SPECTRAL CHARACTERISATION OF AGEING: THE REM-LIKE TRAP MODEL

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ABSTRACT. We review the ageing phenomenon in the context of simplest trap model, Bouchaud's REM-like trap model from a spectral theoretic point of view. We show that the generator of the dynamics of this model can be diagonalised exactly. Using this result, we derive closed expressions for correlation functions in terms of complex contour integrals that permit an easy investigation into their large time asymptotics in the thermodynamic limit. We also give a 'grand canonical' representation of the model in terms of the Markov process on a Poisson point process . In this context we analyse the dynamics on various time scales.

Key words: disordered systems, random dynamics, trap models, ageing, spectral properties.

1. INTRODUCTION

The particular properties of the long term dynamics of many complex and/or disordered systems have been the subject of great interest in the physics, and, increasingly, the mathematics community. The key paradigm here is the notion of *ageing*, a notion that can be characterised in terms of scaling properties of suitable autocorrelation functions. Typically, ageing can be associated to the existence of *infinitely many* time-scales that are inherently relevant to the system. In that respect ageing systems are distinct from *metastable* systems which are characterised by a finite number of well separated time scales corresponding to the live times of different metastable states.

Ageing systems are rather difficult to analyse, both numerically and analytically. Most analytical results, even on the heuristic level, concern either the Langevin dynamics of spherical mean field spin glasses, or *trap models*, a class of artificial Markov processes that in some way is to mimic the long term dynamics of highly disordered systems (see e.g. [8]).

One of the natural questions one is led to ask when being confronted with phenomena related to multiple time scales is whether and how they can be related to *spectral properties*. This relationship has been widely investigated in the context of Markov processes with metastable behaviour (see e.g. [13, 14, 15, 21, 22, 11]), and it would be rather interesting to obtain a spectral characterisation of ageing systems as well, at least in the context of Markov processes. To our knowledge, this problem has not been widely studied so far. The only paper dealing with the problem is the paper [25] by Butaud and Melin that have tackled one of the simplest trap models and on which we will comment below, and [18] and [24] that investigate convergence to equilibrium in the Random Energy Model (REM).

The present paper is intended to make a modest first step into this direction by analysing the relation between spectral properties and ageing rigorously in the REM-like trap model.

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While this model may seem misleadingly simple, it has in the past provided valuable insights into the mechanisms of ageing, and it is our hope that the analysis presented here will provide useful guidelines for further investigations of more complicated models.

The paper will be divided into two parts. In the first we analyse the REM-like trap model in the standard formulation of Bouchaud [9]. In the second part we go a step further and reformulate the model in a slightly different way as a Markov process on a Poisson point process. This formulation makes the relation to the real REM more suggestive (see [3, 4] for a full analysis), and allows in a natural way to study the dynamics of the model on different time scales.

2. The REM–like trap model

Let us begin to recall the definition of *trap models* as introduced by Bouchaud and Dean[9]. Let $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ be a finite graph: $\mathcal{S}$ denotes the set of vertices and $\mathcal{E}$ denotes the set of edges, $\underline{E} := \{E_i,\ i \in \mathcal{S}\}$ be a random field, called *energy landscape* and let $Y(t)$ be a continuous–time random walk on $\mathcal{G}$ with $\underline{E}$–depending transition rates $c_{i,j}$: i.e. $c_{i,j} > 0$ iff $\{i,j\} \in \mathcal{E}$ and

$$\mathbb{P}(Y(t+dt) = j \mid Y(t) = i) = c_{i,j}dt.$$

Setting $\tau_i := \sum_{j \neq i} c_{i,j}$ and $p_{i,j} := c_{i,j}/\tau_i$, the random walk $Y(t)$ can be described as follows: after reaching the site i the system waits an exponential time with expectation τ_i and then it jumps to an adjacent site j with probability $p_{i,j}$. In the trap model, the transition rates are assumed to satisfy the following properties:

$$e^{E_i} c_{i,j} = e^{E_j} c_{j,i}, \quad \forall \{i,j\} \in \mathcal{E}, \tag{2.1}$$

$$\mathbb{E}(\tau_i) = \infty \tag{2.2}$$

where $\mathbb{E}$ denotes the expectation w.r.t. the random field $\underline{E}$. Since in several physical experiments (see [10]) the system initially in equilibrium at high temperature $T \gg T_g$ is quickly cooled under the transition temperature T_g and then its response to an external perturbation is measured, it is reasonable to consider $Y(t)$ with uniform initial distribution. A classical time–time correlation function is given by

$$\Pi(t, t_w) := \mathbb{P}(Y(s) = Y(t_w), \quad \forall s \in [t_w, t_w + t])$$

In order to observe ageing it is necessary to consider a thermodynamic limit, with the size of $\mathcal{G}$ going to infinity, and possible a suitable time–rescaling. Rather recently, there have been a number of rigorous papers devoted to the analysis of trap models on the lattices $\mathbb{Z}$ [19, 20, 5] and $\mathbb{Z}^d$ [6, 12].

In this paper we consider the simples trap model, called the *REM-like trap model* [9] that corresponds to choosing $\mathcal{G}$ to be the complete graph on N vertices i.e.

$$\mathcal{G}_N = (\mathcal{S}_N, \mathcal{E}_N), \quad \mathcal{S}_N := \{1, 2, \dots, N\}, \quad \mathcal{E}_N := \{\,\{i,j\}\, i \neq j \in \mathcal{S}_N\},$$

and to take as energy landscape a family $\underline{E} = \{E_i\ :\ i \in \mathbb{N}\}$ of independent exponential random variables with parameter α such that $0 < \alpha < 1$. Given $N \in \mathbb{N}$, let $Y_N(t)$ be the continuous–time random walk on $\mathcal{G}_N$ with transition rates $c_{i,j} = e^{-E_i}/N$ for $i \neq j$. Setting $x_i = e^{-E_i}$ the infinitesimal generator of the random walk is given by

$$\mathbb{L}_N := \begin{pmatrix} \frac{(N-1)x_1}{N} & -\frac{x_1}{N} & \dots & -\frac{x_1}{N} \\ -\frac{x_2}{N} & \frac{(N-1)x_2}{N} & \dots & -\frac{x_2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_N}{N} & -\frac{x_N}{N} & \dots & \frac{(N-1)x_N}{N} \end{pmatrix} \tag{2.3}$$

The dynamics can be described as follows: after reaching the state i the system waits an exponential time of mean $\frac{N}{N-1}e^{E_i}$ and then jumps with uniform probability to another state. Although strictly speaking the mean waiting time is given by $\frac{N}{N-1}e^{E_i}$, we call $\tau_i := x_i^{-1} = e^{E_i}$ waiting time (the discrepancy is negligible in the thermodynamic limit $N \uparrow \infty$).
Note that τ_i and x_i have distributions respectively given by

$$p(\tau)d\tau = \alpha\tau^{-1-\alpha}d\tau \quad (\tau \geq 1); \qquad p(x)dx = \alpha x^{\alpha-1}dx \quad (0 < x \leq 1),$$

in particular $\mathbb{E}(\tau_i) = \infty$. Moreover, the equilibrium measure is given by $\mu_{eq}(i) = \tau_i/\big(\sum_{j=1}^N \tau_j\big)$. We are interested in the out–of–equilibrium dynamic with uniform initial distribution. $\mathbb{P}_N$ denotes the law of this random walk given a realization of the random variables E_i.

Ageing in the REM-like trap model is manifest from the asymptotic behaviour of the time–time correlation function

$$\Pi_N(t, t_w) := \mathbb{P}_N\big(Y_N(s) = Y_N(t_w),\ \forall s \in [t_w, t_w + t]\big) \tag{2.4}$$

Namely, as shown in [9], for almost all $\underline{E}$ and for all $\theta > 0$

$$\lim_{t_w\uparrow\infty}\lim_{N\uparrow\infty} \Pi_N(\theta t_w, t_w) = \frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{1+\theta}}^1 u^{-\alpha}(1-u)^{\alpha-1}du \tag{2.5}$$

Our main aim here is to show that the ageing behaviour of the system, derived in [9] using renewal arguments, can be obtained solely from *spectral information* about the generator $\mathbb{L}_N$. The method developed below will allow us to get further information on $Y_N(t)$ from the spectral properties of $\mathbb{L}_N$. In particular, given a function h on $(0,\infty)$, it is possible to describe the asymptotic behaviour of $\mathbb{E}_N(h(x_N(t))$ and $\mathbb{E}_N(h(\tau_N(t))$, where $\mathbb{E}_N$ denotes the expectation w.r.t. $\mathbb{P}_N$ and $x_N(t), \tau_N(t)$ are defined as

$$x_N(t) = x_k,\quad \tau_N(t) = \tau_k \quad \text{if} \quad Y_N(t) = k.$$

These results will allow us to investigate the property that the system with high probability visits deeper and deeper traps. i.e. sites with larger and larger waiting time τ_i (see subsection 2.2).

We start by giving a complete description of the eigenvalues and eigenvectors of $\mathbb{L}_N$. Let $\mu = \mu_N$ be the measure on $\mathcal{S}_N$ with $\mu(i) = x_i^{-1} = \tau_i$. Note that $\mathbb{L}_N$ is a symmetric operator on $L^2(\mu)$ and trivially $\mathbb{L}_N\mathbb{I} = 0$ where $\mathbb{I}$ is the vector with all entries equal to 1. The following proposition is based on elementary linear algebra:

Proposition 2.1. *Let $x_1, x_2, \ldots, x_N$ be all distinct. Then, $\mathbb{L}_N$ has N positive simple eigenvalues $0 = \lambda_1 < \lambda_2 \cdots < \lambda_N$ such that*

$$\{\lambda_1, \lambda_2, \ldots, \lambda_N\} = \{\lambda \in \mathbb{C} : \phi(\lambda) = 0\}$$

where $\phi(\lambda)$ is the meromorphic function

$$\phi(\lambda) := \sum_{j=1}^N \frac{\lambda}{x_j - \lambda}, \qquad (\lambda \in \mathbb{C}). \tag{2.6}$$

If the x_i are labelled such that $x_1 < x_2 < \cdots < x_N$, then $x_i < \lambda_{i+1} < x_{i+1}$ for $i = 2, \ldots, N$. Moreover, for any $i = 1, \ldots, N$, the vector $\psi^{(i)} \in \mathbb{R}^N$ defined as

$$\psi_j^{(i)} := \frac{x_j}{x_j - \lambda_i}, \qquad \textit{for } j = 1, \ldots, N$$

is an eigenvector of $\mathbb{L}_N$ *with eigenvalue* λ_i. $\psi^{(1)},\dots,\psi^{(N)}$ *form an orthogonal basis of* $L^2(\mu)$.

Since the x_i have a absolutely continuous distribution, we trivially have the

Corollary 2.2. *The assertions of Proposition 2.1 hold with probability one for all* N.

Proof. Let λ be a generic eigenvalue and let us write a related eigenvector ψ as $\psi = a(1,\dots,1)^t + w$ where $\sum_{j=1}^N w_j = 0$. Since $(\mathbb{L}_N\psi)_j = x_j w_j$, we have to solve the system

$$x_j w_j = \lambda a + \lambda w_j, \qquad \forall j = 1,\dots,N. \tag{2.7}$$

Since $x_1,\dots,x_N$ are distinct, it must be true that $a \neq 0$ (otherwise we get $\psi = 0$). Without loss of generality, we set $a = 1$. Note that $\lambda \neq x_j$ for $j = 1\dots N$, as otherwise (2.7) would imply $\lambda = 0 = x_j$. Therefore we get $w_j = \frac{\lambda}{x_j-\lambda}$. Since it must be true that $\sum_{j=1}^N w_j = 0$, we get that λ is an eigenvalue and ψ, with $\psi_j = \frac{x_j}{x_j-\lambda}$, the corresponding eigenvector, iff $\phi(\lambda) = 0$. This imply that ϕ has at most N zeros. Since $\phi(0) = 0$, and, for real λ, $\lim_{\lambda\downarrow x_i}\phi(\lambda) = -\infty$, $\lim_{\lambda\uparrow x_i}\phi(\lambda) = \infty$, we get that ϕ has exactly N zeros. The conclusion of the proof is trivial. □

Proposition 2.1 has the following simple corollary.

Corollary 2.3. *With probability one, the spectral distribution* $\sigma_N := \mathrm{Av}_{j=1}^N \delta_{\lambda_j}$ *converges weakly to the measure* $\alpha x^{\alpha-1}dx$ *on* $[0,1]$.

Remark. The results of Proposition 2.1 are incompatible with the predictions of [25]. The discrepancy is particularly pronounced in the case of the eigenfunction. The reason for this is an inappropriate use of perturbation expansion in [25]. We will explain this in some detail in an appendix.

We will now show that Proposition 2.1 allows to derive the asymptotics of the autocorrelation functions easily. In fact, it contains far more information on the long time behaviour of the systems some of which we will bring to light later.

