-ikx}}{T(k)}+\frac{R(k)\,e^{ikx}}{T(k)}+o(1),\qquad x\to+\infty.$$

A bound state of (1.1) is a square-integrable solution, and such states occur only at the k-values on $\mathbf{I}^+$ in the upper half complex k-plane $\mathbf{C}^+$ where $T(k)$ has (simple) poles. Note that $\mathbf{I}^+:=i(0,+\infty)$ denotes the positive imaginary axis. Later we will let $\overline{\mathbf{C}^+}:=\mathbf{C}^+\cup\mathbf{R}$ and $\mathbf{I}^-:=i(-\infty,0)$. The behavior at $k=0$ tells us whether the potential in (1.1) is generic or exceptional: The generic case occurs if $T(0)=0$ and the exceptional case occurs if $T(0)\neq 0$. For a review of scattering and bound states of (1.1), the reader is referred to [3-9] and the references therein.

Our paper is organized as follows: In Section 2 we briefly review the effect of adding a bound state to a potential and show that certain integrals of the resulting potential remain unaffected by the bound-state norming constant but affected only by the bound-state energy and in a rather simple manner. In Section 3 we analyze a consequence of the result of Section 2 in the recovery of a real-valued, compactly-supported, square-integrable potential in terms of the data $L(k)/T(k)$. We show that, corresponding to that data, there are a discrete number of potentials, and an additional estimate on the L^2-norm in the data allows the unique identification of a potential among all. We also illustrate the recovery with some explicit examples.

2. Effect of Bound States on Norms of a Potential

Let $V^{[0]}$ denote a potential in the Faddeev class with no bound states. We use $V^{[N]}$ for the potential obtained from $V^{[0]}$ by adding N bound states at $k=i\kappa_j$ with the corresponding bound-state dependency constants γ_j, where we have the ordering $0<\kappa_1<\cdots<\kappa_N$. The superscript $[j]$ refers to quantities associated with the potential $V^{[j]}$; for example, $T^{[j]}$, $R^{[j]}$, and $L^{[j]}$ denote the scattering coefficients, and $f_{\mathrm{l}}^{[j]}$ and $f_{\mathrm{r}}^{[j]}$ denote the left and right Jost solutions. Recall [4,9] that the dependency constants γ_j are defined as

$$\gamma_j := \frac{f_{\mathrm{l}}^{[N]}(i\kappa_j, x)}{f_{\mathrm{r}}^{[N]}(i\kappa_j, x)}, \qquad 1 \leq j \leq N,$$

and the sign of γ_j is such that $(-1)^{N-j}\gamma_j > 0$. It is already known that

$$T^{[N]}(k) = T^{[0]}(k) \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j}, \tag{2.1}$$

$$R^{[N]}(k) = (-1)^N R^{[0]}(k) \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j}, \quad L^{[N]}(k) = (-1)^N L^{[0]}(k) \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j}. \tag{2.2}$$

For the known facts listed in this section, we refer the reader to [4], where it is shown that bound states can be added to a potential via the Darboux transformation. We have

$$V^{[j]}(x) - V^{[j-1]}(x) = -2\,\mu_j'(x), \qquad 1 \leq j \leq N, \tag{2.3}$$

where we have defined

$$\mu_j(x) := \frac{\chi_j'(x)}{\chi_j(x)}, \quad \chi_j(x) := f_{\mathrm{l}}^{[j-1]}(i\kappa_j, x) + |\gamma_j|\, f_{\mathrm{r}}^{[j-1]}(i\kappa_j, x). \tag{2.4}$$

It is known that $\chi_j(x)$ is continuous, strictly positive, and differentiable. In fact, as seen from (2.4) we have

$$\mu_j'(x) = V^{[j-1]}(x) + \kappa_j^2 - \mu_j(x)^2,$$

and hence from (2.3) it follows that

$$V^{[j]}(x) + V^{[j-1]}(x) = 2\left[\mu_j(x)^2 - \kappa_j^2\right]. \tag{2.5}$$

Define

$$I_{j,n}(x) := \left[V^{[j]}(x) - V^{[j-1]}(x)\right]\left[V^{[j]}(x) + V^{[j-1]}(x)\right]^n, \qquad n \geq 0, \quad 1 \leq j \leq N. \tag{2.6}$$

Theorem 2.1. *Let $V^{[j]}$ be the potential obtained from $V^{[0]}$ by adding bound states of energy $-\kappa_1^2, \ldots, -\kappa_j^2$, and assume that $V^{[0]}$ belongs to the Faddeev class without any bound states. We then have*

$$\int_{-\infty}^{\infty} dx\, I_{j,n}(x) = (-1)^{n+1} 2^{n+2} \kappa_j^{2n+1} \frac{n!}{(2n+1)!!}, \qquad n \geq 0, \quad 1 \leq j \leq N, \tag{2.7}$$

where $(2n+1)!! := (1)(3)(5)\cdots(2n+1)$.

