A family of chaotic billiards with variable mixing rates

N. Chernov[1] and H.-K. Zhang[1]

September 11, 2004

Abstract

We describe a one-parameter family of dispersing (hence hyperbolic, ergodic and mixing) billiards where the correlation function of the collision map decays as $1/n^a$ (here n denotes the discrete time), in which the degree $a \in (1, \infty)$ changes continuously with the parameter of the family, β. We also derive an explicit relation between the degree a and the family parameter β.

AMS classification numbers: 37D50, 37A25
Keywords: Decay of correlations, dispersing billiards

1 Introduction

A billiard is a mechanical system in which a point particle moves in a compact container Q and bounces off its boundary ∂Q; in this paper we only consider planar billiards, where $Q \subset \mathbb{R}^2$. The billiard dynamics preserves a uniform measure on its phase space, and the corresponding collision map (generated by the collisions of the particle with ∂Q, see below) preserves a natural (and often unique) absolutely continuous measure on the collision space. The dynamical properties of a billiard are determined by the shape of the boundary ∂Q, and it may vary greatly from completely regular (integrable) to strongly chaotic.

---

[1] Department of Mathematics, University of Alabama at Birmingham;
Email: chernov@math.uab.edu; zhang@math.uab.edu

The dynamics in simple containers (circles, ellipses, rectangles) are completely integrable. The first class of chaotic billiards was introduced by Ya. Sinai in 1970 [14]; he proved that if ∂Q is strictly convex inward, its curvature nowhere vanishes, and the smooth components of ∂Q intersect each other transversally (make no cusps), then the dynamics is hyperbolic (moreover, uniformly hyperbolic), ergodic, mixing and K-mixing. He called such systems *dispersing billiards*, now they are often called *Sinai billiards.* Gallavotti and Ornstein [9] proved that Sinai billiards are Bernoulli systems. Later on the hyperbolicity, ergodicity (as well as Bernoulli property [5, 12]) were established for dispersing billiards with cusps on the boundary [13] and for billiards whose boundary is convex (but not strictly convex) inward – the so called semi-dispersing billiards – under certain conditions [2, 7].

The rates of mixing (precisely defined in the next section) for the collision map in dispersing and semidispersing billiards depend on the shape of the boundary. Assume that

(A) there are no cusps on the boundary and

(B) the boundary curvature does not vanish.

Then the collision map is uniformly hyperbolic, and its mixing properties are very strong – correlations (defined in the next section) decay exponentially [15, 4]. Relaxing the requirements (A) and (B) results in nonuniform hyperbolicity and weaker mixing properties (slower decay of correlations), see below.

If we relax (A), but not (B), then the correlations appear to decay polynomially as $\mathcal{O}(1/n)$. This conjecture is based on heuristic arguments and numerical experiments [10], and the work on proving it rigorously is currently underway.

Here we relax (B) but not (A), i.e. consider dispersing billiards without cusps, but assume that the boundary curvature vanishes at finitely many points (we call them *flat points*). This is a special class of chaotic billiards hardly ever investigated before.

First of all, it is easy to show that if there is no periodic trajectory that hits the boundary at flat points only, then a certain power of the collision map is uniformly hyperbolic, hence correlations decay exponentially. In order to weaken the hyperbolicity and mixing properties, one needs a periodic trajectory making collisions at flat points only. Then the vicinity of that periodic orbit acts as a "trap" where hyperbolicity may remain weak for arbitrarily long times.

For simplicity we assume that there is one such periodic trajectory of period two that runs between two flat points. More precisely, let the boundary ∂Q near those two flat points be given by the equations

$$y = \pm g_\beta(x), \qquad g_\beta(x) = |x|^\beta + 1 \qquad (\beta > 2) \tag{1.1}$$

in some rectangular coordinate system in $\mathbb{R}^2$. The billiard table lies between the "+" and "−" branches of the above function, and elsewhere it is bounded by "regular dispersing" curves, which are strictly convex inward with nowhere vanishing curvature and make no cusps. Note that the curvature of the boundary does vanish at the points $(0, 1)$ and $(0, -1)$, because $\beta > 2$, and the periodic orbit runs between these points along the y axis. The power $\beta > 2$ is the parameter of the so constructed family of billiard tables.

Our main result, stated precisely in the next section, is that the correlations for the collision map decay as $\mathcal{O}(1/n^a)$, where

$$a = \frac{\beta + 2}{\beta - 2}.$$

Therefore, the degree a covers the entire interval from one to infinity. In the limit $\beta \to \infty$, the boundary flattens out, and the correlations decay almost as $1/n$, which is an established result for semi-dispersing billiards with two parallel flat components of the boundary [8]. In the limit $\beta \to 2$, the boundary "curves up" and approaches strictly dispersing case $y = \pm(x^2 + 1)$ with nowhere vanishing curvature; then $a \to \infty$ and so the correlations decay faster than any polynomial function. In the limit $\beta = 2$ the correlations decay exponentially [4]. Thus by varying the parameter β we can adjust the degree of the polynomial decay rate $1/n^a$ to any value $a \in (1, \infty)$.

2 Statement of results

First we recall standard definitions of billiard theory [1, 2, 3, 4]. A billiard is a dynamical system where a point moves freely at unit speed in a domain Q (*the table*) and reflects off its boundary ∂Q (*the wall*) by the rule "the angle of incidence equals the angle of reflection". We assume that $Q \subset \mathbb{R}^2$ and ∂Q is a finite union of C^3 curves (arcs). The phase space of this system is a three dimensional manifold $Q \times S^1$. The dynamics preserves a uniform measure on $Q \times S^1$.

Let $\mathcal{M} = \partial Q \times [-\pi/2, \pi/2]$ be the standard cross-section of the billiard dynamics, we call $\mathcal{M}$ the *collision space*. Canonical coordinates on $\mathcal{M}$ are r and φ, where r is the arc length parameter on ∂Q and $\varphi \in [-\pi/2, \pi/2]$ is the angle of reflection, see Fig. 1. We denote by π the natural projection of M onto ∂Q.

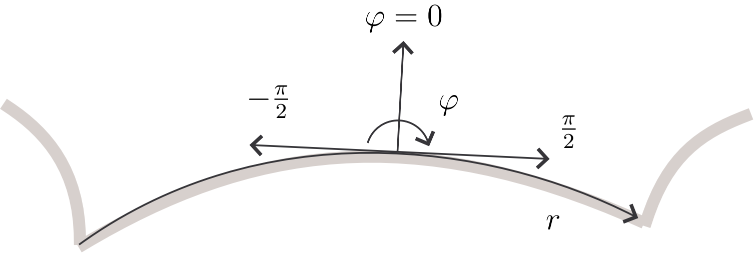

Fig. 1: Orientation of r and φ

The first return map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is called the *collision map* or the *billiard map*, it preserves smooth measure $d\mu = \cos\varphi \, dr \, d\varphi$ on $\mathcal{M}$.