Recall that $p_t(i,j)$, the probability to jump from i to j in an interval of time t, can be expressed as $p_t(i,j) = \left(e^{-t\mathbb{L}_N}\right)_{i,j}$. In particular, by writing ν_t for the probability distribution of $Y_N(t)$ and thinking of the Radon derivative $\frac{d\nu_t}{d\mu}$ as column vector,

$$\frac{d\nu_t}{d\mu} = e^{-t\mathbb{L}_N}\frac{d\nu_0}{d\mu},$$

thus implying

$$\frac{d\nu_t}{d\mu} = \sum_{k=1}^N \frac{< \frac{d\nu_0}{d\mu}, \psi^{(k)} >}{< \psi^{(k)}, \psi^{(k)} >} e^{-t\lambda_k}\psi^{(k)} \tag{2.8}$$

The above formulas are true for an arbitrary initial distribution. Taking ν_0 the uniform distribution, by Proposition 2.1 we get

$$\frac{d\nu_o}{d\mu} = \sum_{k=1}^N \gamma_k \psi^{(k)}, \text{ where } \gamma_k^{-1} :=< \psi^{(k)}, \psi^{(k)} >= \sum_{j=1}^N \frac{x_j}{(x_j-\lambda_k)^2}.$$

Then, by Proposition 2.1 and (2.8),

$$\Pi_N(t,t_w) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{\gamma_k e^{-\lambda_k t_w}}{x_j-\lambda_k} e^{-\frac{N-1}{N}x_j t} \tag{2.9}$$

$$\mathbb{E}_N\left(h(x_N(t))\right) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{\gamma_k e^{-\lambda_k t}}{x_j-\lambda_k} h(x_j) \tag{2.10}$$

The above formulas (that may appear rather ugly at first sight) admit a nice complex integral representation through the following lemma.

Lemma 2.4. *Let γ be a positive oriented loop on $\mathbb{C}$ containing in its interior $\lambda_1,\ldots,\lambda_N$. Let g be an holomorphic function on a domain $D\subset\mathbb{C}$ with $\gamma\subset D$. Then, for any $j=1\ldots,N$,*

$$\sum_{k=1}^{N}\frac{\gamma_k g(\lambda_k)}{x_j-\lambda_k} = \frac{1}{2\pi i}\int_\gamma \frac{g(\lambda)}{\phi(\lambda)(x_j-\lambda)}d\lambda. \tag{2.11}$$

Proof. Let us set $X := \{x_1,\ldots,x_N\}$ and $\Lambda := \{\lambda_1,\lambda_2,\ldots,\lambda_N\}$. Then $\phi(\lambda)$ is an holomorphic function on $\mathbb{C}\setminus X$, where $\phi'(\lambda)=\sum_{j=1}^{N}\frac{x_j}{(x_j-\lambda)^2}$, and in particular $\phi'(\lambda_j)=\gamma_j^{-1}$. Moreover, the function $[\phi(\lambda)(x_j-\lambda)]^{-1}$ that is a priori defined on $\mathbb{C}\setminus(X\cup\Lambda)$ can be analytically continued to X to a meromorphic function with simple poles only at the points of Λ. Now the conclusion follows from a trivial application of the residue theorem. □

We can obviously use Lemma 2.4 to rewrite formulas (2.9) and (2.10) in the form

$$\Pi_N(t,t_w) = \frac{1}{2\pi i}\int_\gamma \frac{e^{-t_w\lambda}}{\lambda}\left(\mathrm{Av}_j\frac{e^{-\frac{N-1}{N}x_j t}}{x_j-\lambda}\Big/\mathrm{Av}_j\frac{1}{x_j-\lambda}\right)d\lambda \tag{2.12}$$

$$\mathbb{E}_N\left(h(x_t)\right) = \frac{1}{2\pi i}\int_\gamma \frac{e^{-t\lambda}}{\lambda}\left(\mathrm{Av}_j\frac{h(x_j)}{x_j-\lambda}\Big/\mathrm{Av}_j\frac{1}{x_j-\lambda}\right)d\lambda \tag{2.13}$$

where Av_j denotes the average over $j=1,2,\ldots,N$.

The above integral representations of $\Pi_N(t,t_w)$ and $\mathbb{E}_N\left(h(x_t)\right)$ have two main advantages. First, the appearance of averages allows to compute their limiting behaviour as $N\uparrow\infty$ easily by using the ergodicity of the random field $\underline{E}$. Second, by means of the residue theorem, their Laplace transform can be easily computed in order to derive the asymptotic behaviour of $\Pi_N(t,t_w)$ and $\mathbb{E}_N\left(h(x_t)\right)$ for $N,t_w,t\gg 1$ (see subsections 2.1, 2.2).

A much more general derivation of the above integral representations is discussed in Appendix C.

2.1. **Ageing behaviour of $\Pi_N(t,t_w)$.**

Proposition 2.5. *Let us define*

$$\Pi(t,t_w) := \frac{1}{2\pi i}\int_\gamma \frac{e^{-t_w\lambda}}{\lambda}\frac{\mathbb{E}_x\left(\frac{e^{-xt}}{\lambda-x}\right)}{\mathbb{E}_x\left(\frac{1}{\lambda-x}\right)}d\lambda. \tag{2.14}$$

where $\mathbb{E}_x$ is the expectation w.r.t. the measure $\alpha x^{\alpha-1}dx$ on $[0,1]$ and γ is any positive oriented complex loop around the interval $[0,1]$. Then

$$\lim_{N\uparrow\infty}\Pi_N(t,t_w) = \Pi(t,t_w) \qquad \forall t,t_w, \qquad a.s.. \tag{2.15}$$

Proof. Recall (2.12) and fix $0 < \delta < 1/2$. Due to analyticity, we can choose the integration contour γ to have distance 1 from the segment $[0,1]$. For each $\lambda \in \gamma$, the random variables $(x_j - \lambda)^{-1}$, $j \in \mathbb{N}$, are i.d.d. and bounded. Therefore, for a suitable positive constant $c > 0$,

$$\mathbb{P}\Big(\Big|\mathrm{Av}_{j=1}^N \frac{1}{x_j - \lambda} - \mathbb{E}_x\big(\frac{1}{\lambda - x}\big)\Big| \geq N^{-\frac{1}{2}+\delta}\big) \leq e^{-cN^{2\delta}} \qquad \forall \lambda \in \gamma. \tag{2.16}$$

Since for each $x \in [0,1]$ and $\lambda \in \gamma$, $|\frac{\partial}{\partial\lambda}(x-\lambda)^{-1}| \leq 1$, a simple chaining argument allows to deduce from the pointwise estimate (2.16) uniform control in λ. With the Borel–Cantelli lemma one can then infer that, a.s.,

$$\sup_{\lambda\in\gamma}\Big|\mathrm{Av}_{j=1}^N \frac{1}{x_j - \lambda} - \mathbb{E}_x\big(\frac{1}{\lambda - x}\big)\Big| \leq c\,N^{-\frac{1}{2}+\delta}, \qquad \forall N \in \mathbb{N}. \tag{2.17}$$

Similar arguments show that, a.s., given $M \in \mathbb{N}$ there exists a constant c_M such that

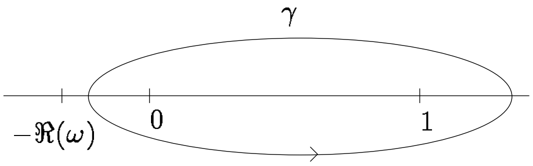
$$\sup_{M-1\leq t\leq M}\ \sup_{\lambda\in\gamma}\Big|\mathrm{Av}_{j=1}^N \frac{e^{-\frac{N-1}{N}x_j t}}{x_j - \lambda} - \mathbb{E}_x\big(\frac{e^{-x_j t}}{\lambda - x_j}\big)\Big| \leq c_M\,N^{-\frac{1}{2}+\delta}, \qquad \forall N \in \mathbb{N}. \tag{2.18}$$

Note that, for each $\lambda \in \gamma$, $\mathrm{Av}_{j=1}^N(x_j - \lambda)^{-1}$ is a convex combination of points of modulus larger or equal than $1/2$, contained in a angular sector with angle non larger than a suitable constant $c < \pi$. In particular, $\big|\mathrm{Av}_{j=1}^N(x_j-\lambda)^{-1}\big| \geq c' > 0$ for all N. From here the assertion of the proposition follows from Lebesgue's Dominated Convergence Theorem. $\square$

Given $\theta > 0$ we are interested in the limit of $\Pi(\theta t_w, t_w)$ as $t_w \uparrow \infty$. This will be done using the Laplace transform of $\Pi(\theta t_w, t_w)$,

$$\hat{\Pi}(\theta,\omega) := \int_0^\infty e^{-\omega t_w}\Pi(\theta t_w, t_w)dt_w, \qquad (\Re(\omega) > 0).$$

The computation of this Laplace transform is trivial if we use the integral expression (2.14).



Let $\omega \in \mathbb{C}$ with $\Re(\omega) > 0$ and fix a positive oriented loop γ around the segment $[0,1]$, such that $\gamma \subset \{z \in \mathbb{C} : \Re(z) > -\Re(\omega)\}$ as in fig. 2.1. Then, $\Re(\omega + \lambda + x\theta) > 0$, for $x \in [0,1]$ and $\lambda \in \gamma$, so that (2.14) implies

$$\hat{\Pi}(\theta,\omega) = \mathbb{E}_x\Big(\frac{1}{2\pi i}\int_\gamma\Big[\lambda(\lambda - x)(\lambda + \omega + \theta x)\mathbb{E}_{\bar{x}}\big(\frac{1}{\lambda - \bar{x}}\big)\Big]^{-1}d\lambda\Big).$$

Here $\mathbb{E}_x$ and $\mathbb{E}_{\bar{x}}$ denote the expectation w.r.t. to the measure $\alpha x^{\alpha-1}dx$ on $[0,1]$.

Let us consider the change of variable $z = \frac{1}{\lambda}$ and write $\hat{\gamma}$ for the path γ with inverted orientation (i.e., positive oriented w.r.t. $\lambda = \infty$). Then we get

$$\hat{\Pi}(\theta,\omega) = \mathbb{E}_x\Big(\frac{1}{2\pi i}\int_{\hat{\gamma}}\Big[(1 - zx)(1 + z\omega + z\theta x)\mathbb{E}_{\bar{x}}\big(\frac{1}{1 - z\bar{x}}\big)\Big]^{-1}dz\Big).$$

Given $x \in [0,1]$, the integrand is a meromorphic function in $\mathbb{C} \setminus [1,\infty)$ that has only a single pole of order 1 inside $\hat{\gamma}$, namely at $z = -(\omega + x\theta)^{-1}$. By the residue theorem we get

$$\hat{\Pi}(\theta,\omega) = \mathbb{E}_x\Big(\frac{1}{\omega + x\theta + x}\Big/\mathbb{E}_{\bar{x}}\big(\frac{\omega + x\theta}{\omega + x\theta + \bar{x}}\big)\Big). \tag{2.19}$$

Lemma 2.6. *The r.h.s. of (2.19) is well defined and holomorphic for any* $\omega \in \mathbb{C}\setminus(-\infty,0]$. *In particular, the function* $\hat{\Pi}(\theta,\omega)$, *defined for* $\Re(\omega) > 0$, *can be analytically continued to the set* $\mathbb{C} \setminus (-\infty,0]$.

Proof. As proved in [16], Chapter 3, the Laplace transform $\hat{\Pi}(\theta,\omega)$ is holomorphic on the set of convergence points. Therefore, we only need to show that the r.h.s. of (2.19) is well defined and holomorphic on $\Im(\omega) \neq 0$. Let us assume $\Im(\omega) > a > 0$. Then, trivially, $\forall x, \bar{x} \in [0,1]$,

$$\frac{\omega + x\theta}{\omega + x\theta + \bar{x}} \in \mathcal{B} := \{z \in \mathbb{C} \,:\, z = |z|e^{i\theta} \text{ with } 0 \leq \theta \leq \theta_0,\ \ |z| \geq c\}$$

for suitable constants c, θ_0 depending on a and such that $\theta_0 < \pi$. Moreover, since $\lim_{|\omega|\uparrow\infty} \frac{\omega+x\theta}{\omega+x\theta+\bar{x}} = 1$,

$$0 < c_1(a) \leq \Big|\frac{\omega + x\theta}{\omega + x\theta + \bar{x}}\Big| \leq c_2(a) \qquad \forall a > 0, \forall \omega \,:\, \Im(\omega) \geq a \tag{2.20}$$

By (2.20) and the geometry of $\mathcal{B}$, we have that $\mathbb{E}_{\bar{x}}\big(\frac{\omega+x\theta}{\omega+x\theta+\bar{x}}\big)$ is well defined and has distance $c_3(a) > 0$ from the origin. Moreover, $|\omega + x\theta + x| \geq \Im(\omega)$. Therefore, the r.h.s. of (2.19) is well defined and, due to the previous estimates and Lebesgue's dominated convergence theorem, it is continuous on $\{\Im(\omega) \neq 0\}$, thus implying continuity on $\mathbb{C} \setminus (-\infty,0]$.