PROOF. Taking the nth power in (2.5) and expanding the result, from (2.3) and (2.5) we get

$$I_{j,n}(x)=-2^{n+1}\frac{d}{dx}\sum_{p=0}^{n}(-1)^{n-p}\kappa_j^{2(n-p)}\binom{n}{p}\frac{\mu_j(x)^{2p+1}}{2p+1}, \tag{2.8}$$

where $\binom{n}{p}:=\dfrac{n!}{p!\,(n-p)!}$ is the binomial coefficient. It is already known that

$$\mu_j(x)=\begin{cases}\kappa_j+o(1), & x\to+\infty,\\ -\kappa_j+o(1), & x\to-\infty.\end{cases} \tag{2.9}$$

Integrating (2.6) on $\mathbf{R}$ and using (2.9), we get

$$\int_{-\infty}^{\infty}dx\, I_{j,n}(x)=(-1)^{n+1}2^{n+2}\kappa_j^{2n+1}\sum_{p=0}^{n}(-1)^p\binom{n}{p}\frac{1}{2p+1}. \tag{2.10}$$

Note that the summation in (2.10) can be evaluated explicitly with the help of

$$\sum_{p=0}^{n}(-1)^p\binom{n}{p}\frac{1}{2p+1}=\int_0^1 dx\,(1-x^2)^n=\frac{n!}{(2n+1)!!},\qquad n\ge 0. \tag{2.11}$$

Thus, using (2.11) in (2.10) we establish (2.7). □

The result in (2.7) is remarkable in the sense that even though the integrand $I_{j,n}(x)$ depends on the bound-state data $\{\kappa_p,\gamma_p\}_{p=1}^{j}$, its integral given in (2.7) is independent of the bound-state data, except for a rather simple κ_j-dependence.

For $n=0$ and $n=1$, respectively, from (2.7) we get

$$\int_{-\infty}^{\infty}dx\,\left[V^{[j]}(x)-V^{[j-1]}(x)\right]=-4\kappa_j,\qquad 1\le j\le N, \tag{2.12}$$

$$\int_{-\infty}^{\infty}dx\,\left[V^{[j]}(x)^2-V^{[j-1]}(x)^2\right]=\frac{16}{3}\kappa_j^3,\qquad 1\le j\le N. \tag{2.13}$$

By summing both sides in each of (2.12) and (2.13) over j, we get the following:

COROLLARY 2.2. *Let $V^{[0]}$ be a potential in the Faddeev class with no bound states; add N bound states with energy $-\kappa_1^2,\dots,-\kappa_N^2$, resulting in the potential $V^{[N]}$. We then have*

$$\int_{-\infty}^{\infty}dx\,\left[V^{[N]}(x)-V^{[0]}(x)\right]=-4\sum_{j=1}^{N}\kappa_j, \tag{2.16}$$

$$\int_{-\infty}^{\infty}dx\,\left[V^{[N]}(x)^2-V^{[0]}(x)^2\right]=\frac{16}{3}\sum_{j=1}^{N}\kappa_j^3. \tag{2.17}$$

Let us indicate some resemblance between the result in (2.7) and the conserved quantities for an evolution equation that is exactly solvable by the inverse scattering transform [12-14]. For example, consider the time-evolution of the scattering data of (1.1) as $T(k)\mapsto T(k)$, $L(k)\mapsto L(k)\,e^{-8ik^3t}$, $\kappa_j\mapsto\kappa_j$, and $\gamma_j\mapsto\gamma_j e^{-8\kappa_j^3 t}$. The potential of (1.1) then evolves as $V(x)\mapsto u(x,t)$, where $u(x,t)$ satisfies the initial-value problem for the Korteweg-de Vries equation (KdV)

$$u_t-6uu_x+u_{xxx}=0,\qquad x\in\mathbf{R},\quad t>0;\quad u(x,0)=V(x).$$

It is known [12-14] that $\int_{-\infty}^{\infty}dx\,u(x,t)$, $\int_{-\infty}^{\infty}dx\,u(x,t)^2$, and an infinite number of other integrals are independent of t even though their integrands contain t explicitly. Such quantities are known as the conserved quantities for the KdV. Consider now, for example, (2.16) and (2.17), and let us time evolve the potentials $V^{[0]}(x)$ and $V^{[N]}(x)$ to obtain the corresponding solutions $u^{[0]}(x,t)$ and $u^{[N]}(x,t)$ of the KdV. Due to the fact that the bound-state energies $-\kappa_j^2$ remain unchanged during the time evolution and that the right hand sides in (2.16) and (2.17) do not contain the dependency constants γ_j, we have

$$\int_{-\infty}^{\infty}dx\,\left[u^{[N]}(x,t)-u^{[0]}(x,t)\right]=-4\sum_{j=1}^{N}\kappa_j,$$

$$\int_{-\infty}^{\infty}dx\,\left[u^{[N]}(x,t)^2-u^{[0]}(x,t)^2\right]=\frac{16}{3}\sum_{j=1}^{N}\kappa_j^3.$$

Other similar conserved quantities for the KdV can be obtained with the help of (2.7).

