PATH INTEGRAL FOR THE RADIAL DIRAC EQUATION

Takashi Ichinose

Department of Mathematics, Faculty of Science, Kanazawa University
Kanazawa, 920–1192, Japan
E-mail: ichinose@kappa.s.kanazawa-u.ac.jp

Dedicated to Professor John R. Klauder on the occasion
of his seventieth birthday with my hearty appreciation

Abstract. For the radial Dirac equation, a countably additive path space measure on the space of continuous paths living on the real half-line is constructed to give a path integral representation of its Green function.

1. Introduction

In this note we want to make a mathematical rigorous approach to the path integral for the radial Dirac equation, namely, the radial part of the Dirac equation in spherical coordinates. The equation is given, for $\mathbf{C}^2$-valued functions $\psi(r,t) = {}^t(\psi_1(r,t), \psi_2(r,t))$, by

$$\frac{\partial}{\partial t}\psi(r,t) = -i[\tau_k + V(r)]\psi(r,t), \tag{1.1}$$

with a real-valued spherically symmetric potential $V = V(r)$, i.e. a function in the real half-line $\mathbf{R}_+ := (0,\infty)$, where the variables (r,t) lie in radial space-time $\mathbf{R}_+ \times \mathbf{R}$. Here τ_k is the free radial Dirac operator with mass m defined for $k = \pm 1, \pm 2, \cdots$, by

$$\tau_k : f \mapsto \Big[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial r} + \begin{pmatrix} m & -k/r \\ -k/r & -m \end{pmatrix}\Big]\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} \tag{1.2a}$$

for suitably smooth $\mathbf{C}^2$-valued functions $f = {}^t(f_1, f_2)$ in $\mathbf{R}_+$, in short,

$$\tau_k = -i\sigma_2\frac{\partial}{\partial r} - \sigma_1\frac{k}{r} + m\sigma_3, \tag{1.2b}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The free radial Dirac operator τ_k arises from the spin-angular momentum decomposition of the free Dirac operator in three space dimensions. The nonzero integer k represents an eigenvalue of the "spin-orbit operator" [see Bjorken and Drell (1964), Thaller (1992), Weidmann (1987)].

The operator τ_k is a symmetric operator with domain $D[\tau_k] = C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$ in $L^2(\mathbf{R}_+;\mathbf{C}^2)$. Here $L^2(\mathbf{R}_+;\mathbf{C}^2)$ is the Hilbert space of the $\mathbf{C}^2$-valued square-integrable functions in $\mathbf{R}_+$ with respect to the Lebesgue measure dr, and $C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$ the locally convex space of the $\mathbf{C}^2$-valued C^∞ functions in $\mathbf{R}_+$ with compact support. It has a singularity at $r = 0$ as in (1.2ab). This is indeed harmless if we consider it as an operator in the L^2 space, but is a problem to construct a path space measure, for we need to consider it as an operator in the L^∞ space [for this see Ichinose and Tamura (1984, 1988, 1987)]. In this context, in Ichinose and Jefferies (2002), we have made an explicit construction of the free propagator, namely, the integral kernel $\mathcal{K}_t(r,\rho)$ of $e^{-it\tau_k}$ for $k = 1$, and shown that, though it turns out to be a distribution *of order zero* in the variables $(r,\rho) \in \mathbf{R}_+ \times \mathbf{R}_+ = {\mathbf{R}_+}^2$, there exists no countably additive path space measure to give a path integral representation to the solution of the Cauchy problem for the radial Dirac equation (1.1).

The aim of this note is to construct a countably additive path space measure to represent by path integral, though not the propagator, the Green function for the radial Dirac equation (1.1). The main idea is to combine our method of constructiong a path space measure developed for the one-dimensional Dirac equation in the papers [Ichinose

(1982, 1984), Ichinose and Tamura (1984, 1988, 1987)], in particular, in their a little more refined review [Ichinose (1993)], with the following simple but intriguing procedure of dealing with the singularity, which was invented by Duru and Kleinert in 1979 and has since been employed by many physicists [see Kleinert (1995, Chap. 12 and 14), Inomata, Katsuraji and Gerry (1992, p.6), Grosche and Steiner (1998, pp. 77–83)] to perform space-time transformations in path integrals.

The free radial Dirac operator τ_k has a singularity at $r = 0$. However, if we multiply this operator τ_k by some (nonnegative) functions $a(r)$ and $b(r)$ from the left and right sides, respectively, then $a(r)\tau_k b(r)$ becomes no more singular. Let us take $a(r) = b(r) = r^{1/2}$, and put $T_k = r^{1/2}\tau_k r^{1/2}$. Then $-iT_k$ becomes

$$\begin{aligned} -iT_k &= -ir^{1/2}\tau_k r^{1/2} = -\sigma_2 r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2} + i(k\sigma_1 - mr\sigma_3) \\ &= -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2} + i\begin{pmatrix} -mr & k \\ k & mr \end{pmatrix} \\ &=: -iT_{k0} - iT_{k1}. \end{aligned} \tag{1.3}$$

We also consider the operator

$$H(k, rV) := T_k + rV(r), \tag{1.4}$$

relevant to the radial Dirac operator in (1.2ab). Then T_k is a symmetric operator with domain $D[T_k] = C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$ in $L^2(\mathbf{R}_+; \mathbf{C}^2)$. For the potentials $V(r)$ we are concerned with, we can show $H(k, rV)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$, as well as T_k. We shall denote their closures or unique selfadjoint extensions again by the same $H(k, rV)$ and T_k.

Then consider, for $H(k, rV)$ instead of $\tau_k + V(r)$, the Cauchy problem

$$\frac{\partial}{\partial t}u(r,t) = -iH(k, rV)u(r,t), \quad u(r,0) = g(r) = {}^t(g_1(r), g_2(r)) \tag{1.5}$$

for $t \in \mathbf{R}$ or the solution $u(r,t) = \big(e^{-itH(k,rV)}g\big)(r)$. Since T_k has no more singularity at $r = 0$, we expect to be able to construct a path space measure associated with the semigroup $e^{-itH(k,rV)}$. Since the resolvent is expressed by the Laplace transform of the semigroup, we shall find the following relation between the desired resolvent kernel of $\tau_k + V$ and the semigroup $e^{-itH(k,rV)}$, though a little formally expressed,

$$\begin{aligned} [(\tau_k + V - \lambda)^{-1}](r, \rho) &= r^{1/2}[(H(k, rV) - \lambda r)^{-1}](r, \rho)\rho^{1/2} \\ &= i\int_0^\infty r^{1/2} e^{-it(H(k,rV) - \lambda r)}\rho^{1/2}\, dt, \end{aligned}$$

for suitable real or complex numbers λ. In this way, we might get, for the original radial Dirac operator $\tau_k + V(r)$, a path integral representation of the resolvent kernel and the Green function if λ might be taken to be zero, though we were unable to find such a representation for the propagator $e^{-it(\tau_k + V)}$ itself.

For the potential $V(x)$, we assume that it is a real-valued function in $\mathbf{R}_+$ such that $V(r) = V_1(r) + V_2(r)$, where $V_1(r) = e/r$ with a real constant e satisfying $|e| \leq \sqrt{k^2 - \frac{1}{4}}$ and $V_2(r)$ is a locally square-integrable function in $\mathbf{R}_+$ which is bounded near $r = 0$.

Note that this class of potentials $V(r)$ contains the Coulomb potential.

In this case it can be shown that the radial Dirac operator $\tau_k + V$ in (1.1) is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$. We denote its unique selfadjoint extension in $L^2(\mathbf{R}_+; \mathbf{C}^2)$ also by the same $\tau_k + V$. Thus this operator has a real spectrum. Further, as to its spectrum, for instance, the following results are known [Weidmann (1987), Thaller (1992, Theorems 4.18, 4.19)]. If $V_2(r) := V_{21}(r) + V_{22}(r)$ satisfies for some $r_0 > 0$ that $V_{21} \in L^1((r_0, \infty))$ and V_{22} is of bounded variation in $[r_0, \infty)$ with $\lim_{r\to\infty} V_{22}(r) = 0$, then the operator $\tau_k + V$ has a purely absolutely continuous spectrum in the real line $\mathbf{R}$ outside the closed interval $[-m, m]$. If $V_1(r)$ is absent (i.e. $e = 0$) and if $\lim_{r\to\infty} |V_2(r)| = \infty$ and $\int_{r_0}^\infty \left| \frac{V_2{}'(r)}{V_2(r)^2} \right| dr < \infty$ for some $r_0 > 0$, then $\tau_k + V$ has the whole real line $\mathbf{R}$ as a purely absolutely continuous spectrum.

We will show the following path integral representation for the resolvent kernel $[(\tau_k + V \mp i\varepsilon)^{-1}](r, \rho)$ for $\varepsilon > 0$ and the Green function for the radial Dirac operator $\tau_k + V(r)$ in (1.1).

The set of all complex 2×2-matrices is denoted by $M_2(\mathbf{C}) = \mathbf{C}^2 \otimes (\mathbf{C}^2)'$. With $[0, \infty) = \overline{\mathbf{R}_+}$ being the closed real half-line, let $C_{00}^\infty([0, \infty)^2; M_2(\mathbf{C}))$ be the locally convex space of the $M_2(\mathbf{C})$-valued C^∞ functions $M(r, \rho)$ in $[0, \infty) \times [0, \infty)$ which have compact support and whose derivatives $\partial r^m \partial \rho^n M(r, \rho)$, for all nonnegative integers m and n, vanish at $(r, 0)$ and $(0, \rho)$ for all $r \geq 0$ and $\rho \geq 0$. Let $C_{00}^\infty([0, \infty)^2; M_2(\mathbf{C}))'$ be its dual space. As $C_0^\infty(\mathbf{R}_+{}^2; M_2(\mathbf{C}))$ is a subspace of $C_{00}^\infty([0, \infty)^2; M_2(\mathbf{C}))$, so $C_{00}^\infty([0, \infty)^2; M_2(\mathbf{C}))'$ is a subspace of the space $\mathcal{D}'(\mathbf{R}_+{}^2; M_2(\mathbf{C}))$ of the $M_2(\mathbf{C})$-valued distributions in $\mathbf{R}_+ \times \mathbf{R}_+$. By $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ we denote respectively the sesquilinear and bilinear inner products of a dual pairing.

The main result of this note is the following theorems. The notation $|0, t|$ stands for the interval $0 \leq s \leq t$ or $0 \geq s \geq t$ according as $t > 0$ or $t < 0$.