We recall Morera's theorem: if $f(\omega)$ is defined and continuous in a open set $\Omega \subset \mathbb{C}$ and if $\int_\gamma f\, d\omega = 0$ for all closed curves γ in Ω, then $f(\omega)$ is holomorphic in Ω. Therefore, using Fubini's and Morera's theorems, one can prove that the function $\mathbb{E}_{\bar{x}}\big(\frac{\omega+x\theta}{\omega+x\theta+\bar{x}}\big)$ is holomorphic on $\mathbb{C} \setminus (-\infty,0]$. The proof can be concluded by a second application of the same theorems. ☐

In what follows, we keep the notation $\hat{\Pi}(\theta,\omega)$ for the analytic continuation of the Laplace transform. The next lemma describe the behaviour of $\hat{\Pi}(\theta,\omega)$ near the origin. Using the Laplace inversion formula, we then derive from this result the asymptotic behaviour of $\Pi(\theta,t_\omega)$ as $t_\omega \uparrow \infty$.

Lemma 2.7. *For any* $\theta > 0$ *let us set*

$$A(\theta) =: \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{\theta+1}}^{1} u^{-\alpha}(1-u)^{\alpha-1} du.$$

Moreover, let us define

$$\mathcal{A} := \{re^{i\phi} \,:\, r \geq 0,\ |\phi| \leq \frac{3}{4}\pi\}. \tag{2.21}$$

Then, for a suitable positive constant $c > 0$

$$\big|\hat{\Pi}(\theta,\omega)\big| \leq c|\omega|^{-1}, \qquad \forall \omega \in \mathcal{A} \,:\, |\omega| \geq 1 \tag{2.22}$$

$$\big|\hat{\Pi}(\theta,\omega) - A(\theta)/\omega\big| \leq c|\omega|^{-\alpha}, \qquad \forall \omega \in \mathcal{A} \,:\, |\omega| \leq 1. \tag{2.23}$$

Proof. The first estimate (2.22) follows trivially from (2.19) and (2.20). Let us prove (2.23) for $\omega \in \mathcal{A}$ and $|\omega| \leq 1$.

In what follows, $c_0, c_1, \dots$ denote some suitable positive constants depending only on θ. Moreover, given $z \in \mathbb{C}$ we denote by $\int_0^z$ and $\int_z^\infty$ the integrals over the paths $\{sz : 0 \leq s \leq 1\}$ and $\{sz : s \geq 1\}$, respectively. We extend the functions $z^{-\alpha}$ and $z^{\alpha-1}$, defined on $(0,\infty)$, to $\mathbb{C} \setminus (-\infty, 0]$ by analytic continuation. Then (2.19) implies

$$\begin{aligned} \omega \hat{\Pi}(\theta,\omega) &= \int_0^{1/\omega} x^{\alpha-1} \Big([1+x(1+\theta)]\,[1+x\theta] \int_0^{1/\omega} \frac{y^{\alpha-1}}{1+x\theta+y} dy \Big)^{-1} dx \\ &= \int_0^{1/\omega} x^{\alpha-1} \Big([1+x(1+\theta)]\,[1+x\theta]^\alpha \int_0^{\frac{1}{\omega(1+x\theta)}} \frac{y^{\alpha-1}}{1+y} dy \Big)^{-1} dx. \end{aligned} \tag{2.24}$$

Let us define

$$\mathcal{B} := \{(\omega, x) \in \mathbb{C}^2 \ \text{ s.t. } \ \omega \in \mathcal{A}, \ x = \frac{s}{\omega} \text{ for some } s : 0 \leq s \leq 1\}.$$

Since $\big(\omega(1+x\theta)\big)^{-1} \in \mathcal{A} \cap \{z : |z| \geq c_0\}$, we obtain

$$\Big| \int_0^{\frac{1}{\omega(1+x\theta)}} \frac{y^{\alpha-1}}{1+y} dy \Big| \geq c_1, \tag{2.25}$$

$$\Big| \int_{\frac{1}{\omega(1+x\theta)}}^\infty \frac{y^{\alpha-1}}{1+y} dy \Big| \leq c_2 \, |\omega(1+x\theta)|^{1-\alpha}. \tag{2.26}$$

Let $B(\theta)$ be defined as

$$B(\theta) := \int_0^\infty \frac{y^{\alpha-1}}{1+y} dy = \int_0^1 u^{-\alpha}(1-u)^{\alpha-1} du = \frac{\pi}{\sin(\pi\alpha)}. \tag{2.27}$$

(note that the above second identity follows from the change of variable $u = y(1+y)^{-1}$, while the last one is well known in the theory of the Gamma function). By means of (2.25) and (2.26) we obtain

$$\begin{aligned} &\Big| \omega \hat{\Pi}(\theta,\omega) - \frac{1}{B(\theta)} \int_0^{\frac{1}{\omega}} \frac{x^{\alpha-1} dx}{\big(1+x(1+\theta)\big)\,\big(1+x\theta\big)^\alpha} \Big| \leq \\ &\qquad |\omega|^{1-\alpha} \int_0^{\frac{1}{\omega}} \frac{|x|^{\alpha-1}\, d|x|}{\big|1+x(1+\theta)\big|\,\big|1+x\theta\big|^{2\alpha-1}} \leq c_3 |\omega|^{1-\alpha}. \end{aligned} \tag{2.28}$$

Since

$$\Big| \int_{\frac{1}{\omega}}^\infty \frac{x^{\alpha-1} dx}{\big(1+x(1+\theta)\big)\,\big(1+x\theta\big)^\alpha} \Big| \leq c_4 |\omega|$$

and, using analyticity and integrability of the singularities around $z = 0$ and $z = \infty$,

$$\int_{s\omega:\, s \geq 0} \frac{x^{\alpha-1} dx}{\big(1+x(1+\theta)\big)\,\big(1+x\theta\big)^\alpha} = \int_0^\infty \frac{x^{\alpha-1} dx}{\big(1+x(1+\theta)\big)\,\big(1+x\theta\big)^\alpha},$$

we get

$$\Big| \omega \hat{\Pi}(\theta,\pi) - \frac{1}{B(\theta)} \int_0^\infty \frac{x^{\alpha-1}}{(1+x(1+\theta))(1+x\theta)^\alpha} dx \Big| \leq c_5 |\omega|^{1-\alpha}.$$

Using the chance of variables $v = x^{-1} + \theta$ and $u = v(1+v)^{-1}$, we obtain

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x(1+\theta))(1+x\theta)^\alpha} dx = \int_{\frac{\theta}{\theta+1}}^1 u^{-\alpha}(1-u)^{\alpha-1} du$$

which implies the assertion of the lemma. ☐

Lemma 2.7 and Proposition A.1 allow us to conclude the proof of the ageing behaviour of $\Pi_N(t, t_w)$:

Proposition 2.8. *For almost all energy landscapes $\underline{E}$, given $\theta > 0$*

$$\lim_{t_w \uparrow \infty} \lim_{N \uparrow \infty} \Pi_N(\theta t_w, t_w) = \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{1+\theta}}^{1} u^{-\alpha}(1-u)^{\alpha-1} du. \tag{2.29}$$

2.2. Visiting deeper and deeper traps.

In this section we use the integral representation (2.13) in order to study the probability that the system at time t is in a deep trap, i.e. in a state with large waiting time. In Proposition 2.9 we first prove that the probability to be in a site with waiting time smaller than $O(1)$ decays as $t^{\alpha-1}$, thus implying the ageing behaviour of other correlation functions described in subsection 3. In the second part, we will investigate the random variable $tx_N(t)$ and show that, for almost all $\underline{E}$, it has a weak limit as $N \uparrow \infty$ and then $t \uparrow \infty$. As consequence, with high probability at time t the system is in a state of waiting time $O(t)$ as stated in Proposition 2.10.

Reasoning as in the proof of Proposition (2.5), we can prove, for almost all energy landscapes $\underline{E}$, that, given a function h on $[0,1]$ that can be uniformly approximated by piecewise C^1 functions,

$$H(t) := \lim_{N \uparrow \infty} \mathbb{E}_N\big(h(x_N(t))\big) = \frac{1}{2\pi i} \int_\gamma \frac{e^{-t\lambda}}{\lambda} \frac{\int_0^1 \frac{h(x)}{\lambda - x} x^{\alpha-1} dx}{\int_0^1 \frac{1}{\lambda - x} x^{\alpha-1} dx} d\lambda \quad \forall t > 0, \tag{2.30}$$

where γ is a positive oriented loop around $[0,1]$.

Since $H(t)$ is a bounded function, the Laplace integral $\hat{H}(\omega) := \int_0^\infty H(t) e^{-\omega t} dt$ is absolutely convergent when $\Re(\omega) > 0$. By the same arguments we used to derive (2.19), it is simple to deduce from the integral representation (2.30) that

$$\hat{H}(\omega) = \frac{1}{\omega} \frac{\int_0^1 \frac{h(x)}{\omega + x} x^{\alpha-1} dx}{\int_0^1 \frac{1}{\omega + x} x^{\alpha-1} dx} \tag{2.31}$$

In the following Proposition we concentrate on the case $h(x) := \mathbb{I}_{x \geq \delta}$. By (2.31), we can give precise information on the asymptotic behaviour of the probability to be at time t in a site with waiting time smaller than $1/\delta$:

Proposition 2.9. *Let*

$$B(\delta) := \frac{\int_\delta^1 x^{\alpha-2} dx}{\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx}, \qquad c(\alpha) := \int_0^\infty y^{\alpha-1} e^{-y} dy \tag{2.32}$$

Then, for almost all energy landscapes $\underline{E}$,

$$\lim_{s \uparrow \infty} s^{1-\alpha} \lim_{N \uparrow \infty} \mathbb{P}_N\big(x_N(s) > \delta\big) = B(\delta)/c(\alpha). \tag{2.33}$$

Finally, we show that with high probability at time t the system is in a trap of depth of order $O(t)$. In particular, the random variables $tx_N(t)$ converge weakly to a nonnegative random variable as $N \uparrow \infty$ and then $t \uparrow \infty$ a.s.. This result corresponds to the convergence of expectation of bounded continuous functions and due to Lemma 2.11 such a convergence can be extended to the larger class of bounded piecewise continuous functions, which is more suitable in order to investigate the phenomenon of visiting deeper and deeper traps:

Proposition 2.10. *Let Z be the only random variable with range in $(0,\infty)$ having Laplace transform*

$$\mathbb{E}\big(e^{-\theta Z}\big) = \frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{\theta+1}}^{1} u^{-\alpha}(1-u)^{\alpha-1}du.$$

Then, for almost all energy landscape $\underline{E}$, given a bounded piecewise continuous function h on $(0,\infty)$,

$$\lim_{t\uparrow\infty}\lim_{N\uparrow\infty}\mathbb{E}_N(h(tx_N(t))) = \lim_{t\uparrow\infty}\frac{1}{2\pi i}\int_\gamma \frac{e^{-t\lambda}}{\lambda}\frac{\int_0^1 \frac{h(xt)}{\lambda-x}x^{\alpha-1}dx}{\int_0^1 \frac{1}{\lambda-x}x^{\alpha-1}dx}d\lambda = \mathbb{E}(h(Z)) \tag{2.34}$$

In particular, for almost all energy landscapes $\underline{E}$,

$$\lim_{t\uparrow\infty}\lim_{N\uparrow\infty}\mathbb{P}\Big(\frac{\tau_N(t)}{t}\geq u\Big) = \mathbb{P}(Z\leq u^{-1}), \qquad \forall u>0.$$

Proof of Proposition 2.9. We have to prove that $\lim_{s\uparrow\infty} s^{1-\alpha}H(s) = B(\delta)/c(\alpha)$ where H is given by (2.30) with $h(x) := \mathbb{I}_{x\geq\delta}$. As in the proof of Lemma 2.6 we can show that the r.h.s. of (2.31) is well defined and holomorphic on $\mathbb{C}\setminus(-\infty,0]$. We keep the notation $\hat{H}$ for this extended function. By the change of variables $x=\omega y$, we get

$$\int_0^1 \frac{x^{\alpha-1}}{\omega+x}dx = \omega^{\alpha-1}\int_{\gamma_\omega}\frac{y^{\alpha-1}}{1+y}dy \tag{2.35}$$

where γ_ω is the oriented path $\{s/\omega\}_{0\leq s\leq 1}$. Let $\hat{\gamma}_\omega$ be the path $\{s/\omega\}_{s\geq 0}$. By analyticity and integrability of the singularities at $z=0, z=\infty$, we have

$$\int_{\hat{\gamma}_\omega}\frac{y^{\alpha-1}}{1+y}dy = \int_0^\infty \frac{y^{\alpha-1}}{1+y}dy$$