Let $f, g \in L^2_\mu(\mathcal{M})$ be two functions. *Correlations* are defined by

$$\mathcal{C}_n(f, g, \mathcal{F}, \mu) = \int_{\mathcal{M}} (f \circ \mathcal{F}^n) \, g \, d\mu - \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu \tag{2.1}$$

It is well known that $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is *mixing* if and only if

$$\lim_{n\to\infty} \mathcal{C}_n(f, g, \mathcal{F}, \mu) = 0 \qquad \forall f, g \in L^2_\mu(\mathcal{M}) \tag{2.2}$$

The rate of mixing of $\mathcal{F}$ is characterized by the speed of convergence in (2.2) for smooth enough functions f and g. We will always assume that f and g are Hölder continuous or piecewise Hölder continuous with singularities that coincide with those of the map $\mathcal{F}^k$ for some k. For example, the length of the free path between successive reflections is one such function.

We say that correlations decay *exponentially* if

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| < \text{const} \cdot e^{-cn}$$

for some $c > 0$ and *polynomially* if

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| < \text{const} \cdot n^{-a}$$

for some $a > 0$. Here the constant factor depends on f and g.

Next we state our results.

Let $Q \subset \mathbb{R}^2$ be a domain bounded by the curves $y = g_\beta(x)$ and $y = -g_\beta(x)$, see (1.1), and several strictly convex (inward) curves with nowhere vanishing curvature and no cusps. An example is shown on Fig. 2 (left).

Our results also apply to billiards bounded by one of the curves (1.1), say $y = g_\beta(x)$, the x-axis and several strictly convex (inward) curves with nowhere vanishing curvature and no cusps. An example is shown on Fig. 2 (right).

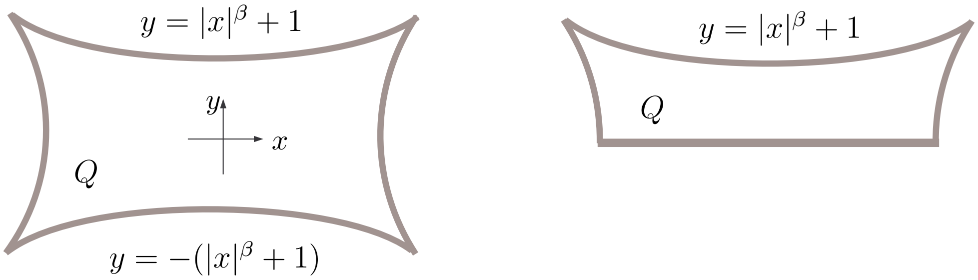

Fig. 2: Dispersing billiards with walls where the curvature vanishes.

Theorem 1. *For the above billiard tables, the correlations (2.1) for the billiard map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ and piecewise Hölder continuous functions f, g on $\mathcal{M}$ decay as*

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq \text{const} \cdot \frac{(\ln n)^{a+1}}{n^a}, \tag{2.3}$$

where $a = (\beta + 2)/(\beta - 2)$.

Remark. The logarithmic factor in (2.3) is a by-product of a general method for correlation analysis developed in [11, 8]. Perhaps it is possible to suppress it by using more powerful Young's techniques [16] but this may require a substantial extra effort.

3 Proof of the main Theorem

We use a general scheme for the analysis of hyperbolic dynamical systems with polynomial decay of correlations developed in [8] (which is an extension of earlier works by Young [16] and Markarian [11]). That scheme has been successfully applied in [8] to various classes of chaotic billiards.

The scheme is based on finding a subset $M \subset \mathcal{M}$ where the map $\mathcal{F}$ is strongly (uniformly) hyperbolic and the subsequent analysis of the return map $F\colon M \to M$, which is defined by

$$F(X) = \mathcal{F}^{N(X)}(X), \qquad N(X) = \min\{i > 0\colon \mathcal{F}^i(X) \in M\}. \tag{3.1}$$

In our case the hyperbolicity is strong everywhere except the vicinity of the two flat points $(0,1)$ and $(0,-1)$ on ∂Q. We fix an $\varepsilon > 0$ and define

$$M = \big(\partial Q \setminus \{|x| < \varepsilon\}\big) \times [-\pi/2, \pi/2] = \mathcal{M} \setminus \pi^{-1}(\{|x| < \varepsilon\}),$$

i.e. we remove from ∂Q a narrow window – the ε-neighborhood of the y axis – that contains both flat points, see Fig. 3. Let $q_1 = (\varepsilon, -g_\beta(\varepsilon))$ denote one of the four points on ∂Q that border the window $|x| < \varepsilon$, and by q_2, q_3, q_4 the other three points (Fig. 3).

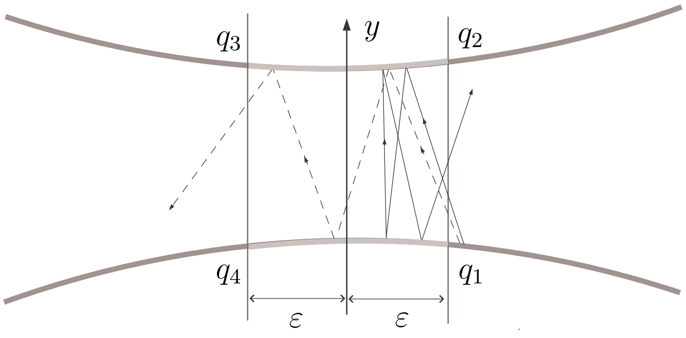

Fig. 3: A window cut through ∂Q to construct M.

In order to prove Theorem 1, according to our general scheme [8], we need to establish two properties of the map F described below.

(F1) First, the map $F\colon M \to M$ enjoys exponential decay of correlations. Moreover, there is a horseshoe $\Lambda \subset M$ with a hyperbolic structure such that the return times to Λ obey an exponential tail bound, see [15, 16, 8] for precise definitions.

(F2) Second, the return times to M under the original map $\mathcal{F}$ defined by

$$R(X; \mathcal{F}, M) = \min\{r \geq 1 :\ \mathcal{F}^r(X) \in M\}$$

satisfy the polynomial tail bound

$$\mu(X \in M:\ R(X;\mathcal{F},M) > n) \leq \text{const}\cdot n^{-a-1} \qquad \forall n \geq 1 \tag{3.2}$$

where $a > 0$ is the constant of Theorem 1.