Theorem 1.1. *Let $V(r)$ be a potential mentioned above.*

(i) *For every $\varepsilon > 0$, there exists a 2×2-matrix distribution-valued, precisely speaking, $C_{00}^\infty([0, \infty)^2; M_2(\mathbf{C}))'$-valued, countably additive path space measure $\mu_{t,0}$ on the Banach space $C(|0, t| \to [0, \infty))$ of the continuous paths $R : |0, t| \to [0, \infty)$ such that the resolvent kernel $[(\tau_k + V \mp i\varepsilon)^{-1}](r, \rho)$ for the radial Dirac operator $\tau_k + V$ in (1.1) admits a path integral representation: for every pair $(f, g) \in C_{00}^\infty([0, \infty); \mathbf{C}^2) \times C_{00}^\infty([0, \infty); \mathbf{C}^2)$,*

$$\begin{aligned}
(f, (\tau_k + V \mp i\varepsilon)^{-1} g) &= \int_0^\infty \int_0^\infty {}^t\overline{f(r)} [(\tau_k + V \mp i\varepsilon)^{-1}](r, \rho) g(\rho)\, dr d\rho \\
&= i \int_0^{\pm\infty} dt \int_{C(|0,t| \to [0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0)) \rangle \\
&\qquad \times R(t)^{1/2} R(0)^{1/2} \exp\Big[- \int_0^t \big(iV(R(s))R(s) \pm \varepsilon R(s) \big)\, ds \Big].
\end{aligned} \tag{1.6}$$

In particular, the resolvent kernel $[(\tau_k + V \mp i\varepsilon)^{-1}](r,\rho)$ *is a distribution of order zero in* $\mathbf{R}_+ \times \mathbf{R}_+$.

(ii) *The measure* $\mu_{t,0}$ *is concentrated on the set of the continuous paths* $R : |0,t| \to [0,\infty)$ *for which there exists a finite partition:* $0 = t_0 \lesseqslantgtr t_1 \lesseqslantgtr \cdots \lesseqslantgtr t_{n+1} = t$ *of the interval* $|0,t|$ *such that for* $t_{h-1} \lesseqqgtr s \lesseqqgtr t_h,\ 1 \le h \le n+1$,

$$R(s) = R(0)e^{\pm[\sum_{p=1}^{h-1}(-1)^{p-1}(t_p - t_{p-1}) + (-1)^{h-1}(s-t_{h-1})]}. \tag{1.7}$$

Therefore each of these continuous paths $R(\cdot)$ *is, for some finite* n, *an* n*-vertex piecewise smooth curve in radial space-time, starting from* $R(0)$ *at time* 0 *and reaching* $R(t)$ *at time* t, *and exponentially growing or decreasing in each partioned short time interval.*

We denote by $G_\pm(r,\rho)$ the Green function for the radial Dirac operator $\tau_k + V$ to be given as the limit of the integral kernel of the resolvent $(\tau_k + V \mp i\varepsilon)^{-1}$ as $\varepsilon \to +0$, if this limit exists.

Theorem 1.2. *For the same potential* $V(r)$ *as in Theorem 1.1, suppose that* 0 *is not an eigenvalue of the radial Dirac operator* $\tau_k + V$. *Suppose that the Green function* $G_\pm(r,\rho)$ *for the radial Dirac operator* $\tau_k + V$ *in (1.1) exists. Then it is a distribution of order zero in* $(r,\rho) \in \mathbf{R}_+ \times \mathbf{R}_+$, *and admits a path integral representation: for every pair* $(f,g) \in C_{00}^\infty([0,\infty);\mathbf{C}^2) \times C_{00}^\infty([0,\infty);\mathbf{C}^2)$,

(i)

$$\begin{aligned}\int_0^\infty \int_0^\infty {}^t\overline{f(r)} G_\pm(r,\rho) g(\rho)\, dr d\rho \\ = i \lim_{\varepsilon\to+0} \int_0^{\pm\infty} dt \int_{C(|0,t|\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \\ \times R(t)^{1/2} R(0)^{1/2} \exp\Big[-\int_0^t \big(iV(R(s))R(s) \pm \varepsilon R(s)\big)\, ds\Big];\end{aligned} \tag{1.8}$$

(ii)

$$\begin{aligned}\int_0^\infty \int_0^\infty {}^t\overline{f(r)} G_\pm(r,\rho) g(\rho)\, dr d\rho \\ = i \lim_{\varepsilon\to+0} \int_0^{\pm\infty} dt \int_{C(|0,t|\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \\ \times R(t)^{1/2} R(0)^{1/2} \exp\Big[-i\int_0^t V(R(s))R(s)\, ds \mp \varepsilon t\Big].\end{aligned} \tag{1.9}$$

If we formally take the delta functions at the two points $r > 0$ and $\rho > 0$ respectively for f and g, the formula (1.8)/(1.9) looks like

$$\begin{aligned}G_\pm(r,\rho) = i \int_0^{\pm\infty} dt \int_{C(|0,t|\to[0,\infty)), R(0)=\rho, R(t)=r} \\ \times r^{1/2}\rho^{1/2} \exp\Big[-i\int_0^t V(R(s))R(s)\, ds\Big] d\mu_{t,0}(R).\end{aligned} \tag{1.10}$$

It should be emphasized that our path integral concerns the three-dimensional Dirac equation, though in the spherical coordinates. For the Dirac equation in the one-dimensional space, i.e. the whole real line **R**, a path integral measure to represent the propagator was constructed first in Ichinose (1982, 1984), Ichinose and Tamura (1984) and then further studied in Ichinose and Tamura (1988, 1987) and Ichinose (1993). The problem was treated in Blanchard, Combe, Sirugue and Sirugue-Collin (1985) from a different point of view based on Poisson process, and there are further related works Gaveau (1984), Gaveau, Jacobson, Kac and Schulman (1984), Combe, Sirugue and Sirugue-Collin (1987) and Zastawniak (1988) on the subject. The method used in the present note is also connected with the Poisson process, though we do not explicitly appeal to it. This note is only to give the first step to describe the idea, and application to some problem in quantum field theory will be discussed elsewhere.

To prove the theorems, we investigate, in Section 2, the Cauchy problem (1.5) in the L^∞ norm for the relevant operator to our radial Dirac operator, and also show essential selfadjointness of both the radial Dirac operator (1.2ab) and its relevant operator (1.4). In Section 3, we construct, by means of the Riesz representation theorem, a countably additive path space measure associated with the semigroup for the Cauchy problem (1.5), on a big space of paths living on the closed real half-line over each finite time interval $[0, t]$. Then it is shown that this measure is in fact concentrated on the set of continuous, piecewise smooth paths with a finite number of vertices in radial space-time. In Section 4, we shall show, together with this measure constructed, a path integral representation of Feynman–Kac type first for this semigroup associated with (1.5) and then through the procedure mentioned for the resolvent kernel and the Green function for the radial Dirac operator we are concerned with.

2. The radial Dirac operator and its relevant operator

To construct our path measure, we need to investigate the Cauchy problems (1.1) for the radial Dirac operator $\tau_k + V$ and (1.5) for its relevant operator $H(k, rV) = r^{1/2}(\tau_k + V)r^{1/2} = T_k + rV$ in (1.4).

For the potential $V(r)$*, it is assumed throughout this section that* $V(r)$ *is a real-valued function in* $\mathbf{R}_+$ *such that* $V(r) = V_1(r) + V_2(r)$*, where* $V_1(r) = e/r$ *with* $|e| \leq \sqrt{k^2 - \frac{1}{4}}$ *and* $V_2(r)$ *is a function in* $L^2_{loc}(\mathbf{R}_+)$ *which is bounded near* $r = 0$*.*

We have so far taken for granted that the radial Dirac operator $\tau_k + V$ in (1.1) and its relevant operator $H(k, rV)$ in (1.4) are essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$. We will confirm these facts here. For some related results on the former $\tau_k + V$, we refer also to Weidmann (1987) and Arnold, Kalf and Schneider (1997). The L^2-norm in $L^2(\mathbf{R}_+; \mathbf{C}^2)$ is denoted by $\|\cdot\|_2$.

First consider the radial Dirac operator $\tau_k + V$ in (1.1). There exists a unique solution of the Cauchy problem (1.1).

Proposition 2.1. *The operator* $\tau_k + V$ *is essentially selfadjoint on* $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$*.*

Proof. Put $D_1 := \tau_k + V_1$, so that $\tau_k + V = D_1 + V_2$. It is known by Weidmann (1987), D_1 is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$. So we consider V_2 is a perturbation from D_1. We are going to use the method by Jörgens (1972, Theorem 5.6). To show essential selfadjointness of $D_1 + V_2$, we need to show the following conditions.

(i) For each $\varphi \in C_0^\infty(\mathbf{R}_+)$, the commutator $[D_1, \varphi] = D_1\varphi - \varphi D_1$ is a bounded operator on $L^2(\mathbf{R}_+; \mathbf{C}^2)$.

(ii) For every nonempty bounded open interval $(r_1, r_2) \subseteq \mathbf{R}_+$, there exist constants $a > 0$ and $0 \leq b < 1$ such that

$$\|V_2 v\|_{L^2((r_1,r_2);\mathbf{C}^2)} \leq a\|v\|_2 + b\|D_1 v\|_2, \quad v \in C_0^\infty(\mathbf{R}_+; \mathbf{C}^2).$$

(iii) Let $\chi(r)$ be a C^∞-function in $\mathbf{R}_+$ with $0 \leq \chi(r) \leq 1$ such that $\chi(r) = 1,\ 0 < r \leq 1$ and $\chi(r) = 0,\ r \geq 2$. Put $\varphi_n(r) = \chi(r/n)$, so that $\varphi_n(r)$ converges to 1 pointwise as $n \to \infty$. Then $\|[D_1, \varphi_n]v + [V_2, \varphi_n]v\|_2$ is bounded for every $v \in D[(D_1 + V_2)^*]$.

But these conditions can be checked as follows.

Indeed, (i) is clear because $[D_1, \varphi] = -i\sigma_2\varphi'(r)$.