Let us define $\mathcal{A} := \{re^{i\theta}\,:\, 0<r<\infty,\ |\theta|\leq \frac{3}{4}\pi\}$. Then, for a suitable constant c_1,

$$\Big|\int_{\hat{\gamma}_\omega\setminus\gamma_\omega}\frac{y^{\alpha-1}}{1+y}dy\Big| \leq c_1|\omega|^{1-\alpha}, \qquad \forall\omega\in\mathcal{A}\,:\,|\omega|\leq 1,$$

implying

$$\int_0^1\frac{x^{\alpha-1}}{\omega+x}dx = \omega^{\alpha-1}\Big(\int_0^\infty\frac{y^{\alpha-1}}{1+y}dy + O(|\omega|^{1-\alpha})\Big), \tag{2.36}$$

where $A = B + O(1/N)$ is understood to mean that there exists $C<\infty$ such that $|A-B|\leq C/N$. Trivially,

$$\int_\delta^1 \frac{x^{\alpha-1}}{\omega+x}dx = \big(1+O(|\omega|)\big)\int_\delta^1 x^{\alpha-2}dx. \tag{2.37}$$

Let us note that the estimate of error terms in (2.36) and in (2.37) is uniform in $\omega\in\mathcal{A}$, $|\omega|\leq 1$. Then, from (2.31), (2.36), (2.37) we get

$$|\omega^\alpha\hat{H}(\omega) - B(\delta)/c(\alpha)| \leq c_2|\omega|^{1-\alpha}, \quad \forall\omega\in\mathcal{A}\,:\,|\omega|\leq 1. \tag{2.38}$$

Since trivially $|\hat{H}(\omega)|\leq c_3|\omega|^{-1}$ for $\omega\in\mathcal{A}$ with $|\omega|>1$, the assertion of the proposition follows from Proposition A.1. □

Proof of Proposition 2.10. As discussed before (2.30), one can show, for almost all energy landscapes $\underline{E}$, that, given a piecewise continuous function h on $(0,\infty)$,

$$\Phi_t(h) := \lim_{N\uparrow\infty} \mathbb{E}_N(h(tx_N(t))) = \frac{1}{2\pi i}\int_\gamma \frac{e^{-t\lambda}}{\lambda}\frac{\int_0^1 \frac{h(xt)}{\lambda-x}x^{\alpha-1}dx}{\int_0^1 \frac{1}{\lambda-x}x^{\alpha-1}dx}d\lambda \qquad \forall t\geq 0,$$

where γ is a positive oriented loop around $[0,1]$. Note that Φ_t defines a positive linear functional on the space of continuous functions on $(0,\infty)$, decaying at ∞ and satisfying $\Phi_t(1)=1$. Therefore, the Riesz–Markov representation theorem (see Theorem IV.18 in [26]) implies that $\Phi_t(h)=\mu_t(h)$, for a unique Borel probability measure μ_t on $[0,\infty)$. In particular, there exists a random variable Z_t on $(0,\infty)$ such that

$$\lim_{N\uparrow\infty} tx_N(t) \to Z_t \text{ weakly },\forall t>0 \qquad \text{a.s..}$$

If we take $h(t)=e^{-t\theta}$, then $\Phi_t(h)=\mu_t(h)=\Pi(\theta t,t)$, with Π defined as in (2.14). That means that $\Pi(\theta t,t)$ is the Laplace transform of Z_t. As proved in subsection 2.1,

$$\lim_{t\uparrow\infty}\Pi(\theta t,t)=\frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{\theta+1}}^1 u^{-\alpha}(1-u)^{\alpha-1}du := f(\theta).$$

We state that $f(\theta)$ is the Laplace transform of a random variable Z with range in $[0,\infty)$. To this aim we apply the criterion given by Theorem 1, Section XIII.4 in [17]. By (2.27), $f(0)=1$. Moreover, $f^{(1)}(\theta)=-\frac{\sin(\pi\alpha)}{\pi}\theta^{-\alpha}(1+\theta)^{-1}$ thus implying (by trivial computations) that $(-1)^k f^{(k)}(\theta)\geq 0$. This completes the proof of our statement.

Since the Laplace transform of Z_t converges to the Laplace transform of Z as t goes to ∞, we have that Z_t converges weakly to Z, thus implying the limit (2.34) whenever h is a bounded continuous functions on $(0,\infty)$. Finally, due to Theorem 5.2 in [7], the limit remains valid if h is a bounded measurable function whose set of discontinuity points has zero measure w.r.t. the distribution of Z. Therefore, Lemma 2.11 allows to prove (2.34) for h bounded and piecewise continuous. □

Lemma 2.11. *The distribution function $F(z):=\mathbb{P}(Z\leq z)$ of the positive random variable Z is continuous.*

Proof. Trivially F is increasing and right continuous. Therefore, it has a countable set of points of discontinuity. Moreover, by the Laplace inversion formula (see [17], XIII.4), if x is a point of continuity, then

$$F(x)=\lim_{a\to\infty}\sum_{n\leq ax}\frac{(-a)^n}{n!}f^{(n)}(a).$$

Given $s=0,1,2,\dots$ and $\gamma>0$ let $c_s(\gamma)>0$ be such that $D_a^s a^{-\gamma}=(-1)^s c_s(\gamma)a^{-\gamma-s}$. Then the Leibniz formula implies

$$(-1)^n D_a^n\left(a^{-\alpha}(1+a)^{-1}\right)=\sum_{s=0}^n c_s(\alpha)c_{n-s}(1)a^{-\alpha-s}(1+a)^{-1-n+s}\leq(-1)^n D_a^n a^{-\alpha-1}.$$

Since $f^{(1)}(a)=-\frac{\sin(\pi\alpha)}{\pi}a^{-\alpha}(1+a)^{-1}$, the above estimate implies

$$(-1)^n f^{(n)}(a)=|f^{(n)}(a)|\leq\prod_{k=1}^{n-1}(k+\alpha)\theta^{-n-\alpha}, \qquad \forall n\geq 2.$$

In particular, given two points of continuity $0 < x < z$ we have

$$F(z) - F(x) \leq \limsup_{a \to \infty} a^{-\alpha} \sum_{ax < n \leq az} \frac{1}{n} \prod_{k=1}^{n-1} (1 + \frac{\alpha}{k}). \tag{2.39}$$

One can prove that the sequence $\prod_{k=1}^{n-1} e^{-\frac{\alpha}{n}}(1+\frac{\alpha}{k})$ is convergent (see [1], Chapter 5, Section 2.4). Let us denote its limit by c_α and let γ be Euler's constant

$$\gamma := \lim_{n \uparrow \infty} \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \Big).$$

Then we can write

$$\frac{1}{n} \prod_{k=1}^{n-1} (1 + \frac{\alpha}{k}) = e^{\alpha \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n-1)\right)} \frac{(n-1)^\alpha}{n} \prod_{k=1}^{n-1} e^{-\frac{\alpha}{k}} (1 + \frac{\alpha}{k}). \tag{2.40}$$

In particular, in (2.40) we can substitute $\prod_{k=1}^{n-1} e^{-\frac{\alpha}{k}}(1 + \frac{\alpha}{k})$ with c_α with an error term in (2.39) bounded by

$$\text{const.}\, a^{-\alpha}(az - ax)(ax)^{-1+\alpha} \Big| \prod_{k=1}^{n-1} e^{-\frac{\alpha}{k}}(1 + \frac{\alpha}{k}) - c_\alpha \Big| \leq c(x,z) \Big| \prod_{k=1}^{n-1} e^{-\frac{\alpha}{k}}(1 + \frac{\alpha}{k}) - c_\alpha \Big|$$

which is negligible as $a \uparrow \infty$. Therefore

$$F(z) - F(x) \leq \limsup_{a \to \infty} c_\alpha a^{-\alpha} e^{\gamma\alpha} \sum_{ax < n \leq az} (n-1)^{-1+\alpha} \leq c'(z^\alpha - x^\alpha) \tag{2.41}$$

for a suitable positive constant c'. Since (2.41) is valid almost everywhere and F is monotonic, we have that F is continuous. □

3. OTHER CORRELATION FUNCTIONS

In this section we study the asymptotic behaviour of some other time-time correlation functions $\Pi_N^{(1)}(t, t_w)$, $\Pi_N^{(2)}(t, t_w)$ for which deep traps play a special role. This section is mainly a preparation of what is to follow in the second part of the paper.

Given $\delta > 0$, we define the set of sites with small waiting time as $D_N := \{i : x_i \geq \delta,\ i = 1, \ldots, N\}$. Moreover, we set

$$\Pi_N^{(1)}(t, t_w) := \mathbb{P}_N\big(Y_N(u) \in D_N \quad \forall u \in (t_w, t_w + t] \text{ s.t. } Y_N(u) \neq Y_N(u^-)\,\big) \tag{3.1}$$

$$\Pi_N^{(2)}(t, t_w) := \mathbb{P}_N\big(Y_N(u) \in D_N \cup \{Y_N(t_w)\} \quad \forall u \in (t_w, t_w + t] \text{ s.t. } x_N(u) \neq x_N(u^-)\,\big). \tag{3.2}$$

Given a subset $A \subset \mathcal{S}_N$, $i \in A$ and $s > 0$, let $\varphi_{N,A}(i,s)$ be defined as

$$\varphi_{N,A}(i,s) := \mathbb{P}_N\big(Y_N(u) \in A\ \forall u \in [0,s] \,|\, Y_N(0) = i\big),$$

then

$$\Pi_N^{(1)}(t, t_w) = \Pi_N(t, t_w) + \sum_{j=1}^N \mathbb{P}_N(Y_N(t_w) = j) \int_0^t ds\, \frac{x_j e^{-s x_j}}{N} \sum_{i \in D_N} \varphi_{N, D_N}(i, t-s), \tag{3.3}$$

$$\Pi_N^{(2)}(t, t_w) = \sum_{j=1}^N \mathbb{P}_N(x_N(t_w) = x_j)\, \varphi_{N, D_N \cup \{j\}}(j, t) \tag{3.4}$$

where the first identity can be derived by conditioning on the first jump performed after the waiting time t_w and by recalling the following realization of the dynamics: after arriving at the state i, the system waits an exponential time with parameter x_i and after that it jumps to a site in $\mathcal{S}_N$ with uniform probability.

The following proposition is mainly a consequence of the phenomenon of visiting deep traps with higher and higher probability. To this aim recall Proposition 2.9 or simply, as a consequence of Proposition 2.10, that

$$\lim_{t\uparrow\infty}\lim_{N\uparrow\infty}\mathbb{P}_N(x_N(t)>\epsilon)=0, \qquad \forall\epsilon>0. \tag{3.5}$$

Proposition 3.1. *For almost all* $\underline{x}$

$$\lim_{t_w\uparrow\infty}\sup_{t\geq 0}\left|\Pi_N^{(i)}(t,t_w)-\Pi_N(t,t_w)\right|=0 \qquad \text{for } i=1,2 \tag{3.6}$$

Proof. We consider first the case $i=1$.

We claim that for any $u>0$ and $i\in D_N$,

$$\varphi_{N,D_N}(i,u)\leq \exp\Big(-\delta\, u\big(1-\frac{|D_N|}{N}\big)\Big). \tag{3.7}$$

In order to prove such a bound, we introduce a new random walk $Y_N^*(t)$ having generator $\mathbb{L}^*$ defined as the r.h.s. of (2.3), where x_i is replaced by δ if $i\in D_N$. By a simple coupling argument one gets

$$\varphi_{N,D_N}(i,u)\leq\varphi^*_{N,D_N}(i,u)$$

where the function φ^*_{N,D_N} is the analogous of $\varphi_{N,D_N}(i,u)$ for the random walk $Y_N^*(t)$. At this point, it is enough to observe that φ^*_{N,D_N} equals the r.h.s. of (3.7).

Now fix $\epsilon>0$. Then, due to (3.3) and (3.7),

$$\begin{aligned}
&\left|\Pi_N^{(1)}(t,t_w)-\Pi_N(t,t_w)\right| \\
&\leq \mathbb{P}_N(x_N(t_w)\geq\epsilon)+\sum_{j\,:\,x_j<\epsilon}\mathbb{P}_N(Y_N(t_w)=j)\frac{x_j}{N}\sum_{i\in D_N}\int_0^t\varphi_{N,D_N}(i,t-s)\,ds \\
&\leq \mathbb{P}_N(x_N(t_w)\geq\epsilon)+\epsilon\int_0^t e^{-\delta\, u\left(1-\frac{|D_N|}{N}\right)}\,du.
\end{aligned} \tag{3.8}$$

By the law of large numbers,

$$\lim_{N\uparrow\infty}\int_0^\infty e^{-\delta\, u\left(1-\frac{|D_N|}{N}\right)}<\infty, \qquad \text{a.s..}$$

The proposition now follows from the fact that ϵ is arbitrary and from (3.5).