3. Recovery of the Potential from $L(k)/T(k)$

In [1] the recovery of a bounded, real-valued, compactly-supported potential is considered in terms of the data $\mathcal{D}(k):=L(k)/T(k)$ known for $k\in\mathbf{R}$. In the class of such potentials corresponding to the same $\mathcal{D}(k)$, it was shown (cf. Theorem 2.3 of [1]) that there exists a positive constant C such that if V_1 and V_2 are two potentials with L^2-norms not exceeding C then $V_1\equiv V_2$. The uniqueness and the reconstruction were obtained by transforming the problem into an equivalent time-domain problem; however, the value of C was left unspecified. In this section, we show how the value of C can be specified.

Recently, we have analyzed [2] the recovery of the potential V of (1.1) from $\mathcal{D}(k)$ when V belongs to the Faddeev class. In this inverse problem, the construction of V is equivalent to the construction of the data $\{L(k),N,\{\kappa_j\},\{\gamma_j\}\}$, where L is the left reflection coefficient, N is the number of bound states, the set $\{-\kappa_j^2\}_{j=1}^N$ corresponds to the bound-state energies, and the set $\{\gamma_j\}_{j=1}^N$ corresponds to the bound-state dependency constants. We have four cases to consider:

(a) No information is available on the support of V, and the only data available is $\mathcal{D}(k)$.
(b) In addition to $\mathcal{D}(k)$, it is known that the support of V is confined to a half line. In this case, there is no loss of generality in assuming that $V\equiv 0$ for $x<0$.
(c) In addition to $\mathcal{D}(k)$, it is known that the support of V is confined a finite interval. In this case, there is no loss of generality in assuming that $V\equiv 0$ for $x\notin[0,1]$.
(d) In addition to $\mathcal{D}(k)$ and knowledge that $V\equiv 0$ for $x\notin[0,1]$, it is known that V is square integrable and some information related to the L^2-norm

is available. Such additional information may be in the form of a positive constant C which acts as an upper bound on the L^2-norm.

Let us consider the construction of V or equivalently of $\{L(k), N, \{\kappa_j\}, \{\gamma_j\}\}$ in each of these four cases. For the analysis in the first three cases we refer the reader to [2] and give a brief summary below. Our results show that in case (c), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, we are able to determine all the corresponding potentials, there are a discrete number of such potentials, the L^2-norm of each such potential is readily evaluated with the help of (2.17), and appropriate additional information on the L^2-norm enables us to further restrict the set of potentials corresponding to $\mathcal{D}(k)$. We also explain how the constant C in Theorem 2.3 of [1] arises: That constant allows us to identify the potential with the smallest L^2-norm among all those corresponding to the same $\mathcal{D}(k)$. By analyzing the inverse problem stated in (d), we show how to determine the precise values of C that can be used in [1].

Case (a): Recovery of V from $\mathcal{D}$ with no Support Information.
If no information other than $\mathcal{D}(k)$ is available, we have the following:

(a.i) If $\mathcal{D}(k)$ is bounded at $k = 0$, then there is no restriction on N and hence $N \in \{0, 1, 2, \dots\}$. Note that this case corresponds to the exceptional case for (1.1).

(a.ii) If $\mathcal{D}(k)$ is unbounded at $k = 0$, then $\lim_{k\to 0}[2ik\,\mathcal{D}(k)]$ is either a positive constant or a negative constant. Thus, either $\mathcal{D}(k) \to -\infty$ or $\mathcal{D}(k) \to +\infty$ as $k \to 0$ on $\mathbf{I}^+$. In the former case N must be even, i.e. $N \in \{0, 2, 4, \dots\}$; in the latter case N must be odd, i.e. $N \in \{1, 3, 5, \dots\}$. Note that both these correspond to the generic case for (1.1).

(a.iii) For each N-value resulting from (i) or (ii), given $\mathcal{D}(k)$ there corresponds a $2N$-parameter family of potentials where the parameter set is $\{\kappa_j, \gamma_j\}_{j=1}^N$. There are no restrictions on the κ_j other than $0 < \kappa_1 < \cdots < \kappa_N$. There are no restrictions on the γ_j other than $(-1)^{N-j}\gamma_j > 0$.