To see (ii), let (r_1, r_2) be a bounded open interval in $\mathbf{R}_+$. By assumption, $V_2(r)$ is bounded for $r \leq r_0$ for some $r_0 > 0$. First suppose that $r_1 \geq r_0$, so that $(r_1, r_2) \subseteq (r_0, r_2)$, and V_2 is in $L^2((r_0, r_2))$ as well as in $L^2((r_1, r_2))$. Then for $v \in C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$, we have by the Schwarz inequality

$$\|V_2 v\|_{L^2((r_1,r_2);\mathbf{C}^2)} \leq \|V_2 v\|_{L^2((r_0,r_2);\mathbf{C}^2)} \leq \|V_2\|_{L^2((r_0,r_2)} \sup_{r_0 \leq r \leq r_2} |v(r)|.$$

On the other hand, for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\sup_{r_0 \leq r \leq r_2} |v(r)| \leq \varepsilon\|v'\|_{L^2((r_0,\infty);\mathbf{C}^2)} + C(\varepsilon)\|v\|_{L^2((r_0,r_2);\mathbf{C}^2)}.$$

Moreover we have with (1.2b)

$$\begin{aligned}
&\|v'\|_{L^2((r_0,\infty);\mathbf{C}^2)} \\
&= \| -i\sigma_2 \frac{\partial}{\partial r} v\|_{L^2((r_0,\infty);\mathbf{C}^2)} = \|[\tau_k + (\sigma_1 \frac{k}{r} - m\sigma_3)]v\|_{L^2((r_0,\infty);\mathbf{C}^2)} \\
&= \|[D_1 + (\sigma_1 \frac{k}{r} - m\sigma_3) - V_1]v\|_{L^2((r_0,\infty);\mathbf{C}^2)} \\
&\le \|D_1 v\|_{L^2((r_0,\infty);\mathbf{C}^2)} + \|(\sigma_1 \frac{k}{r} - m\sigma_3)v\|_{L^2((r_0,\infty);\mathbf{C}^2)} + \|V_1 v\|_{L^2((r_0,\infty);\mathbf{C}^2)} \\
&\le \|D_1 v\|_2 + \Big(m + \frac{|k|+|e|}{r_0}\Big)\|v\|_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|V_2 v\|_{L^2((r_1,r_2);\mathbf{C}^2)} \\
&\qquad \le \varepsilon \|V_2\|_{L^2((r_0,r_2))}\|D_1 v\|_2 + \Big[\varepsilon\Big(m + \frac{|k|+|e|}{r_0}\Big) + C(\varepsilon)\Big]\|V_2\|_{L^2((r_0,r_2))}\|v\|_2.
\end{aligned}$$

So choose ε so small that

$$b := \varepsilon \|V_2\|_{L^2((r_0,r_2)} < 1$$

and then put

$$a := \Big[\varepsilon\Big(m + \frac{|k|+|e|}{r_0}\Big) + C(\varepsilon)\Big]\|V_2\|_{L^2((r_0,r_2))}$$

to get (ii). Next suppose $r_1 \le r_0$, so that $(r_1, r_2) = (r_1, r_0] \cup (r_0, r_2)$, though (r_0, r_2) is empty if $r_0 \ge r_2$. We need to see (ii) on $(r_1, r_0]$, since we can see similarly to the above on (r_0, r_2). On the interval $(r_1, r_0]$, we have $\|V_2 v\|_{L^2((r_1,r_2);\mathbf{C}^2)} \le \sup_{r_1 \le r \le r_2} |V_2(r)|\|v\|_2$. Thus in either case we have shown (ii).

(iii) holds because $[V_2, \varphi_n] = 0$ and

$$\|[D_1, \varphi_n]v\|_2 = \| -i\sigma_2 \varphi_n' v\|_2 \le \frac{1}{n} \sup_{1 \le r \le 2} |\chi'(r)|\|v\|_2$$

for every $v \in L^2(\mathbf{R}_+; \mathbf{C}^2)$. This proves Proposition 2.1.

Next, consider the Cauchy problem for (1.5) with T_k in (1.3) and $V(r) \equiv 0$

$$\frac{\partial}{\partial t} u(r,t) = -iT_k u(r,t) = \Big[-\sigma_2\Big(r\frac{\partial}{\partial r} + \frac{1}{2}\Big) + i(k\sigma_1 - mr\sigma_3)\Big]u(r,t), \quad u(r,0) = g(r), \tag{2.1}$$

where the initial data $g = {}^t(g_1, g_2)$ is a $\mathbf{C}^2$-valued continuous function in $[0, \infty)$ with compact support. This equation is a hyperbolic system with local propagation speed $c(r) = r$. We can easily show that the solution $u(\cdot, t)$ satisfies for every $t \in \mathbf{R}$

$$\|u(\cdot, t)\|_2 = \|g\|_2. \tag{2.2}$$

Moreover, this Cauchy problem is L^∞ well-posed with the estimate to be given below. It is this fact that plays a crucial role in this note.

To describe it, put $N := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_0 + i\sigma_1)$, where σ_0 is the 2×2-identity matrix. N is a unitary matrix such that $N\sigma_2 N^{-1} = -\sigma_3$. Note also that $N\sigma_1 N^{-1} = \sigma_1$ and $N\sigma_3 N^{-1} = \sigma_2$. Then there exists a unique solution $u(r,t)$ of (2.1), which is given as the solution of the integral equation

$$u(r,t) = (e^{-itT_{k0}}g)(r,t) - i\int_0^t \big(e^{-i(t-s)T_{k0}}(T_{k1}u)(\cdot,s)\big)(r)ds. \tag{2.3}$$

Solving the equation (2.3), we shall show the following proposition.

Proposition 2.2. *If $g = {}^t(g_1, g_2)$ is a $\mathbf{C}^2$-valued continuous function in $[0,\infty)$ with compact support, e.g. support in a bounded closed interval $[0,S]$ for $S > 0$, then the solution $u(\cdot,t)$ has support in the bounded interval $[e^{-|t|}\inf \operatorname{supp} g, e^{|t|}\sup\operatorname{supp} g]$, and satisfies*

$$\|Nu(\cdot,t)\|_\infty = \|\exp[-itT_k]Ng\|_\infty \le e^{|t|(mSe^{|t|}+|k|+1/2)}\|Ng\|_\infty. \tag{2.4}$$

Proof. We may assume that $t > 0$. With (1.3) in mind, put for notational simplicity

$$\begin{aligned} A &:= -iNT_{k0}N^{-1} = \sigma_3 r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2}, \\ B &:= -iNT_{k1}N^{-1} = ik\sigma_1 + (-imr)\sigma_2 =: B_1 + B_2, \\ C &:= -iNT_kN^{-1} = \sigma_3 r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2} + (ik\sigma_1 - imr\sigma_2) = A + B. \end{aligned} \tag{2.5}$$

Here and in the following, keep in mind that B_2 and and hence B are matrices dependent on the variable r.

Then we have $Nu(r,t) = (e^{tC}Ng)(r)$, and (2.3) becomes

$$e^{tC} = e^{tA} + \int_0^t e^{(t-s)A}Be^{sC}ds, \tag{2.6}$$

these operators being applied to Ng. Then for the first term on the right-hand side of (2.6), it is easy to see that, for a $\mathbf{C}^2$-valued continuous $f = {}^t(f_1, f_2)$ on $[0,\infty)$ with support in the bounded closed interval $[0,S]$,

$$(e^{tA}f)(r) = E(t/2)\begin{pmatrix} f_1(re^t) \\ f_2(re^{-t}) \end{pmatrix}, \quad E(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \tag{2.7}$$

so that $e^{tA}f$ has support in $[0, e^tS]$. We shall have with $f = Ng$

$$\begin{aligned} &\|e^{tA}Ng\|_\infty \le e^{t/2}\|Ng\|_\infty, \\ &\|e^{tC}Ng\|_\infty \le e^{t(mSe^t+|k|+1/2)}\|Ng\|_\infty, \end{aligned} \tag{2.8}$$

if g is continuous in $[0,\infty)$ and has support in the bounded closed interval $[0,S]$ for $S>0$.

We have only to prove the latter of (2.8), which amounts to the same thing as (2.4). It will be seen from the integral equation (2.6) for e^{tC}. Iteration of the equation (2.6) yields the Dyson series expansion

$$e^{tC} = e^{tA} + \sum_{n=1}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} e^{(t-t_n)A} B e^{(t_n-t_{n-1})A} B \cdots B e^{t_1 A}\, dt_1. \tag{2.9}$$

We shall see below that this expansion, if applied to a $\mathbf{C}^2$-valued bounded continuous function f with support in the bounded interval $[0,S]$, is convegent in the L^∞ norm.

We want to rewrite the integrand of the above n-ple integral in (2.9).

First we observe that, since the Pauli matrices σ_1, σ_2 and σ_3 anti-commute with one another, namely, $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}$ for $j,k=1,2,3$, we have with $E(t)$ in (2.7)

$$e^{tA}B_1 = B_1 e^{-tA}, \quad e^{tA}B_2 = B_2 E(-t) e^{-tA}. \tag{2.10}$$

It follows that

$$e^{tA}B = e^{tA}(B_1+B_2) = [B_1 + B_2E(-t)]e^{-tA}. \tag{2.11}$$

Here we show (2.10). The first equation is seen because $e^{tA}\sigma_1 = \sigma_1 e^{-tA}$. The second one is because by (2.7)

$$\begin{aligned}
(e^{tA}B_2 f)(r) &= -im(e^{tA}(r\sigma_2 f))(r) = -im\sigma_2\big(e^{-tA}(\rho f)\big)(r) \\
&= -im\int_0^\infty \begin{pmatrix} e^{-t/2}\delta(\rho - re^{-t}) & 0 \\ 0 & e^{t/2}\delta(\rho - re^{t}) \end{pmatrix} \rho \begin{pmatrix} f_1(\rho) \\ f_2(\rho) \end{pmatrix} d\rho \\
&= -imr\sigma_2 \begin{pmatrix} e^{-3t/2} f_1(re^{-t}) \\ e^{3t/2} f_2(re^{t}) \end{pmatrix} \\
&= \Big[B_2 \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} e^{-tA} f\Big](r) = [B_2E(-t)e^{-tA}f](r).
\end{aligned}$$

To treat the integrand of the n-ple integral on the right-hand side of (2.9), we put $\Delta t_j = t_j - t_{j-1}$ with $t_{n+1}=t$ and $t_0 = 0$. Then we have

$$\begin{aligned}
&e^{\Delta t_{n+1}A} B e^{\Delta t_n A} B \cdots B e^{\Delta t_1 A} \\
&= [B_1 + B_2E(-\Delta t_{n+1})] e^{-\Delta t_{n+1}A} e^{\Delta t_n A} B \cdots B e^{\Delta t_1 A} \\
&= [B_1 + B_2E(-\Delta t_{n+1})] e^{-(\Delta t_{n+1} - \Delta t_n)A} B \cdots B e^{\Delta t_1 A} \\
&= [B_1 + B_2E(-\Delta t_{n+1})][B_1 + B_2E(\Delta t_{n+1} - \Delta t_n)] e^{(\Delta t_{n+1} - \Delta t_n + \Delta t_{n-1})A} B \cdots B e^{\Delta t_1 A} \\
&= \overleftarrow{\prod}{}_{p=2}^{n+1} [B_1 + B_2E((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1}\Delta t_q)] e^{\sum_{h=1}^{n+1}(-1)^{h-1}\Delta t_h A},
\end{aligned} \tag{2.12}$$

where $\overleftarrow{\prod}_{p=2}^{n+1} M_p = M_{n+1}M_n \cdots M_2$. Hence the expansion (2.9) becomes

$$\begin{aligned} e^{tC} &= e^{tA} + \sum_{n=1}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \\ &\qquad \times \overleftarrow{\prod}_{p=2}^{n+1} \Big[B_1 + B_2 E\Big((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1} \Delta t_q\Big)\Big] \exp\Big[\Big(\sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h\Big) A\Big] \\ &=: \sum_{n=0}^{\infty} C_n(t). \end{aligned} \tag{2.13}$$

For $C_n(t)$ in the expansion (2.13) we have the following

Lemma 2.3. (i) *The integral kernel* $C_n(t)(r,\rho)$ *of* $C_n(t)$ *has support on the set of* $(r,\rho) \in [0,\infty) \times [0,\infty)$ *with* $\rho = re^{\pm[\sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h]}$.