To deal with case (ii), one proceeds in essentially the same way, decomposing the path of the process at its returns to the point x_i, and summing over the number of these returns. One finds easily that the case when the process does not leave x_i for the entire period t dominates, leading to the assertion of the proposition. We leave the details to the reader. □

4. The REM–like trap model on a Poisson point process.

In this section we consider a slightly different formulation of the REM like trap model that betrays more directly its connection to the REM dynamics (see [3, 4]) and that offers a somewhat more natural insight in the rôle of time scales in the analysis of ageing systems. Let us consider a Poisson point process $\mathcal{P}=\sum_i\delta_{E_i}$ on $\mathbb{R}$ with intensity measure $\alpha e^{-\alpha E}dE$, where $0<\alpha<1$. Note that such processes arise naturally as the extremal process of sequences of random variables. Trivially, a.s. the support of $\mathcal{P}$ is an infinite set of points, whose maximum is almost surely bounded from above. Thus we can label the points in the support of $\mathcal{P}$ in decreasing order: $E_1 > E_2 > \dots$. Then, the energy landscape $\underline{E}$ is defined as $\underline{E} = (E_1, E_2, \dots)$. We want to define a random process on the support of this point process that jumps "uniformly" from any point to any other point in the support. To do this, we need to introduce a cut–off. Here we fix an energy threshold E and set

$$N_E = \max\{i \,:\, E_i \geq E\}.$$

Note that N_E is a Poisson random variable with expectation $e^{-\alpha E}$. Moreover, the probability that $N_E \geq 1$ can be made as small as desired when E is chosen small enough, as we assume in what follows.

Let $\mathcal{G}_E = (\mathcal{S}_E, \mathcal{E}_E)$ be the graph with

$$\mathcal{S}_E := \{1, 2, \dots, N_E\}, \qquad \mathcal{E}_E := \{\, \{i,j\} \,:\, i \neq j \in \mathcal{S}_E\}.$$

Since here we want to investigate the effect of *time rescaling*, we introduce a time unit $\tau_0 = e^{E_0}$. Then, the continuous–time random walk $Y_E(t)$ is the random walk on $\mathcal{G}_E$ having uniform initial distribution and such that, after arriving at site $i \in \mathcal{E}_E$, it waits an exponential time with mean $\frac{N_E}{N_E-1} e^{E_i}/\tau_0$ and then jumps with uniform probability to a different site of $\mathcal{S}_E$. In particular, the Markov generator $\mathbb{L}_E$ for the above defined random walk is given by $\mathbb{L}_N$ in (2.3) with $N := N_E$ and $x_i := \tau_0 e^{-E_i}$ (since for $E \ll 0$, $\frac{N_E}{N_E-1} \sim 1$ when referring to waiting time we disregard the coefficient $\frac{N_E}{N_E-1}$ as in section 2). Although $Y_E(t)$ depends on τ_0, our notation does not refer to such a dependence. In what follows we denote by $\mathbb{P}_E$ the probability measure on the paths space determined by $Y_E(\cdot)$, and by $\mathbb{E}_E$ the corresponding expectation.

Note that the physical waiting time (the absolute one) for the system at state i is given by $T_i := e^{E_i}$ while in the above dynamics the waiting time is $\tau_i := T_i/\tau_0$, thus in agreement with the choice to consider τ_0 as our new time unit. In what follows we consider, when taking the thermodynamic limit $E \downarrow -\infty$, three different kinds of time rescaling: τ_0 fixed, $\tau_0 := e^E$ (that is $E_0 = E$) and $\tau_0 \downarrow 0$ after $E \downarrow -\infty$.

As in section 2 we are interested in the asymptotic behaviour of time–time correlation functions. In particular, let us introduce here the correlation function

$$\Pi_E(t, t_w) := \mathbb{P}_E\big(Y_E(s) = Y_E(t_w),\ \forall s \in [t_w, t_w + t]\big).$$

We will prove that when τ_0 is fixed the system exhibits fast relaxation, thus excluding ageing behaviour (see Proposition 4.2). At the other extreme, the scaling $\tau_0 = e^E$ corresponds to the implicit choice made in the standard Bouchaud model considered in the previous sections. In fact with this choice the system can be thought of as a *grand canonical* version of the original REM–like trap model and all the results of the previous sections carry over. Finally, we consider the third scaling: $\tau_0 \downarrow 0$ after $E \downarrow -\infty$. In Proposition 4.6 we show that when performing such limits the correlation function $\Pi_E(t, t_w)$ converges to $f(\theta)$ where $\theta = t/t_w$ and $f(\theta)$ denotes the r.h.s. of identity (2.29), that is the limiting behaviour of the correlation function $\Pi_E(t, t_w)$ is trivial. At this point a simple consideration is fundamental. If we assume that the physical instruments in the laboratory have sensibility up to the time unit τ_0, then it is natural to disregard jumps into states with physical waiting time $T_i = e^{E_i}$ much smaller than τ_0. Therefore, a time–time correlation function much more interesting than $\Pi_E(t, t_w)$ is the following one, where $\delta > 0$ is fixed

$$\Pi_E^{(1)}(t, t_w) = \mathbb{P}_E\big(\, x_E(u) \geq \delta \quad \forall u \in (t_w, t_w + t] :\ x_E(u) \neq x_E(u^-) \,\big).$$

where $x_E(t) := x_k$ whenever $Y_E(t) = k$. In Section 5 we prove that $\Pi_E^{(1)}(t, t_w)$ exhibits ageing behaviour: $\Pi_E^{(1)}(\theta t_w, t_w)$ converges to the above $f(\theta)$ after taking the (ordered) limits $E \downarrow -\infty$, $\tau_0 \downarrow 0$ and $t_w \uparrow \infty$.

Finally, in this section we discuss the asymptotic spectral behaviour for the above time rescalings. We will show that ageing appears whenever the limiting spectral density has a singularity of order $O(x^{\alpha-1})$ at 0.

Let us recall some properties of the Ppp $\sum_i \delta_{x_i}$ with intensity measure $\alpha\tau_0^{-\alpha}x^{\alpha-1}dx$ on $(0,\infty)$ which will be frequently used below. Given $M > 0$ the truncated Ppp $\sum_{x_i \le M} \delta_{x_i}$ can be realized as follows. Let n_M be a Poisson variable with expectation $\left(\frac{M}{\tau_0}\right)^\alpha = \int_0^M \alpha\tau_0^{-\alpha}x^{\alpha-1}dx$ and let X_i, $i \in \mathbb{N}$, be i.i.d. random variables on $[0, M]$ with probability distribution $p(X)dX = \alpha M^{-\alpha}X^{\alpha-1}dX$. Then

$$\sum_{x_i \le M} \delta_{x_i} \sim \sum_{i=1}^{n_M} \delta_{X_i}, \tag{4.1}$$

in the sense that the above point processes have the same distribution. In particular, by taking $M = \tau_0 e^{-E}$, we get

$$\sum_{i \le N_E} \delta_{x_i} \sim \sum_{i=1}^{n_E^*} \delta_{X_i}, \tag{4.2}$$

where n_E^* is a Poisson variable with expectation $e^{-\alpha E}$ and $X_i, i \in \mathbb{N}$, are i.i.d. random variables on $[0, \tau_0 e^{-E}]$ with probability distribution $p(X)dX = e^{\alpha E}\alpha\tau_0^{-\alpha}X^{\alpha-1}dX$.

Notation It is convenient to introduce the random walks $x_E(t)$, $\tau_E(t)$ defined as

$$x_E(t) := x_k, \quad \tau_E(t) := \tau_k \qquad \text{if } Y_E(t) = k.$$

We denote by γ_E the positive oriented loop having support

$$\mathrm{supp}(\gamma_E) = \{x \pm i \,:\, x \in [-1, \tau_0 e^{-E}]\} \cup \{-1 + bi \,:\, |b| \le 1\} \cup \{\tau_0 e^{-E} + 1 + bi \,:\, |b| \le 1\}.$$

Moreover, we call γ_∞ the infinite open path, oriented from $\infty + i$ to $\infty - i$, having support

$$\mathrm{supp}(\gamma_\infty) = \{x \pm i \,:\, x \ge -1\} \cup \{-1 + bi \,:\, |b| \le 1\}$$

Finally, for given E, $0 = \lambda_1^{(E)} < \lambda_2^{(E)} < \cdots < \lambda_{N_E}^{(E)}$, are the N_E distinct eigenvalues of the infinitesimal generator $\mathbb{L}_E$ (see Proposition 2.1).

4.1. τ_0 fixed.

Let us first observe that $\sum_{i=1}^\infty \tau_i < \infty$ for almost all $\underline{E}$. In fact, since the Ppp $\sum_i \delta_{\tau_i}$ has intensity measure $\alpha\tau_0^\alpha\tau^{-(1+\alpha)}d\tau$ on $(0,\infty)$,

$$\mathbb{E}(|\{i : \tau_i \ge 1\}|) = \int_1^\infty \alpha\tau_0^\alpha\tau^{-(1+\alpha)}d\tau < \infty, \qquad \mathbb{E}(\sum_{i:\,\tau_i<1} \tau_i) = \int_0^1 \alpha\tau_0^\alpha\tau^{-\alpha}d\tau < \infty.$$

Whenever $\sum_{i=1}^\infty \tau_i < \infty$, it is simple to derive the asymptotic spectral behaviour of the system from Proposition 2.1 and to show its fast relaxation, thus implying the absence of ageing:

Proposition 4.1. *For almost all* $\underline{E}$,

$$\lim_{E\downarrow-\infty} \sum_{j=1}^{N_E} \delta_{\lambda_j^{(E)}} = \sum_{j=1}^\infty \delta_{\lambda_j} \qquad \textit{vaguely in } \mathcal{M}([0,\infty)), \tag{4.3}$$

where $\mathcal{M}([0,\infty))$ *denotes the space of locally bounded measure on* $[0,\infty)$ *and*

$$\{0=\lambda_1<\lambda_2<\lambda_3<\dots\}=\{\lambda\in\mathbb{C}\,:\,\sum_{k=1}^{\infty}\frac{\lambda}{\lambda-x_k}=0\}$$

Proof. In what follows we assume that $\sum_{i=1}^{\infty}\tau_i<\infty$, which is true a.s.. Then the function $\phi_\infty(\lambda):=\sum_{k=1}^{\infty}\frac{\lambda}{x_k-\lambda}$ is well defined on $\mathbb{C}\setminus\{x_i\,:\,i\geq 1\}$ and has non negative zeros $0=\lambda_1<\lambda_2<\dots$, such that $x_{i-1}<\lambda_i<x_i$ for any $i>1$. At this point it is enough to show that

$$\lim_{E\downarrow-\infty}\lambda_i^{(E)}=\lambda_i,\qquad\forall i=1,2,\dots.$$

The assertion is trivial for $i=1$. Suppose that $N_E\geq i>1$ and set $\psi_E(\lambda):=\sum_{k=1}^{N_E}\frac{1}{x_k-\lambda}$. Due to Proposition 2.1, $\lambda_i^{(E)}$ is the unique zero of $\psi_E(\lambda)$ in the interval (x_{i-1},x_i). In particular,

$$\psi_E(\lambda_i)=\psi_E(\lambda_i)-\psi_E(\lambda_i^{(E)})=\int_{\lambda_i^{(E)}}^{\lambda_i}\dot{\psi}_E(\lambda)d\lambda.$$

Since $\dot{\psi}_E(\lambda)\geq\frac{1}{(x_i-x_{i-1})^2}$ for all $\lambda\in(x_{i-1},x_i)$, we get

$$\left|\lambda_i^{(E)}-\lambda_i\right|\leq(x_i-x_{i-1})^2|\psi_E(\lambda_i)|$$

and therefore the assertion follows by observing that the identity $\phi_\infty(\lambda_i)=0$ implies

$$\left|\psi_E(\lambda_i)\right|\leq\sum_{k=N_E+1}^{\infty}\frac{1}{x_k-x_i}\downarrow 0\qquad\text{as } E\downarrow-\infty.$$

□

Proposition 4.2. *For almost all* $\underline{E}$*,*

$$\lim_{t\uparrow\infty}\lim_{E\downarrow-\infty}\mathbb{P}_E(x_E(t)=x_j)=\frac{\tau_j}{\sum_{k=1}^{\infty}\tau_k}\qquad\forall j=1,2,\dots\tag{4.4}$$

thus implying

$$\lim_{t_w\uparrow\infty}\lim_{E\downarrow-\infty}\Pi_E(\theta t_w,t_w)=0\;\;\forall\theta>0,\tag{4.5}$$

$$\lim_{t_w\uparrow\infty}\lim_{E\downarrow-\infty}\Pi_E(t,t_w)=\frac{\sum_{i=1}^{\infty}\tau_i e^{-x_i t}}{\sum_{i=1}^{\infty}\tau_i}\;\;\forall t>0.\tag{4.6}$$

