On the summability of formal solutions of linear partial differential equations*

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Abstract

We investigate the existence and the Borel summability of formal power series in one variable with holomorphic coefficients, solutions to linear partial differential equations with non Kowalevskaya type in the complex domain. Sufficient conditions are given in terms of the shape of the partial differential equations and initial conditions. Existence and asymptotic behaviour of local sectorial holomorphic solutions to these linear partial differential equations are obtained as byproduct.

Key words: asymptotic expansions, Cauchy problems, formal power series, Gevrey estimates, κ-summability, linear partial differential equations in the complex domain. 2000 MSC:35C10, 35C20.

1 Introduction

In this paper, we study linear partial differential equations

$$(E) \qquad \partial_{z_1}^{\beta_1}\partial_{z_2}^{\beta_2}\hat{X}(t,z_1,z_2) = P(t,z_1,z_2,\partial_t,\partial_{z_1},\partial_{z_2})\hat{X}(t,z_1,z_2),$$

where P is a polynomial in ∂_t, ∂_{z_1}, ∂_{z_2} with holomorphic coefficients in the variables t, z_1, z_2 in some neighborhood of the origin in $\mathbb{C}^3$.

Our approach is based on resummation methods of formal series solutions of (E), known as Borel summability, starting with the fundamental contributions of Écalle, see [9]. This point of view has shown a great success in the framework of differential equations in the complex domain, see [2],[18],[25]. But in the context of partial differential equations, very little is known. Direct approaches with integral representations have been attempted for linear partial differential equations with constant coefficients, see [3],[15],[16],[17], and also fixed point methods have been introduced for more general linear and even non-linear partial differential equations, see [7],[22].

The goal of this paper is to provide a general approach to contruct Borel summable formal series solutions of equations of the type (E). In a first part of the paper, we give sufficient conditions for the existence and unicity of formal series in the variable t,

$$\hat{X}(t,z_1,z_2) = \sum_{j=0}^{+\infty} \frac{t^j}{j!} X_j(z_1,z_2),$$

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which are solutions of (E), where $X_j(z_1, z_2)$, $j \geq 0$, are analytic functions in some neighborhood of the origin in $\mathbb{C}^2$ (Proposition 2).

In a second part, we investigate κ-summability properties of these constructed formal series. We give new sufficient conditions under which these constructed formal series solutions of (E) are κ-summable, where κ depends on the polynomial P and β_1, β_2 (Theorem 1). In the proof, we use a key proposition (Proposition 1), already introduced in a previous paper, see [4]. We will also make use, in a crucial way, of a version of the Cauchy-Kowalevskaya theorem obtained by L. Hörmander (see [14]), stated below.

Theorem H *Consider a linear partial differential equation of the form*

$$\partial_x^\beta u(x) = \sum_{|\alpha| \leq |\beta|} a_\alpha(x) \partial_x^\alpha u(x), \tag{1}$$

where a_α are analytic functions in some neighborhood of the origin in $\mathbb{C}^n$, with initial conditions

$$\partial_{x_j}^k u = \varphi_{j,k}, \tag{2}$$

for $x_j = 0$, if $0 \leq k \leq \beta_j - 1$, for all $1 \leq j \leq n$, where $\varphi_{j,k}$ are analytic functions in some neighborhood of the origin. Assume that β does not belong to the convex hull in $\mathbb{R}_+^n$ of the indices α for which a_α is not the zero function. Then, the problem (1), (2) *has a unique solution u that is analytic in a neighborhood of the origin.*

The main idea of the proof will consist in applying this latter theorem to an auxiliary linear partial equation constructed from (E), using a majorazing series principle. Following the same steps as in the proof of Theorem 1, we also deduce the important following property. Showing that formal solutions $\hat{X}$ of (E) are κ-summable can be reduced to showing that formal solutions of an auxiliary partial differential equation of the same type as (E) are Gevrey (Proposition 3).

One important consequence of the main result of the paper (Theorem 1), is the existence and asymptotic behaviour of holomorphic solutions $X(t, z_1, z_2)$ of (E) (where X is the κ-sum of $\hat{X}$), for t in a small sector and (z_1, z_2) in a neighborhood of the origin. Similar existence results on local sectorial holomorphic solutions to parabolic partial differential equations are known using methods from functional analysis like holomorphic semi-groups, see [1] and more recently using the notion of maximal regularity, see [24].

We can also mention possible implications concerning the appearence of singularities at the edge of the domain of summation in the complex plane. Such phenomenons of obstruction to analytic continuation of solutions is of high interest in the case of non-linear partial differential equations, see for instance [8] and references therein, and remains an active field of research for linear partial differential equations, see [11],[12],[20].

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2 Summability criterions

In this section, we recall a proposition already obtained in a previous paper [4], which gives necessary and sufficient conditions for the summability of formal series with variable coefficients.

