Note on the norm convergence of the unitary Trotter product formula

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Abstract

When the unitary Trotter product formula converges in operator norm is discussed, and the obtained results are applied to the unitary group generated by the Dirac operator and the relativistic Schrödinger operator with suitable potentials.

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1 Introduction and Results

Let A and B be selfadjoint operators in a Hilbert space $\mathcal{H}$ with domains $D(A)$ and $D(B)$ such that the operator sum $H = A+B$ is essentially selfadjoint on $D(A)\cap D(B)$. Denote the unique selfadjoint extension, i.e. the closure of $A+B$ by the same H. Here we include the case where the extension is not strict, i.e. when $A+B$ itself is already selfadjoint. Then it was proved by Trotter [T] that the unitary product formula in strong operator topology,

$$\text{s-}\lim_{n\to\infty}(e^{-itA/n}e^{-itB/n})^n = e^{-itH}, \tag{1.1}$$

$$\text{s-}\lim_{n\to\infty}(e^{-itB/2n}e^{-itA/n}e^{-itB/2n})^n = e^{-itH}, \tag{1.2}$$

holds uniformly on each t-bounded interval in $\mathbf{R}$.

The aim of this note is to show the following theorem on the norm convergence of the unitary Trotter product formula with optimal error bounds, under some conditions on the domains of A and B. We denote by $(\cdot,\cdot)$ the inner product of $\mathcal{H}$ and by $\|\cdot\|$ the norm of bounded operators on $\mathcal{H}$.

We assume that there exists a dense subspace $\mathcal{D}\subset D(A)\cap D(B)$ such that
(d1) e^{-itA}, $e^{-itB} : \mathcal{D}\to\mathcal{D}$ for $t\in\mathbf{R}$.

($d2$) There exists a symmetric operator E on $\mathcal{D}$ such that

$$i((Bu, Av) - (Au, Bv)) = (Eu, v)$$

for every $u,\, v \in \mathcal{D}$.

($d3$) There exist symmetric operators E_A and E_B on $\mathcal{D}$ such that

$$i((Eu, Av) - (Au, Ev)) = (E_A u, v), \quad i((Eu, Bv) - (Bu, Ev)) = (E_B u, v)$$

for every $u,\, v \in \mathcal{D}$, where $E : D(E) \subset \mathcal{H} \to \mathcal{H}$ is the operator defined in ($d2$).

($d4$) It holds that $e^{-itH} : \mathcal{D} \to \mathcal{D}$ for $t \in \mathbf{R}$.

Remark 1.1. The above operators E, E_A and E_B are formally represented in the form of commutators $E = i[A, B] = i(AB - BA)$ and

$$E_A = i[A, E] = -[A, [A, B]], \quad E_B = i[B, E] = -[B, [A, B]].$$

Theorem 1.1. *Let A and B be selfadjoint operators with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively. Suppose that $H = A + B$ is essentially selfadjoint on $D(A) \cap D(B)$ and denote by the same H its unique selfadjoint realization. If (d1) and (d2) are fulfilled and if E in (d2) admits a bounded extension on $\mathcal{H}$, then*

$$\|(e^{-i(t/n)A}e^{-i(t/n)B})^n - e^{-itH}\| = O(n^{-1}), \tag{1.3}$$

$$\|(e^{-i(t/n)B/2}e^{-i(t/n)A}e^{-i(t/n)B/2})^n - e^{-itH}\| = O(n^{-1}), \tag{1.4}$$

as $n \to \infty$, uniformly on each bounded t-interval in $\mathbf{R}$.

Theorem 1.2. *Let H be as in Theorem 1.1. If (d1)–(d3) are fulfilled and if E_A and E_B admit bounded extensions on $\mathcal{H}$, then*

$$\|(e^{-i(t/n)B/2}e^{-i(t/n)A}e^{-i(t/n)B/2})^n - e^{-itH}\| = O(n^{-2}), \tag{1.5}$$

as $n \to \infty$, uniformly on each bounded t-interval in $\mathbf{R}$.

Theorem 1.3. *Let H be again as in Theorem 1.1. If (d1)–(d4) are fulfilled and if E as well as E_A and E_B admits a bounded extension on $\mathcal{H}$, then*

$$\|(e^{-i(t/n)A}e^{-i(t/n)B})^n - e^{-itH} - i(t/2n)[e^{-itH}, A]\| = O(n^{-2}), \tag{1.6}$$

$$\|(e^{-i(t/n)B}e^{-i(t/n)A})^n - e^{-itH} - i(t/2n)[e^{-itH}, B]\| = O(n^{-2}), \tag{1.7}$$

and

$$\|\frac{1}{2}((e^{-i(t/n)A}e^{-i(t/n)B})^n + (e^{-i(t/n)B}e^{-i(t/n)A})^n) - e^{-itH}\| = O(n^{-2}) \tag{1.8}$$

as $n \to \infty$, uniformly on each bounded t-interval in $\mathbf{R}$.

If the conditions of the theorems are not satisfied, the norm convergence of the unitary Trotter product formula does not always hold. In [I2] we have observed there are examples with $A = -i\frac{d}{dx}$ and $B = V(x)\times$ in $L^2(\mathbf{R})$ for which the unitary product formula in operator norm holds as well as does not hold. The optimality of these error bounds has already been discussed in [ITTZ]. To prove our theorems, we shall use the method of commutator used in our recent paper [IT2] to get optimal error bounds, for the selfadjoint Trotter–Kato product formula with Schrödinger operators, on norm convergence as well as on pointwise uniform approximation to the integral kernels concerned.

In passing, the selfadjoint Trotter–Kato product formula in the strong operator topology was proved by Trotter [T] when the operator sum of nonnegative selfadjoint operators A and B is essentially selfadjoint on $D(A) \cap D(B)$, and by Kato [K] for their form sum. In [IT1], [ITTZ], its convergence has been proved even in the operator norm for the operator sum of nonnegative selfadjoint operators A and B which is selfadjoint on $D(A) \cap D(B)$, with optimal error bound $O(n^{-1})$. A further extension is also given in [INZ] to the case of the form sum of two nonnegative selfadjoint operators with some auxiliary conditions. The norm convergence has also been proved in [CNZ] in the case where one of the operators A and B is nonnegative selfadjoint and the second is an accretive operator which is relatively bounded with respect to the first with relative bound less than 1, with error bound $O((\log n)/n)$. We note here that the method of the present note can give one more supplement to the norm convergence of the selfadjoint Trotter–Kato product formula with optimal error bounds: under the same hypotheses as in each of Theorems 1.1–1.3 with additional assumption that both A and B are nonnegative, the selfadjoint product formula in operator norm also holds, namely, the same assertion as in each of these theorems is valid with e^{-itA}, e^{-itB} and e^{-itH} replaced by e^{-tA}, e^{-tB} and e^{-tH}.