It is shown in [8] that Theorem 1 follows from (F1) and (F2). Also, the proof of (F1) is reduced in [8] to the verification of the following property of unstable manifolds:

Let $W \subset M$ denote an unstable manifold (it is a smooth curve since $\dim M = 2$). Since the map F has singularities (described below) the image $F(W)$ may consist of finitely or countably many unstable manifolds. Let W_i, $i \geq 1$, denote the preimages of the smooth components of $F(W)$, i.e. the subcurves $W_i \subset W$ on which the map F is smooth. Next, for every point $X \in W_i$ denote by $\Lambda(X)$ the Jacobian of the map F restricted to W_i, i.e. the local factor of expansion (stretching factor) of the curve W_i under the map F at the point X. Put

$$\Lambda_i = \min_{X \in W_i} \Lambda(X)$$

In order to prove (F1) we need to verify that

$$\liminf_{\delta\to 0} \sup_{W:\,|W|<\delta} \sum_i \Lambda_i^{-1} < 1, \tag{3.3}$$

where the supremum is taken over unstable manifolds W of length $< \delta$.

The reduction of (F1) to (3.3) is carried out in [8] for very general 2D hyperbolic maps that include our family of dispersing billiards.

Thus it remains to prove (F2) and (3.3). This requires detailed investigation of the singularities of the map F. The definition (3.1) makes it clear that F is singular at X whenever $\mathcal{F}(X)$ or $N(X)$ is singular. The singularities of the original map $\mathcal{F}$ are well studied [4] and the estimate (3.3) is proved for unstable manifolds affected by those, so we focus on the singularities of $N(X)$.

The value $n = N(X) - 1$ is the number of bounces the billiard trajectory of the point $X \in M$ experiences in the window $|x| < \varepsilon$ before returning to M. For large $N(X)$, the trajectory of X runs almost parallel to the y axis for a long time, and we distinguish two types of such trajectories, see Fig. 3. The trajectories of the first type enter the window, almost approach its central axis (the y axis), but then turn back and exit on the same side they entered (the solid line on Fig. 3). The trajectories of the other type move through the window, cross the y axis, and exit on the opposite side (the dashed line on Fig. 3). These two types of trajectories are separated by points whose trajectories converge to the y axis and never return to M. The singularities of $N(X)$ occur at points where the number of bounces in the window $|x| < \varepsilon$ changes from n to $n+1$ or $n-1$.

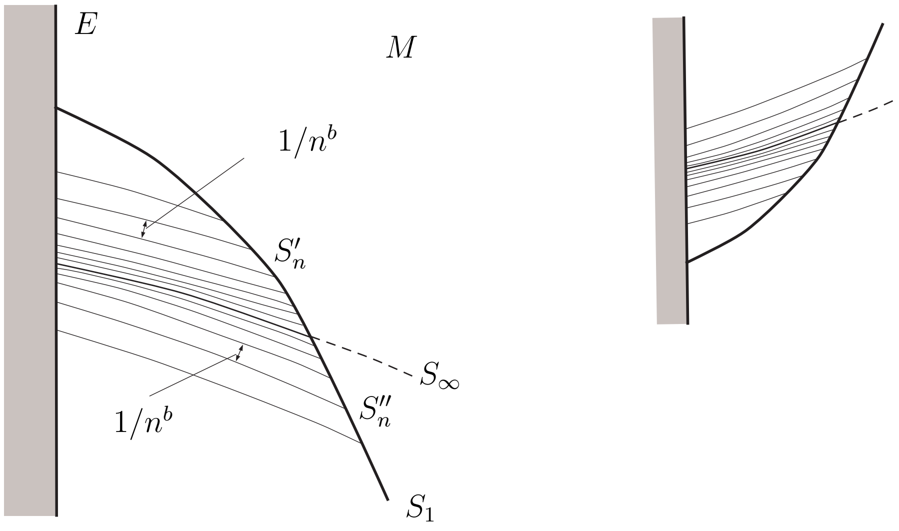

Fig. 4: Singularities of the map F (left) and F^{-1} (right).

Figure 4 (left) shows the structure of singularity lines of the map F near the point q_1, in the r, φ coordinates. The bold vertical line E on the left is $\pi^{-1}(q_1)$, the edge of M. The bold steeply decreasing curve S_1 terminating on E consists of points $\{X : \mathcal{F}(X) \in \pi^{-1}(q_2)\}$, which hit the point q_2 of ∂Q under the map $\mathcal{F}$. The points above S_1 are mapped by $\mathcal{F}$ to the right of q_2, so they do not leave M. The points below the curve S_1 are mapped by $\mathcal{F}$ to the left of q_2, and then they enter the window $|x| < \varepsilon$.

The decreasing curve S_∞ which crosses S_1 and terminates on E consists of points whose trajectories converge to the y axis (thus S_∞ is the stable manifold of the periodic orbit running along the y axis). The dashed part of S_∞ (to the right of S_1) does not enter the window immediately, but will do so in one or a few iterations.

The region above S_∞ but below S_1 consists of points whose trajectories enter the window but turn back without reaching the y axis (like the solid trajectory on Fig. 3). This region is divided into infinitely many strips by decreasing curves S_n', $n \geq 1$, which correspond to the discontinuities of the function $N(X)$: the curve S_n' separates the region $C_n' \colon = \{N(X) = n\}$ from the similar region C_{n+1}'. The curves S_n' are almost parallel to S_∞ and accumulate toward S_∞ from above.

The region below S_∞ consists of points whose trajectories enter the window and manage to move through it crossing the y axis (like the dashed trajectory on Fig. 3). This region is divided into infinitely many strips by decreasing curves S_n'', $n \geq 1$, which correspond to the discontinuities of the function $N(X)$: the curve S_n'' separates the region $C_n'' \colon = \{N(X) = n\}$ from the similar region C_{n+1}''. The curves S_n'' are almost parallel to S_∞ and accumulate toward S_∞ from below.

Due to the time-reversibility of the billiard dynamics, the singularities of the map F^{-1} have a similar structure. In fact, the picture shown on Fig. 4 (left) must be flipped about the horizontal line $\varphi = 0$ to become the illustration of singularity curves of F^{-1} near the same point q_1, see a scaled-down version of it shown on Fig. 4 (right).

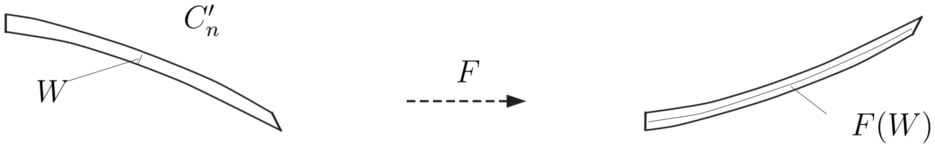

Fig. 5: The transformation of C_n' under F.