We consider a sequence $(X_j)_{j \geq 0}$ of holomorphic functions $X_j : \mathcal{D}_{1,\rho} \times \mathcal{D}_{2,\rho} \longrightarrow \mathbb{C}$ where $\mathcal{D}_{1,\rho} = \times_{i=1}^{r} D_{\rho_i}$, $\mathcal{D}_{2,\rho} = \times_{i=r+1}^{m} D_{\rho_i}$ with $D_{\rho_i} = \{z \in \mathbb{C} : |z| < \rho_i\}$, $1 \leq i \leq m$ and $\rho = (\rho_1, \ldots, \rho_m) \in \mathbb{R}_+^m$.

For notations, if $z = (x_1, \dots, x_r) \in \mathbb{C}^r$ and $n = (n_1, \dots, n_r) \in \mathbb{N}^r$ we will write $\partial_z^n = \partial_{x_1}^{n_1} \cdots \partial_{x_r}^{n_r}$, $n! = n_1! \cdots n_r!$ and $|n| = n_1 + \cdots + n_r$.

Definition 1 *The formal series*

$$(\hat{X}) \qquad \hat{X}(t, z_1, z_2) = \sum_{j=0}^{\infty} \frac{t^j}{j!} X_j(z_1, z_2)$$

is said to be $k-$summable in the direction $d \in \mathbb{R}$ if

i) *There exist $\rho \in \mathbb{R}_+^m$ and $R(\rho) > 0$, such that the following formal series, called formal Borel transform with respect to the variable t of order k,*

$$W(t, z_1, z_2) = \sum_{j=0}^{\infty} \frac{X_j(z_1, z_2) t^j}{j! \Gamma(1 + j/k)},$$

is absolutely convergent for $(z_1, z_2) \in \mathcal{D}_{1,\rho} \times \mathcal{D}_{2,\rho}$ and $|t| < R(\rho)$,

ii) *There exists $\delta > 0$ such that for all $(z_1, z_2) \in \mathcal{D}_{1,\rho} \times \mathcal{D}_{2,\rho}$, the series $W(t, z_1, z_2)$ can be analytically continued with respect to t in a sector $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $K > 0$ such that*

$$\sup_{(z_1,z_2) \in \bar{\mathcal{D}}_{1,\rho} \times \bar{\mathcal{D}}_{2,\rho}} |W(t, z_1, z_2)| \leq C e^{K|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the Laplace transform of order k of $w(t, z_1, z_2)$,

$$(X) \qquad X(t, z_1, z_2) = t^{-k} \int_0^{\infty(\gamma)} W(\tau, z_1, z_2) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $k-$ sum of the formal series $\hat{X}(t, z_1, z_2)$.

Proposition 1 *The following statements are equivalent.*

i) *The formal series $\hat{X}$ is $k-$summable with respect to t in direction d.*

ii) *For all $n \in \mathbb{N}^r$, the series*

$$W_n(t, z_2) = \partial_{z_1}^n W(t, z_1, z_2)_{|z_1=0}$$

all converge for $|t| \leq \tau$, $z_2 \in \mathcal{D}_{2,\rho}$, for some $\tau > 0$ that is independent of n. Moreover, there exists $\delta_1 > 0$ so that for all $z_2 \in \mathcal{D}_{2,\rho}$, all the functions $W_n(t, z_2)$ can be holomorphically continued with respect to t into a sector $S_{d,\delta_1} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta_1\}$. Finally, for every $\delta_2 < \delta_1$, there exist constants $C, K > 0$, independent of n, so that

$$|W_n(t, z_2)| \leq C^{|n|} n! e^{K|t|^k},$$

for all $t \in S_{d,\delta_3}$, $z_2 \in \mathcal{D}_{2,\rho}$ and $n \in \mathbb{N}^r$.

iii) *For all* $n \in \mathbb{N}^r$*, the formal series*

$$\hat{X}_n(t,z_2) = \partial_{z_1}^n \hat{X}(t,z_1,z_2)_{|z_1=0}$$

are k*-summable in the direction* d*. Moreover, the sums* $X_n(t,z_2)$ *are holomorphic in a region* $G \times \mathcal{D}_{2,\rho}$*, independent of* n*, where* G *has bissecting direction* d *and opening greater than* π/k*, and for every closed subsector* $\bar{S} \subset G$*, there exist constants* $C, K > 0$*, independent of* n*, such that*

$$\sup_{t\in\bar{S}, z_2\in\mathcal{D}_{2,\rho}} |\partial_t^l X_n(t,z_2)| \leq CK^{|n|+l} n! l! \Gamma(1+l/k)$$

for all $n \in \mathbb{N}^r$ *and* $l \in \mathbb{N}$*.*

Proof First, we show that **i**) and **ii**) are equivalent. The fact that **i**) implies **ii**) follows directly from the standard multi-dimensional Cauchy formula. Indeed, for all $n \in \mathbb{N}^r$, the series W_n can be written in term of W as follows,

$$W_n(t,z_2) = \frac{n!}{(2\pi i)^r} \oint_C \frac{W(t,z_1,z_2)}{x_1^{n_1+1}\cdots x_r^{n_r+1}} dx_1 \cdots dx_r,$$

with $z_1 = (x_1, \ldots, x_n)$, where $C = \times_{i=1}^r C_i$, and C_i are circles of radius $0 < \tau < \min_i \rho_i$. The fact that **ii**) implies **i**) follows directly from the standard multi-dimensional Taylor expansion of $W(t,z_1,z_2)$ with respect to z_1. Indeed, the series W can be written in term of the W_n, $n \in \mathbb{N}^r$, as follows

$$(TW) \qquad W(t,z_1,z_2) = \sum_{n\in\mathbb{N}^r} \frac{W_n(t,z_2)}{n!} x_1^{n_1} \cdots x_r^{n_r},$$

for all $(z_1,z_2) \in \mathcal{D}_{1,\rho} \times \mathcal{D}_{2,\rho}$ and $t \in D_\tau \cup S_{d,\delta_1}$.