For some account for the unitary and selfadjoint Trotter product formulas in the strong operator topology, we refer to Reed–Simon [RS].

In Section 2, we prove the theorems, and in Section 3, give as applications some examples that the norm convergence of the unitary product formula holds as well as does not hold. In particular, it should be noted that the unitary Trotter product formula in operator norm holds for the Dirac operator and the relativistic Schrödinger operator with suitable potentials.

2 Proof of Theorems

First we prepare some lemmas and then prove theorems.

We make use of the commutator relation

$$[e^{-itX}, Y] = i \int_0^t e^{-isX}[Y, X]e^{-i(t-s)X}\, ds. \tag{2.1}$$

Put

$$G(t) = e^{-itA}e^{-itB}, \tag{2.2}$$

$$K(t) = e^{-itB/2}e^{-itA}e^{-itB/2}. \tag{2.3}$$

Lemma 2.1. Let u, $v \in \mathcal{D}$, and assume that $(d1)$–$(d3)$ are fulfilled. Then

(1) $((Ae^{-itB} - e^{-itB}A)u, v) = -\int_0^t (Ee^{-i(t-s)B}u, e^{isB}v)ds.$

(2) $((Be^{-itA}-e^{-itA}B)u,v)=\int_0^t (Ee^{-i(t-s)A}u,e^{isA}v)ds.$

(3) $((Ee^{-itB}-e^{-itB}E)u,v)=-\int_0^t (E_Be^{-i(t-s)B}u,e^{isB}v)ds.$

(4) $((Ee^{-itA}-e^{-itA}E)u,v)=-\int_0^t (E_Ae^{-i(t-s)A}u,e^{isA}v)ds.$

Proof. We show only the relation (1). The others are shown in exactly the same way.

Put $f(t)=((Ae^{-itB}-e^{-itB}A)u,v)$. Then

$$\begin{aligned} f(t) &= -(e^{-isB}Ae^{-i(t-s)B}u,v)\Big|_{s=0}^{t} = -\int_0^t (d/ds)(e^{-isB}Ae^{-i(t-s)B}u,v)ds \\ &= -\int_0^t i\Big((B(e^{-i(t-s)B}u),A(e^{isB}v))-(A(e^{-i(t-s)B}u),B(e^{isB}v))\Big)ds. \end{aligned}$$

Since by $(d1)$, both $e^{-i(t-s)B}u$ and $e^{isB}v$ are in $\mathcal{D}$, we have

$$f(t)=-\int_0^t (Ee^{-i(t-s)B}u,e^{isB}v)ds$$

by $(d2)$, showing the desired relation.

Lemma 2.2. *On the same hypothesis as in Theorem 1.1, one has*

$$\|e^{-itH}-G(t)\|=O(|t|^2),\quad t\to 0. \tag{2.4}$$

Proof. Put $g(t)=(G(t)u,v)$ with $u,\ v\in\mathcal{D}$ fixed arbitrarily. We calculate $g'(t)=(d/dt)g(t)$ as

$$\begin{aligned} g'(t) &= (-iAG(t)u,v)+(e^{-itA}(-iB)e^{-itB}u,v) \\ &= (-iHG(t)u,v)+(-i[e^{-itA},B]e^{-itB}u,v). \end{aligned}$$

Since by assumption with Lemma 2.1 (2) the second term in the last member is bounded by $O(|t|)\|u\|\|v\|$ and since u and v range over the dense subspace $\mathcal{D}$, we have by Duhamel's principle

$$e^{-itH}-G(t)=i\int_0^t e^{-i(t-s)H}[e^{-isA},B]e^{-isB}ds.$$

Hence follows the desired bound.

Lemma 2.3. *On the same hypothesis as in Theorem 1.2, one has*

$$\|e^{-itH}-K(t)\|=O(|t|^3),\quad t\to 0. \tag{2.5}$$

Proof. Put $k(t)=(K(t)u,v)$ with $u,\ v\in\mathcal{D}$ fixed arbitrarily. We calculate $k'(t)=(d/dt)k(t)$. It has the decomposition $k'(t)=k_0(t)+k_1(t)+k_2(t)$, where

$$\begin{aligned} k_0(t) &= (-i(B/2)K(t)u,v)=(-ie^{-itA}e^{-it(B/2)}u,e^{it(B/2)}(B/2)v), \\ k_1(t) &= (e^{-it(B/2)}(-iA)e^{-itA}e^{-it(B/2)}u,v)=(-iAe^{-itA}e^{-it(B/2)}u,e^{it(B/2)}v), \\ k_2(t) &= (e^{-it(B/2)}e^{-itA}(-i(B/2))e^{-it(B/2)}u,v)=(-i(B/2)e^{-it(B/2)}u,e^{itA}e^{it(B/2)}v). \end{aligned}$$

We apply Lemma 2.1 (1) to

$$k_1(t) = -i(e^{-it(B/2)}A(e^{-itA}e^{-it(B/2)}u), v).$$

Then we get

$$k_1(t) = (-iAK(t)u, v) + r_1(t),$$

where

$$\begin{aligned} r_1(t) &= -\frac{i}{2}\int_0^t (e^{-is(B/2)}Ee^{-i(t-s)(B/2)}(e^{-itA}e^{-it(B/2)}u), v)ds \\ &= -\frac{i}{2}\int_0^t (Ee^{-is(B/2)}(e^{-itA}e^{-it(B/2)}u), e^{i(t-s)(B/2)}v)ds \end{aligned}$$

by making a change of variable. Similarly we apply Lemma 2.1 (2) to

$$k_2(t) = -\frac{i}{2}(e^{-itA}B(e^{-it(B/2)}u), e^{it(B/2)}v)$$

to get

$$k_2(t) = (-i(B/2)K(t)u, v) + r_2(t),$$

where

$$r_2(t) = \frac{i}{2}\int_0^t (e^{-isA}E(e^{-i(t-s)A}e^{-it(B/2)}u), e^{it(B/2)}v)ds.$$