Proof. In what follows we assume that $\underline{E}$ satisfies $\sum_i\tau_i<\infty$. By setting $h(x)=\mathbb{I}_{x=x_j}$ in (2.13) we get the integral representation

$$\mathbb{P}_E(x_E(t)=x_j)=\frac{1}{2\pi i}\int_{\gamma_E}\frac{e^{-\lambda t}}{\lambda(x_j-\lambda)}\Big(\sum_{k=1}^{N_E}\frac{1}{x_k-\lambda}\Big)^{-1}d\lambda.\tag{4.7}$$

By applying the residue theorem (see the arguments used in order to derive (2.19)), it is simple to compute the Laplace transform $\hat{F}_E(\omega)=\int_0^\infty\mathbb{P}_E(x_E(t)=x_j)e^{-\omega t}dt$ for $\Re(\omega)>0$:

$$\hat{F}_E(\omega)=\Big(\omega(\omega+x_j)\sum_{k=1}^{N_E}\frac{1}{\omega+x_k}\Big)^{-1}\tag{4.8}$$

We note that, almost surely, there exists $c>0$ such that

$$\left|\hat{F}_E(\omega)\right|\leq c\frac{1}{|\omega|},\qquad\forall E,\quad\forall\omega\in\mathcal{A}:=\{re^{i\theta}\,:\,0<r<\infty,\ |\theta|\leq\frac{3}{4}\pi\}\tag{4.9}$$

This follows easily from the bound below where $\omega = a + ib$ and N is any positive integer:

$$|\sum_{j=1}^{N} \frac{1}{\omega + x_j}| \geq \begin{cases} \frac{1}{|\omega+x_1|} & \text{if } a \geq 0 \\ \frac{b}{(a+x_1)^2+b^2} & \text{if } a < 0. \end{cases}$$

Let us now introduce the path $\tilde{\gamma}$ consisting of the parabolic arcs $\{-t \pm it^2 \,:\, t \geq 1\}$ and the circular arc of radius $\sqrt{2}$ around the origin connecting (in anti–clockwise way) $-1-i$ to $-1+i$. The orientation of $\tilde{\gamma}$ is such that $-1+i$ comes before $-1+i$. Then, by means of (4.9), the inverse formula of Laplace transform and the dominated convergence theorem, we get

$$\lim_{E\downarrow-\infty} \mathbb{P}_E(x_E(t) = x_j) = \int_{\tilde{\gamma}} e^{t\omega} \hat{F}(\omega) d\omega, \qquad F(\omega) := \Big(\omega(\omega + x_j) \sum_{k=1}^{\infty} \frac{1}{\omega + x_k}\Big)^{-1}.$$

Note that $F(\omega)$ is the limit of $F_E(\omega)$ as $E \downarrow \infty$, in particular it satisfies (4.9). Moreover, $F(\omega)$ is the Laplace transform of $\lim_{E\downarrow-\infty} \mathbb{P}_E(x_E(t) = x_j)$ and trivially

$$\Big|\omega F(\omega) - \frac{\tau_j}{\sum_{k=1}^{\infty} \tau_k}\Big| \leq c|\omega|, \qquad \forall |\omega| \leq 1 :\, \omega \in \mathcal{A}.$$

At this point (4.4) follows from Proposition A.1. Moreover, from (4.4) and the identity

$$\Pi_E(t, t_w) = \sum_{j=1}^{N_E} \mathbb{P}_E(\, x_E(t_w) = x_j \,) e^{-\frac{N_E - 1}{N_E} x_j t}$$

it is simple to infer (4.5) and (4.6). ☐

4.2. $\tau_0 = e^E$.

Note that by choosing $\tau_0 = e^E$, the random variables $X_1, X_2, \dots$ introduced in (4.2) are i.i.d. with distribution $p(X)dX = \alpha X^{\alpha-1} dX$ on $[0,1]$. Therefore, due to (4.2), we can think of $Y_E(t)$ as the *grand canonical* version of the Bouchaud's REM–like trap model. In particular, it exhibits the same asymptotic spectral density and the same ageing behaviour:

Proposition 4.3. *For almost all* $\underline{E}$,

$$\lim_{E\downarrow-\infty} \frac{1}{N_E} \sum_{j=1}^{N_E} \delta_{\lambda_j^{(E)}} = \alpha x^{\alpha-1} dx \qquad \textit{weakly in } \mathcal{M}([0,1]).$$

Proof. By approximating continuous functions on $[0,1]$ with step functions having rational values and jumps at rational points, it is enough to prove that, given $0 \leq a < b \leq 1$,

$$\lim_{E\downarrow-\infty} \frac{1}{N_E} |\{j \,: 1 \leq j \leq N_E,\ \lambda_j^{(E)} \in [a,b]\}| = b^\alpha - a^\alpha \qquad \text{a.s.}$$

We set

$$A_E := \big|\{j :\ 1 \leq j \leq N_E,\ x_j \in [a,b]\}\big| = \big|\{j \,:\, j \geq 1, e^{-E_j} \in [e^{-E}a, e^{-E}b]\}\big|.$$

Then, due to Proposition 2.1, we only need to prove that

$$\lim_{E\downarrow-\infty} \frac{A_E}{N_E} = b^\alpha - a^\alpha \qquad \text{a.s.}$$

To this aim observe that N_E and A_E are Poisson variables with expectation (or equivalently variance) respectively equals to $e^{-\alpha E}$ and $e^{-\alpha E}(b^\alpha - a^\alpha)$. Given a positive integer n we set $E(n) = -\frac{2}{\alpha}\ln n$, i.e. $e^{-\alpha E(n)} = n^2$. It is simple to derive from Chebyshev inequality and Borel–Cantelli lemma that

$$\lim_{n\uparrow\infty}\frac{N_{E(n)}}{e^{-\alpha E(n)}} = 1, \qquad \lim_{n\uparrow\infty}\frac{A_{E(n)}}{e^{-\alpha E(n)}} = b^\alpha - a^\alpha \qquad \text{a.s.}$$

By a simple argument based on monotonicity, one can extend the first limit to $\lim_{E\downarrow-\infty}\frac{N_E}{e^{-\alpha E}} = 1$ a.s. In order to extend the second limit to general E we observe that, whenever $E(n+1) < E \leq E(n)$,

$$\left|A_E - A_{E(n)}\right| \leq \left|\left\{ j \,:\, e^{-E_j} \in [a\, e^{-E(n)}, a\, e^{-E(n+1)}] \cup [b\, e^{-E(n)}, b\, e^{-E(n+1)}] \right\}\right|.$$

Since the r.h.s. is a Poisson variable with expectation of order $O(n)$, by means of Chebyshev inequality and Borel–Cantelli lemma we obtain

$$\lim_{n\uparrow\infty}\sup_{E(n+1)<E\leq E(n)}\frac{\left|A_E - A_{E(n)}\right|}{e^{-\alpha E(n)}} = 0 \qquad \text{a.s.}$$

thus allowing to prove that $\lim_{E\downarrow-\infty}\frac{A_E}{e^{-\alpha E}} = b^\alpha - a^\alpha$ a.s. □

Proposition 4.4. *For almost all $\underline{E}$,*

$$\lim_{E\downarrow-\infty}\Pi_E(t,t_w) = \frac{1}{2\pi i}\int_\gamma \frac{e^{-t_w\lambda}}{\lambda}\frac{\int_0^1 \frac{e^{-xt}}{\lambda - x}x^{\alpha-1}dx}{\int_0^1 \frac{1}{\lambda-x}x^{\alpha-1}dx}d\lambda, \qquad \forall t, t_w \tag{4.10}$$

In particular, for almost all $\underline{E}$, given $\theta > 0$

$$\lim_{t_w\uparrow\infty}\lim_{E\downarrow-\infty}\Pi_E(\theta t_w, t_w) = \frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{1+\theta}}^1 u^{-\alpha}(1-u)^{\alpha-1}du. \tag{4.11}$$

Proof. Our starting point is (4.2) and the following inequality, valid for any bounded function f with $\mathbb{E}(f(X_i)) = 0$:

$$\mathbb{P}\left(\left|\mathrm{Av}_{j=1}^k f(X_j)\right| \geq \delta\right) \leq 2\exp\left(-\frac{k\delta^2}{4\|f\|_\infty}\right), \qquad \forall\delta > 0, k = 1, 2, \dots.$$

In particular, by conditioning on n_E^* (see (4.2) we get

$$\mathbb{P}\left(\left|\mathrm{Av}_{j=1}^{n_E^*} f(X_j)\right| \geq \delta\right) \leq 2\exp\left\{-e^{-\alpha E}\left(1 - e^{-\frac{\delta^2}{4\|f\|_\infty}}\right)\right\} \tag{4.12}$$

It is simple to derive (4.10) from the above estimate, Borel–Cantelli lemma and the integral representation

$$\Pi_E(t,t_w) = \frac{1}{2\pi i}\int_\gamma \frac{e^{-t_w\lambda}}{\lambda}\frac{\mathrm{Av}_{j=1}^{N_E}\frac{e^{-x_j t}}{x_j-\lambda}}{\mathrm{Av}_{j=1}^{N_E}\frac{1}{x_j-\lambda}}d\lambda \tag{4.13}$$

where γ is a positive oriented closed path around $[0,1]$ (see 2.12). Note that the r.h.s. of (4.10) corresponds to the function $\Pi(t,t_w)$ introduced in Proposition 2.5. Therefore, the conclusion of the proof follows from Propositions 2.5 and 2.8. □

4.3. $\tau_0 \downarrow 0$ after $E \downarrow -\infty$.

In this scaling regime, we show that the vague limit of the suitably rescaled spectral density is given by the measure $\alpha x^{\alpha-1}dx$ on $[0,\infty)$ and we recover the ageing property of the correlation function as before. What is more, however, is that due to the fact that we are effectively already at 'infinite times' on the microscopic scale, we get a pure ageing function even before taking t and t_w to infinity:

Proposition 4.5. *For almost all $\underline{E}$,*

$$\lim_{\tau_0\downarrow 0}\lim_{E\downarrow -\infty} \tau_0^\alpha \sum_{j=1}^{N_E} \delta_{\lambda_j^{(E)}} = \alpha x^{\alpha-1}dx \qquad \textit{vaguely in } \mathcal{M}([0,\infty)).$$

Proposition 4.6. *For almost all energy landscape $\underline{E}$, given positive t, t_w,*

$$\lim_{E\downarrow -\infty} \Pi_E(t,t_w) = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{e^{-t_w\lambda}}{\lambda}\,\frac{\tau_0^\alpha \sum_{j=1}^\infty \frac{e^{-x_j t}}{x_j-\lambda}}{\tau_0^\alpha \sum_{j=1}^\infty \frac{1}{x_j-\lambda}}\,d\lambda \tag{4.14}$$

and

$$\lim_{\tau_0\downarrow 0}\lim_{E\downarrow -\infty} \Pi_E(t,t_w) = \frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{1+\theta}}^1 u^{-\alpha}(1-u)^{\alpha-1}du \qquad \textit{where } \theta = \frac{t}{t_w}. \tag{4.15}$$

Remark The integral in (4.14) exists due to Lemma 4.8.

Due to Proposition 2.1, Proposition 4.5 follows if one is able to prove that $\tau_0^\alpha \sum_{j=1}^{N_E} \delta_{x_j}$ converges vaguely to $\alpha x^{\alpha-1}dx$ on $[0,\infty)$ when taking the (ordered) limits $E \downarrow -\infty$, $\tau_0 \downarrow 0$. This is the content of Lemma 4.7 below concerning the self–average property of Poisson point processes with finite intensity measure (compare it with Lemma 4.16 in [4]). Finally, the proof of Proposition 4.6 is based on (and given after) the technical Lemmata 4.7, 4.8.

Lemma 4.7. *Let $M > 0$ and let f be a bounded continuous function on $[0,M]$. Then there exists $\delta > 0$ such that for almost all energy landscape $\underline{E}$*

$$\Big|\tau_0^\alpha \sum_{x_i\le M} f(x_i) - \int_0^M f(x)\alpha x^{\alpha-1}dx\Big| \le c\,\tau_0^\delta, \qquad \forall \tau_0 > 0 \tag{4.16}$$

for a suitable positive constant c.