From the data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$, one uniquely constructs

$$T^{[0]}(k) = \exp\left(\frac{-1}{2\pi i}\int_{-\infty}^{\infty} ds\,\frac{\log\left(1 + |\mathcal{D}(s)|^2\right)}{s - k - i0^+}\right), \qquad k \in \overline{\mathbf{C}^+}. \tag{3.1}$$

Then, with the help of (2.1), it is seen that the set $\{\mathcal{D}(k), N, \{\kappa_j\}\}$ leads to the left reflection coefficient given by

$$L(k) = \mathcal{D}(k)\,T^{[0]}(k)\prod_{j=1}^{N}\frac{k + i\kappa_j}{k - i\kappa_j}, \qquad k \in \mathbf{R}. \tag{3.2}$$

Note that $T^{[0]}(k)$ appearing in (3.1) and (3.2) corresponds to the transmission coefficient for the potential $V^{[0]}$, which is obtained by removing all the N bound states from V. The left and right reflection coefficients, $L^{[0]}(k)$ and $R^{[0]}(k)$, respectively, corresponding to $V^{[0]}$ are uniquely determined only in the generic case as [cf. (2.2)]

$$L^{[0]}(k) = (-1)^N\mathcal{D}(k)\,T^{[0]}(k), \quad R^{[0]}(k) = (-1)^{N-1}\mathcal{D}(-k)\,T^{[0]}(k), \qquad k \in \mathbf{R}, \tag{3.3}$$

because only in the generic case $(-1)^N$ is uniquely determined from $\mathcal{D}(k)$. In the exceptional case, the value of $(-1)^N$ cannot be determined from $\mathcal{D}(k)$ and hence there are two choices for $V^{[0]}$, which we denote by $V_1^{[0]}$ and $V_2^{[0]}$, respectively, with the corresponding scattering coefficients determined in terms of $\mathcal{D}(k)$ and $T^{[0]}(k)$ in (3.1) as follows:

$$T_1^{[0]}(k)=T^{[0]}(k),\quad L_1^{[0]}(k)=\mathcal{D}(k)\,T^{[0]}(k),\quad R_1^{[0]}(k)=-\mathcal{D}(-k)\,T^{[0]}(k),$$
$$T_2^{[0]}(k)=T^{[0]}(k),\quad L_2^{[0]}(k)=-\mathcal{D}(k)\,T^{[0]}(k),\quad R_2^{[0]}(k)=\mathcal{D}(-k)\,T^{[0]}(k).$$

For the comparison of two potentials with the same transmission coefficient but with reflection coefficients differing in sign, the reader is referred to [9-11]. As the next proposition shows, even though $V_1^{[0]}\not\equiv V_2^{[0]}$, some of their characteristic features are related.

PROPOSITION 3.1. *Let $V_1^{[0]}$ and $V_2^{[0]}$ be two exceptional potentials in the Faddeev class with no bound states, and assume that $T_1^{[0]}\equiv T_2^{[0]}$, $L_1^{[0]}\equiv -L_2^{[0]}$, and $R_1^{[0]}\equiv -R_2^{[0]}$, i.e. their reflection coefficients differ in sign and their transmission coefficients are the same for all $k\in\mathbf{R}$. Then we have the following:*

(i) $\displaystyle\int_{-\infty}^{\infty}dx\,\left[V_2^{[0]}(x)-V_1^{[0]}(x)\right]\left[V_2^{[0]}(x)+V_1^{[0]}(x)\right]^n=0,\qquad n\ge 0.$

(ii) *$V_1^{[0]}$ vanishes on a half line if and only if $V_2^{[0]}$ vanishes on the same half line. Consequently, $V_1^{[0]}$ vanishes outside some interval if and only if $V_2^{[0]}$ vanishes outside that interval.*

(iii) *If $V_1^{[0]}$ vanishes on $\mathbf{R}^-$ and is continuous on the interval $(0,\delta)$ for some $\delta>0$, then $V_1^{[0]}(0^+)=-V_2^{[0]}(0^+)$. Similarly, if $V_1^{[0]}$ vanishes on $\mathbf{R}^+$ and is continuous on the interval $(-\delta,0)$ for some $\delta>0$, then $V_1^{[0]}(0^-)=-V_2^{[0]}(0^-)$.*

PROOF. From (2.24) and (2.25) of [11] we have

$$V_2^{[0]}(x)=V_1^{[0]}(x)-2\,\rho_1'(x)=2\,\rho_1(x)^2-V_1^{[0]}(x),\tag{3.4}$$

where

$$\rho_1(x):=\frac{f_{1\mathrm{l}}^{[0]\prime}(0,x)}{f_{1\mathrm{l}}^{[0]}(0,x)}=\frac{f_{1\mathrm{r}}^{[0]\prime}(0,x)}{f_{1\mathrm{r}}^{[0]}(0,x)},$$

with $f_{1\mathrm{l}}^{[0]}(k,x)$ and $f_{1\mathrm{r}}^{[0]}(k,x)$ being the left and right Jost solutions for the potential $V_1^{[0]}$. It is known that $f_{1\mathrm{l}}^{[0]}(0,x)$ is continuous and strictly positive and that $\rho_1(x)=o(1/x)$ as $x\to\pm\infty$. Hence, with the help of (3.4) we get [cf. (2.8)]

$$\left[V_2^{[0]}(x)-V_1^{[0]}(x)\right]\left[V_2^{[0]}(x)+V_1^{[0]}(x)\right]^n=-\frac{2^{n+1}}{2n+1}\left[\rho_1(x)^{2n+1}\right]',\qquad n\ge 0,$$