We apply again Lemma 2.1 (3), (4) to $r_1(t)$ and $r_2(t)$. Then

$$r_1(t) = -r_0(t) + r_{10}(t), \quad r_2(t) = r_0(t) + r_{20}(t),$$

where $r_0(t) = (i/2)t(Ee^{-itA}e^{-it(B/2)}u, e^{it(B/2)}v)$ and

$$r_{10}(t) = (R_{10}(t)u, v), \quad r_{20}(t) = (R_{20}(t)u, v)$$

with

$$\begin{aligned} R_{10}(t) &= -\frac{i}{4}\int_0^t\int_0^s e^{-i(t-\sigma)(B/2)}E_Be^{-i\sigma(B/2)}e^{-itA}e^{-it(B/2)}d\sigma ds, \\ R_{20}(t) &= -\frac{i}{2}\int_0^t\int_0^s e^{-it(B/2)}e^{-i\sigma A}E_Ae^{-i(t-\sigma)A}e^{-it(B/2)}d\sigma ds. \end{aligned}$$

By assumption, $R_{10}(t)$ and $R_{20}(t)$ are bounded, and

$$\|R_{10}(t)\| + \|R_{20}(t)\| = O(|t|^2), \quad t \to 0.$$

Putting $w(t) = K(t)u$, we see that $w(t)$ satisfies

$$(w'(t), v) = (-iHw(t), v) + r(t)$$

with

$$r(t) = r_{10}(t) + r_{20}(t) = ((R_{10}(t) + R_{20}(t))u, v).$$

Since u and v range over the dense subspace $\mathcal{D}$, we have

$$e^{-itH} - K(t) = -\int_0^t e^{-i(t-s)H}(R_{10}(s) + R_{20}(s))ds$$

by Duhamel's principle. This yields the desired bound.

The proof of Theorem 1.3 needs the following lemma.

Lemma 2.4. Assume that $(d1)$ and $(d2)$ are fulfilled. If E admids a bounded extension, then

$$\|G(t) - K(t)\| = O(|t|^2), \quad t \to 0.$$

Proof. We write

$$G(t) = K(t) + [e^{-itA}, e^{-it(B/2)}]e^{-it(B/2)}.$$

We have by formula (2.1) and Lemma 2.1 (1)

$$\begin{aligned}
[e^{-itA}, e^{-it(B/2)}] &= e^{-isA}e^{-it(B/2)}e^{-i(t-s)A}\Big|_{s=0}^{t} \\
&= i\int_0^t e^{-isA}[e^{-it(B/2)}, A]e^{-i(t-s)A}ds
\end{aligned}$$

and

$$[e^{-it(B/2)}, A] = \frac{1}{2}\int_0^t e^{-is(B/2)}Ee^{-i(t-s)(B/2)}ds.$$

By assumption, E admits a bounded extension, and hence it follows that

$$\|\,[e^{-it(B/2)}, A]\,\| = O(|t|).$$

This proves the lemma.

Now we prove theorems.

Proof of Theorem 1.1. We show (1.3). With $\tau = t/n$ and $D_1(\tau) = e^{-i\tau H} - G(\tau)$, we have

$$e^{-itH} - G(\tau)^n = e^{-in\tau H} - G(\tau)^n = \sum_{k=1}^{n} X_k(\tau),$$

where

$$X_k(\tau) = e^{-i(k-1)\tau H}D_1(\tau)G(\tau)^{n-k}, \quad 1 \le k \le n.$$

Since by Lemma 2.2 $\|X_k(\tau)\| \le \|D_1(\tau)\| \le O(|\tau|^2)$ for every k, we have $\|e^{-itH} - G(\tau)^n\| = O(|\tau|)$, showing (1.3). We can show (1.4) similarly, noting that Lemma 2.2 together with Lemma 2.4 implies that $\|e^{-itH} - K(t)\| = O(|t|^2)$. This proves the theorem.

Proof of Theorem 1.2. With $\tau = t/n$ and $D_2(\tau) = e^{-i\tau H} - K(\tau)$, we have

$$e^{-itH} - K(\tau)^n = e^{-in\tau H} - K(\tau)^n = \sum_{k=1}^{n} Y_k(\tau),$$

where

$$Y_k(\tau) = e^{-i(k-1)\tau H} D_2(\tau) K(\tau)^{n-k}, \quad 1 \le k \le n.$$

Since by Lemma 2.3 $\|Y_k(\tau)\| \le \|D_2(\tau)\| \le O(|\tau|^3)$ for every k, we have $\|e^{-itH} - K(\tau)^n\| = O(|\tau|^2)$, showing (1.5). This proves the theorem.

Proof of Theorem 1.3. We put $\tau = t/n$. We denote by $O_p(n^{-\rho})$ the class of bounded operators whose norm obeys the bound $O(n^{-\rho})$ as $n \to \infty$.

By the assumption $(d1)$–$(d4)$ and by analogous arguments used to prove Lemma 2.1, we can show that $[B, e^{-itH}]$ is a bounded operator, because

$$[B, e^{-itH}] = i\int_0^t e^{-isH}[H, B]e^{-i(t-s)H} ds = i\int_0^t e^{-isH} E e^{-i(t-s)H} ds.$$

We have also

$$\begin{aligned}
&[e^{-itH}, e^{-i\tau B}] = i\int_0^\tau e^{-i\sigma B}[B, e^{-itH}]e^{-i(\tau-\sigma)B} d\sigma = O_p(\tau),\\
&[E, e^{-i\tau B}] = i\int_0^\tau e^{-i\sigma B}[B, E]e^{-i(\tau-\sigma)B} d\sigma = -\int_0^\tau e^{-i\sigma B} E_B e^{-i(\tau-\sigma)B} d\sigma = O_p(\tau).
\end{aligned}$$

Therefore we have by Theorem 1.2 and Lemma 2.4

$$\begin{aligned}
G(t/n)^n &= e^{-i\tau A} e^{-i\tau(B/2)} K(\tau)^{n-1} e^{-i\tau(B/2)}\\
&= e^{-i\tau A} e^{-i\tau(B/2)} e^{-i(n-1)\tau H} e^{-i\tau(B/2)} + O_p(n^{-2})\\
&= e^{-itH} + e^{-i\tau A} e^{-i\tau(B/2)} [e^{-i(n-1)\tau H}, e^{-i\tau(B/2)}] + O_p(n^{-2}).
\end{aligned}$$