(ii) *If* $f(r) = {}^t(f_1(r), f_2(r))$ *is continuous in* $[0,\infty)$ *and has support in* $[0,S]$, *each* $C_n(t)f$ *has support in the bounded interval* $[e^{-|t|} \inf \operatorname{supp} f, e^{|t|} \sup \operatorname{supp} f]$. *It holds that*

$$\|C_n(t)f\|_\infty \le \frac{(mSe^{|t|} + |k|)^n}{n!} |t|^n e^{|t|/2} \|f\|_\infty. \tag{2.14}$$

Proof of Lemma 2.3. We may assume that $t > 0$. Note by (2.7) that the integral kernel of e^{tA} is given by

$$e^{tA}(r,\rho) = \begin{pmatrix} e^{t/2}\delta(\rho - re^t) & 0 \\ 0 & e^{-t/2}\delta(\rho - re^{-t}) \end{pmatrix} = E(t/2) \begin{pmatrix} \delta(\rho - re^t) & 0 \\ 0 & \delta(\rho - re^{-t}) \end{pmatrix}, \tag{2.15}$$

so that the integral kernel of $e^{-itT_{k0}} \equiv N^{-1}e^{tA}N$ becomes

$$\begin{aligned} K_0(t;r,\rho) &= N^{-1} \begin{pmatrix} e^{t/2}\delta(\rho - re^t) & 0 \\ 0 & e^{-t/2}\delta(\rho - re^{-t}) \end{pmatrix} N \\ &= N^{-1}E(t/2) \begin{pmatrix} \delta(\rho - re^t) & 0 \\ 0 & \delta(\rho - re^{-t}) \end{pmatrix} N. \end{aligned} \tag{2.16}$$

By (2.15) with the expression of $C_n(t)$ in (2.13), we have the assertion (i) and the first half of (ii), because $|\sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h| \le \sum_{h=1}^{n+1} (t_h - t_{h-1}) = t$.

Now we show the second half of (ii). We can show for $n \geq 1$,

$$\begin{aligned}
&\|C_n(t)f\|_\infty \\
&\leq \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \\
&\qquad \times \Big\| \overleftarrow{\prod}{}_{p=2}^{n+1} \big[B_1 + B_2 E\big((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1} \Delta t_q\big)\big] \exp\Big[\Big(\sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h\Big) A\Big] f \Big\|_\infty \\
&= \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \\
&\qquad \times \sup_{r>0} \Big\| \overleftarrow{\prod}{}_{p=2}^{n+1} \big[B_1 + B_2(r) E\big((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1} \Delta t_q\big)\big] \\
&\qquad\qquad \times E\Big(\frac{1}{2} \sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h\Big) \begin{pmatrix} f_1(re^{\sum_{h=1}^{n+1}(-1)^{h-1}\Delta t_h}) \\ f_2(re^{-\sum_{h=1}^{n+1}(-1)^{h-1}\Delta t_h}) \end{pmatrix} \Big\|_{\ell^\infty} \\
&\leq \frac{(mSe^t + |k|)^n}{n!} t^n e^{t/2} \|f\|_\infty.
\end{aligned} \tag{2.17}$$

Here the ℓ^∞-norm of a vector $z = {}^t(z_1, z_2) \in \mathbf{C}^2$ is denoted by $\|z\|_{\ell^\infty} = \max\{|z_1|, |z_2|\}$, and we have used

$$\Big\| \overleftarrow{\prod}{}_{p=2}^{n+1} \big[B_1 + B_2 E\big((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1} \Delta t_q\big)\big] \Big\| \leq (|k| + mre^t)^n, \tag{2.18}$$

and $|\sum_{h=1}^{n+1} (-1)^{h-1} \Delta t_h| \leq t$ and $\|B_1 + B_2 E\big((-1)^p \sum_{q=p}^{n+1} (-1)^{q-1} \Delta t_q\big)\| \leq |k| + mre^t$. This proves (2.14), ending the proof of Lemma 2.3.

Now we are in a position to finish the proof of Proposition 2.2. By Lemma 2.3, (2.14), we can conclude the expansion (2.9) or (2.13) is convergent in L^∞ norm, which yields the latter half of (2.8) or the estimate (2.4), as well as the support property of the solution $u(\cdot, t)$. This completes the proof of Proposition 2.2.

Proposition 2.4. *Let $W(r) = rV(r)$ with $V(r)$ as in Proposition 2.1. Then the operator $H(k, W) = T_k + W(r)$ in (1.4) (as well as T_k alone) is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$.*

Proof. We use the method of "invariance of domain" by Chernoff (1973, 1977).

In order to make the notations clearer, inside this proof, we denote the closure of the symmetric operator $H(k, W)$ with domain $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$ by $\widetilde{H}(k, W)$. $\widetilde{H}(k, W)$ is also symmetric, and evidently its domain $D[\widetilde{H}(k, W)]$ includes the space $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$.

To show that $H(k, W)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+; \mathbf{C}^2)$ or its closure $\widetilde{H}(k, W)$ is selfadjoint, we need to prove that if $\phi \in D[H(k, W)^*]$ satisfies $H(k, W)^*\phi = \pm i\phi$, then $\phi = 0$. Consider the case $H(k, W)^*\phi = -i\phi$. This means that ϕ is in $L^2(\mathbf{R}_+; \mathbf{C}^2)$ and, in view of (1.3), as a distribution

$$0 = H(k, W)^*\phi + i\phi = -i\sigma_2\Big(r\frac{\partial}{\partial r} + \frac{1}{2}\Big)\phi + T_{k1}\phi + W(r)\phi + i\phi.$$

Let $\mathcal{D}$ be the subspace of the members of $D[\widetilde{H}(k,W)]$ with compact support in $\mathbf{R}_+$, so that $C_0^\infty(\mathbf{R}_+;\mathbf{C}^2) \subseteq \mathcal{D}$. Take any g in $\mathcal{D}$ and consider the Cauchy problem for $\widetilde{H}(k,W)$

$$\frac{\partial}{\partial t}u(r,t) = -i\widetilde{H}(k,W)u(r,t), \quad u(r,0) = g(r). \tag{2.19}$$

Taken into account that $\widetilde{H}(k,W) \subseteq H(k,W)^* = \widetilde{H}(k,W)^*$, this means in view of (1.2b) and (1.3) that the solution $u(r,t)$ in $D[\widetilde{H}(k,W)]$ and so in $L^2(\mathbf{R}_+;\mathbf{C}^2)$ satisfies, as a distribution, the differential equation

$$\frac{\partial}{\partial t}u(r,t) = -i\Big[-i\sigma_2\Big(r\frac{\partial}{\partial r} + \frac{1}{2}\Big) + (-k\sigma_1 + mr\sigma_3) + W(r)\Big]u(r,t).$$

Then by the analogous arguments used to prove Lemma 2.3, we can see that there exists a unique solution $u(\cdot,t)$ of (2.19), which belongs to the space $\mathcal{D}$. In the same reason as we saw (2.2), we also have $\|u(\cdot,t)\|_2 = \|g\|_2$ in L^2-norm.

Put $F(t) = (\phi, u(\cdot,t))$. Then $F(t)$ is bounded in $t \in \mathbf{R}$, because

$$|F(t)| \le \|\phi\|_2\|u(\cdot,t)\|_2 = \|\phi\|_2\|g\|_2.$$

Since $H(k,W)^* = \widetilde{H}(k,W)^*$, we find by differentiating in t

$$F'(t) = (\phi, -i\widetilde{H}(k,W)u(\cdot,t)) = (H(k,W)^*\phi, -iu(\cdot,t)) = (\phi, u(\cdot,t)) = F(t).$$

It follows that $F(t) = F(0)e^t$. Since $F(t)$ is bounded, we must have $F(0) = 0$, that is, $(\phi, g) = 0$. Since $\mathcal{D}$ is dense, we can conclude that $\phi = 0$. A similar argument can be made for the case $H(k,W)^*\phi = i\phi$. This proves Proposition 2.4.

3. Construction of the path space measure

We consider only the case $t > 0$; the case $t < 0$ will be treated similarly. We want to construct our path space measure $\mu_{t,0}$ on the space of the 1-dimensional continuous paths $R : [0, t] \to [0, \infty)$ living on the closed half-line $[0, \infty) = \overline{\mathbf{R}_+}$, by modifying the method Nelson (1964) used to construct the Wiener measure. The argument will go analogously with that employed in Ichinose and Tamura (1988, 1987), in particular, in Ichinose (1993) for the Dirac equation in one-dimensional space $\mathbf{R}$.

We write the integral kernel of the unitary group $e^{-itT_k} \equiv N^{-1}e^{tC}N$ with generator (1.3) or (2.5) as $K(t; r, \rho) := e^{-itT_k}(r, \rho)$.