We show that **ii**) and **iii**) are equivalent. The fact that **ii**) implies **iii**) follows from the Taylor expansion of $X(t,z_1,z_2)$, the sum of $\hat{X}$, with respect of z_1, which yields

$$(TX) \qquad X(t,z_1,z_2) = \sum_{n\in\mathbb{N}^r} \frac{X_n(t,z_2)}{n!} x_1^{n_1} \cdots x_r^{n_r},$$

and the expansion (TW), which gives the relation

$$X_n(t,z_2) = t^{-k} \int_0^{\infty(\gamma)} W_n(\tau,z_2) e^{-(\tau/t)^k} d\tau^k,$$

for all $n \in \mathbb{N}^r$. From the latter relations, one can show that the estimates on W_n implies those for X_n. On the other hand, suppose that **iii**) holds. The estimates on X_n allow to defined the series $X(t,z_1,z_2)$ by the formula (TX). We define $W(t,z_1,z_2)$ to be the Borel transform of order k of $X(t,z_1,z_2)$, that is

$$W(t,z_1,z_2) = \frac{k}{2\pi i} \int_{\partial V} e^{(t/\zeta)^k} X(\zeta,z_1,z_2) d\zeta/\zeta.$$

The Taylor expansions of $W(t,z_1,z_2)$ and $X(\zeta,z_1,z_2)$ with respect to z_1 yields the relations

$$W_n(t,z_2) = \frac{k}{2\pi i} \int_{\partial V} e^{(t/\zeta)^k} X_n(\zeta,z_2) d\zeta/\zeta,$$

for all $n \in \mathbb{N}^r$. Finally, from the latter relations, one can show that the estimates on X_n implies those for W_n. □

3 Summability of formal solutions of linear partial differential equations

In this section we study formal series $\hat{X}(t,z_1,z_2)$ of the form $(\hat{X})$ which are solutions of the linear partial differential equation

$$(PDE) \qquad \partial_{z_1}^{S_1}\partial_{z_2}^{S_2}\hat{X}(t,z_1,z_2) = -(\sum_{q=0}^{Q} P_q(\partial_{z_1},\partial_{z_2})\partial_t^q)\hat{X}(t,z_1,z_2),$$

where $Q \geq 1$ and P_q denote polynomials of the form

$$P_q(Z_1,Z_2) = \sum_{(k_1,k_2)\in\mathcal{S}_q} b_{q,k_1,k_2}(t,z_1,z_2)Z_1^{k_1}Z_2^{k_2},$$

where $\mathcal{S}_q = \{(k_1,k_2) \in \mathbb{N}^2 : 0 \leq k_1+k_2 \leq K_q\}$, whose coefficients are holomorphic functions on $D_{2R(\rho)} \times D_{\rho_1} \times D_{\rho_2}$ written in the form

$$b_{q,k_1,k_2}(t,z_1,z_2) = \sum_{l_1,l_2\geq 0} b_{q,k_1,k_2,l_1,l_2}(t)z_1^{l_1}z_2^{l_2}.$$

We denote by K_q the total degree of P_q. From the Cauchy formula, we know that for all $q,k_1,k_2,l_0,l_1,l_2 \geq 0$, we get

$$(CI) \qquad \sup_{t\in\bar{D}_{R(\rho)}} \frac{|\partial_t^{l_0} b_{q,k_1,k_2,l_1,l_2}(t)|}{l_0!} = |b|_{q,k_1,k_2,l_1,l_2,l_0},$$

with the property that the series

$$|b|_{q,k_1,k_2}(t^K,z_1,z_2) = \sum_{l_0,l_1,l_2\geq 0} |b|_{q,k_1,k_2,l_1,l_2,l_0} t^{Kl_0} z_1^{l_1} z_2^{l_2}$$

are, for all integers $K > 0$, convergent in a neighborhood of the origin in $\mathbb{C}^3$.

In the next lemma we describe formal series solutions of (PDE) in term of differential relations.