Here we calculate the commutator $F := [e^{-i(n-1)\tau H}, e^{-i\tau(B/2)}]$ in the last member above. If we take account of the fact that $[B, e^{-i(n-1)\tau H}]$ is bounded, as seen above, we have

$$\begin{aligned}
F &= \frac{i}{2}\int_0^\tau e^{-i\sigma(B/2)}[B, e^{-i(n-1)\tau H}]e^{-i(\tau-\sigma)(B/2)} d\sigma\\
&= \frac{i\tau}{2} e^{-i\tau(B/2)}[B, e^{-i(n-1)\tau H}] + O_p(n^{-2}).
\end{aligned}$$

Hence

$$\begin{aligned}
G(t/n)^n &= e^{-itH} + (i\tau/2)e^{-i\tau A}e^{-i\tau B}[B, e^{-i(n-1)\tau H}] + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)K(\tau)[B, e^{-i(n-1)\tau H}] + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)e^{-i\tau H}[B, e^{-i(n-1)\tau H}] + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)e^{-i\tau H}(Be^{-i(n-1)\tau H} - e^{-i(n-1)\tau H}B) + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)(e^{-i\tau H}Be^{-i(n-1)\tau H} - e^{-itH}B) + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)[B, e^{-itH}] + O_p(n^{-2})\\
&= e^{-itH} + (i\tau/2)[e^{-itH}, A] + O_p(n^{-2})
\end{aligned}$$

by Lemmas 2.3 and 2.4, because $\|\,[e^{-i\tau H}, B]\,\| = O(n^{-1})$. This yields the desired relation.

As a concluding remark, we want to comment that the theorems actually hold if there exists a dense subspace $\mathcal{D}$ for which all the equations in the proof above can make sense. The assumption $(d1)$–$(d3)$ and/or $(d4)$ is one of the sufficient for this.

3 Applications

In [I2] we have shown that if $A = -i\frac{d}{dx}$, $B = x^k$, with k being a positive integer, in $L^2(\mathbf{R})$, and if H is the unique selfadjoint extension of $A + B$, then

$$\|(e^{-itA/n}e^{-itB/n})^n - e^{-itH}\| = \begin{cases} 2, & k \geq 2, \\ O(n^{-1}), & k = 1, \end{cases}$$

as $n \to \infty$. This means that the unitary Trotter product formula in operator norm holds for $k = 1$, but not for $k \geq 2$. It holds also, if $B = V(x)$ is a multiplication operator by a real-valued C^1 smooth function in $\mathbf{R}$ with bounded derivative $V'(x)$, that

$$\|(e^{-itA/n}e^{-itB/n})^n - e^{-itH}\| = O(n^{-1}), \quad n \to \infty.$$

More generally, the unitary Trotter product formula in operator norm holds, if, in $L^2(\mathbf{R}^d)^n$, A is a selfadjoint $n \times n$-system of differential operators of the first order and B a multiplication operator by a real-valued C^1 or C^2 smooth function $V(x)$ which has bounded derivatives $\partial_x^\alpha V(x)$ for $|\alpha| = 1$ or $|\alpha| = 2$. In particular, it holds for the Dirac operator $A = D_0 \equiv -i\sum_{j=1}^3 \alpha_j \partial_{x_j} + m\beta$, with mass $m \geq 0$, in $L^2(\mathbf{R}^3)^4$, and $B = V(x)$ a real-valued C^1 or C^2 smooth function. Let H be the unique selfadjoint extension of $A + B = D_0 + V(x)$.

Proposition 3.1[*Dirac operator on* $\mathcal{H} = L^2(\mathbf{R}^3)^4$]. (i) *Assume that* $V(x)$ *is a* C^1 *smooth function with bounded derivatives* $\partial_x V(x)$*. Then, as* $n \to \infty$*, one has*

$$\|(e^{-itV/n}e^{-itD_0/n})^n - e^{-itH}\| = O(n^{-1}), \tag{3.1}$$

$$\|(e^{-itD_0/n}e^{-itV/n})^n - e^{-itH}\| = O(n^{-1}). \tag{3.2}$$

(ii) *Assume that* $V(x)$ *is a real-valued* C^2 *smooth function with bounded derivatives* $\partial_x^\alpha V(x)$ *for* $|\alpha| = 2$*. Then, as* $n \to \infty$*, one has*

$$\|(e^{-itV/2n}e^{-itD_0/n}e^{-itV/2n})^n - e^{-itH}\| = O(n^{-2}). \tag{3.3}$$

(iii) *Assume that* $V(x)$ *is a real-valued* C^2 *smooth function with bounded derivatives* $\partial_x^\alpha V(x)$ *for* $1 \leq |\alpha| \leq 2$*. Then, as* $n \to \infty$*, one has*

$$\|(e^{-i(t/n)D_0}e^{-i(t/n)V})^n - e^{-itH} - i(t/2n)[e^{-itH}, D_0]\| = O(n^{-2}), \tag{3.4}$$

$$\|(e^{-i(t/n)V}e^{-i(t/n)D_0})^n - e^{-itH} - i(t/2n)[e^{-itH}, V]\| = O(n^{-2}), \tag{3.5}$$

$$\|\frac{1}{2}((e^{-i(t/n)D_0}e^{-i(t/n)V})^n + (e^{-i(t/n)V}e^{-i(t/n)D_0})^n) - e^{-itH}\| = O(n^{-2}). \tag{3.6}$$

In all formulas (3.1)–(3.6) *above, the convergence is uniform on each bounded* t*-interval in* $\mathbf{R}$*.*

Proof. Take $\mathcal{D} = \{u \in L^2(\mathbf{R}^3)^4;\ \langle x\rangle^L u \in H^1(\mathbf{R}^3)^4 \text{ for every } L \geq 0\}$, where $\langle x\rangle = (1 + x^2)^{1/2}$. Then $(d1)$ is easy to check, and the operators E, E_A and E_B act on $\mathcal{D}$ as the commutators $i[D_0, V]$, $-[D_0, [D_0, V]]$ and $-[V, [D_0, V]]$ as in Remark 1.1. It is clear, because $[D_0, V]$ is essentially the derivatives $\partial_x V(x)$ and the $[D_0, [D_0, V]]$ is essentially the derivatives $\partial_x^\alpha V(x)$ for $|\alpha| = 2$, and $[V, [D_0, V]] = 0$. We can see also that $(d4)$ is satisfied. Therefore Proposition 3.1 follows from Theorems 1.1, 1.2 and 1.3.