Furthermore, F maps each region C_n' onto a symmetric region made by the singularity curves of F^{-1} (near the point q_1 or q_2). Similarly, F maps each region C_n'' onto a symmetric region made by the singularity curves of F^{-1} near the point q_3 or q_4. The action of F on C_n' is schematically shown on Fig. 5: long sides of C_n' are transformed into short sides of $F(C_n')$, while short sides of C_n' are transformed into long sides of $F(C_n')$. Unstable manifolds $W \subset C_n'$ (which are short increasing curves in the r, φ coordinates) are mapped onto long unstable curves stretching across $F(C_n')$ completely, see Fig. 5. Let h_n denote the height of the region C_n (of course, it is not uniform across C_n', but we can take the maximum height, for example). Then, since the length of $F(C_n')$ is $\mathcal{O}(1)$, the factor of expansion of unstable manifolds $W \subset C_n'$ is[1]

$$\Lambda_n \sim 1/h_n. \tag{3.4}$$

(this, of course, requires the distortions be uniformly bounded on $W \subset C_n'$, which follows from general results [4, 8]). A similar analysis applies to the region C_n''.

The qualitative description of the singularity curves for the map F outlined above is the result of rather straightforward (albeit somewhat meticulous) geometric considerations, which we omit. In order to determine the rates of the decay of correlations we need certain quantitative estimates on the measure of the regions C_n' and C_n'' and on the factor of expansion of unstable manifolds $W \subset C_n'$ and $W \subset C_n''$ under the map F.

Proposition 2. *Unstable manifolds $W \subset C_n'$ and $W \subset C_n''$ are expanded under the map F by a factor $\Lambda_n \sim n^b$, where $b = a + 2$. Accordingly, see (3.4), the height (and hence the measure) of the regions C_n' and C_n'' is $\sim n^{-b}$.*

The proposition will be proven in the next section. Here we complete the proof (F2) and (3.3), thus deriving Theorem 1.

It is immediate that

$$\mu(X \in M :\ R(X;\mathcal{F},M) > n) = \sum_{m>n} \mu(C_m' \cup C_m'') \le \text{const} \cdot n^{-a-1}$$

which proves (F2).

Next, every unstable manifold $W \subset M$ is a smooth monotonically increasing curve in the r, φ coordinates. Hence for every $n \ge 1$ the intersection $W \cap C_n'$ is at most one curve, and the same is true for $W \cap C_n'$. If W crosses the separating line S_∞, then it intersects C_n' and C_n'' for all $n \ge n_\delta$, where n_δ grows to ∞ as $|W| = \delta$ converges to 0. Then

$$\sum_i \Lambda_i^{-1} < \text{const} \sum_{n=n_\delta}^{\infty} \frac{1}{n^{a+2}} < \frac{\text{const}}{n_\delta^{a+1}},$$

which is less than 1 for all sufficiently small $\delta > 0$. If W does not cross S_∞, but crosses S_n' or S_n'' with sufficiently large n, the analysis is similar. If W

---

[1] Our notation $A \sim B$ has the following meaning: there is a constant $C = C(Q) > 1$ such that $C^{-1} < A/B < C$.

only crosses S_n' or S_n'' with small n, then a standard trick - the use of a higher iterate of F – applies, see [8]. $\square$

Remark. To establish an upper bound on correlations, we only need an upper bound on the measures in (3.2). Thus it will be enough to obtain a lower bound on Λ_n in Proposition 2. This is what we do in the next section: we prove that $\Lambda_n \geq \text{const}\, n^b$. While our arguments can be easily extended to obtain an upper bound $\Lambda_n \leq \text{const}\, n^b$ as well, we do not pursue this goal.

4 Proof of Proposition 2

Given an unstable manifold $W \subset C_n'$ (or $W \subset C_n''$) and a point $X \in W$, the map $F = \mathcal{F}^n$ expands W at X by the factor [2, 4]

$$\Lambda_n(X) = \prod_{m=0}^{n-1} \bigl(1 + \tau(X_m)\mathcal{B}(X_m)\bigr) \tag{4.1}$$

where $X_m = \mathcal{F}^m(X)$, and for every point $Y = (r, \varphi) \in \mathcal{M}$ we denote by $\tau(Y)$ the time between the collisions at the points Y and $\mathcal{F}(Y)$ and

$$\mathcal{B}(Y) = \frac{1}{\cos\varphi}\left(\frac{d\varphi}{dr} + \mathcal{K}(r)\right), \tag{4.2}$$

where $d\varphi/dr$ denotes the slope of the unstable manifold $W(Y)$ passing through Y and $\mathcal{K}(r)$ the curvature of the boundary ∂Q at the point r.

We note that $\mathcal{B}(Y)$ is the geometric curvature of the orthogonal cross-section of the family of trajectories on the billiard table Q coming from $W(Y)$, see [1, 2, 4] for more details. The expansion factor (4.1) is measured in the so called p-norm defined by

$$|V|_p = \cos\varphi\, |dr| \tag{4.3}$$

for tangent vectors $V = (dr, d\varphi) \in \mathcal{T}_X\mathcal{M}$. The p-norm is equivalent to the Euclidean norm

$$|V| = \bigl[(dr)^2 + (d\varphi)^2\bigr]^{1/2} \tag{4.4}$$

along the trajectory of $\mathcal{F}^m(X)$, $1 \leq m \leq n$, as we will prove below.

The value of $\mathcal{B}(Y)$ is positive for all $Y \in \mathcal{M}$. The initial value $\mathcal{B}(X)$, $X \in W$, is bounded away from zero and infinity:

$$\mathcal{B}_{\min} \le \mathcal{B}(X) \le \mathcal{B}_{\max},$$

where $\mathcal{B}_{\min} > 0$ is determined by our choice of ε. For the computation of $\mathcal{B}(X_m)$ we have a recurrent formula

$$\mathcal{B}(X_m) = \frac{2\mathcal{K}(r_m)}{\cos\varphi_m} + \frac{1}{\tau(X_{m-1}) + 1/\mathcal{B}(X_{m-1})}, \tag{4.5}$$

where $(r_m, \varphi_m) = X_m$. Let x_m denote the x coordinate of the collision point $r_m \in \partial Q$, then it is easy to compute

$$\mathcal{K}(r_m) = \frac{\beta(\beta-1)|x_m|^{\beta-2}}{\left(1+\beta^2|x_m|^{2(\beta-1)}\right)^{3/2}}. \tag{4.6}$$

We note that $\mathcal{K}(r_m)$ approaches zero, as x_m approaches zero, and we will see later that $\mathcal{B}(X_m)$ approach zero as well.

Next we consider the trajectory of a point $X \in C_n''$ (the case $X \in C_n'$ is easier and will be treated later). Due to an obvious symmetry of the table Q about the x-axis it is convenient to fold Q in half and reflect its upper part $y > 0$ onto its lower half $y < 0$, then our trajectory will bounce between the x-axis and the lower side of Q, see Fig. 6.

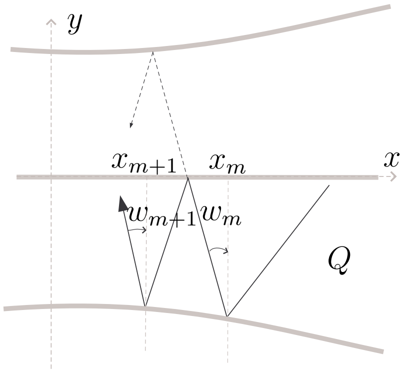

Fig. 6: The m-th collision and the parameters

Let n' be uniquely defined by $x_{n'+1} < 0 < x_{n'}$. First we consider the interval $1 \le m \le n'$, i.e. where $x_m > 0$.