Though in this paper we have meant $\mathbf{R}_+ = (0, \infty)$, the open real half-line, we will denote by $\dot{\mathbf{R}}_+ = [0, \infty) \cup \{\infty\}$ the one-point compactification of the closed half-line $[0, \infty)$. For each fixed $t > 0$ let $\mathcal{R}_{t,0} = \prod_{[0,t]} \dot{\mathbf{R}}_+ = (\dot{\mathbf{R}}_+)^{[0,t]}$ be the uncountably many copies of $\dot{\mathbf{R}}_+$. By the Tychonoff theorem $\mathcal{R}_{t,0}$ is a compact Hausdorff space in the product topology. It may be regarded as the space of all paths $R : [0, t] \to \dot{\mathbf{R}}_+$, possibly discontinuous and possibly passing through zero and/or the infinity ∞. Let $C(\mathcal{R}_{t,0})$ be the Banach space of the complex-valued continuous functions on $\mathcal{R}_{t,0}$ with the maximum norm $\| \cdot \|$, and $C_{\text{fin}}(\mathcal{R}_{t,0})$ the subspace of those Ψ in $C(\mathcal{R}_{t,0})$ for which there exist a finite partition of the interval $[0, t]$:

$$0 = s_0 < s_1 < \cdots < s_\ell = t, \tag{3.1}$$

and a complex-valued bounded continuous function $F(x_0, x_1, \cdots, x_\ell)$ on $(\dot{\mathbf{R}}_+)^{\ell+1}$ such that

$$\Psi(R) = F(R(s_0), R(s_1), \cdots, R(s_\ell)). \tag{3.2}$$

Define, for each fixed $t > 0$, a functional $L_{t,0}(\Psi; f, g)$ which is linear in $\Psi \in C_{\text{fin}}(\mathcal{R}_{t,0})$ and sesquilinear in $(f, g) \in C_{00}^\infty([0, \infty); \mathbf{C}^2) \times C_{00}^\infty([0, \infty); \mathbf{C}^2)$, $f = {}^t(f_1, f_2)$, $g = {}^t(g_1, g_2)$, by

$$\begin{aligned} L_{t,0}(\Psi; f, g) = \int_0^\infty dr_\ell \int_0^\infty dr_{\ell-1} \cdots \int_0^\infty dr_0 \, {}^t\overline{f(r_\ell)} K(s_\ell - s_{\ell-1}; r_\ell, r_{\ell-1}) \\ \times K(s_{\ell-1} - s_{\ell-2}; r_{\ell-1}, r_{\ell-2}) \cdot \cdots \cdot K(s_1 - s_0; r_1, r_0) F(r_0, r_1, \cdots, r_\ell) g(r_0). \end{aligned} \tag{3.3}$$

The integral on the right-hand side of this equation is equal to

$$\begin{aligned} &\int_0^\infty dr_\ell \int_0^\infty dr_{\ell-1} \cdots \int_0^\infty dr_0 \, {}^t\overline{Nf(r_\ell)} \big(NK(s_\ell - s_{\ell-1}; r_\ell, r_{\ell-1})N^{-1}\big) \\ &\quad \times \big(NK(s_{\ell-1} - s_{\ell-2}; r_{\ell-1}, r_{\ell-2})N^{-1}\big) \cdot \cdots \cdot \big(NK(s_1 - s_0; r_1, r_0)N^{-1}\big) \\ &\quad \times F(r_0, r_1, \cdots, r_\ell) Ng(r_0). \end{aligned}$$

Therefore (3.3) can also be written by definition of e^{tC} as

$$\begin{aligned} &L_{t,0}(\Psi; f, g) \\ &= \int_0^\infty \int_0^\infty {}^t\overline{Nf(r_\ell)} e^{\Delta s_\ell C} e^{\Delta s_{\ell-1} C} \cdots e^{\Delta s_1 C} F(r_0, \cdots, r_{\ell-1}, r_\ell) Ng(r_0) \, dr_\ell dr_0, \end{aligned} \tag{3.4}$$

$$\Delta s_j = s_j - s_{j-1}, \ 1 \le j \le \ell,$$

where $e^{\Delta s_j C}$ is a linear operator with kernel $NK(s_j - s_{j-1}; r_j, r_{j-1})N^{-1}$ transforming functions of r_{j-1} to functions of r_j. Substituting e^{tC} in (2.13) with t replaced by Δs_j into (3.4), we get

$$\begin{aligned}
&L_{t,0}(\Psi; f, g) \\
&= \sum_{n=0}^{\infty} \sum_{n_1+\cdots+n_\ell=n} \int_0^\infty \int_0^\infty {}^t\overline{Nf(r_\ell)} C_{n_\ell}(\Delta s_\ell) \cdots C_{n_2}(\Delta s_2) C_{n_1}(\Delta s_1) \\
&\qquad \times [F(r_0, r_1, \cdots, r_\ell) N g(r_0)]\, dr_\ell dr_0 \\
&=: \sum_{n=0}^{\infty} L_{t,0}^n(\Psi; f, g).
\end{aligned} \tag{3.5}$$

To construct the measure $\mu_{t,0}$ on $\mathcal{R}_{t,0}$, the following lemma is of crucial importance.

Lemma 3.1. (i) *For each $t > 0$, $L_{t,0}(\Psi; f, g)$ and $L_{t,0}^n(\Psi; f, g)$ are independent of the choice of F corresponding to $\Psi \in C_{\mathrm{fin}}(\mathcal{R}_{t,0})$.*

(ii) *If one of f and g in $C_{00}^\infty([0,\infty); \mathbf{C}^2)$ has support in the bounded interval $[0, S]$, then the following inequalities hold:*

$$|L_{t,0}(\Psi; f, g)| \le \|N\|_1 \|N\|_\infty e^{t(mSe^t + |k| + 1/2)} \|\Psi\| \|f\|_1 \|g\|_\infty, \tag{3.6}$$

$$|L_{t,0}^n(\Psi; f, g)| \le \|N\|_1 \|N\|_\infty \frac{(mSe^t + |k|)^n}{n!} t^n e^{t/2} \|\Psi\| \|f\|_1 \|g\|_\infty, \tag{3.7}$$

for all $n = 0, 1, 2, \cdots$. Here $\|N\|_1$ and $\|N\|_\infty$ are the norms of the unitary matrix N, respectively, as a linear operator on $\mathbf{C}^2$ with ℓ^1-norm and ℓ^∞-norm, which are less than or equal to $\sqrt{2}$.

Proof. The first assertion (i) is due to the semigroup property of e^{-itT_k} or e^{tC}. The proof of the second one (ii) will make use of the inequality for $C_{n_j}(\Delta s_j)$ to be obtained from (2.14) with n_j, Δs_j in place of n, t to get

$$\begin{aligned}
|L_{t,0}^n(\Psi; f, g)| &\le \sum_{n_1+\cdots+n_\ell=n} \prod_{p=1}^{\ell} \Big[\frac{(mSe^{\Delta s_p} + |k|)^{n_p}}{n_p!} (\Delta s_p)^{n_p} e^{\Delta s_p/2} \Big] \|F\|_\infty \|Nf\|_1 \|Ng\|_\infty \\
&\le (mSe^t + |k|)^n \frac{(\sum_{p=1}^{\ell} \Delta s_p)^n}{n!} e^{(1/2)\sum_{q=1}^{\ell} \Delta s_q} \|F\|_\infty \|Nf\|_1 \|Ng\|_\infty \\
&= \frac{(mSe^t + |k|)^n}{n!} t^n e^{t/2} \|\Psi\| \|Nf\|_1 \|Ng\|_\infty.
\end{aligned}$$

Here we have used that for $1 \le j \le \ell$, $\Delta s_j \le \sum_{p=1}^{\ell} \Delta s_p = t$, and that

$$\sum_{n_1+\cdots+n_\ell=n} \frac{(\Delta s_1)^{n_1} (\Delta s_2)^{n_2} \cdots (\Delta s_\ell)^{n_\ell}}{n_1!\, n_2! \cdots n_\ell!} = \frac{(\sum_{p=1}^{\ell} \Delta s_p)^n}{n!} = \frac{t^n}{n!}.$$

Since $\|Nf\|_1 \leq \|N\|_1\|f\|_1$ and $\|Ng\|_\infty \leq \|N\|_\infty\|g\|_\infty$, this yields (3.7). Hence by summing up, we also obtain (3.6) from (3.5). This proves Lemma 3.1.

A consequence of this lemma is the following. Since $C_{\mathrm{fin}}(\mathcal{R}_{t,0})$ is dense in $C(\mathcal{R}_{t,0})$ by the Stone-Weierstrass theorem, the inequalities (3.6) and (3.7) are also valid for all $\Psi \in C(\mathcal{R}_{t,0})$. This means that for each fixed pair (f, g), $L_{t,0}(\,\cdot\,; f, g)$ and $L^n_{t,0}(\,\cdot\,; f, g)$ are complex-valued continuous linear functional on $C(\mathcal{R}_{t,0})$. By the Riesz representation theorem, there exist complex-valued regular Borel measures $\mu_{t,0;f,g}$ and $\mu^n_{t,0;f,g}$ on $\mathcal{R}_{t,0}$ for all $n = 0, 1, 2, \cdots$, such that for all $\Psi \in C(\mathcal{R}_{t,0})$,

$$L_{t,0}(\Psi; f, g) = \int_{\mathcal{R}_{t,0}} d\mu_{t,0;f,g}(X)\,\Psi(R), \tag{3.8}$$

$$L^n_{t,0}(\Psi; f, g) = \int_{\mathcal{R}_{t,0}} d\mu^n_{t,0;f,g}(X)\,\Psi(R). \tag{3.9}$$

Hence, since the left-hand sides of (3.8) and (3.9) are continuous sesquilinear forms in (f, g) in the product space $C^\infty_{00}([0,\infty); \mathbf{C}^2) \times C^\infty_{00}([0,\infty); \mathbf{C}^2)$, we can also see, with $M_2(\mathbf{C})$ being the set of all 2×2-matrices, that there exist 2×2-matrix distribution-valued, precisely, $C^\infty_{00}([0,\infty)^2; M_2(\mathbf{C}))'$-valued, regular Borel measures $\mu_{t,0}(\cdot)$ and $\mu^n_{t,0}(\cdot)$ on $\mathcal{R}_{t,0}$ such that for every pair $(f, g) \in C^\infty_{00}([0,\infty); \mathbf{C}^2) \times C^\infty_{00}([0,\infty); \mathbf{C}^2)$,

$$\langle {}^t\overline{f}, \mu_{t,0}(\cdot)g\rangle = \mu_{t,0;f,g}(\cdot), \tag{3.10}$$

$$\langle {}^t\overline{f}, \mu^n_{t,0}(\cdot)g\rangle = \mu^n_{t,0;f,g}(\cdot). \tag{3.11}$$

Here we note that we have

$$C^\infty_{00}([0,\infty)^2; M_2(\mathbf{C})) = C^\infty_{00}([0,\infty); \mathbf{C}^2)\hat{\otimes}C^\infty_{00}([0,\infty); \mathbf{C}^2),$$

and the space of the continuous sesquilinear forms on the product space $C^\infty_{00}([0,\infty); \mathbf{C}^2) \times C^\infty_{00}([0,\infty); \mathbf{C}^2)$ coincides with the dual space

$$C^\infty_{00}([0,\infty)^2; M_2(\mathbf{C}))' = C^\infty_{00}([0,\infty); \mathbf{C}^2)'\hat{\otimes}C^\infty_{00}([0,\infty); \mathbf{C}^2)'.$$

Here the tensor products on the right-hand side are completed in the π- or ε-tensor product topology, because $C^\infty_{00}([0,\infty); \mathbf{C}^2)$ and $C^\infty_{00}([0,\infty); \mathbf{C}^2)'$ are nuclear spaces, since they are subspaces of the nuclear spaces $C^\infty_0(\mathbf{R}; \mathbf{C}^2)$ and $\mathcal{D}'(\mathbf{R}; \mathbf{C}^2)$, respectively [e.g. Pietsch (1970, Chaps. 5, 6), Trèves (1967, Chap. 50)].