Lemma 1 *Rewrite the formal series* $(\hat{X})$ *in a formal series with respect to the variables* z_1,z_2,

$$\hat{X}(t,z_1,z_2) = \sum_{n_1,n_2\geq 0} \frac{\hat{X}_{n_1,n_2}(t)}{n_1!n_2!} z_1^{n_1} z_2^{n_2}.$$

Then, $\hat{X}(t,z_1,z_2)$ *is a formal series solution of* (PDE) *if and only if the following differential relation* (DR) *is satisfied,*

$$\frac{\hat{X}_{m_1+S_1,m_2+S_2}(t)}{(m_1)!(m_2)!} = -\sum_{q=0}^{Q}\sum_{(k_1,k_2)\in\mathcal{S}_q}\sum_{l_1+n_1=m_1}\sum_{l_2+n_2=m_2} b_{q,k_1,k_2,l_1,l_2}(t)\frac{\partial_t^q \hat{X}_{n_1+k_1,n_2+k_2}(t)}{(n_1)!(n_2)!},$$

for all $m_1,m_2 \geq 0$.

Proof We first have that

$$\partial_{z_1}^{k_1}\partial_{z_2}^{k_2}\partial_t^q \hat{X}(t,z_1,z_2) = \sum_{n_1,n_2\geq 0} \frac{\partial_t^q \hat{X}_{n_1+k_1,n_2+k_2}(t)}{(n_1)!(n_2)!} z_1^{n_1} z_2^{n_2}.$$

So that

$$P_q(\partial_{z_1},\partial_{z_2})\partial_t^q \hat{X}(t,z_1,z_2) =$$

$$\sum_{m_1,m_2\geq 0} \left(\sum_{(k_1,k_2)\in\mathcal{S}_q} \sum_{l_1+n_1=m_1} \sum_{l_2+n_2=m_2} b_{q,k_1,k_2,l_1,l_2}(t) \frac{\partial_t^q \hat{X}_{n_1+k_1,n_2+k_2}(t)}{(n_1)!(n_2)!} \right) z_1^{m_1} z_2^{m_2}.$$

The lemma then easily follows. □

Example: For the heat equation

$$\partial_{z_1}^2 \hat{X}(t,z_1,z_2) = (-\partial_{z_2}^2 + \partial_t)\hat{X}(t,z_1,z_2),$$

the differential relation writes,

$$\hat{X}_{m_1+2,m_2}(t) = -\hat{X}_{m_1,m_2+2}(t) + \partial_t \hat{X}_{m_1,m_2}(t).$$

In the next proposition, we give sufficient conditions for the existence and uniqueness of formal series $(\hat{X})$ that are solutions of (PDE).

Proposition 2 *We assume that the tuple (S_1,S_2) does not belong to the convex hull in $\mathbb{R}_+^2$ of the set of tuples (k_1,k_2) for which b_{q,k_1,k_2} is not the zero function, for all $0\leq q\leq Q$ and that*

$$S_1 + S_2 \geq \max_{0\leq q\leq Q} K_q.$$

Moreover, assume that sequences $(\hat{X}_{\mu_1,\mu_2}(t))_{\mu_2\geq 0}$, for all $0\leq\mu_1\leq S_1-1$ and sequences $(\hat{X}_{\mu_1,\mu_2}(t))_{\mu_1\geq 0}$, for all $0\leq\mu_2\leq S_2-1$ are given.

Then, there exists a unique sequence $(\hat{X}_{\mu_1,\mu_2}(t))_{\mu_1,\mu_2}$ of formal series which satisfies the differential relation (DR). As a result, for given sequences $(\hat{X}_{\mu_1,\mu_2}(t))_{\mu_2\geq 0}$, for all $0\leq\mu_1\leq S_1-1$ and sequences $(\hat{X}_{\mu_1,\mu_2}(t))_{\mu_1\geq 0}$, for all $0\leq\mu_2\leq S_2-1$, there exists a unique formal series $(\hat{X})$, solution of (PDE).

Proof By assumption, (S_1,S_2) does not belong to the convex hull in $\mathbb{R}_+^2$ of the set of tuples (k_1,k_2) for which b_{q,k_1,k_2} is not the zero function, for all $0\leq q\leq Q$. From the classical Hahn-Banach separation theorem, we deduce that there exist $\rho_1,\rho_2\in\mathbb{R}$ such that

$$(Co) \qquad \rho(S_1,S_2) = \rho_1 S_1 + \rho_2 S_2 > \rho_1 k_1 + \rho_2 k_2 = \rho(k_1,k_2),$$

for all (k_1,k_2) for which b_{q,k_1,k_2} is not the zero function, for all $0\leq q\leq Q$.

On the other hand, from the hypothesis $S_1+S_2\geq\max_{0\leq q\leq Q}K_q$, we can replace $\rho(k_1,k_2)$ by $\rho(k_1,k_2)+s(k_1+k_2)$ in the inequalities (Co), for all $s>0$. In such a way, we may assume in the following that $\rho_1,\rho_2>0$ in the inequalities (Co).

Take $(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathbb{N}^2$, we say by definition, that (μ_1,μ_2) is lower than (ν_1,ν_2) and is denoted by $(\mu_1,\mu_2)<_\rho(\nu_1,\nu_2)$ if $\rho(\mu_1,\mu_2)<\rho(\nu_1,\nu_2)$. The relation $<_\rho$ is a partial ordering on $\mathbb{N}^2$.