Indeed, there is a very peculiar case of the Stark Hamiltonian $H = A + B := -\Delta + x_1$ in $L^2(\mathbf{R}^d)$ with $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ where Theorem 1.2 holds because the double commutators $[A, [A, B]]$ and $[B, [A, B]]$ are bounded in this case. But the unitary Trotter product formula in operator norm will not in general hold for the Schrödinger operator $-\Delta + V(x)$ with nontrivial potential $V(x)$, since it does not satisfy the conditions of the theorems. However, *it does hold* for the relativistic Schrödinger operator $H = A + B := H_0 + V \equiv \sqrt{-\Delta + m^2} + V(x)$ in $L^2(\mathbf{R}^d)$ with some real-valued potentials $V(x)$. It is known [II] that H is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ if the potential $V(x)$ is a function in $L^2_{loc}(\mathbf{R}^d)$ satisfying $V(x) \geq -c_1 e^{c_2|x|}$ with c_1, c_2 being nonnegative constants, in particular, under the assumption of the following proposition. Denote the unique selfadjoint extension of $H_0 + V$ by the same H.

Proposition 3.2 [*massive* (*resp. massless*) *relativistic Schrödinger operator on* $\mathcal{H} = L^2(\mathbf{R}^d)$]. (i) *Assume in case* $m > 0$ (*resp.* $m = 0$) *that* $V(x)$ *is a real-valued* C^2 *smooth function with bounded derivatives* $\partial_x^\alpha V(x)$ *for* $1 \leq |\alpha| \leq 2$ (*resp.* $0 \leq |\alpha| \leq 2$). *Then, as* $n \to \infty$, *one has*

$$\|(e^{-itH_0/n}e^{-itV/n})^n - e^{-itH}\| = O(n^{-1}), \tag{3.7}$$

$$\|(e^{-itV/n}e^{-itH_0/n})^n - e^{-itH}\| = O(n^{-1}). \tag{3.8}$$

(ii) *Assume in case* $m > 0\,(resp.\, m = 0)$ *and that* $V(x)$ *is a real-valued* C^4 *smooth function with bounded derivatives* $\partial_x^\alpha V(x)$ *for* $1 \leq |\alpha| \leq 4$ *(resp.* $0 \leq |\alpha| \leq 4$*). Then, as* $n \to \infty$, *one has*

$$\|(e^{-itV/2n}e^{-itH_0/n}e^{-itV/2n})^n - e^{-itH}\| = O(n^{-2}), \tag{3.9}$$

$$\|(e^{-i(t/n)H_0}e^{-i(t/n)V})^n - e^{-itH} - i(t/2n)[e^{-itH}, H_0]\| = O(n^{-2}), \tag{3.10}$$

$$\|(e^{-i(t/n)V}e^{-i(t/n)H_0})^n - e^{-itH} - i(t/2n)[e^{-itH}, V]\| = O(n^{-2}), \tag{3.11}$$

$$\|\frac{1}{2}((e^{-i(t/n)H_0}e^{-i(t/n)V})^n + (e^{-i(t/n)V}e^{-i(t/n)HH_0})^n) - e^{-itH}\| = O(n^{-2}). \tag{3.12}$$

In all formulas (3.7)–(3.12) *above, the convergence is uniform on each bounded t-interval in* $\mathbf{R}$.

In Proposition 3.2 we have made the assumption on the potential $V(x)$ as minimal as possible. For the proof of this proposition, we will not use $H_0 = \sqrt{-\Delta + m^2}$ as the pseudodifferential operator, since its symbol $\sqrt{\xi^2 + m^2}$ is not smooth in the massless case $(m = 0)$, so that standard pseudodifferential calculus does not allow us to treat the issue well even if $V(x)$ is C^∞ smooth. Instead we use the following expression as an integral operator:

$$\begin{aligned}(H_0 u)(x) : &= mu(x) - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}}\, y \cdot \partial_x u(x)] n(y) dy \\ &= mu(x) - \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} [u(x+y) - u(x)] n(y) dy \end{aligned} \tag{3.13}$$

for $u \in \mathcal{S}(\mathbf{R}^d)$ (e.g. [I1, Eq. (2.7)], cf. [LL, Chap. 7]). Here $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y| < 1\}$, i.e. $I_{\{|y|<1\}}(z) = 1$, if $|z| < 1$, and $= 0$, if $|z| \geq 1$. The integral kernel $n(y)$ is a positive function dependent on $m \geq 0$ and given by

$$\begin{aligned} n(y) &= 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} |y|^{-(d+1)/2} K_{(d+1)/2}(m|y|), \quad m > 0, \\ n(y) &= \pi^{-(d+1)} \Gamma\Big(\frac{d+1}{2}\Big) |y|^{-(d+1)}, \quad m = 0, \end{aligned} \tag{3.14}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind of order ν and $\Gamma(z)$ the gamma function. It holds that for $m \geq 0$,

$$n_{0,\kappa} := \int_{0<|y|<1} |y|^\kappa n(y) dy < \infty, \quad 1 < \kappa \leq 2, \tag{3.15}$$

and for $m > 0$ and $m = 0$, respectively,

$$\begin{aligned} n_{+;\infty,\kappa} &:= \int_{|y|\geq 1} |y|^\kappa n(y) dy < \infty, \quad \kappa \geq 0, \quad m > 0, \\ n_{0;\infty,0} &:= \int_{|y|\geq 1} n(y) dy < \infty, \quad m = 0. \end{aligned} \tag{3.16}$$

Further, for $1 \leq j \leq d$, $y_j n(y)$ is a Calderón–Zygmund kernel. So, if we define a linear operator K_j by the singular integral

$$(K_j g)(x) = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq |y| < 1} g(x + y) y_j n(y) dy, \tag{3.17}$$

for $g \in \mathcal{S}(\mathbf{R}^d)$, then by the Calderón–Zygmund theorem (e.g. [S, Chap. II, Sect. 3, Theorem 2, p. 35]), the limit of the integral on the right of (3.17) exists, and K_j becomes a bounded operator on $L^2(\mathbf{R}^d)$, i.e.

$$\|K_j g\| \leq C\|g\|, \quad g \in \mathcal{S}(\mathbf{R}^d), \quad 1 \leq j \leq d, \tag{3.18}$$

with a constant $C > 0$, where $\|\cdot\|$ on both sides is the L^2 norm. When $d = 1$, we write simply K for K_1.