Proof. Let $X_1, X_2, \dots$ and n_M be as in (4.1). Due to (4.1) and since $Var(n_M) = \mathbb{E}(n_M) = (M/\tau_0)^\alpha$,

$$\mathbb{P}\Big(\Big|\frac{|\{j\,:\,x_j\le M\}|}{(M/\tau_0)^\alpha} - 1\Big| \ge \epsilon\Big) \le (\tau_0/M)^\alpha \epsilon^{-2}$$

In particular, given $\gamma, s > 0$ with $2s - \gamma\alpha < -1$, by means of Borel–Cantelli lemma we obtain that for almost all energy landscape $\underline{E}$ there exists $c > 0$ such that

$$\Big|\frac{|\{j\,:\,x_j\le M\}|}{(M/\tau_0)^\alpha} - 1\Big| \le c\,k^{-s}, \qquad \forall k = 1,2,\dots \text{ where } \tau_0 := k^{-\gamma}.$$

Due to the above estimate,

$$\Big|\tau_0^\alpha \sum_{x_i\le M} f(x_i) - M^\alpha \mathrm{Av}_{x_i\le M} f(x_i)\Big| \le ck^{-s}\|f\|_\infty, \qquad \forall k \in 1,2,\dots \text{ where } \tau_0 := k^{-\gamma} \tag{4.17}$$

where $\mathrm{Av}_{x_i\le M}$ denotes the average over the set $\{x_i \le M\}$. As done for (4.12), if $0<\rho<1$,

$$\mathbb{P}\Big(\Big|\mathrm{Av}_{x_i\le M} f(x_i) - M^{-\alpha}\int_0^M f(x)\alpha x^{\alpha-1}dx\Big| \ge \rho\Big) \le 2\exp\{-(\frac{M}{\tau_0})^{\alpha}(1-e^{-c\rho^2})\} \le 2e^{-c'\rho^2\tau_0^{-\alpha}}$$

In particular, by Borel–Cantelli lemma, for almost all $\underline{E}$,

$$\Big|\mathrm{Av}_{x_i\le M} f(x_i) - M^{-\alpha}\int_0^M f(x)\alpha x^{\alpha-1}dx\Big| \le c\,k^{-s}, \qquad \forall k=1,2,\dots \text{ where } \tau_0 := k^{-\gamma} \tag{4.18}$$

if s is chosen small enough. At this point (4.17) and (4.18) imply the assertion of the lemma, if $\tau_0 = k^{-\gamma}$ for some $k=1,2,\dots$ The general case $\tau_0>0$ follows easily from the uniform continuity of f. □

Lemma 4.8. *For almost all energy landscapes $\underline{E}$, there are positive constants τ_0^*, c_1, c_2 having the following properties. If $\tau_0 \le \tau_0^*$, $N \ge |\{j : x_j \le 1\}|$ and $\lambda \in \gamma_\infty$ (or $\lambda = a+ib$ with $|b|\le 1$ and $a \ge x_j+1$ for all $j\le N$) then*

$$\Big|\tau_0^\alpha \sum_{j=1}^{N} \frac{1}{x_j-\lambda}\Big| \ge c_1|\lambda|^{-2}. \tag{4.19}$$

Moreover, if $\tau_0 \le \tau_0^*$ and $M\ge 1$, then

$$\tau_0^\alpha \sum_{j=1}^{\infty} \frac{1}{|x_j-\lambda|} \le c_2|\lambda|^{\alpha-1}\ln(1+|\lambda|), \qquad \textit{if } \lambda\in\gamma_\infty \tag{4.20}$$

$$\tau_0^\alpha \sum_{x_j\le M}^{\infty} \frac{1}{|x_j-\lambda|} \le c_2 M^{\alpha-1}\ln M, \qquad \textit{if } \lambda\in\gamma_\infty \textit{ or } \Re(\lambda)=M+1 \tag{4.21}$$

where $\sum_{x_j\le M}$ means $\sum_{j\ge 1 : x_j \le M}$.

Proof. It is convenient to introduce the non rescaled Ppp $\sum_i \delta_{y_i}$, with $y_i := e^{-E_i}$, having intensity measure $\alpha y^{\alpha-1}dy$ on $(0,\infty)$. Moreover, we fix here $\beta>2$ and $0<\gamma<\beta/2-1$ and we define

$$N_n := |\{j : n^{\beta/\alpha} \le y_j < (n+1)^{\beta/\alpha}\}|.$$

for n positive integer. Then a simple application of Borel–Cantelli lemma implies that, for almost all energy landscapes $\underline{E}$,

$$\Big|\frac{N_n}{(n+1)^\beta - n^\beta} - 1\Big| \le c\,n^{-\gamma}, \qquad \forall n = 1,2,\dots \tag{4.22}$$

In fact, by Chebyshev inequality and since N_n is a Poisson variable with $Var(N_n) = \mathbb{E}(N_n) = (n+1)^\beta - n^\beta$,

$$\mathbb{P}\big(|N_n/\mathbb{E}(N_n) - 1| > n^{-\gamma}\big) \le c\,n^{2\gamma+1-\beta}$$

Moreover, by Borel–Cantelli lemma and a simple argument based on monotonicity, it is simple to prove that there exists $\delta>0$ such that, for almost all $\underline{E}$,

$$\Big|\frac{|\{j : y_j \le u\}|}{u^\alpha} - 1\Big| \le \kappa\,u^{-\delta} \qquad \forall u\ge 1. \tag{4.23}$$

for a suitable positive constant κ.

In what follows we write $\lambda = a + ib$. Then

$$\Big|\sum_{j=1}^{N} \frac{1}{x_j - \lambda}\Big|^2 = \Big(\sum_{j=1}^{N} \frac{x_j - a}{(x_j - a)^2 + b^2}\Big)^2 + \Big(\sum_{j=1}^{N} \frac{b}{(x_j - a)^2 + b^2}\Big)^2 \tag{4.24}$$

In order to prove (4.19) we assume (4.22) and (4.23) to be valid and let $0 < \tau_0 \leq \tau_0^* \leq 1$ where τ_0^* is such that $u^\alpha - \kappa u^{-\gamma} > 0$ for all $u \geq 1/\tau_0^*$. In particular, $\{x_j : x_j \leq 1\} \neq \emptyset$. By (4.24), if $N \geq |\{j : x_j \leq 1\}|$ and $\lambda \in \gamma_\infty$

$$\tau_0^\alpha \Big|\sum_{j=1}^{N} \frac{1}{x_j - \lambda}\Big| \geq \begin{cases} \tau_0^\alpha \sum_{x_j \leq 1} \frac{|b|}{(x_j - a)^2 + b^2} \geq c\,|\lambda|^{-2} \tau_0^\alpha |\{x_j : x_j \leq 1\}| & \text{if } |b| \geq \frac{1}{2} \\ \tau_0^\alpha \sum_{x_j \leq 1} \frac{x_j + 1}{(x_j + 1)^2 + b^2} \geq c\,\tau_0^\alpha |\{x_j : x_j \leq 1\}| & \text{if } |b| < \frac{1}{2} \end{cases}$$

At this point, (4.19) for $\lambda \in \gamma_\infty$ follows from (4.23). The case $\lambda = a + ib$, with $|b| \leq 1$ and $a \geq x_j + 1$ for all $j \leq N$, can be treated similarly.

It is simple to derive (4.20) and (4.21) from estimates (4.25), ..., (4.31) below valid for almost all $\underline{E}$:

if $a \leq 100$, $1 \leq M$ and $\lambda = a + ib \in \gamma_\infty$ then

$$\tau_0^\alpha \sum_{j=1}^{\infty} \frac{1}{|x_j - \lambda|} \leq c, \tag{4.25}$$

$$\tau_0^\alpha \sum_{x_j \geq M} \frac{1}{|x_j - \lambda|} \leq c\,M^{\alpha - 1}; \tag{4.26}$$

if τ_0 is small enough and $a \geq 100$ then

$$\tau_0^\alpha \sum_{x_j \leq \frac{a}{2}} \frac{1}{|x_j - a|} \leq c\,a^{\alpha - 1} \tag{4.27}$$

$$\tau_0^\alpha \sum_{\frac{a}{2} \leq x_j \leq a - 1} \frac{1}{|x_j - a|} \leq c\,a^{\alpha - 1} \ln a \tag{4.28}$$

$$\tau_0^\alpha \sum_{a - 1 \leq \lambda \leq a + 1} \frac{1}{|x_j - \lambda|} \leq c\,a^{\alpha - 1} \quad \text{if } \lambda = \alpha + ib \in \gamma_\infty \tag{4.29}$$

$$\tau_0^\alpha \sum_{a + 1 \leq x_j \leq 2a} \frac{1}{|x_j - a|} \leq c\,a^{\alpha - 1} \ln a, \tag{4.30}$$

$$\tau_0^\alpha \sum_{x_j \geq M} \frac{1}{|x_j - a|} \leq c\,a^{-2\alpha} M^{\alpha - 1}, \quad \text{if } M \geq 2a \tag{4.31}$$

Let $\lambda \in \gamma_\infty$ with $a \leq 100$. Then, due to (4.23),

$$\tau_0^\alpha \sum_{x_j \leq 1} \frac{1}{|x_j - \lambda|} \leq c\,\tau_0^\alpha |\{j : y_j \leq \frac{1}{\tau_0}\}| \leq c'$$

while, due to (4.22),

$$\tau_0^\alpha \sum_{x_j \geq 1} \frac{1}{|x_j - \lambda|} \leq c\,\tau_0^\alpha \sum_{x_j \geq 1} \frac{1}{x_j} \leq c' \tau_0^{\alpha - 1} \sum_{n \geq \lfloor \tau_0^{-\alpha/\beta} \rfloor} n^{\beta - 1 - \frac{\beta}{\alpha}} \leq c'' \tag{4.32}$$

thus proving (4.25). The proof of (4.26) follows the same arguments of (4.32). (4.27) is a simple consequence of (4.23). The l.h.s. of (4.29) can be bounded by $\tau_0^\alpha |\{j : a-1 \le \tau_0 y_j \le a+1\}|$ and (4.22) allows to conclude the proof of (4.29).

The proof of (4.28), (4.30) and (4.31) can be easily derived from the following estimate. Let $1 \le A \le B$ with $B \le a-1$ or $A \ge a+1$, then (4.22) implies

$$\begin{aligned}\tau_0^\alpha \sum_{A \le x_i \le B} \frac{1}{|x_i - a|} &\le c\,\tau_0^\alpha \sum_{n=n_-}^{n=n_+} \frac{n^{\beta-1}}{|\tau_0 n^{\beta/\alpha} - 1|} \le c'\tau_0^\alpha \int_u^v \frac{x^{\beta-1}}{|a - \tau_0 x^{\beta/\alpha}|} dx \\ &= c'\, a^{-1-\alpha} \int_{u(\frac{\tau_0}{a})^{\alpha/\beta}}^{v(\frac{\tau_0}{a})^{\alpha/\beta}} \frac{y^{\beta-1}}{|1 - y^{\beta/\alpha}|} dy\end{aligned}$$

where $n_- = \lfloor (A/\tau_0)^{\alpha/\beta} \rfloor - 1$, $n_+ = \lfloor (B/\tau_0)^{\alpha/\beta} \rfloor + 1$, $u = n_- - 1$, $v = n_+ + 1$ (we assume τ_0 small enough in order to exclude the singular point in the above intervals of sum and integral). ☐

Proof of Proposition 4.6. In order to avoid confusion we underline here the dependence on $\tau_0 = e^{E_0}$ by writing $\Pi_{E,E_0}(t, t_w)$ in place of $\Pi_E(t, t_w)$. Our starting point is given by the integral representation (2.12):

$$\Pi_{E,E_0}(t, t_w) = \frac{1}{2\pi i} \int_{\gamma_E} \frac{e^{-t_w \lambda}}{\lambda} \frac{\tau_0^\alpha \sum_{j=1}^{N_E} \frac{e^{-x_j t}}{x_j - \lambda}}{\tau_0^\alpha \sum_{j=1}^{N_E} \frac{1}{x_j - \lambda}} d\lambda. \tag{4.33}$$

Let us choose $\underline{E}$ satisfying Lemma 4.8. Then, due to the exponential decaying factor $e^{-t_w \lambda}$ and to Lemma 4.8, if $\tau_0 \le \tau_0^*$ and E is small enough such that $\tau_0 e^{-E} \ge 1$, the path integral γ_E in (4.33) can be substituted with γ_∞. At this point, (4.14) follows from Lemma 4.8 and the Dominated Convergence Theorem.