It follows from the expansion (3.5) that

$$\mu_{t,0;f,g}(\cdot) = \sum_{n=0}^{\infty} \mu^n_{t,0;f,g}(\cdot) \tag{3.12}$$

in the sense of the expansion of the measure or

$$\mu_{t,0}(\cdot) = \sum_{n=0}^{\infty} \mu^n_{t,0}(\cdot) \tag{3.13}$$

in the sense of the expansion of the 2×2-matrix distribution-valued measure.

Finally we shall observe on which set of the paths the measures $\mu_{t,0}$ and $\mu_{t,0}^n$ are concentrated.

Lemma 3.2. *For each* $n = 0, 1, 2, ...$, *the measure* $\mu_{t,0}^n$ *is concentrated on the set of the continuous, exactly* n*-vertex piecewise smooth paths* $R : [0,t] \to [0,\infty)$, *in fact, with each of their segments being a particular smooth curve such that for an* n*-partition:* $0 = t_0 < t_1 < \cdots < t_{n+1} = t$ *of the interval* $[0,t]$ *and for* $t_{h-1} \leq s \leq t_h,\ 1 \leq h \leq n+1$,

$$R(s) = R(0)e^{\pm[\sum_{p=1}^{h-1}(-1)^{p-1}(t_p - t_{p-1}) + (-1)^{h-1}(s - t_{h-1})]}.$$

Therefore in each n*-partitioned short interval* (t_{p-1}, t_p) *each segement is exponentially growing or decreasing.*

Consequently, the measure $\mu_{t,0}$ *is concentrated on the union of these sets of paths on which the* $\mu_{t,0}^n$ *are concentrated.*

Proof. This basically follows from the very expression (2.15) or (2.16) that the integral kernel of e^{tA} or $e^{-itT_{k0}}$ has. Inspecting through the expression of $C_{n_j}(\Delta s_j)$ in (2.13) with n_j, Δs_j in place of n, t what is the support of the integral kernel of $C_{n_j}(\Delta s_j)$, we shall be able to see through $L_{t,0}^n(\Psi; f, g)$ and $L_{t,0}(\Psi; f, g)$ in (3.5) the statement of this lemma.

We will sketch how it goes. We have to investigate what the product operator

$$C_{n_\ell}(\Delta s_\ell) \cdots C_{n_2}(\Delta s_2) C_{n_1}(\Delta s_1)$$

in the integrand in the second member of (3.5) is. This operator corresponds to the partition (3.1) of the interval $[0,t]$. Each factor $C_{n_j}(\Delta s_j)$, $1 \leq j \leq \ell$, is, as seen from the n-th term the expansion (2.9)/(2.13), an n_j-ple integral with n_j, Δs_j in place of n, t, and with n_j integration variables $t_{j,q}$ (in place of t_q), moving in the n_j-plex :

$$s_{j-1} \equiv t_{j,0} \leq t_{j,1} \leq \cdots \leq t_{j,n_j} \leq t_{j,n_j+1} \equiv s_j. \tag{3.14}$$

Then we have from (2.13), with $\Delta t_{j,q} := t_{j,q} - t_{j,q-1},\ 1 \leq q \leq n_j + 1$,

$$
\begin{aligned}
&C_{n_\ell}(\Delta s_\ell) \cdots C_{n_2}(\Delta s_2) C_{n_1}(\Delta s_1) \\
&= \int_{s_{\ell-1}}^{s_\ell} \int_{s_{\ell-1}}^{t_{\ell,n_\ell}} \cdots \int_{s_{\ell-1}}^{t_{\ell,2}} dt_{\ell,n_\ell} dt_{\ell,n_\ell-1} \cdots dt_{\ell,1} \\
&\times \int_{s_{\ell-2}}^{s_{\ell-1}} \int_{s_{\ell-2}}^{t_{\ell-1,n_{\ell-1}-1}} \cdots \int_{s_{\ell-2}}^{t_{\ell-1,2}} dt_{\ell-1,n_{\ell-1}} dt_{\ell-1,n_{\ell-1}-1} \cdots dt_{\ell-1,1} \\
&\times \cdots \\
&\times \int_{s_1}^{s_2} \int_{s_1}^{t_{2,n_2}} \cdots \int_{s_1}^{t_{2,2}} dt_{2,n_2} dt_{2,n_2-1} \cdots dt_{2,1} \int_{s_0}^{s_1} \int_{s_0}^{t_{1,n_1}} \cdots \int_{s_0}^{t_{2,2}} dt_{1,n_1} dt_{1,n_1-1} \cdots dt_{1,1} \\
&\qquad \times \overleftarrow{\prod}_{p_\ell=2}^{n_\ell+1} \Big[B_1 + B_2 E\Big((-1)^{p_\ell} \sum_{q=p_\ell}^{n_\ell+1} (-1)^{q-1} \Delta t_{\ell,q}\Big)\Big] \exp\Big[\Big(\sum_{h=1}^{n_\ell+1} (-1)^{h-1} \Delta t_{\ell,h}\Big) A\Big] \\
&\qquad \times \overleftarrow{\prod}_{p_{\ell-1}=2}^{n_{\ell-1}+1} \Big[B_1 + B_2 E\Big((-1)^{p_{\ell-1}} \sum_{q=p_{\ell-1}}^{n_{\ell-1}+1} (-1)^{q-1} \Delta t_{\ell-1,q}\Big)\Big] \\
&\qquad \times \exp\Big[\Big(\sum_{h=1}^{n_{\ell-1}+1} (-1)^{h-1} \Delta t_{\ell-1,h}\Big) A\Big] \\
&\qquad \times \cdots \\
&\qquad \times \overleftarrow{\prod}_{p_2=2}^{n_2+1} \Big[B_1 + B_2 E\Big((-1)^{p_2} \sum_{q=p_2}^{n_2+1} (-1)^{q-1} \Delta t_{2,q}\Big)\Big] \exp\Big[\Big(\sum_{h=1}^{n_2+1} (-1)^{h-1} \Delta t_{2,h}\Big) A\Big] \\
&\qquad \times \overleftarrow{\prod}_{p_1=2}^{n_1+1} \Big[B_1 + B_2 E\Big((-1)^{p_1} \sum_{q=p_1}^{n_1+1} (-1)^{q-1} \Delta t_{1,q}\Big)\Big] \exp\Big[\Big(\sum_{h=1}^{n_1+1} (-1)^{h-1} \Delta t_{1,h}\Big) A\Big].
\end{aligned}
\tag{3.15}
$$

Here note that this integral (3.15) is a just n-ple integral with $n = n_1 + \cdots + n_\ell$, and there are no integrations in the variables

$$s_{\ell-1} \equiv t_{\ell,0} \equiv t_{\ell-1,n_{\ell-1}+1},\ s_{\ell-2} \equiv t_{\ell-1,0} \equiv t_{\ell-2,n_{\ell-2}+1},\ \cdots, s_1 \equiv t_{1,0} \equiv t_{0,n_1+1}.$$

Applying the formula (2.11) to the first product of the integrand on the right-hand side of this equation (3.15), we see the integrand on the right-hand side of (3.15) is equal to

$$
\begin{aligned}
&\overleftarrow{\prod}_{p_\ell=2}^{n_\ell+1}\big[B_1+B_2E\big((-1)^{p_\ell}\sum_{q=p_\ell}^{n_\ell+1}(-1)^{q-1}\Delta t_{\ell,q}\big)\big]\\
&\times\overleftarrow{\prod}_{p_{\ell-1}=2}^{n_{\ell-1}+1}\big[B_1+B_2E\big((-1)^{n_{\ell-1}+2-p_{\ell-1}}\sum_{h=1}^{n_\ell+1}(-1)^{h-1}\Delta t_{\ell,h}\\
&+(-1)^{p_{\ell-1}}\sum_{q=p_{\ell-1}}^{n_{\ell-1}+1}(-1)^{q-1}\Delta t_{\ell-1,q}\big)\big]\\
&\quad\times\exp\Big[\Big((-1)^{n_{\ell-1}}\sum_{h=1}^{n_\ell+1}(-1)^{h-1}\Delta t_{\ell,h}+\sum_{h=1}^{n_{\ell-1}+1}(-1)^{h-1}\Delta t_{\ell-1,h}\Big)A\Big]\\
&\times\cdots\\
&\times\overleftarrow{\prod}_{p_2=2}^{n_2+1}\big[B_1+B_2E\big((-1)^{p_2}\sum_{q=p_2}^{n_2+1}(-1)^{q-1}\Delta t_{2,q}\big)\big]\exp\Big[\Big(\sum_{h=1}^{n_2+1}(-1)^{h-1}\Delta t_{2,h}\Big)A\Big]\\
&\times\overleftarrow{\prod}_{p_1=2}^{n_1+1}\big[B_1+B_2E\big((-1)^{p_1}\sum_{q=p_1}^{n_1+1}(-1)^{q-1}\Delta t_{1,q}\big)\big]\exp\Big[\Big(\sum_{h=1}^{n_1+1}(-1)^{h-1}\Delta t_{1,h}\Big)A\Big].
\end{aligned}
$$