We denote by w_m the angle made by the y-axis and the velocity vector after the mth collision. Note that $(\beta x^{\beta-1}, 1)$ is the inward normal vector to ∂Q at the point r_m. Elementary geometric considerations yield the following relations:

$$\begin{aligned} w_m - w_{m+1} &= 2\arctan(\beta x_{m+1}^{\beta-1}) \\ x_m - x_{m+1} &= 2\tan w_m + (x_m^\beta + x_{m+1}^\beta)\tan w_m. \end{aligned} \tag{4.7}$$

Using Taylor expansion we obtain

$$\begin{aligned} w_m - w_{m+1} &= 2\beta x_{m+1}^{\beta-1} - R_{w,m+1} \\ x_m - x_{m+1} &= 2w_m + R_{x,m+1}, \end{aligned} \tag{4.8}$$

where

$$R_{w,m+1} = \tfrac{2}{3}\beta^3 x_{m+1}^{3(\beta-1)} + \mathcal{O}(x_{m+1}^{5(\beta-1)}) > 0 \tag{4.9}$$

and

$$R_{x,m+1} = \tfrac{2}{3}w_m^3 + (x_m^\beta + x_{m+1}^\beta)w_m + \mathcal{O}(x_m^\beta w_m^3 + w_m^5) > 0 \tag{4.10}$$

(the positivity of $R_{w,m+1}$ and $R_{x,m+1}$ is guaranteed by the smallness of ε). Note that both $\{x_m\}$ and $\{w_m\}$ are decreasing sequences of positive numbers for $m = 1, \dots, n'$.

Lemma 3. *Let $n'' \in [1, n']$ be uniquely defined by the condition*

$$w_{n''-1} > 2w_{n'} > w_{n''}. \tag{4.11}$$

Then for all $n'' \le m \le n'$

$$x_m \sim (n' - m) w_{n'} \tag{4.12}$$

and

$$n' - n'' \sim w_{n'}^{\frac{2-\beta}{\beta}} \tag{4.13}$$

(recall our convention on the usage of "$\sim$" in the previous section).

Proof. Due to (4.8) and (4.11), for any $m \in [n'', n')$ we have

$$2w_{n'} \le 2w_m \le x_m - x_{m+1} \le 3w_m \le 6w_{n'}$$

hence

$$2(n'-m)w_{n'} \le x_m \le 6(n'-m+1)w_{n'} \tag{4.14}$$

(note that $0 \le x_{n'} \le 3w_{n'}$). Next, due to (4.14) and (4.8)

$$2^{\beta-1}(n'-m-1)^{\beta-1}w_{n'}^{\beta-1} \le w_m - w_{m+1} \le 2\beta\, 6^{\beta-1}(n'-m)^{\beta-1}w_{n'}^{\beta-1}$$

therefore

$$w_m \sim w_{n'} + (n'-m)^{\beta}w_{n'}^{\beta-1}$$

Substituting $m = n''$, then $m = n''-1$ and using (4.11) implies (4.13) and completes the proof of the lemma. □

We note that (4.12) and (4.13) imply

$$x_{n''} \sim w_{n'}^{\frac{2}{\beta}}, \tag{4.15}$$

hence $x_{n''} \ll \varepsilon$ and thus $n'' \gg 1$. Next we consider the case $1 < m \le n''$.

Lemma 4. *For all* $1 < m \le n''$ *we have*

$$x_m^{\beta} \sim w_m^2 \sim m^{\frac{2\beta}{2-\beta}}. \tag{4.16}$$

Moreover,

$$n'' \sim w_{n'}^{\frac{2-\beta}{\beta}}. \tag{4.17}$$

Proof. Due to (4.8) and the mean value theorem, for some $x_* \in (x_{m+1}, x_m)$

$$\begin{aligned} x_m^{\beta} - x_{m+1}^{\beta} &= \beta x_*^{\beta-1}(x_m - x_{m+1}) \\ &= 2\beta x_*^{\beta-1}w_m + \mathcal{O}\big(x_m^{\beta-1}w_m^3 + x_m^{2\beta-1}w_m\big). \end{aligned}$$

Similarly, for some $w_* \in (w_{m+1}, w_m)$

$$\begin{aligned} w_m^2 - w_{m+1}^2 &= 2w_*(w_m - w_{m+1}) \\ &= 4\beta x_{m+1}^{\beta-1}w_* + \mathcal{O}\big(x_m^{3(\beta-1)}w_m\big). \end{aligned}$$

This easily implies

$$1 \le \frac{w_m^2 - w_{m+1}^2}{x_m^\beta - x_{m+1}^\beta} \le 5.$$

Also, (4.15) and (4.11) imply that $w_{n''}^2 \sim x_{n''}^\beta$, i.e.

$$C' \le \frac{w_{n''}^2}{x_{n''}^\beta} \le C''$$

where we can assume $C' < 1$ and $C'' > 5$. Now the first relation in (4.16) follows easily.

Next, denote $z_m = x_m^{\frac{\beta-2}{2}}$. Then (4.8) and the mean value theorem imply

$$\begin{aligned} z_m - z_{m+1} &\sim x_m^{\frac{\beta-4}{2}} (x_m - x_{m+1}) \\ &\sim x_m^{\frac{\beta-4}{2}} w_m \\ &\sim x_m^{\beta-2} = z_m^2 \end{aligned}$$

(we used the first relation in (4.16)). Now let $Z_m = 1/z_m$, then

$$Z_{m+1} - Z_m \sim Z_{m+1}/Z_m \sim 1$$

(we note that $x_m - x_{m+1} \sim x_m^{\frac{\beta}{2}} \ll x_m$, hence $x_m/x_{m+1} \approx 1$). Since $x_0 \ge \varepsilon$,

$$Z_0 \le \varepsilon^{-\frac{\beta-2}{2}} = \text{const}, \tag{4.18}$$

and we obtain

$$Z_m \sim m \qquad \text{and} \qquad z_m \sim 1/m, \tag{4.19}$$

which proves the second relation in (4.16). Now (4.17) is immediate due to (4.15). □

Equations (4.13) and (4.17) imply $n'' \sim n' - n''$ and $n' \sim w_{n'}^{\frac{2-\beta}{\beta}}$. A similar analysis can be done for the remaining part of the trajectory, $n' < m < n$, which shows that $n - n' \sim w_{n'+1}^{\frac{2-\beta}{\beta}}$. Since $w_{n'} \approx w_{n'+1}$, we obtain $n - n' \sim n'$, and so

$$n'' \sim n \qquad \text{and} \qquad w_{n'} \sim n^{\frac{\beta}{2-\beta}}. \tag{4.20}$$

Lemma 5. *For all* $m < n'$ *we have*

$$w_m^2 - 2x_m^\beta < w_{m+1}^2 - 2x_{m+1}^\beta$$

i.e. $\{w_m^2 - 2x_m^\beta\}$ *is an increasing sequence for* $m = 1, \dots, n'$.