and integrating both sides over $\mathbf{R}$ we obtain (i). To prove (ii), notice that there is no loss of generality in choosing the half line as $\mathbf{R}^-$. If $V_1^{[0]}\equiv 0$ for $x<0$, then $f_{1\mathrm{r}}^{[0]}(0,x)=1$ for $x\le 0$, and hence $\rho_1(x)=0$ for $x\le 0$. Thus, from (3.4) it follows that $V_2^{[0]}\equiv 0$ for $x<0$ as well. Conversely, it follows that $V_1^{[0]}\equiv 0$ on $\mathbf{R}^-$ whenever $V_2^{[0]}\equiv 0$ there; thus, we have proved (ii). From (3.4) we see that the first statement in (iii) holds whenever $\rho_1(0^+)=0$, which is the case due to the continuity of $\rho_1(x)$ at $x=0$ and $\rho_1(x)=0$ for $x\le 0$, which is satisfied when $V_1^{[0]}\equiv 0$ for $x<0$. The second statement in (iii) is obtained in a similar manner. □

Note that Proposition 3.1(i) holds even when n is a noninteger. By letting $n=0$ and $n=1$ there and using the fact that a potential in the Faddeev class is integrable, we obtain

$$(3.5)\quad \int_{-\infty}^{\infty} dx\, V_1^{[0]}(x) = \int_{-\infty}^{\infty} dx\, V_2^{[0]}(x), \qquad \int_{-\infty}^{\infty} dx\, \left[V_1^{[0]}(x)^2 - V_2^{[0]}(x)^2\right] = 0.$$

For smooth potentials, we refer the reader to (2.101) of [10] for results similar to (3.5) and their generalizations.

Case (b): Recovery of V from $\mathcal{D}$ with Half-line Support.

If $\mathcal{D}(k)$ is given for $k \in \mathbf{R}$ and if it is also known that $V \equiv 0$ for $x < 0$, then, in addition to all the results given in case (a), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

(b.iii) $\mathcal{D}(k)$ has a unique analytic extension to $k \in \mathbf{C}^+$ and such an extension is uniquely determined by our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. The value of N must satisfy $N \le Z + 1$, where Z denotes the number of zeros of $\mathcal{D}(k)$ on $\mathbf{I}^+$. In fact, from the proof of Proposition 3.1 in [2] it follows that if $\mathcal{D}(k)$ has multiple zeros on $\mathbf{I}^+$, then Z is actually the number of distinct zeros of odd multiplicity, without counting the multiplicities.

(b.iv) For each N-value resulting from restrictions (a.i), (a.ii), and (b.iii), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, there corresponds an N-parameter family of potentials where the parameter set is $\{\kappa_j\}_{j=1}^N$. The κ_j satisfy the restrictions $0 < \kappa_1 < \cdots < \kappa_N$ and $(-1)^{N-j}\mathcal{D}(i\kappa_j) > 0$ for $j = 1, \dots, N$. The latter restriction confines the κ_j to subintervals whose endpoints are uniquely determined by the zeros of $\mathcal{D}(k)$ on $\mathbf{I}^+$. The dependency constants γ_j are uniquely determined as $\gamma_j = \mathcal{D}(i\kappa_j)$ and hence they are not free parameters. The left reflection coefficient $L(k)$ given in (3.2) becomes meromorphic in $\mathbf{C}^+$ with simple poles at $k = i\kappa_j$ for $j = 1, \dots, N$. Thus, (3.2) now holds for $k \in \overline{\mathbf{C}^+}$ and the left reflection coefficient $L^{[0]}(k)$ given in (3.3) becomes analytic in $\mathbf{C}^+$.

Case (c): Recovery of V from $\mathcal{D}$ with Compact Support.

If $\mathcal{D}(k)$ for $k \in \mathbf{R}$ is given and if it is also known that $V \equiv 0$ for $x \notin [0, 1]$, then, in addition to all the results in cases (a) and (b), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

(c.iii) The quantity $k\,\mathcal{D}(k)$ has a unique analytic extension to the entire complex plane, and such an extension is uniquely determined by our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. Moreover, as in (b.iii) the value of N must satisfy $N \le Z + 1$, where Z is the number of zeros of $\mathcal{D}(k)$ on $\mathbf{I}^+$ having odd multiplicities, without counting the multiplicities.

(c.iv) For each N-value resulting from restrictions (a.i), (a.ii), and (c.iii), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, there correspond a discrete number of potentials where the discrete parameter set is $\{\kappa_j\}_{j=1}^N$. The set $\{\kappa_j\}_{j=1}^N$ must be a subset of $\{\beta_m\}$ and satisfy the additional restrictions $0 < \kappa_1 < \cdots < \kappa_N$ and $(-1)^{N-j}\mathcal{D}(i\kappa_j) > 0$ for $j = 1, \dots, N$. Here, each $k = -i\beta_m$ corresponds to a zero of $1/T^{[0]}(k)$ on $\mathbf{I}^-$, where $T^{[0]}(k)$ is the quantity in (3.1), and $k/T^{[0]}(k)$ is now entire on $\mathbf{C}$ and uniquely constructed via (3.1) from our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. The values $k = -i\beta_m$ correspond to the (real) resonances of $V^{[0]}$. For an answer to the question whether the set $\{\beta_m\}$ is a finite set or an infinite set, we refer the reader to [15]. Informally speaking, if $V^{[0]} \in C_0^\infty[0, 1]$ and the order of the zero of $V^{[0]}$

at $x = 0$ or at $x = 1$ is infinite, then the set $\{\beta_m\}$ may be infinite; otherwise, it is a finite set.