From inequalities (Co), we deduce that

$$(k_1+m_1-l_1,k_2+m_2-l_2)<_\rho (S_1+m_1,S_2+m_2),$$

for all $m_1,m_2,l_1,l_2\geq 0$ and all (k_1,k_2) for which b_{q,k_1,k_2} is not the zero function, for all $0\leq q\leq Q$.

In this way, the term $\hat{X}_{m_1+S_1,m_2+S_2}$ in the differential relation (DR), which can be called in this case differential recurrence, can be uniquely expressed in terms of $\hat{X}_{\mu_1,\mu_2}$ with $(\mu_1,\mu_2)<_\rho (m_1+S_1,m_2+S_2)$ for all $m_1,m_2\geq 0$. The result of the proposition follows. □

The next result gives sufficient conditions for the existence of κ-summable formal series solutions $(\hat{X})$ of (PDE), for well chosen κ.

Theorem 1 *Consider the linear partial differential equation,*

$$(PDE) \qquad \partial_{z_1}^{S_1}\partial_{z_2}^{S_2}\hat{X}(t,z_1,z_2)=-(\sum_{q=0}^{Q}P_q(\partial_{z_1},\partial_{z_2})\partial_t^q)\hat{X}(t,z_1,z_2).$$

Assume that the conditions of the proposition 2 are fulfilled. Assume moreover that there exists an integer $K>1$ such that

$$S_1+S_2\geq \max_{0\leq q\leq Q}(qK+K_q),$$

and $(0,S_1,S_2)$ does not belong to the convex hull in $\mathbb{R}_+^3$ of the set of tuples (qK,k_1,k_2), for which b_{q,k_1,k_2} is not the zero function.

We assume that the series $\hat{X}_{\mu_1,\mu_2}(t)$ for all $\mu_2\geq 0$, $0\leq\mu_1\leq S_1-1$ and the series $\hat{X}_{\mu_1,\mu_2}(t)$ for all $\mu_1\geq 0$, $0\leq\mu_2\leq S_2-1$, are $1/(K-1)$-summable with respect to t in a direction $d\in\mathbb{R}$.

Moreover, we assume that their sums $X_{\mu_1,\mu_2}(t)$ are holomorphic in a sectorial region $G\subset D_{R(\rho)}$, independent of μ_1,μ_2, which has bisecting direction d and opening greater than $(K-1)\pi$ and satisfy the following estimates.
For all closed sector $\bar{S}\subset G$, there exist $C,\tilde{K}>0$ such that

$$\sup_{t\in\bar{S}}|\partial_t^{n_0}X_{n_1,n_2}(t)|\leq C\tilde{K}^{n_0+n_1+n_2}(Kn_0)!n_1!n_2!,$$

for all $0\leq n_1\leq S_1-1$ and all $n_0,n_2\in\mathbb{N}$ and for all $0\leq n_2\leq S_2-1$ and all $n_0,n_1\in\mathbb{N}$.

Then the formal series $\hat{X}(t,z_1,z_2)$, solution of (PDE), is $1/(K-1)$-summable with respect to t in the direction $d\in\mathbb{R}$.

Proof The goal of the proof is to show that the conditions **iii)** in Proposition 1 are fulfilled. First of all, under the given assumptions, it follows from the relation (DR) and Proposition 2 (using the general results on derivation of k-summable series, see [2]), that the formal series $\hat{X}_{\mu_1,\mu_2}(t)$ are $1/(K-1)$-summable in the direction $d\in\mathbb{R}$, for all $\mu_1,\mu_2\geq 0$. Moreover, their sums $X_{\mu_1,\mu_2}(t)$ are holomorphic in the sectorial region G.

The rest of the proof will consist in proving that the estimates in **iii)** of Proposition 1 are satisfied for the formal series $\hat{X}_{\mu_1,\mu_2}(t)$.

We consider now the sequence of positive real numbers

$$w_{n_0,n_1,n_2}=\sup_{t\in\bar{S}}|\partial_t^{n_0}X_{n_1,n_2}(t)|,$$

for $n_0,n_1,n_2\geq 0$, where $\bar{S}$ is a closed sector in G.

From the Leibniz formula, we have that

$$\partial_t^{n_0}(b_{q,k_1,k_2,l_1,l_2}(t)\frac{\partial_t^q X_{n_1+k_1,n_2+k_2}(t)}{n_1!n_2!}) = \sum_{h=0}^{n_0} \frac{n_0!}{(n_0-h)!h!}\partial_t^{n_0-h} b_{q,k_1,k_2,l_1,l_2}(t)\frac{\partial_t^{h+q} X_{n_1+k_1,n_2+k_2}(t)}{n_1!n_2!}.$$

On the other hand, by classical estimates, we have that

$$\frac{n_0!}{h!} \leq \frac{(Kn_0)!}{(Kh)!},$$

for all $n_0 \geq 0$ and all $0 \leq h \leq n_0$. From the latter estimates and (CI), we deduce that

$$\sup_{t\in\bar{S}}\left|\partial_t^{n_0}(b_{q,k_1,k_2,l_1,l_2}(t)\frac{\partial_t^q X_{n_1+k_1,n_2+k_2}(t)}{n_1!n_2!})\right| \\ \leq \sum_{h=0}^{n_0}\frac{(Kn_0)!}{(Kh)!}|b|_{q,k_1,k_2,l_1,l_2,n_0-h}\frac{\sup_{t\in\bar{S}}\left|\partial_t^{h+q}X_{n_1+k_1,n_2+k_2}(t)\right|}{n_1!n_2!}$$