Proof. By the convexity of the function x^β,

$$2x_m^\beta - 2x_{m+1}^\beta \geq 2\beta x_{m+1}^{\beta-1}(x_m - x_{m+1}).$$

Now due to (4.8)–(4.10)

$$2\beta x_{m+1}^{\beta-1}(x_m - x_{m+1}) > 2w_m(w_m - w_{m+1}) > w_m^2 - w_{m+1}^2. \qquad \square$$

Lemma 5 implies $w_m^2 - 2x_m^\beta < w_{n'}^2$, hence

$$w_m < \sqrt{2x_m^\beta + w_{n'}^2} < \sqrt{2}\, x_m^{\frac{\beta}{2}} + \tfrac{1}{2}\, x_m^{-\frac{\beta}{2}} w_{n'}^2. \tag{4.21}$$

Next we derive a more precise estimate on the x coordinate:

Lemma 6. *For all* $1 \leq m \leq n''$ *we have*

$$x_m^{\frac{2-\beta}{2}} \leq Lm + C_1 \ln m + C_2 m \Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}} + C_3 \tag{4.22}$$

where $L = (\beta - 2)\sqrt{2}$ *and* $C_1, C_2, C_3 > 0$ *are some constants.*

Proof. Due to (4.8) and (4.21)

$$x_m - x_{m+1} < 2\sqrt{2}\, x_m^{\frac{\beta}{2}} + x_m^{-\frac{\beta}{2}} w_{n'}^2 + C x_m^{\frac{3\beta}{2}}$$

for some large $C > 0$ (we used the fact $w_m^2 \sim x_m^\beta$). As before, we put $z_m = x_m^{\frac{\beta-2}{2}}$. We consider two cases. If $\beta \geq 4$, then the function $x^{\frac{\beta-2}{2}}$ is convex down, and

$$\begin{aligned} z_m - z_{m+1} &\leq \tfrac{\beta-2}{2}\, x_m^{\frac{\beta-4}{2}} (x_m - x_{m+1}) \\ &\leq L\, x_m^{\beta-2} + \tfrac{\beta-2}{2}\, x_m^{-2} w_{n'}^2 + C x_m^{2\beta-2} \\ &\leq L\, z_m^2 + \tfrac{\beta-2}{2}\, z_m^{-\frac{4}{\beta-2}} w_{n'}^2 + C z_m^{\frac{4\beta-4}{\beta-2}}. \end{aligned}$$

If $\beta < 4$, then the function $x^{\frac{\beta-2}{2}}$ is convex up, and

$$\begin{aligned} z_m - z_{m+1} &\le \tfrac{\beta-2}{2}\, x_{m+1}^{\frac{\beta-4}{2}}(x_m - x_{m+1}) \\ &\le L\, x_m^{\frac{\beta}{2}}\, x_{m+1}^{\frac{\beta-4}{2}} + \tfrac{\beta-2}{2}\, x_{m+1}^{-2} w_{n'}^2 + C x_m^{2\beta-2} \\ &\le L\, z_m^{\frac{\beta}{\beta-2}}\, z_{m+1}^{\frac{\beta-4}{\beta-2}} + \tfrac{\beta-2}{2}\, z_{m+1}^{-\frac{4}{\beta-2}} w_{n'}^2 + C z_m^{\frac{4\beta-4}{\beta-2}} \end{aligned}$$

As before, let $Z_m = 1/z_m$. Then in the case $\beta > 4$ we have

$$\begin{aligned} Z_{m+1} - Z_m &\le L\, \frac{Z_{m+1}}{Z_m} + \frac{\beta-2}{2}\, Z_{m+1}^{\frac{2\beta}{\beta-2}} w_{n'}^2 + C Z_{m+1}^{-\frac{2\beta}{\beta-2}} \\ &\le L + L\, \frac{Z_{m+1} - Z_m}{Z_m} + \frac{\beta-2}{2}\, Z_{m+1}^{\frac{2\beta}{\beta-2}} w_{n'}^2 + C Z_{m+1}^{-\frac{2\beta}{\beta-2}} \end{aligned}$$

Solving the last inequality for $Z_{m+1} - Z_m$ and using (4.19) and (4.20) gives

$$Z_{m+1} - Z_m \le L + \frac{C'}{m} + C''\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}}$$

for some large $C', C'' > 0$. Summing up over m implies (4.22) with $C_3 = \varepsilon^{-\frac{\beta-2}{2}}$, see (4.18).

In the other case, $\beta < 4$, we have

$$\begin{aligned} Z_{m+1} - Z_m &\le L\bigg(\frac{Z_{m+1}}{Z_m}\bigg)^{\frac{2}{\beta-2}} + \frac{\beta-2}{2}\, Z_{m+1}^{\frac{2\beta}{\beta-2}} w_{n'}^2 + C Z_{m+1}^{-\frac{2\beta}{\beta-2}} \\ &\le L + G\, \frac{Z_{m+1} - Z_m}{Z_m} + \frac{\beta-2}{2}\, Z_{m+1}^{\frac{2\beta}{\beta-2}} w_{n'}^2 + C Z_{m+1}^{-\frac{2\beta}{\beta-2}} \end{aligned}$$

with $G = \frac{3L}{\beta-2}$, and the subsequent analysis is similar to the previous case. □

Corollary 7. *For all $1 \le m \le n''$ we have*

$$2\mathcal{K}(r_m) \ge D\bigg[m + C_1' \ln m + C_2' m\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}} + C_3'\bigg]^{-2} \tag{4.23}$$

where $D = \frac{\beta(\beta-1)}{(\beta-2)^2}$ and $C_1', C_2', C_3' > 0$ are some constants.