Then by repeated use of (2.11) we reach

$$
\begin{aligned}
&C_{n_\ell}(\Delta s_\ell)\cdots C_{n_2}(\Delta s_2)C_{n_1}(\Delta s_1)\\
&=\int_{s_{\ell-1}}^{s_\ell}\int_{s_{\ell-1}}^{t_{\ell,n_\ell}}\cdots\int_{s_{\ell-1}}^{t_{\ell,2}}\int_{s_{\ell-2}}^{s_{\ell-1}}\int_{s_{\ell-2}}^{t_{\ell-1,n_{\ell-1}-1}}\cdots\int_{s_{\ell-2}}^{t_{\ell-1,2}}\\
&\times\cdots\\
&\times\int_{s_1}^{s_2}\int_{s_1}^{t_{2,n_2}}\cdots\int_{s_1}^{t_{2,2}}\int_{s_0}^{s_1}\int_{s_0}^{t_{1,n_1}}\cdots\int_{s_0}^{t_{1,2}}\\
&\quad\times dt_{\ell,n_\ell}dt_{\ell,n_\ell-1}\cdots dt_{\ell,1}\;dt_{\ell-1,n_{\ell-1}}dt_{\ell-1,n_{\ell-1}-1}\cdots dt_{\ell-1,1}\\
&\qquad\times\cdots\times dt_{2,n_2}dt_{2,n_2-1}\cdots dt_{2,1}\;dt_{1,n_1}dt_{1,n_1-1}\cdots dt_{1,1}\\
&\quad\times B(t;n_1,n_2,\ldots,n_\ell)\exp\Big[\Big((-1)^{n_1+n_2+\cdots n_{\ell-1}}\sum_{h=1}^{n_\ell+1}(-1)^{h-1}\Delta t_{\ell,h}\\
&\quad+(-1)^{n_1+n_2+\cdots n_{\ell-2}}\sum_{h=1}^{n_{\ell-1}+1}(-1)^{h-1}\Delta t_{\ell-1,h}+\cdots+(-1)^{n_1+n_2}\sum_{h=1}^{n_3+1}(-1)^{h-1}\Delta t_{3,h}\\
&\qquad+(-1)^{n_1}\sum_{h=1}^{n_2+1}(-1)^{h-1}\Delta t_{2,h}+\sum_{h=1}^{n_1+1}(-1)^{h-1}\Delta t_{1,h}\Big)A\Big],
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
&B(t; n_1, n_2, \dots, n_\ell)\\
&= \overleftarrow{\prod}{}_{p_\ell=2}^{n_\ell+1} \big[B_1 + B_2 E\big((-1)^{p_\ell} \sum_{q=p_\ell}^{n_\ell+1} (-1)^{q-1} \Delta t_{\ell,q}\big)\big]\\
&\quad \times \overleftarrow{\prod}{}_{p_{\ell-1}=2}^{n_{\ell-1}+1} \big[B_1 + B_2 E\big((-1)^{n_{\ell-1}+2-p_{\ell-1}} \sum_{h=1}^{n_\ell+1} (-1)^{h-1} \Delta t_{\ell,h}\\
&\qquad + (-1)^{p_{\ell-1}} \sum_{q=p_{\ell-1}}^{n_{\ell-1}+1} (-1)^{q-1} \Delta t_{\ell-1,q}\big)\big]\\
&\quad \times \overleftarrow{\prod}{}_{p_{\ell-2}=2}^{n_{\ell-2}+1} \big[B_1 + B_2 E\big((-1)^{n_{\ell-2}+2-p_{\ell-2}+n_{\ell-1}} \sum_{h=1}^{n_\ell+1} (-1)^{h-1} \Delta t_{\ell,h}\\
&\quad + (-1)^{n_{\ell-2}+2-p_{\ell-2}} \sum_{h=1}^{n_{\ell-1}+1} (-1)^{h-1} \Delta t_{\ell-1,h} + (-1)^{p_{\ell-2}} \sum_{q=p_{\ell-2}}^{n_{\ell-1}+1} (-1)^{q-1} \Delta t_{\ell-2,q}\big)\big] \qquad (3.17)\\
&\quad \times \cdots\\
&\quad \times \overleftarrow{\prod}{}_{p_1=2}^{n_1+1} \Big[B_1 + B_2 E\Big((-1)^{n_1+2-p_1+n_2+\cdots+n_{\ell-1}} \sum_{h=1}^{n_\ell+1} (-1)^{h-1} \Delta t_{\ell,h}\\
&\qquad + (-1)^{n_1+2-p_1+n_2+\cdots+n_{\ell-2}} \sum_{h=1}^{n_{\ell-1}+1} (-1)^{h-1} \Delta t_{\ell-1,h}\\
&\qquad + \cdots + (-1)^{n_1+2-p_1+n_2} \sum_{h=1}^{n_3+1} (-1)^{h-1} \Delta t_{3,h}\\
&\qquad + (-1)^{n_1+2-p_1} \sum_{h=1}^{n_2+1} (-1)^{h-1} \Delta t_{2,h} + (-1)^{p_1} \sum_{q=p_1}^{n_1+1} (-1)^{q-1} \Delta t_{1,q}\Big)\Big].
\end{aligned}
$$

In the exponential power of the second factor on the right-hand side of (3.16) above, observe the neighbouring summations to find the first term in each summation and the last term in its next summation have the same sign, so that for each j with $1 \le j \le \ell$,

$$
\begin{aligned}
&(-1)^{n_1+\cdots+n_{j-1}} \Delta t_{j,1} + (-1)^{n_1+\cdots+n_{j-2}} (-1)^{n_{j-1}} \Delta t_{j-1,n_{j-1}+1}\\
=&(-1)^{n_1+\cdots+n_{j-1}} (\Delta t_{j,1} + \Delta t_{j-1,n_{j-1}+1})\\
=&(-1)^{n_1+\cdots+n_{j-1}} \big((t_{j,1} - t_{j,0}) + (t_{j-1,n_{j-1}+1} - t_{j-1,n_{j-1}})\big)\\
=&(-1)^{n_1+\cdots+n_{j-1}} (t_{j,1} - t_{j-1,n_{j-1}}),
\end{aligned}
$$

because $t_{j,0} = t_{j-1,n_{j-1}+1} = s_j$. The same is true for the neighbouring summations in the variables of the matrix $E(\cdot)$ on the right-hand side of (3.17). We see by the partition (3.1) with (3.14) the $n = n_1 + \cdots + n_\ell$ integration variables move in the set of

ℓ n_j-pleces:

$$\begin{aligned}\Delta(s_1,s_2,\dots,s_\ell):\quad 0 &\equiv s_0\equiv t_{1,0}\le t_{1,1}\le t_{1,2}\le\cdots\le t_{1,n_1}\le t_{1,n_1+1}\equiv s_1\\ &\equiv t_{2,0}\le t_{2,1}\le t_{2,2}\le\cdots\le t_{2,n_2}\le t_{2,n_2+1}\equiv s_2\\ &\equiv t_{3,0}\le t_{3,1}\le\cdots\le t_{\ell-1,n_{\ell-1}}< t_{\ell-1,n_{\ell-1}+1}\equiv s_{\ell-1}\\ &\equiv t_{\ell,0}\le t_{\ell,1}\le t_{\ell,2}\le\cdots\le t_{\ell,n_\ell}\le t_{\ell,n_\ell+1}\equiv s_\ell\equiv t.\end{aligned}\tag{3.18}$$

So, we change the n integration variables $\{t_{j,q};\ 1\le q\le n_j,\ 1\le j\le\ell\}$ to $\{t_1,t_2,\dots,t_n\}$, namely, put for $0\le h\le n_1+n_2+\cdots+n_\ell+1=n+1$,

$$t_h=\begin{cases} t_{1,0}=s_0=0, & h=0,\\ t_{1,h}, & 1\le h\le n_1,\\ t_{2,h-n_1}, & n_1+1\le h\le n_1+n_2,\\ t_{3,h-(n_1+n_2)}, & n_1+n_2+1\le h\le n_1+n_2+n_3,\\ \cdots, & \\ t_{j,h-(n_1+n_2+\cdots+n_{j-1})}, & n_1+n_2+\cdots+n_{j-1}+1\le h\le n_1+n_2+\cdots+n_j,\\ \cdots, & \\ t_{\ell-1,h-(n_1+n_2+\cdots+n_{\ell-2})}, & n_1+n_2+\cdots+n_{\ell-2}+1\le h\le n_1+n_2+\cdots+n_{\ell-1},\\ t_{\ell,h-(n_1+n_2+\cdots+n_{\ell-1})}, & n_1+n_2+\cdots+n_{\ell-1}+1\le h\le n_1+n_2+\cdots+n_\ell,\\ t_{\ell,n_\ell+1}=s_\ell=t, & h=n_1+n_2+\cdots+n_\ell+1=n+1.\end{cases}\tag{3.19}$$

Then we see from (3.18) the new integration variables $t_1,t_2,\dots,t_n$ move in the set

$$\begin{aligned}\Delta(s_1,s_2,\dots,s_\ell):\quad 0\equiv &s_0\equiv t_0\le t_1\le t_2\le\cdots\le t_{n_1}\le\\ &s_1\le t_{n_1+1}\le\cdots\le t_{n_1+n_2}\le\\ &s_2\le t_{n_1+n_2+1}\le\cdots\le t_{n_1+\cdots+n_{\ell-1}}\le\\ &s_{\ell-1}\le t_{n_1+\cdots+n_{\ell-1}+1}\le\cdots\le t_n\le s_\ell\equiv t.\end{aligned}\tag{3.20}$$

Thus

$$\begin{aligned}&C_{n_\ell}(\Delta s_\ell)\cdots C_{n_2}(\Delta s_2)C_{n_1}(\Delta s_1)\\ &=\int\cdots\int_{\Delta(s_1,s_2,\dots,s_\ell)}dt_n dt_{n-1}\cdots dt_2 dt_1\\ &\qquad\times B(t;n_1,n_2,\dots,n_\ell)\exp\Big[\Big(\sum_{h=1}^{n+1}(-1)^{h-1}(t_h-t_{h-1})\Big)A\Big],\end{aligned}\tag{3.21}$$

where (3.17) turns out to be, with $\Delta t_h := t_h - t_{h-1}$,

$$\begin{aligned}
&B(t; n_1, n_2, \dots, n_\ell) \\
&= \overset{\longleftarrow}{\prod}{}_{p_\ell=2}^{n_\ell+1} \Big[B_1 + B_2 E\Big((-1)^{n_1+\cdots+n_{\ell-1}+p_\ell} \sum_{h=n_1+\cdots+n_{\ell-1}+p_\ell}^{n+1} (-1)^{h-1}\Delta t_h\Big)\Big] \\
&\quad \times \overset{\longleftarrow}{\prod}{}_{p_{\ell-1}=2}^{n_{\ell-1}+1} \Big[B_1 + B_2 E\Big((-1)^{n_1+\cdots+n_{\ell-2}+p_{\ell-1}} \sum_{h=n_1+\cdots+n_{\ell-2}+p_{\ell-1}}^{n+1} (-1)^{h-1}\Delta t_h\Big)\Big] \\
&\quad \times \overset{\longleftarrow}{\prod}{}_{p_{\ell-2}=2}^{n_{\ell-2}+1} \Big[B_1 + B_2 E\Big((-1)^{n_1+\cdots+n_{\ell-3}+p_{\ell-2}} \sum_{h=n_1+\cdots+n_{\ell-3}+p_{\ell-2}}^{n+1} (-1)^{h-1}\Delta t_h\Big)\Big] \\
&\quad \times \cdots \\
&\quad \times \overset{\longleftarrow}{\prod}{}_{p_1=2}^{n_1+1} \Big[B_1 + B_2 E\Big((-1)^{p_1} \sum_{h=p_1}^{n+1} (-1)^{h-1}\Delta t_h\Big)\Big].
\end{aligned} \tag{3.22}$$

We can see by this expression (3.21) with (3.22) through (3.5) that the measure $\mu_{t,0}^n$ is concentrated on those paths mentioned in the statement of Lemma 3.2. This end the proof of this lemma.