This H_0 becomes a selfadjoint operator in $L^2(\mathbf{R}^d)$ with domain $H^1(\mathbf{R}^d)$.

Proof of Proposition 3.2. We take $\mathcal{D} = \{u \in L^2(\mathbf{R}^d);\ \langle x\rangle^2 u \in H^2(\mathbf{R}^d)\}$ for massive case, and $\mathcal{D} = H^2(\mathbf{R}^d)$ for massless case. Then $(d1)$ is easy to check, and the operators E, E_A and E_B act on $\mathcal{D}$ as the commutators as in Remark 1.1. We can prove as in Lemma 3.1 below that $E = i[H_0, V]$, $E_A = -[V, [H_0, V]]$ and $E_B = -[H_0, [H_0, V]]$ are bounded under the assumption in the proposition. It can also be seen that $(d4)$ is satisfied. Thus Proposition 3.2 will be concluded from Theorems 1.1, 1.2 and 1.3.

Before going to the lemma, let us notice the difference of assumption on the potential $V(x)$ in Proposition 3.2 between the massive and massless cases. In fact, in the former case the function $V(x)$ itself is assumed to be bounded, but not in the latter (For this see also the bounds obtained for the three commutators following Lemma 3.1 below). To this end, note that in the proof the dense subspace $\mathcal{D}$ in the massive case is taken smaller than in the massless case.

Lemma 3.1. *In both massive and massless cases, if $V(x)$ satisfies the same assumption as in Proposition 3.2, then*

$$[H_0, V],\ [V, [H_0, V]],\ [H_0, [H_0, V]] : \mathcal{S}(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$$

admit bounded extensions on $L^2(\mathbf{R}^d)$.

Proof. For simplicity, we show only in the case of dimension $d = 1$. The proof in the case of dimension d is essentially the same. In fact, we shall show these three operators applied to a function u in $\mathcal{S}(\mathbf{R})$ exist with operator norms obeying the following bounds.

$$\begin{array}{llr}
\text{(i)} & \|[H_0, V]\| \le C_1\|V'\|_\infty + C_2\|V''\|_\infty\,, & m > 0, \quad (3.19)\\
 & \|[H_0, V]\| \le C_0\|V\|_\infty + C_1\|V'\|_\infty + C_2\|V''\|_\infty\,, & m = 0; \quad (3.20)\\
\text{(ii)} & \|[V, [H_0, V]]\| \le C_1\|V'\|_\infty^2\,, & m > 0, \quad (3.21)\\
 & \|[V, [H_0, V]]\| \le C_0\|V\|_\infty^2 + C_1\|V'\|_\infty^2\,, & m = 0; \quad (3.22)\\
\text{(iii)} & \|[H_0, [H_0, V]]\| \le C_2\|V''\|_\infty + C_3\|V'''\|_\infty + C_4\|V^{(4)}\|_\infty\,, & m > 0, \quad (3.23)\\
 & \|[H_0, [H_0, V]]\| \le \sum_{j=0}^{4} C_j\|V^{(j)}\|_\infty\,, & m = 0. \quad (3.24)
\end{array}$$

Here $C_0, C_1, C_2, \ldots$, denote constants dependent on those constants $n_{0,\kappa}$ in (3.15), $n_{+;\infty,\kappa}$, $n_{0;\infty,0}$ in (3.16), and C in (3.18). The C_j with the same j may be different.

Proof of (i). Let $u \in \mathcal{S}(\mathbf{R})$, whose L^2 norm is denoted by $\|u\|$. We have

$$\begin{aligned}
&([H_0, V]u)(x)\\
= \;& -\int_{|y|>0} [(V(x+y) - V(x))u(x+y) - I_{\{|y|<1\}}yV'(x)u(x)]n(y)dy\\
= \;& -\int_{0<|y|<1} yV'(x)(u(x+y) - u(x))n(y)dy\\
& -\int_{0<|y|<1} (V(x+y) - V(x) - yV'(x))u(x+y)n(y)dy\\
& -\int_{|y|\ge 1} (V(x+y) - V(x))u(x+y)n(y)dy\\
=: \;& u_1(x) + u_2(x) + u_3(x). \qquad (3.25)
\end{aligned}$$

For the function u_1 in (3.25), we use (3.17) to rewrite $u_1(x) = -V'(x)(Ku)(x)$. Then its L^2 norm obeys by (3.18)

$$\|u_1\| \le \|V'\|_\infty \|Ku\| \le C\|V'\|_\infty \|u\|.$$

For u_2, we have

$$\begin{aligned}
\|u_2\| &\le \frac{1}{2}\|V''\|_\infty \Big[\int \Big(\int_{0<|y|<1} |u(x+y)||y|^2 n(y)dy\Big)^2 dx\Big]^{1/2}\\
&\le \frac{1}{2}\|V''\|_\infty \int_{0<|y|<1} |y|^2 n(y)dy\, \|u\| \le \frac{1}{2} n_{0,2}\|V''\|_\infty \|u\|.
\end{aligned}$$

For u_3, in case $m > 0$, we have by the Schwarz inequality

$$\begin{aligned}\|u_3\| &\le \|V'\|_\infty\Big[\int\Big(\int_{|y|\ge 1}|u(x+y)||y|n(y)dy\Big)^2 dx\Big]^{1/2}\\ &\le \|V'\|_\infty\int_{|y|\ge 1}|y|n(y)dy\,\|u\| \le n_{+;\infty,1}\|V'\|_\infty\|u\|,\end{aligned}$$

and in case $m = 0$,

$$\begin{aligned}\|u_3\| &\le 2\|V\|_\infty\Big[\int\Big(\int_{|y|\ge 1}|u(x+y)|n(y)dy\Big)^2 dx\Big]^{1/2}\\ &\le 2\|V\|_\infty\int_{|y|\ge 1}n(y)dy\,\|u\| \le 2n_{0;\infty,0}\|V\|_\infty\|u\|.\end{aligned}$$

Thus putting all the estimates $\|u_j\|$, $j = 1, 2, 3$, above together, we have shown (i).