Proof. Equation (4.6) implies

$$\mathcal{K}(r_m) = \beta(\beta-1)x_m^{\beta-2} + \mathcal{O}\big(x_m^{3\beta-4}\big).$$

To estimate the main term we use (4.22), and the remainder term is $\mathcal{O}\big(m^{-\frac{6\beta-8}{\beta-2}}\big)$ by (4.16), so it can be incorporated into the right hand side of (4.23) by choosing sufficiently large constants $C_1', C_2', C_3' > 0$. □

Lemma 8. *For all* $1 \le m < n''$ *we have*

$$\mathcal{B}(X_{m-1}) \ge A\Big[m + C_4 \ln m + C_5 m\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}} + C_6\Big]^{-1} \tag{4.24}$$

where $A > 0$ *satisfies* $2A^2 - A = D$*, hence* $A = \frac{\beta-1}{\beta-2}$*, and* $C_4, C_5, C_6 > 0$ *are large constants.*

Proof. We use induction on m. For $m = 1$ the validity of (4.24) is guaranteed by choosing C_6 large enough. Assume that (4.24) is valid for some $m < n''-1$. Due to (4.5) and (4.23) it is enough to verify

$$\frac{D}{\Big[m + C_1' \ln m + C_2' m\big(\frac{m}{n}\big)^{\frac{2\beta}{\beta-2}} + C_3'\Big]^2} + \frac{A}{A\tau_m + m + C_4 \ln m + C_5 m\big(\frac{m}{n}\big)^{\frac{2\beta}{\beta-2}} + C_6}$$
$$> \frac{A}{m + 1 + C_4 \ln(m+1) + C_5(m+1)\big(\frac{m+1}{n}\big)^{\frac{2\beta}{\beta-2}} + C_6}$$

provided $C_4, C_5, C_6 > 0$ are large enough. Here

$$\tau_m = \tau(X_m) = 2 + \mathcal{O}(w_m) = 2 + \mathcal{O}\big(m^{\frac{\beta}{\beta-2}}\big).$$

It is easy to see that

$$\frac{A}{m + 1 + C_4 \ln(m+1) + C_5(m+1)\big(\frac{m+1}{n}\big)^{\frac{2\beta}{\beta-2}} + C_6}$$
$$- \frac{A}{A\tau_m + m + C_4 \ln m + C_5 m\big(\frac{m}{n}\big)^{\frac{2\beta}{\beta-2}} + C_6} < \frac{2A^2 - A}{\Theta}$$

where Θ denotes the product of the two denominators. Thus it is enough to verify

$$\frac{D}{\Big[m + C_1' \ln m + C_2' m\big(\frac{m}{n}\big)^{\frac{2\beta}{\beta-2}} + C_3'\Big]^2} > \frac{2A^2 - A}{\Theta}.$$

We recall that $2A^2 - A = D$. Thus it is enough to verify

$$\Theta > \Big[m + C_1' \ln m + C_2' m\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}} + C_3'\Big]^2. \tag{4.25}$$

The leading term m^2 appears on both sides and cancels out. Keeping only the largest non-cancelling terms on both sides of (4.25) we obtain

$$2\,C_4 m\ln m + 2\,C_5 m^2\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}} > 2\,C_1' m\ln m + 2\,C_2' m^2\Big(\frac{m}{n}\Big)^{\frac{2\beta}{\beta-2}},$$

which can be ensured by choosing C_4 and C_5 large enough. This implies (4.25) and then Lemma 8. □

Corollary 9.

$$\mathcal{B}(X_{m-1}) \geq \frac{A}{m} + \frac{C_4'\ln m}{m^2} + \frac{C_5' m}{n^2} + \frac{C_6'}{m^2}, \tag{4.26}$$

where $C_4', C_5', C_6' > 0$ *are large constants.*

Proof. This follows from (4.24) by Taylor expansion and because $\frac{2\beta}{\beta-2} > 2$. □

Now we are ready to estimate the expansion factor $\Lambda_n(X)$ given by (4.1).

Lemma 10. *We have*

$$\prod_{m=0}^{n''-1}\big(1+\tau(X_m)\mathcal{B}(X_m)\big) \geq Cn^{\frac{2\beta-2}{\beta-2}} \tag{4.27}$$

where $C > 0$ *is a constant.*

Proof. Note that $\tau(X_m) > 2$. Hence, due (4.26), we have

$$\ln\left[\prod_{m=0}^{n''-1}\big(1+\tau(X_m)\mathcal{B}(X_m)\big)\right] > \sum_{m=1}^{n''}\left[\frac{2A}{m} + \frac{2C_4'\ln m}{m^2} + \frac{2C_5' m}{n^2} + \frac{C_7}{m^2}\right]$$

with some large constant $C_7 > 0$. Therefore,

$$\ln\left[\prod_{m=0}^{n''-1}\big(1+\tau(X_m)\mathcal{B}(X_m)\big)\right] > 2A\ln n'' + \text{const} > 2A\ln n + \text{const},$$

where the last inequality follows from (4.20). Lastly, note that $2A = \frac{2\beta-2}{\beta-2}$, which completes the proof of the lemma. □

The bound (4.27) implies

$$\Lambda_n^{(1)}(X) \colon= \prod_{m=0}^{n'-1}\big(1+\tau(X_m)\mathcal{B}(X_m)\big) \geq Cn^{\frac{2\beta-2}{\beta-2}} \tag{4.28}$$

Lemma 11.

$$\Lambda_n^{(2)}(X) := \prod_{m=n'}^{n-1} \bigl(1 + \tau(X_m)\mathcal{B}(X_m)\bigr) \geq C n^{\frac{\beta}{\beta-2}} \tag{4.29}$$

where $C > 0$ is a constant.

Proof. This can be obtained by a detailed analysis of the dynamics on the interval (n', n) similar to the one done for the interval $(0, n')$, but we will use a shortcut: the time-reversibility of the billiard dynamics will allow us to derive (4.29) directly from (4.28).

Let V^u and V^s be two unit vectors tangent to the unstable and stable manifolds, respectively, at the point X. Since the angle between V^u and V^s is bounded away from zero, the area of the parallelogram Π spanned by V^u and V^s is of order one (uniformly in n).

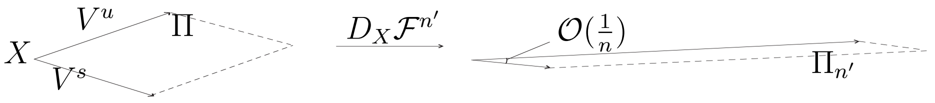

Fig. 7:

Consider the parallelogram $\Pi_{n'} = D_X\mathcal{F}^{n'}(\Pi)$ spanned by the vectors $V^u_{n'} = D_X\mathcal{F}^{n'}(V^u)$ and $V^s_{n'} = D_X\mathcal{F}^{n'}(V^s)$. Since the map $\mathcal{F}^{n'}$ preserves the measure $d\mu = \cos\varphi\, dr\, d\varphi$, we have

$$\cos\varphi_{n'} \ \text{Area}(\Pi_{n'}) = \cos\varphi \ \text{Area}(\Pi).$$

Note that $\cos\varphi \approx 1$ and $\cos\varphi_{n'} \approx 1$, hence

$$\text{Area}(\Pi_{n'}) \sim \text{Area}(\Pi) \sim 1.$$

On the other hand,

$$\text{Area}(\Pi_{n'}) = |V^u_{n'}|\, |V^s_{n'}| \, \sin\gamma_{n'}$$

where $|V^u_{n'}|$ and $|V^s_{n'}|$ denote the lengths of these vectors in the Euclidean norm (4.4) and $\gamma_{n'}$ denotes the angle between them.