If we differentiate n_0 times the relation (DR) with respect to t, we deduce the inequalities (MRI),

$$\frac{w_{n_0,m_1+S_1,m_2+S_2}}{(Kn_0)!m_1!m_2!} \leq \sum_{q=0}^{Q}\sum_{(k_1,k_2)\in\mathcal{S}_q}\sum_{l_1+n_1=m_1}\sum_{l_2+n_2=m_2}\sum_{h=0}^{n_0}\frac{|b|_{q,k_1,k_2,l_1,l_2,n_0-h}}{(Kh)!}\frac{w_{h+q,n_1+k_1,n_2+k_2}}{(n_1)!(n_2)!}$$

for all $n_0, m_1, m_2 \geq 0$.

We choose now an integer $K > 1$ such that

$$S_1 + S_2 \geq \max_{0\leq q\leq Q}(qK + K_q),$$

and we assume that $(0, S_1, S_2)$ does not belong to the convex hull in $\mathbb{R}_+^3$ of the set of tuples (qK, k_1, k_2) for which b_{q,k_1,k_2} is not the zero function. We consider the auxiliary partial differential equation (CK),

$$\partial_{z_1}^{S_1}\partial_{z_2}^{S_2}E(t,z_1,z_2) = \left(\sum_{q=0}^{Q}(\sum_{(k_1,k_2)\in\mathcal{S}_q}|b|_{q,k_1,k_2}(t^K,z_1,z_2)\partial_{z_1}^{k_1}\partial_{z_2}^{k_2})\partial_t^{qK}\right)E(t,z_1,z_2),$$

which satisfies the conditions of Theorem H from the introduction. From Theorem H, we deduce that there exists a unique function $E(t,z_1,z_2)$, solution of (CK), with E holomorphic for $|t| < R(\rho)$ and $(z_1,z_2) \in \mathcal{D}_\rho$ if

$$E(t,0,z_2),\ldots,\partial_{z_1}^{S_1-1}E(t,0,z_2)$$

are prescribed holomorphic functions on $\{|t| < R(\rho)\} \times D_{\rho_2}$ and

$$E(t,z_1,0),\ldots,\partial_{z_2}^{S_2-1}E(t,z_1,0)$$

are prescribed holomorphic functions on $\{|t| < R(\rho)\}\times D_{\rho_1}$. In the rest of the proof, we prescribe the functions $\partial_{z_1}^j E(t,0,z_2)$, $0 \leq j \leq S_1 - 1$ to be of the form $\partial_{z_1}^j E(t,0,z_2) = V_j(t^K, z_2)$, where V_j are holomorphic in a neighborhood of the origin. We also prescribe the functions $\partial_{z_2}^j E(t,z_1,0)$, $0 \leq j \leq S_2 - 1$ to be of the form $\partial_{z_2}^j E(t,z_1,0) = W_j(t^K, z_1)$, where W_j are holomorphic in a neighborhood of the origin. From the latter assumptions and due the fact that the coefficients of the equation (CK) are also of the form $\alpha(t^K, z_1, z_2)$, we deduce that the solution E of (CK) can be written in the form $E(t, z_1, z_2) = V(t^K, z_1, z_2)$, where V is holomorphic in a neighborhood of the origin.

If we expand $V(t^K, z_1, z_2)$ into a Taylor series in the variables t, z_1, z_2,

$$V(t^K, z_1, z_2) = \sum_{n_0,n_1,n_2 \geq 0} v_{n_0,n_1,n_2} \frac{t^{Kn_0} z_1^{n_1} z_2^{n_2}}{(Kn_0)!n_1!n_2!},$$

we get that

$$\begin{aligned} &|b|_{q,k_1,k_2}(t^K, z_1, z_2)\partial_{z_1}^{k_1}\partial_{z_2}^{k_2}\partial_t^{qK} V(t^K, z_1, z_2) \\ &\quad = \sum_{m_0,m_1,m_2} \Big(\sum_{l_1+n_1=m_1} \sum_{l_2+n_2=m_2} \sum_{l_0+n_0=m_0} |b|_{q,k_1,k_2,l_1,l_2,l_0} \frac{v_{n_0+q,n_1+k_1,n_2+k_2}}{(Kn_0)!n_1!n_2!} \Big) t^{Km_0} z_1^{m_1} z_2^{m_2}. \end{aligned}$$