Case (d): Recovery of V from $\mathcal{D}$ with Compact Support and L^2-norm.

Let us assume that $\mathcal{D}(k)$ is given for $k \in \mathbf{R}$ and it is known that $V \equiv 0$ for $x \notin [0, 1]$, $V \in L^2[0, 1]$, and $||V|| \le C$, where we denote the L^2-norm of V as $||V|| := \sqrt{\int_{-\infty}^{\infty} dx\, V(x)^2}$. We will determine the precise values of C that assure a unique or nonunique determination of V from $\mathcal{D}$.

As outlined below (3.3) in case (a), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, we are able to uniquely determine $V^{[0]}$ when $\mathcal{D}$ is singular at $k = 0$, and we determine two distinct potentials $V_1^{[0]}$ and $V_2^{[0]}$ if $\mathcal{D}$ is finite at $k = 0$. In the latter case, we know from (3.5) that $||V_1^{[0]}|| = ||V_2^{[0]}||$. Thus, $\mathcal{D}(k)$ uniquely determines the L^2-norm of $V^{[0]}$, even though there are two distinct choices for $V^{[0]}$ in the exceptional case. Let us denote that unique value by $||V^{[0]}||$.

As seen from (c.iv), for each allowed allowed integer N, $\mathcal{D}(k)$ uniquely [2] determines a discrete number of ordered sets $\{\kappa_j\}_{j=1}^N$ with the ordering $0 < \kappa_1 < \cdots < \kappa_N$ related to the bound states of $V^{[N]}$. Let us define

$$C_0 := ||V^{[0]}||; \quad C_N := \left[||V^{[0]}||^2 + \frac{16}{3} \sum_{j=1}^{N} \kappa_j^3 \right]^{1/2}.$$

Thus, for each N, C_N consists of a sequence of values. Clearly, C_0 consists of a single number. By listing all the elements in C_N for all allowed N-values, we obtain a discrete set of ordered positive numbers consisting of various κ_j values, and we denote that discrete set by $\{C_N\}$. This set is a subset of $\{\beta_m\}$, as indicated in (c.iv). The smallest number in the ordered set $\{C_N\}$ is strictly less than the next larger number due to the fact that each set $\{\kappa_j\}_{j=1}^N$ with the largest allowable N consists of distinct positive elements. This allows us to determine the value of C in the inequality $||V|| \le C$ in order to determine a unique potential V corresponding to our data $\mathcal{D}$. By choosing C as greater than or equal to the smallest number in the set $\{C_N\}$ but strictly less than the next larger element, we will uniquely determine the potential V. Next we illustrate this determination with some explicit examples.

As our scattering data let us use $\mathcal{D}(k) = \dfrac{-\epsilon\, e^{ik} \sin\sqrt{k^2+\epsilon}}{2ik\sqrt{k^2+\epsilon}}$, where ϵ is a positive parameter. In fact, one corresponding potential is the square well of depth ϵ supported on the interval $[0, 1]$. For each value of ϵ, let us obtain all the potentials corresponding to $\mathcal{D}(k)$ with support confined to $[0, 1]$ and specify their L^2-norms. We have $\lim_{k\to 0}[2ik\,\mathcal{D}(k)] = -\sqrt{\epsilon}\,\sin\sqrt{\epsilon}$, and hence the exceptional case occurs when $\sqrt{\epsilon}/\pi$ is a positive integer and the generic case occurs otherwise. The zeros of $\mathcal{D}(k)$ on $\mathbf{I}^+$ occur when $\sin\sqrt{k^2+\epsilon} = 0$, and hence these are all simple zeros occurring at $k = i\sqrt{\epsilon - (j-1)^2\pi^2}$ for $j = 1, \dots, Z$, with Z being equal to $\lfloor\sqrt{\epsilon}/\pi\rfloor$, i.e. the greatest integer less than or equal to $\sqrt{\epsilon}/\pi$. As $k \to \infty$ on $\mathbf{I}^+$, we have $\mathcal{D}(k) \to 0^+$. As $k \to 0$ on $\mathbf{I}^+$, we get $(-1)^Z\mathcal{D}(k) \to 0^+$ in the exceptional case, and $(-1)^Z\mathcal{D}(k) \to +\infty$ in the generic case. Define

$$\frac{1}{\tau(k)} := e^{ik}\left[\cos\sqrt{k^2+\epsilon} + \frac{2k^2+\epsilon}{2ik\sqrt{k^2+\epsilon}}\,\sin\sqrt{k^2+\epsilon}\right].$$