Proof of (ii). We can show

$$([V,[H_0,V]]u)(x) = \int_{|y|>0}(V(x+y)-V(x))^2u(x+y)n(y)dy. \tag{3.26}$$

Then, in case $m > 0$, we have by the Schwarz inequality

$$\begin{aligned}\|[V,[H_0,V]]u\| &\le \|V'\|_\infty^2\Big[\int\Big(\int_{|y|>0}|u(x+y)||y|^2n(y)dy\Big)^2 dx\Big]^{1/2}\\ &\le \|V'\|_\infty^2\int_{|y|>0}|y|^2n(y)dy\,\|u\| \le (n_{0,2}+n_{+;\infty,2})\|V'\|_\infty^2\|u\|.\end{aligned}$$

Similarly, in case $m = 0$,

$$\begin{aligned}\|[V,[H_0,V]]u\| &\le \|V'\|_\infty^2\int_{0<|y|<1}|y|^2n(y)dy\,\|u\| + 4\|V\|_\infty^2\int_{|y|\ge 1}n(y)dy\|u\|\\ &\le [n_{0,2}\|V'\|_\infty^2 + 4n_{0;\infty,0}\|V\|_\infty^2]\|u\|.\end{aligned}$$

This shows (ii).

Proof of (iii). Below we sometimes write A for H_0. We begin with showing $E_A := -[H_0,[H_0,V]]$ has the following expression: for $u \in \mathcal{S}(\mathbf{R})$,

$$\begin{aligned}(E_Au)(x) &= -\lim_{\varepsilon\downarrow 0}\int_{|y|\ge\varepsilon}\int_{|z|\ge\varepsilon}(V(x+y+z)-2V(x+y)+V(x))\\ &\qquad\qquad\times u(x+y+z)n(y)n(z)dydz.\end{aligned} \tag{3.27}$$

For the proof, we note first by (3.25) that the symmetric operator $E = i[H_0,V]$ acts as $(Eu)(x) = \lim_{\varepsilon\to 0}(E_\varepsilon u)(x)$ for $u \in \mathcal{S}(\mathbf{R})$, where

$$(E_\varepsilon u)(x) = -i\int_{|y|\ge\varepsilon}(V(x+y)-V(x))u(x+y)n(y)dy.$$

By definition, $(E_A u, v) = i((Eu, Av) - (Au, Ev))$ for u, $v \in \mathcal{S}(\mathbf{R})$. If we write the second equality in (3.13) as

$$(Au)(x) = (H_0 u)(x) = mu(x) - \lim_{\varepsilon \to 0} (A_\varepsilon u)(x),$$

then it follows that

$$(E_A u, v) = \lim_{\varepsilon \to 0} (p_\varepsilon + q_\varepsilon).$$

where $p_\varepsilon = -i(E_\varepsilon u, A_\varepsilon v)$ and $q_\varepsilon = i(A_\varepsilon u, E_\varepsilon v)$. We can calculate the two scalar products p_ε and q_ε as

$$\begin{aligned} p_\varepsilon &= -\int\int\int_{D_\varepsilon} (V(x+y) - V(x))u(x+y)(\bar{v}(x+z) - \bar{v}(x))n(y)n(z)dydzdx, \\ q_\varepsilon &= -\int\int\int_{D_\varepsilon} (V(x+z) - V(x))(u(x+y) - u(x))\bar{v}(x+z)n(y)n(z)dydzdx, \end{aligned}$$

where $D_\varepsilon = \{x \in \mathbf{R}, |y| \geq \varepsilon, |z| \geq \varepsilon\}$. The sum $p_\varepsilon + q_\varepsilon$ admits the decomposition $r_\varepsilon + r_{1\varepsilon} + r_{2\varepsilon}$, where

$$r_\varepsilon = -\int\int\int_{D_\varepsilon} (V(x+y) - 2V(x) + V(x+z))u(x+y)\bar{v}(x+z)n(y)n(z)dydzdx,$$

and

$$\begin{aligned} r_{1\varepsilon} &= \int\int\int_{D_\varepsilon} (V(x+y) - V(x))u(x+y)\bar{v}(x)n(y)n(z)dydzdx, \\ r_{2\varepsilon} &= \int\int\int_{D_\varepsilon} (V(x+z) - V(x))u(x)\bar{v}(x+z)n(y)n(z)dydzdx. \end{aligned}$$

We consider the integral r_ε, and make a change of variables $(x+z, z) \to (x, z)$. It is easy to see that the integration domain D_ε is invariant under this transformation, and we get

$$r_\varepsilon = -\int\int\int_{D_\varepsilon} (V(x-z+y) - 2V(x-z) + V(x))u(x-z+y)\bar{v}(x)n(y)n(z)dydzdx,$$

We further change $-z \to z$ to obtain that

$$r_\varepsilon = -\int\int\int_{D_\varepsilon} (V(x+y+z) - 2V(x+z) + V(x))u(x+y+z)\bar{v}(x)n(y)n(z)dydzdx.$$

Similarly we have

$$r_{2\varepsilon} = \int\int\int_{D_\varepsilon} (V(x) - V(x-z))u(x-z)\bar{v}(x)n(y)n(z)dydzdx = -r_{1\varepsilon},$$

so that $(E_A u, v) = \lim_{\varepsilon \to 0} r_\varepsilon$. Thus we have shown that E_A acts on $\mathcal{S}(\mathbf{R})$ as in (3.27).

Thus we have from (3.27)

$$\begin{aligned} & ([H_0, [H_0, V]]u)(x) \\ = \; & \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \int_{|z| \geq \varepsilon} (V(x+y+z) - V(x+y) - V(x+z) + V(x)) \\ & \qquad \qquad \times u(x+y+z)n(y)n(z)dydz \\ = \; & \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq |y| < 1} \int_{\varepsilon \leq |z| < 1} + \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq |y| < 1} \int_{|z| \geq 1} + \lim_{\varepsilon \downarrow 0} \int_{|y| \geq 1} \int_{\varepsilon \leq |z| < 1} + \int_{|y| \geq 1} \int_{|z| \geq 1} \\ =: \; & v_{00}(x) + v_{0\infty}(x) + v_{\infty 0}(x) + v_{\infty\infty}(x). \end{aligned} \tag{3.28}$$

By the mean value theorem we have

$$\begin{aligned}
& V(x+y+z)-V(x+y)-V(x+z)+V(x)\\
= \;& z(V'(x+y)-V'(x))+z^2\int_0^1(1-\theta)[V''(x+y+\theta z)-V''(x+\theta z)]\,d\theta\\
= \;& z(V'(x+y)-V'(x)-yV''(x))+yzV''(x)\\
& +z^2\int_0^1(1-\theta)[V''(x+y+\theta z)-V''(x+\theta z)-yV'''(x+\theta z)]\,d\theta\\
& +z^2y\int_0^1(1-\theta)V'''(x+\theta z)\,d\theta, \qquad (3.29)
\end{aligned}$$

and the same is valid with y and z exchanged.