4. Path integral representation

We first establish the path integral representation to the solution $u(r,t)$ of the Cauchy problem (1.5) for the operator $H(k, rV)$ in (1.4), and next give the proof of Theorems 1.1 and 1.2.

We begin with showing the following path integral representation of Feynman–Kac type for the semigroup $e^{-itH(k,rV)}$.

Theorem 4.1. *For $e^{-itH(k,rV)}$ it holds that*

$$\begin{aligned}
&(f, e^{-itH(k,rV)} g) \\
&= \int_{C([0,t]\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \exp\Big[-i\int_0^t V(R(s))R(s)\, ds\Big].
\end{aligned} \tag{4.1}$$

Proof. For $s > 0$, define a linear operator $U(s)$ by

$$(U(s)g)(r) = \big(e^{-isT_k} e^{-isW} g\big)(r) = \int_0^\infty K(s; r, \rho) e^{-isV(\rho)\rho} g(\rho)\, d\rho \tag{4.2}$$

with $W(r) = V(r)r$ for $g \in C_{00}^\infty([0,\infty); \mathbf{C}^2)$; recall $K(s; r, \rho)$ is the integral kernel of the semigroup $e^{-isT_k} \equiv N^{-1}e^{sC}N$ for T_k in (1.3). $U(s)$ is a contraction operator on $L^2(\mathbf{R}_+; \mathbf{C}^2)$ because e^{-isT_k} is a unitary operator, and also bounded in L^∞ operator norm because of (2.4)/(2.8).

Let n be an arbitrary positive integer and put $t_{h-1} = (h-1)t/n,\ h = 1, 2, \cdots, n+1$. Then we find, with $r_n = r$,

$$\begin{aligned} &[U(t/n)^n g](r) \\ &= \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n \text{ times}} \overset{\longleftarrow}{\prod}{}_{h=1}^{n} K(t/n; r_h - r_{h-1}) e^{-i\sum_{h=1}^n V(r_{h-1}) r_{h-1}(t/n)} g(r_0)\, dr_0 \cdots dr_{n-1}. \end{aligned} \tag{4.3}$$

Then the left-hand side converges, by the Trotter product formula (e.g Reed and Simon (1980)), to $e^{-itH(k,rV)}g$ in $L^2(\mathbf{R}_+; \mathbf{C}^2)$, since we have seen in Proposition 2.4 that $H(k, rV) = T_k + rV$ is essentially selfadjoint. For the right-hand side, we find by (3.3) and (3.8) that taking the sesquilinear inner product of (4.3) with $f \in C_{00}^\infty([0,\infty); \mathbf{C}^2)$ yields

$$\begin{aligned} &(f, U(t/n)^n g) \\ &= \int_{C([0,t]\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \exp\Big[-i \sum_{h=1}^{n} V(R(t_{h-1})) R(t_{h-1})(t/n)\Big], \end{aligned} \tag{4.4}$$

the right-hand side of which converges by the Lebesgue bounded convergence theorem, as $n \to \infty$, to

$$\int_{C([0,t]\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \exp\Big[-i \int_0^t V(R(s)) R(s)\, ds\Big].$$

Thus we have shown Theorem 4.1.

Now, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1.

In Theorem 4.1, considering, instead of $e^{-itH(h,rV)}$, $e^{-it(H(k,rV)-\lambda r)}$, where $\lambda = \pm i\varepsilon$ with ε being a non-zero real number, so that λ is not in the spectrum of the operator $\tau_k + V$, we can get by Theorem 4.1 the expression (4.1) corresponding to this case. Then taking into account the relation between the radial Dirac operator $\tau_k + V(r)$ concerned and $H(k, rV)$ in (1.4), we have only to take the Laplace transform of both sides to get (1.6) in Theorem 1.1.

Proof of Theorem 1.2.

The case (i), (1.8), is an immediate consequence of Theorem 1.1. As for the case (ii), multiply both sides of the equation (4.1) by $e^{\mp\varepsilon t}$ with the same ε as above. Then we find the expression for $e^{-itH(k,rV)\mp\varepsilon t}$. Note that $H(k, rV)$ has also a real spectrum, since by Proposition 2.4, it is a selfadjoint operator. Then taking the Laplace tansform and letting $\varepsilon \to 0$, we have the formula (1.9).

This completes the proof of the main theorems in Section 1.

Acknowledgememts. I am grateful to Professors Akira Inomata and Hagen Kleinert for valuable discussions about the procedure described in Introduction. Especially, I would like to thank Professor Inomata for kindly explaining the issue with its rich background in the method of path integration. I would also like to thank Professors Joachim Weidmann and Osanobu Yamada for helpful discussions about the essential selfadjointness of the radial Dirac operator in Section 2. Last but not least, I would like to thank Professor John R. Klauder for carefully reading the draft of this manuscript to give me valuable comments.

The preliminary version of this work was announced at the Workshop "Mathematical Problems in Quantum Mechanics", a satellite meeting to the 14th International Congress on Mathematical Physics (M∩Φ) 2003, Lisbon. I would like to heartily thank its organizers (Pavel Exner, Pedro Freitas, Jan Philip Solovej) for their invitation, in particular, Professor Freitas, for his kind hospitality and generous support during my stay at IST, University of Lisbon.

This work is partially supported by the Grant-in-Aid for Scientific Research (B) No. 13440044 and No. 16340038, Japan Society for the Promotion of Science.

Arnold, V, Kalf, H. and Schneider, A., “Separated Dirac operators and asymptotically constant linear systems,” Math. Proc. Camb. Phil. Soc. **121**, 141–146(1997).

Bjorken, J. D. and Drell, S. D., *Relativistic Quantum Mechanics* McGraw-Hill, New York 1964.

Blanchard, Ph., Combe, Ph., Sirugue, M. and Sirugue-Collin, M., “Probabilistic solution of the Dirac equation, Path integral representation for the solution of the Dirac equation in presence of an electromagnetic field,” Bielefeld, BiBoS Preprint Nos. **44**, **66**, 1985.

Chernoff, P., “Essential self-adjointness of powers of generators of hyperbolic equations,” J. Functional Analysis **12**, 401–414 (1973); “Schrödinger and Dirac operators with singular potentials and hyperbolic equations,” Pacific J. Math. **72**, 361–382 (1977).

Combe, Ph., Sirugue, M. and Sirugue-Collin, M., “Point processes and quantum physics: Some developments and results,” Proceedings of the 8th Internat. Congress on Math. Phys. (M$\cap\Phi$), (M. Mebkhout and R. Sénéor, eds.), Marseille, 1986, World Scientific, Singapore 1987, pp. 421–430.

Duru, H. and Kleinert, H., “ Solution of the path integral for the H-atom,” Phys. Lett. **B 84**, 185–188 (1979); “ Quantum mechanics of H-atom from path integrals,” Fortschr. Phys. **30**, 401–435 (1982).

Gaveau, B., “Representation formulas of the Cauchy problem for hyperbolic systems generalizing Dirac system,” J. Functional Analysis **58**, 310–319 (1984).

Gaveau, B., Jacobson, T., Kac , M. and Schulman , L. S., “Relativistic extension of the analogy between quantum mechanics and Brownian motion,” Phys. Rev. Lett. **53**, 419–422 (1984).

Grosche, Ch. and Steiner, F., *Handbook of Feynman Path Integrals*, Springer Tracts in Modern Physics, **145**, Springer, Berlin, Heidelberg 1998.

Ichinose, T., “Path integral for the Dirac equation in two space-time dimensions,” Proc. Japan Acad. Ser. **A**, Math. Sci. **58**, no. 7, 290–293 (1982).

Ichinose, T., “Path integral for a hyperbolic system of the first order,” Duke Math. J. **51**, 1–36 (1984).

Ichinose, T., “Path integral for the Dirac equation,” Sugaku Expositions **6**, no. 1, American Mathematical Society, 13–31(1993).

Ichinose, T. and Jefferies, B., “The propagator of the radial Dirac equation,” J. Math. Phys. **43**, no. 8, 3963–3983 (2002).

Ichinose, T. and Tamura, H., “Propagation of a Dirac particle. A path integral approach,” J. Math. Phys. **25**, 1810–1819 (1984).

Ichinose, T. and H. Tamura, H., “The Zitterbewegung of a Dirac particle in two-dimensional space-time,” J. Math. Phys. **29**, 103–109 (1988).

Ichinose, T. and Tamura, H., “Path integral approach to relativistic quantum mechanics — Two-dimensional Dirac equation,” Progr. Theoret. Phys. Suppl. No. **92**, 144–175 (1987).

Inomata, A., Katsuraji, H. and Gerry, C. C., *Path Integrals and Coherent States of SU(2) and SU(1,1)*, World Scientific, Singapore 1992.

Jörgens, K., "Perturbations of the Dirac operator," Lecture Notes in Math. **280**, pp. 87–102, Springer, Berlin 1972.

Kleinert, H., *Path Integrals in Quantum Mechanics Statistics and Polymer Physics*, 2nd edition, World Scientific, Singapore 1990, 1995.

Nelson, E., "Feynman integrals and the Schrödinger equation," J. Math. Phys. **5**, 332–343 (1964).

Pietsch, A., *Nuclear Locally Convex Spaces*, Springer, Berlin, Heidelberg, New York 1970.

Reed, M. and Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis*, Revised and enlarged ed., Academic Press, New York, London 1980.

Thaller, B., *The Dirac Equation*, Texts and Monographs in Physics, Springer Verlag, Berlin-Heidelberg-New York 1992.

Trèves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York-London 1967.

Weidmann, J., *Spectral Theory of Ordinary Differential Operators*, Lect. Notes in Math. **1258**, Springer, Berlin 1987.

Zastawniak, T., "Path integrals for the telegrapher's and Dirac equations; the analytic family of measures and the underlying Poisson process," Bull. Polish Acad. Sci. Math. **36**, 341–356 (1988).