From this, we deduce a multivariate linear recurrence (MR) satisfied by the coefficients v_{n_0,n_1,n_2},

$$\frac{v_{n_0,m_1+S_1,m_2+S_2}}{(Kn_0)!m_1!m_2!} = \sum_{q=0}^{Q} \sum_{(k_1,k_2)\in\mathcal{S}_q} \sum_{l_1+n_1=m_1} \sum_{l_2+n_2=m_2} \sum_{h=0}^{n_0} \frac{|b|_{q,k_1,k_2,l_1,l_2,n_0-h}}{(Kh)!} \frac{v_{h+q,n_1+k_1,n_2+k_2}}{(n_1)!(n_2)!}$$

for all $n_0, m_1, m_2 \geq 0$. From the holomorphy of V, we deduce the following estimates for the coefficients v_{n_0,n_1,n_2}. There exist $C, \tilde{K} > 0$, such that

$$(Es) \qquad |v_{n_0,n_1,n_2}| \leq C\tilde{K}^{|n|+n_0}(Kn_0)!n!,$$

where $n = (n_1, n_2)$, for all $n \in \mathbb{N}^2$ and $n_0 \in \mathbb{N}$.

From Theorem H, we see that the sequence $(v_{n_0,n_1,n_2})_{n_0,n_1,n_2}$ is uniquely determined by the sequences $(v_{n_0,n_1,n_2})_{n_0,n_2}$, $0 \leq n_1 \leq S_1 - 1$ and the sequences $(v_{n_0,n_1,n_2})_{n_0,n_1}$, $0 \leq n_2 \leq S_2 - 1$. We choose now the latter sequences as follows.

$$v_{n_0,n_1,n_2} := w_{n_0,n_1,n_2},$$

for all $0 \leq n_1 \leq S_1 - 1$, $n_0, n_2 \geq 0$ and for all $0 \leq n_2 \leq S_2 - 1$, $n_0, n_1 \geq 0$.

From the estimates in the assumptions of the theorem 1, we deduce that the series

$$E(t, 0, z_2), \ldots, \partial_{z_1}^{S_1-1} E(t, 0, z_2)$$

are convergent on $\{|t| < R(\rho)\} \times D_{\rho_2}$ and

$$E(t, z_1, 0), \ldots, \partial_{z_2}^{S_2-1} E(t, z_1, 0)$$

are convergent on $\{|t| < R(\rho)\} \times D_{\rho_1}$. Moreover, due to the relations (MRI) and (MR), we deduce that

$$w_{n_0,n_1,n_2} \leq v_{n_0,n_1,n_2},$$

for all n_0, n_1, n_2.

From the estimates (Es), we finally deduce that there exist $C, \tilde{K} > 0$, such that the sequence w_{n_0,n_1,n_2} satisfies

$$w_{n_0,n_1,n_2} \leq C\tilde{K}^{|n|+n_0}(Kn_0)!n!,$$

where $n = (n_1, n_2)$, for all $n \in \mathbb{N}^2$ and $n_0 \in \mathbb{N}$.

Finally, we conclude that the condition **iii)** *in Proposition 1 is satisfied for the formal series* $\hat{X}_{n_1,n_2}(t)$, so that the formal series $\hat{X}(t, z_1, z_2)$, solution of (PDE), is $1/(K-1)$-summable with respect to t in the direction $d \in \mathbb{R}$. □

Example: In the case of the heat equation

$$\partial_{z_1}^2 \hat{X}(t, z_1, z_2) = (-\partial_{z_2}^2 + \partial_t)\hat{X}(t, z_1, z_2),$$

one can construct formal series solutions

$$\hat{X}(t, z_1, z_2) = \sum_{n_1,n_2 \geq 0} \frac{\hat{X}_{n_1,n_2}(t)}{n_1!n_2!} z_1^{n_1} z_2^{n_2},$$

which are 1-summable with respect to t in a direction $d \in \mathbb{R}$, under the following assumptions: The series $\hat{X}_{0,\mu_2}(t)$ and $\hat{X}_{1,\mu_2}(t)$ are 1-summable with respect to t in a direction $d \in \mathbb{R}$, for all $\mu_2 \geq 0$. Moreover, their sums $X_{0,\mu_2}(t)$ and $X_{1,\mu_2}(t)$ are holomorphic in a sectorial region $G \subset D_{R(\rho)}$, independent of μ_2, which has bisecting direction d and opening greater than π and satisfy the following estimates. For all closed sector $\bar{S} \subset G$, there exist $C, \tilde{K} > 0$ such that

$$\sup_{t \in \bar{S}} |\partial_t^{n_0} X_{n_1,n_2}(t)| \leq C\tilde{K}^{n_0+n_1+n_2}(n_0!)^2 n_1!n_2!,$$

for all $n_1 \in \{0, 1\}$ and all $n_0, n_2 \in \mathbb{N}$. We recover partially the result obtained in [4].

We recall the following definition of formal series which are Gevrey of order (q_0, q_1, q_2).

Definition 2 *A formal series*

$$A(t, z_1, z_2) = \sum_{n_0,n_1,n_2} a_{n_0,n_1,n_2} t^{n_0} z_1^{n_1} z_2^{n_2},$$

is said to be Gevrey of order (q_0, q_1, q_2) *if there exist* $C, \tilde{K} > 0$ *such that*

$$|a_{n_0,n_1,n_2}| \leq C\tilde{K}^{n_0+n_1+n_2}\Gamma(1 + q_0 n_0)\Gamma(1 + q_1 n_1)\Gamma(1 + q_2 n_2),$$

for all $n_0, n_1, n_2 \geq 0$.