Note that $\tau(k)$ corresponds to the transmission coefficient of the square-well potential of depth ϵ supported on $[0,1]$. It is known that $1/\tau(k)$ has exactly $Z+1$ (simple) zeros on $\mathbf{I}^+$, which we denote by ξ_j with the ordering $0<\xi_1<\cdots<\xi_{Z+1}$. The quantity in (3.1) is obtained as

$$\frac{1}{T^{[0]}(k)}=\frac{1}{\tau(k)}\prod_{j=1}^{Z+1}\frac{k+i\xi_j}{k-i\xi_j}. \tag{3.6}$$

Example 3.1. When $\epsilon=5$, we are in the generic case and $Z=0$. Hence, $N\le 1$, but $\mathcal{D}(k)\to+\infty$ on $\mathbf{I}^+$ indicates that N must be odd. Thus, $N=1$ is the only allowed value. In this case, $1/T^{[0]}(k)$ given in (3.6) has two zeros on $\mathbf{I}^-$ at $k=-i\beta_j$ with $\beta_1=1.5433\overline{4}$ and $\beta_2=1.585\overline{7}$. We use an overline to indicate roundoff. In (3.6) we have $\xi_1=\beta_2$. Corresponding to $\mathcal{D}(k)$ we have two potentials $V_1^{[1]}$ and $V_2^{[1]}$, having bound states at $k=i\beta_1$ and $k=i\beta_2$, respectively. Note that $V_2^{[1]}$ is the square well of depth ϵ. We have $||V_1^{[1]}||=4.8312\overline{6}$ and $||V_2^{[1]}||=5$. Thus, knowledge of any C satisfying $||V_1^{[1]}||\le C<||V_2^{[1]}||$ helps us to identify $V_1^{[1]}$ or $V_2^{[1]}$ as the unique potential corresponding to $\mathcal{D}(k)$. The left reflection coefficients $L_1^{[1]}$ and $L_2^{[1]}$ corresponding to $V_1^{[1]}$ and $V_2^{[1]}$, respectively, are obtained from (3.2) as

$$L_j^{[1]}(k)=\mathcal{D}(k)\,T^{[0]}(k)\,\frac{k+i\beta_j}{k-i\beta_j},\qquad j=1,2.$$

Note that $V_1^{[1]}$ and $V_2^{[1]}$ can uniquely be constructed [11] from $L_1^{[1]}$ and $L_2^{[1]}$, respectively, because they vanish for $x<0$.

Example 3.2. When $\epsilon=\pi^2$, we are in the exceptional case and $Z=0$. Hence, both $N=0$ and $N=1$ are allowed. In this case $1/T^{[0]}(k)$ given in (3.6) has only one zero on $\mathbf{I}^-$ at $k=-i\beta_1$ with $\beta_1=2.52258\overline{8}$. Thus, in (3.6) we have $\xi_1=\beta_1$. Corresponding to $\mathcal{D}(k)$ we have two potentials $V^{[0]}$ and $V^{[1]}$, the former with no bound states and the latter with one bound state at $k=i\beta_1$. Note that $V^{[1]}$ is the square well of depth ϵ. We have $||V^{[0]}||=3.3853\overline{7}$ and $||V^{[1]}||=\pi^2$. Thus, knowledge of any C satisfying $||V^{[0]}||\le C<||V^{[1]}||$ helps us to identify either $V^{[0]}$ or $V^{[1]}$ as the unique potential corresponding to $\mathcal{D}(k)$. The left reflection coefficients $L^{[0]}$ and $L^{[1]}$ corresponding to $V^{[0]}$ and $V^{[1]}$, respectively, are obtained from (3.2) as

$$L^{[0]}(k)=\mathcal{D}(k)\,T^{[0]}(k),\quad L^{[1]}(k)=\mathcal{D}(k)\,T^{[0]}(k)\,\frac{k+i\beta_1}{k-i\beta_1}.$$

Having $L^{[0]}$ and $L^{[1]}$ at hand, the potentials $V^{[0]}$ and $V^{[1]}$ can be uniquely constructed.

Example 3.3. When $\epsilon=20$, we are in the generic case and $Z=1$. Hence, $N\le 2$, but $\mathcal{D}(k)\to-\infty$ on $\mathbf{I}^+$ indicates that N must be even. Thus, $N=0$ and $N=2$ are the only possibilities. In this case $1/T^{[0]}(k)$ given in (3.6) has two zeros on $\mathbf{I}^-$ at $k=-i\beta_1$ with $\beta_1=\xi_1=1.9302\overline{1}$ and $k=-i\beta_2$ with $\beta_2=\xi_2=3.9255\overline{6}$. When $N=2$, the only potential $V^{[2]}$ corresponding to $\mathcal{D}(k)$ is the square well of depth ϵ with support $[0,1]$. When $N=0$, the corresponding potential $V^{[0]}$ is uniquely determined from $\mathcal{D}(k)$ and its left reflection coefficient $L^{[0]}(k)$ is obtained from (3.2) as

$$L^{[0]}(k)=\mathcal{D}(k)\,T^{[0]}(k)\,\frac{(k-i\beta_1)(k-i\beta_2)}{(k+i\beta_1)(k+i\beta_2)}.$$

In this case we have $||V^{[0]}|| = 6.2463\overline{5}$ and $||V^{[2]}|| = 20$. Thus, an appropriate specification of the upper limit on the L^2-norm of the potential allows the unique identification of $V^{[0]}$ or $V^{[2]}$ from $\mathcal{D}(k)$.