For the function v_{00} in (3.28) we have by (3.29)

$$\begin{aligned}
& v_{00}(x)\\
= \;& \lim_{\varepsilon\downarrow 0}\Big[\int_{\varepsilon\le|y|<1}\int_{\varepsilon\le|z|<1} z[V'(x+y)-V'(x)-yV''(x)]u(x+y+z)n(y)n(z)dydz\\
& +V''(x)\int_{\varepsilon\le|y|<1}\Big(\int_{\varepsilon\le|z|<1} u(x+y+z)yn(y)dy\Big)zn(z)dz\\
& +\int_{\varepsilon\le|z|<1} z^2n(z)dz\int_{\varepsilon\le|y|<1}\Big(\int_0^1(1-\theta)\\
& \quad\times[V''(x+y+\theta z)-V''(x+\theta z)-yV'''(x+\theta z)]d\theta\Big)u(x+y+z)n(y)dy\\
& +\int_{\varepsilon\le|z|<1} z^2n(z)dz\Big(\int_0^1(1-\theta)(V'''(x+\theta z)d\theta\Big)\int_{\varepsilon\le|y|<1} u(x+y+z)yn(y)dy\Big].
\end{aligned}$$

Therefore by (3.17) and (3.18), its L^2 norm obeys

$$\begin{aligned}
\|v_{00}\| \;\le\;& \frac{1}{2}\|V'''\|_\infty\int_{0<|y|<1}|y|^2n(y)dy\|Ku\|+\|V''\|_\infty\|K^2u\|\\
&+\;\frac{1}{4}\|V^{(4)}\|_\infty\Big[\int\Big(\int_{0<|y|<1}\int_{0<|z|<1}|u(x+y+z)||y|^2n(y)|z|^2n(z)dydz\Big)^2dx\Big]^{1/2}\\
&+\;\frac{1}{2}\|V'''\|_\infty\Big[\int\Big(\int_{0<|z|<1}z^2n(z)dz(Ku)(x+z)\Big)^2dx\Big]^{1/2}\\
\le\;& \Big[C^2\|V''\|_\infty+Cn_{0,2}\|V'''\|_\infty+\frac{1}{4}n_{0,2}^2\|V^{(4)}\|_\infty\Big]\|u\|.
\end{aligned}$$

For $v_{0\infty}$ and $v_{\infty 0}$, we have by (3.29) and (3.17)

$$\begin{aligned}
v_{0\infty} \;=\;& \int_{|z|\ge 1}(V'(x+z)-V'(x))(Ku)(x+z)n(z)dz\\
&+\int_{|z|\ge 1}n(z)dz\int_{0<|y|<1}\int_0^1(1-\theta)[V''(x+z+\theta y)-V''(x+\theta y)]d\theta\\
&\qquad\times u(x+y+z)y^2n(y)dy,\\
v_{\infty 0} \;=\;& \int_{|y|\ge 1}(V'(x+y)-V'(x))(Ku)(x+y)n(y)dy\\
&+\int_{|y|\ge 1}n(y)dy\int_{0<|z|<1}\int_0^1(1-\theta)[V''(x+y+\theta z)-V''(x+\theta z)]d\theta\\
&\qquad\times u(x+y+z)z^2n(z)dz.
\end{aligned}$$

Therefore, in case $m > 0$, by (3.18),

$$\begin{aligned}
\|v_{0\infty}\| &\le \|V''\|_\infty \int_{|z|\ge 1} |z|n(z)dz\|Ku\| \\
&\quad +\|V'''\|_\infty \Big[\int\Big(\int_{|z|\ge 1} |z|n(z)dz \int_{0<|y|<1} |u(x+y+z)|y^2n(y)dy\Big)^2 dx\Big]^{1/2} \\
&\le [Cn_{+;\infty,1}\|V''\|_\infty + n_{+;\infty,1}n_{0,2}\|V'''\|_\infty]\|u\|.
\end{aligned}$$

In case $m = 0$,

$$\begin{aligned}
\|v_{0\infty}\| &\le 2\|V'\|_\infty \int_{|z|\ge 1} n(z)dz\|Ku\| \\
&\quad +2\|V''\|_\infty \Big[\int\Big(\int_{|z|\ge 1} n(z)dz \int_{0<|y|<1} |u(x+y+z)|y^2n(y)dy\Big)^2 dx\Big]^{1/2} \\
&\le [2Cn_{0;\infty,0}\|V'\|_\infty + 2n_{0;\infty,0}n_{0,2}\|V''\|_\infty]\|u\|.
\end{aligned}$$

The same is valid for $v_{\infty 0}$.
For $v_{\infty\infty}$, in case $m > 0$,

$$\begin{aligned}
\|v_{\infty\infty}\| &\le \|V''\|_\infty \Big[\int\Big(\int_{|y|\ge 1}\int_{|z|\ge 1} |u(x+y+z)||y|n(y)|z|n(z)dydz\Big)^2 dx\Big]^{1/2} \\
&\quad +\frac{1}{2}\|V'''\|_\infty \Big[\int\Big(\int_{|y|\ge 1}\int_{|z|\ge 1} |u(x+y+z)||y|n(y)|z|^2n(z)dydz\Big)^2 dx\Big]^{1/2} \\
&\le [n_{+;\infty,1}^2\|V''\|_\infty + \frac{1}{2}n_{+;\infty,1}n_{+;\infty,2}\|V'''\|_\infty]\|u\|.
\end{aligned}$$

In case $m = 0$,

$$\begin{aligned}
\|v_{\infty\infty}\| &\le 4\|V\|_\infty \Big[\int\Big(\int_{|y|\ge 1}\int_{|z|\ge 1} |u(x+y+z)|n(y)n(z)dydz\Big)^2 dx\Big]^{1/2} \\
&\le 4n_{0;\infty,0}^2\|V\|_\infty\|u\|.
\end{aligned}$$

Thus putting all the estimates $\|v_{00}\|$, $\|v_{0\infty}\|$, $\|v_{\infty 0}\|$ and $\|v_{\infty\infty}\|$ above together, we have shown (iii). Therefore we have proved Lemma 3.1, completing the proof of Proposition 3.2.

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