Next we estimate $\gamma_{n'}$. It easily follows from (4.5) that

$$\mathcal{B}(X_{n'}) \geq \left[\sum_{m=0}^{n'-1} \tau(X_m) + 1/\mathcal{B}(X_0)\right]^{-1} \sim \frac{1}{n'} \sim \frac{1}{n}.$$

Now (4.2) implies that the slope of the vector $V^u_{n'}$ is

$$\frac{d\varphi}{dr} = \cos\varphi_{n'}\ \mathcal{B}(X_{n'}) - \mathcal{K}(r_{n'}).$$

We note that $\cos\varphi_{n'} \approx 1$ and $\mathcal{K}(r_{n'}) \sim n^{-\beta}$ due to (4.6), because $x_{n'} < 3w_{n'} \sim n^{\frac{\beta}{2-\beta}}$, cf. (4.13). Therefore,

$$\frac{d\varphi}{dr} > \frac{C}{n}$$

for some constant $C > 0$. Hence the vector $V^u_{n'}$ makes an angle $\geq C/n$ with the horizontal r-axis. By the time reversibility, the vector $V^s_{n'}$ makes an angle $\leq -C/n$ with the horizontal r-axis, see Fig. 7, hence $\sin\gamma_{n'} > c/n$ for some constant $c > 0$, and we obtain

$$|V^u_{n'}|\,|V^s_{n'}| < cn$$

for some constant $c > 0$.

Next, the Euclidean norm $|V|$ defined by (4.4) is uniformly equivalent to the p-norm (4.3) for both stable and unstable vectors in our considerations. Indeed, $\cos\varphi \approx 1$ and $|d\varphi| \leq C\,|dr|$ for some constant $C > 0$, as it easily follows from (4.2). Therefore, we obtain

$$|V^u_{n'}|_p\,|V^s_{n'}|_p < cn$$

for some constant $c > 0$. Obviously,

$$|V^u_{n'}|_p = \Lambda^{(1)}_{n'}(X)\,|V^u|_p \sim \Lambda^{(1)}_{n'}(X).$$

Now it is time for a little trick. By the time reversibility of the billiard dynamics, the contraction of stable vectors during the time interval $(0, n')$ is the same as the expansion of the corresponding unstable vectors during the time interval (n', n), hence

$$|V^s_{n'}|_p \sim \left[\Lambda^{(2)}_{n'}(X)\right]^{-1} |V^s|_p \sim \left[\Lambda^{(2)}_{n'}(X)\right]^{-1}.$$

Therefore,

$$\Lambda^{(2)}_{n'}(X) > c\Lambda^{(1)}_{n'}(X)/n \tag{4.30}$$

for some constant $c > 0$. Now (4.30) and (4.28) imply (4.29). □

Combining (4.28) and (4.29) gives

$$\Lambda_n(X) = \Lambda^{(1)}_n(X)\,\Lambda^{(2)}_n(X) \geq Cn^{\frac{3\beta-2}{\beta-2}}.$$

This proves Proposition 2 for $W \subset C''_n$ because, in its notation, we have

$$b = a + 2 = \frac{3\beta - 2}{\beta - 2}.$$

We now consider the remaining case $W \subset C'_n$, which correspond to trajectories that start near the point q_1, enter the window $|x| < \varepsilon$, but turn around before reaching the central line $x = 0$ and come back into the vicinity of q_1 or q_2 (as shown by the solid line on Fig. 3).

In that case n' can be defined as the turning point, i.e. by $x_{n'} < x_{n'-1}$ and $x_{n'} < x_{n'+1}$. Observe that if $X' = (r', \varphi') \in C'_n$, then there exists another point $X = (r, \varphi) \in C''_n$ with $r = r'$ and $\varphi < \varphi'$, whose trajectory goes through the window, as it is clear from Fig. 4. Since $\varphi' < \varphi$, it follows that the x-coordinate x_m of the point $\mathcal{F}^m(X)$ will be always smaller than the x-coordinate x'_m of the point $\mathcal{F}^m(X')$, for all $1 \leq m \leq n$. This observation and the bound (4.22) that we have proved for x_m implies that the same bound holds for x'_m and for all $1 \leq m \leq n''$. The rest of the proof of Proposition 2 for $X' \in C'_n$ is identical to that of the case $X \in C''_n$.

Proposition 2 is now proven. □

References

[1] Bunimovich, L. A.; Sinai, Ya. G. & Chernov, N. I. *Markov partitions for two-dimensional hyperbolic billiards,* Russian Math. Surveys **45** (1990) 105–152.

[2] Bunimovich L. A.; Sinai, Ya. G. & Chernov, N. I., *Statistical properties of two-dimensional hyperbolic billiards,* Russian Math. Surveys **46** (1991) 47–106.

[3] Chernov, N., *Entropy, Lyapunov exponents and mean-free path for billiards*, J. Statist. Phys. **88** (1997), 1–29.

[4] Chernov, N., *Decay of correlations in dispersing billiards*, J. Statist. Phys. **94** (1999), 513–556.

[5] Chernov, N. and Haskell, C., *Nonuniformly hyperbolic K-systems are Bernoulli*, Ergodic Theory and Dynamical Systems **16** (1996), 19–44.

[6] Chernov, N. and Markarian, R., *Introduction to the Ergodic Theory of Chaotic Billiards*, 2nd Ed., IMPA, Rio, Brasil, 2003.

[7] Chernov, N. and Troubetzkoy, S., *Ergodicity of billiards in polygons with pockets*, Nonlinearity **11** (1998), 1095–1102.

[8] Chernov, N. and Zhang, H.-K., *Billiards with polynomial mixing rates*, manuscript, archived in http://www.ma.utexas.edu/mp_arc/ #04-261.

[9] Gallavotti, G. and Ornstein, D., *Billiards and Bernoulli schemes*, Comm. Math. Phys. **38** (1974), 83–101.

[10] Machta, J., *Power law decay of correlations in a billiard problem*, J. Statist. Phys. **32** (1983), 555–564.

[11] Markarian, R., *Billiards with polynomial decay of correlations*, Er. Th. Dynam. Syst. **24** (2004), 177–197.

[12] Ornstein, D. and Weiss, B., *On the Bernoulli nature of systems with some hyperbolic structure*, Ergod. Th. Dynam. Syst. **18** (1998), 441–456.

[13] Reháček, J., On the ergodicity of dispersing billiards. Random Comput. Dynam. **3** (1995), 35–55.

[14] Sinai, Ya. G., *Dynamical systems with elastic reflections. Ergodic properties of diepersing billiards,* Russian Math. Surveys **25** (1970) 137–189.

[15] Young L.-S., *Statistical properties of systems with some hyperbolicity including certain billiards*, Ann. Math., **147** (1998), 585–650.

[16] Young, L.-S., *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.