Following the same steps as in the proof of Theorem 1, one can show the following proposition.

Proposition 3 *Assume that the conditions of the proposition 2 are fulfilled. We assume that the series* $\hat{X}_{\mu_1,\mu_2}(t)$ *for all* $\mu_2 \geq 0$, $0 \leq \mu_1 \leq S_1 - 1$ *and the series* $\hat{X}_{\mu_1,\mu_2}(t)$ *for all* $\mu_1 \geq 0$, $0 \leq \mu_2 \leq S_2 - 1$, *are* $1/(\kappa - 1)$*-summable with respect to* t *in a direction* $d \in \mathbb{R}$, *for a real number* $\kappa > 1$. *Consider the unique formal series*

$$V(t, z_1, z_2) = \sum_{n_0,n_1,n_2 \geq 0} v_{n_0,n_1,n_2} \frac{t^{n_0} z_1^{n_1} z_2^{n_2}}{n_0!n_1!n_2!},$$

where

$$v_{n_0,n_1,n_2} := \sup_{t \in \bar{S}} |\partial_t^{n_0} X_{n_1,n_2}(t)|,$$

for all $0 \leq n_1 \leq S_1 - 1$, $n_0, n_2 \geq 0$ *and for all* $0 \leq n_2 \leq S_2 - 1$, $n_0, n_1 \geq 0$, *which satisfies the partial differential equation*

$$(*) \qquad \partial_{z_1}^{S_1}\partial_{z_2}^{S_2} V(t,z_1,z_2) = \left(\sum_{q=0}^{Q} (\sum_{(k_1,k_2)\in\mathcal{S}_q} |b|_{q,k_1,k_2}(t,z_1,z_2)\partial_{z_1}^{k_1}\partial_{z_2}^{k_2})\partial_t^q \right) V(t,z_1,z_2),$$

where $(0, S_1, S_2)$ *is assumed not to belong to the convex hull in* $\mathbb{R}_+^3$ *of the set of tuples* (q, k_1, k_2) *for which* b_{q,k_1,k_2} *is not the zero function and*

$$S_1 + S_2 \geq \max_{0\leq q\leq Q} (q + K_q),$$

Then, if the formal series V *are Gevrey of order* $((\kappa-1), 0, 0)$, *then the formal series* $\hat{X}$, *solution of* (PDE), *constructed from the sequence* $(\hat{X}_{\mu_1,\mu_2})_{\mu_1,\mu_2\geq 0}$, *is* $1/(\kappa-1)$*-summable in direction* d.

Remark It has to be mentioned (see [21]), that M. Miyake and Y. Hashimoto have given criterions, involving a Newton polygon associated to partial differential equations $(*)$ with polynomial coefficients in t and initial conditions for V, under which the formal series V can be shown to be Gevrey of some orders $((\kappa-1), 0, 0)$, where κ depends on slopes of this Newton polygon.

More precisely, one defines a Newton polygon (NP) attached to an equation $(*)$ as the convex hull in $\mathbb{R}^2$ of $\mathbb{R}_-$ and all the sets

$$Q(\sigma, q, k_1 - S_1, k_2 - S_2) = \{(u, \sigma - q) \in \mathbb{R}^2 : u \leq q + k_1 + k_2 - S_1 - S_2\},$$

for tuples (σ, q, k_1, k_2) such that $|b|_{\sigma,q,k_1,k_2}(z_1, z_2)$ is not the zero function, with

$$|b|_{q,k_1,k_2}(t,z_1,z_2) = \sum_{\sigma\in I_{q,k_1,k_2}} |b|_{\sigma,q,k_1,k_2}(z_1,z_2)t^{\sigma},$$

where I_{q,k_1,k_2} are a finite subsets of $\mathbb{N}$. As a particular case of Theorem A, p. 21 in [21], one gets a Cauchy-Kowalevskaya theorem for linear partial differential equations in a class of Gevrey formal series.

Consider an equation $(*)$ with polynomial coefficients in t, under the hypotheses of the proposition 3, and initial conditions $(**)$,

$$(\partial_{z_1}^k V)(t, 0, z_2) = \varphi_{1,k}(t, z_2),$$

for all $0 \leq k \leq S_1 - 1$, and

$$(\partial_{z_2}^k V)(t, z_1, 0) = \varphi_{2,k}(t, z_1),$$

for all $0 \leq k \leq S_2 - 1$, where $\varphi_{1,k}$, $\varphi_{2,k}$ are formal series which are Gevrey of order $((\kappa-1), 0)$.

Consider $\kappa \in [1, 1+1/k_-]$, where k_- is the slope of the side of the Newton polygon (NP), lying in the lower half plane, with the origin as end point. Assume that $k_- \in [0, +\infty)$. By convention, $1/0 = +\infty$. Then, the problem $(*)$, $(**)$ has a unique formal series solution $V(t, z_1, z_2)$ that is Gevrey of order $((\kappa-1), 0, 0)$.

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