Example 3.4. When $\epsilon = 130$, the allowed values for N are 0, 2, and 4. In this case $1/T^{[0]}(k)$ given in (3.6) has six zeros on $\mathbf{I}^-$ at $k = -i\beta_j$ with $\beta_1 = 4.8729\overline{5}$, $\beta_2 = 8.2260\overline{7}$, $\beta_3 = 8.3286\overline{5}$, $\beta_4 = 10.087\overline{9}$, $\beta_5 = 10.740\overline{7}$, $\beta_6 = 11.08\overline{5}$. For $N = 0$, the only potential corresponding to $\mathcal{D}(k)$ has norm $||V^{[0]}|| = 23.96\overline{8}$. For $N = 2$, there are five potentials corresponding to $\mathcal{D}(k)$ with norms $||V_1^{[2]}|| = 64.50\overline{9}$, $||V_2^{[2]}|| = 65.366\overline{8}$, $||V_3^{[2]}|| = 91.956\overline{6}$, $||V_4^{[2]}|| = 115.38\overline{7}$, $||V_5^{[2]}|| = 120.19\overline{7}$, where $V_1^{[2]}$ has bound states $\{-\beta_1^2, -\beta_2^2\}$, $V_2^{[2]}$ has $\{-\beta_1^2, -\beta_3^2\}$, $V_3^{[2]}$ has $\{-\beta_1^2, -\beta_6^2\}$, $V_4^{[2]}$ has $\{-\beta_4^2, -\beta_6^2\}$, and $V_5^{[2]}$ has $\{-\beta_5^2, -\beta_6^2\}$. For $N = 4$, there are four potentials corresponding to $\mathcal{D}(k)$ with norms $||V_1^{[4]}|| = 130$, $||V_2^{[4]}|| = 130.43\overline{2}$, $||V_3^{[4]}|| = 134.28\overline{7}$, $||V_4^{[4]}|| = 134.70\overline{5}$, where $V_1^{[4]}$ has bound states $\{-\beta_1^2, -\beta_2^2, -\beta_4^2, -\beta_6^2\}$, $V_2^{[4]}$ has $\{-\beta_1^2, -\beta_3^2, -\beta_4^2, -\beta_6^2\}$, $V_3^{[4]}$ has $\{-\beta_1^2, -\beta_2^2, -\beta_5^2, -\beta_6^2\}$, and finally $V_4^{[4]}$ has $\{-\beta_1^2, -\beta_3^2, -\beta_5^2, -\beta_6^2\}$. Thus, some appropriate knowledge on the L^2-norm of the potential allows us to pick a unique potential among all these 16 potentials corresponding to the same $\mathcal{D}(k)$. Note that $V_1^{[4]}$ is the square well of depth ϵ.

References

[1] W. Rundell and P. Sacks, *On the determination of potentials without bound state data*, J. Comput. Appl. Math. **55** (1994), 325–2347.

[2] T. Aktosun and V. Papanicolaou, *Recovery of a potential from the ratio of reflection and transmission coefficients*, J. Math. Phys. **44** (2003), 4875–4883.

[3] L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Amer. Math. Soc. Transl. (Ser. 2) **65** (1967), 139–166.

[4] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.

[5] R. G. Newton, *The Marchenko and Gel'fand-Levitan methods in the inverse scattering problem in one and three dimensions*, Conference on inverse scattering: theory and application, ed. by J. B. Bednar et al., SIAM, Philadelphia, 1983, pp. 1–74.

[6] A. Melin, *Operator methods for inverse scattering on the real line*, Comm. Partial Differential Equations **10** (1985), 677–766.

[7] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.

[8] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, 2nd ed., Springer, New York, 1989.

[9] T. Aktosun and M. Klaus, *Chapter 2.2.4, Inverse theory: problem on the line*, Scattering, ed. by E. R. Pike and P. C. Sabatier, Academic Press, London, 2001, pp. 770–785.

[10] A. Degasperis and P. C. Sabatier, *Extension of the one-dimensional scattering theory, and ambiguities*, Inverse Problems **3** (1987), 73–109.

[11] T. Aktosun, M. Klaus, and C. van der Mee, *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation*, J. Math. Phys. **34** (1993), 2651–2690.

[12] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467–490.

[13] G. L. Lamb, Jr., *Elements of soliton theory*, Wiley, New York, 1980.

[14] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.

[15] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (1987), 277–296.

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