In order to prove (4.15), given a positive integer M, we set

$$g_{M,E_0}(t, t_w) := \frac{1}{2\pi i} \int_{\Gamma_M} \frac{e^{-t_w \lambda}}{\lambda} \frac{\tau_0^\alpha \sum_{x_j \le M} \frac{e^{-x_j t}}{x_j - \lambda}}{\tau_0^\alpha \sum_{x_j \le M} \frac{1}{x_j - \lambda}}\, d\lambda$$

where Γ_M is the positive oriented path having support

$$\operatorname{supp}(\Gamma_M) = \{\lambda \in \mathbb{C} : |\lambda - x| = 1 \text{ for some } x \in [0, M]\}.$$

Then, by applying Lemma 4.8, whenever $\tau_0 \le \tau_0^*$

$$\big| \lim_{E \downarrow -\infty} \Pi_{E,E_0}(t, t_w) - g_{M,E_0}(t, t_w) \big| \le c M^{\alpha-1} \ln M, \qquad \forall M \in \mathbb{N}_+.$$

Let us assume that $\underline{E}$ satisfies (4.16) for all $M \in \mathbb{N}_+$ and for $f(x) = \frac{e^{-xt}}{x-\lambda}$ or $f(x) = \frac{1}{x-\lambda}$, for all λ in a countable dense set of Γ_M and for all rational positive t. Then, by a chaining argument, we get

$$\lim_{E_0 \downarrow -\infty} g_{M,E_0}(t, t_w) = g_M(t, t_w), \qquad \forall t, t_w > 0$$

where

$$g_M(t, t_w) = \frac{1}{2\pi i} \int_{\Gamma_M} \frac{e^{-t_w \lambda}}{\lambda} \frac{\int_0^M \frac{e^{-xt}}{\lambda - x} x^{\alpha-1} dx}{\int_0^M \frac{1}{\lambda - x} x^{\alpha-1} dx} d\lambda, \tag{4.34}$$

thus implying

$$\limsup_{E_0 \downarrow -\infty} \big| \lim_{E \downarrow -\infty} \Pi_{E,E_0}(t, t_w) - g_M(t, t_w) \big| \le M^{\alpha-1} \ln M, \qquad \forall M \in \mathbb{N}_+.$$

At this point let us observe (see the proof of Lemma 4.8) that there exist $c, c' > 0$ such that for all $M \in \mathbb{N}_+$:

$$\begin{aligned} &\Big|\int_0^M \frac{x^{\alpha-1}}{\lambda - x} ds\Big| \geq c\,|\lambda|^{-2}, \qquad \forall \lambda \in \Gamma_M \cup \gamma_\infty \\ &\int_0^M \frac{x^{\alpha-1}}{|\lambda - x|} dx \leq c', \qquad \forall \lambda \in \Gamma_M \cup \gamma_\infty \\ &\int_M^\infty \frac{x^{\alpha-1}}{|\lambda - x|} dx \leq c' M^{\alpha-1} \ln M, \qquad \forall \lambda \in \gamma_\infty . \end{aligned} \tag{4.35}$$

From the above estimates we infer

$$\big|g_M(t, t_w) - g(t, t_w)\big| \leq c\, M^{\alpha-1} \ln M \tag{4.36}$$

where

$$g(t, t_w) := \frac{1}{2\pi i} \int_{\gamma_\infty} \frac{e^{-t_w \lambda}}{\lambda} \frac{\int_0^\infty \frac{e^{-xt}}{\lambda - x} x^{\alpha-1} dx}{\int_0^\infty \frac{1}{\lambda - x} x^{\alpha-1} dx} d\lambda \tag{4.37}$$

Using the analytic properties of the integrand in the r.h.s. of (4.37), one can show that $g(t, t_w) = g(t/t_w, 1)$. In order to compute $g(\theta, 1)$, we observe that for a suitable positive constant c $|g(\theta s, s) - g_M(\theta s, s)| \leq cM^{\alpha-1} \ln M$, for any $s \geq 1$ (in fact, the constant c in (4.36) can be chosen uniformly if $t_w \geq 1$). By the results of Subsection 2.1 (compare (4.34) with $\Pi(t, t_w)$ in Proposition 2.5) we get

$$\lim_{s \uparrow \infty} g_M(\theta s, s) = \text{ r.h.s. of (4.15)}$$

thus concluding the proof. □

5. Other correlation function when $\tau_0 \downarrow 0$ after $E \downarrow -\infty$

As stated in Proposition 4.6, the standard time–time correlation function $\Pi_E(t, t_w)$ has trivial behaviour after taking the limits $E \downarrow -\infty$, $\tau_0 \downarrow 0$. For physical reasons, it is more natural to disregard jumps into states with physical waiting time $T_i = e^{E_i}$ much smaller than τ_0, since we assume that the physical instruments in the laboratory have sensibility up to the time unit τ_0. Therefore, let us fix $\delta > 0$ and consider here the more natural time–time correlation function

$$\Pi_E^{(1)}(t, t_w) = \mathbb{P}_E\big(\, x_E(u) \geq \delta \quad \forall u \in (t_w, t_w + t] :\; x_E(u) \neq x_E(u^-)\,\big).$$

The main result of this section if the following one:

Proposition 5.1. *For almost all $\underline{E}$*

$$\lim_{t_w \uparrow \infty} \sup_{t > 0} \overline{\lim_{E_0 \downarrow -\infty}} \; \overline{\lim_{E \downarrow -\infty}} \big|\, \Pi_E^{(1)}(t, t_w) - \Pi_E(t, t_w) \,\big| = 0. \tag{5.1}$$

In particular, for almost all $\underline{E}$,

$$\lim_{t_w \uparrow \infty} \lim_{E_0 \downarrow -\infty} \lim_{E \downarrow -\infty} \Pi_E^{(1)}\big(\theta t_w, t_w\big) = \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{1+\theta}}^1 u^{-\alpha}(1-u)^{\alpha-1} du, \qquad \forall \theta > 0. \tag{5.2}$$

Note that the correlation function $\Pi_E^{(1)}(t, t_w)$ is the analogous of $\Pi_N^{(1)}(t, t_w)$ of Proposition 3.1. As for the proof of Proposition 3.1, a useful observation is that, given $\delta > 0$,

$$\lim_{t \uparrow \infty} \lim_{\tau_0 \downarrow 0} \lim_{E \downarrow -\infty} \mathbb{P}_E\big(\, x_E(t) > \delta \,\big) = 0 \qquad \text{a.s.}. \tag{5.3}$$

We can prove a stronger result concerning to the phenomenon that with high probability the system visits deeper and deeper traps. In fact, note that by (2.13)

$$\mathbb{P}_E\big(x_E(t)>\delta\big)=\frac{1}{2\pi i}\int_{\gamma_E}\frac{e^{-t\lambda}}{\lambda}\frac{\sum_{j=1}^{N_E}\frac{\mathbb{I}_{x_j\geq\delta}}{x_j-\lambda}}{\sum_{j=1}^{N_E}\frac{1}{x_j-\lambda}}d\lambda.$$

Then, by reasoning as in the proof of Proposition 4.8 and using the results of Section 2.2, one can easily show the analogous of Proposition 2.9:

Proposition 5.2. *For almost all energy landscapes $\underline{E}$,*

$$\lim_{t\uparrow\infty}\lim_{\tau_0\downarrow 0}\lim_{E\downarrow-\infty}t^{1-\alpha}\mathbb{P}_E\big(x_E(t)>\delta\big)=\frac{B(\delta)}{c(\alpha)} \tag{5.4}$$

where

$$B(\delta):=\frac{\int_\delta^\infty x^{\alpha-2}dx}{\int_0^\infty\frac{x^{\alpha-1}}{1+x}dx},\qquad c(\alpha):=\int_0^\infty y^{\alpha-1}e^{-y}dy$$

Proof of Proposition 5.1. Trivially, (5.2) follows from (5.1) and Proposition 4.6. Since $E_0\downarrow-\infty$ after $E\downarrow-\infty$, we assume that $E<E_0$ and define

$$D_{E,E_0}:=\{i\,:E\leq E_i\leq E_0\}.$$

This set corresponds to the small traps where we allow the particle to jump in. By (5.3) and the same arguments used in the proof of Proposition 3.1 for $i=1$ (with exclusion of the last step since here $\lim_{E\downarrow-\infty}\frac{|D_{E,E_0}|}{N_E}=1$ a.s.) it is simple to derive the assertion of the proposition from Lemma 5.3. □

Lemma 5.3. *For almost all energy landscapes $\underline{E}$ there exist positive constants p,c (independent of E,E_0) satisfying the following property. Whenever $|\{i\,:\,E_i>E_0\}|>0$,*

$$\limsup_{E\downarrow-\infty}\frac{1}{N_E}\sum_{i\in D_{E,E_0}}\varphi_{E,E_0}(i,t)\leq c\,e^{-pt},\qquad\forall t>0, \tag{5.5}$$

where $D_{E,E_0}:=\{i\,:E\leq E_i\leq E_0\}$, for $E<E_0$, and the function φ_{E,E_0} is defined as

$$\varphi_{E,E_0}(i,u):=\mathbb{P}_E(Y_E(u)\in D_{E,E_0}\;\forall u\in[0,s]\,|\,Y_E(0)=i), \tag{5.6}$$

Proof. Let us assume that $E<E_0$, $|\{i\,:\,E_i>E_0\}|>0$ and, without loss of generality, $\delta=1$.

We fix $\ell>0$ such that $e^{-\alpha\ell}<\frac{1}{2}$ and define

$$W_{1,E}:=\{i\,:\,E\leq E_i<E+\ell\},\quad N_{1,E}:=|W_{1,E}|$$
$$W_{2,E}:=\{i\,:\,E+\ell\leq E_i\leq E_0\},\quad N_{2,E}:=|W_{2,E}|$$

Note that $D_{E,E_0}=W_{1,E}\cup W_{2,E}$ and that $N_E,N_{1,E},N_{2,E}$ are Poisson variables having expectations $e^{-\alpha E}$, $e^{-\alpha E}(1-e^{-\alpha\ell})$ and $e^{-\alpha E-\alpha\ell}-e^{-\alpha E_0}$. In particular, for almost all $\underline{E}$,

$$\lim_{E\downarrow-\infty}\frac{N_E}{e^{-\alpha E}}=1,\qquad p_1:=\lim_{E\downarrow-\infty}\frac{N_{1,E}}{N_E}=1-e^{-\alpha\ell},\qquad p_2:=\lim_{E\downarrow-\infty}\frac{N_{2,E}}{N_E}=e^{-\alpha\ell}<\frac{1}{2} \tag{5.7}$$

We observe that $\tilde{n}:=N_E-N_{1,E}-N_{2,E}$ is a positive integer independent of E and $x_i\geq e^{E_0-E-\ell}$ if $i\in W_{1,E}$, while $x_i\geq 1$ if $i\in W_{1,E}$. Let us introduce a new random walk

$Y_E^*(t)$ on $\mathcal{S}_E$ whose infinitesimal generator $\mathbb{L}_E^*$ is defined as $\mathbb{L}_E$ with x_i replaced by x_i^* defined as

$$x_i^* = \begin{cases} x_i & \text{if } i \notin D_{E,E_0}, \\ A_E := e^{E_0 - E - \ell} & \text{if } i \in W_{1,E}, \\ 1 & \text{if } i \in W_{2,E} \end{cases}$$

We denote by $\mathbb{P}_E^*$ the probability on path space associated to $Y_E^*(t)$ when having initial uniform distribution and we set

$$\varphi^*_{E,E_0}(i,u) := \mathbb{P}^*_E(Y^*_E(u) \in D_{E,E_0}\ \forall u \in [0,s] \,|\, Y^*_E(0) = i). \tag{5.8}$$

By a simple coupling argument, one gets $\varphi_{E,E_0}(i,u) \leq \varphi^*_{E,E_0}(i,u)$. In particular

$$\frac{1}{N_E} \sum_{i \in D_{E,E_0}} \varphi_{E,E_0}(i,t) \leq \Phi := \frac{1}{N_E} \sum_{i \in D_{E,E_0}} \varphi^*_{E,E_0}(i,t), \qquad \forall i \in D_{E,E_0}, \forall t \geq 0. \tag{5.9}$$

At this point it remains to estimate Φ. In order to simplify notation we write simply D, N, N_1, N_2, A, by dropping the index E. Moreover, we consider the following realization of the dynamics of Y_E^*: after arriving at a site i, the system waits an exponential time of parameter x_i^* and the it jumps to a point of $\mathcal{S}_E$ with uniform probability. In particular, jumps can be *degenerate*, i.e. initial and final sites can coincide.

We claim that

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3$$

where

$$\Phi_1 := \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \binom{k_1+k_2}{k_1} \Big(\frac{N_1}{N}\Big)^{k_1} \Big(\frac{N_2}{N}\Big)^{k_2+1} A^{k_1} \int_0^t du\, e^{-Au} \frac{u^{k_1-1}}{(k_1-1)!} e^{-(t-u)} \frac{(t-u)^{k_2}}{k_2!} \tag{5.10}$$

$$\Phi_2 := \sum_{k_1=0}^{\infty} \sum_{k_2=1}^{\infty} \binom{k_1+k_2}{k_1} \Big(\frac{N_1}{N}\Big)^{k_1+1} \Big(\frac{N_2}{N}\Big)^{k_2} A^{k_1} \int_0^t du\, e^{-Au} \frac{u^{k_1}}{k_1!} e^{-(t-u)} \frac{(t-u)^{k_2-1}}{(k_2-1)!} \tag{5.11}$$

$$\Phi_3 := \frac{N_1}{N} \exp\Big\{-At\Big(1 - \frac{N_1}{N}\Big)\Big\} + \frac{N_2}{N} \exp\Big\{-t\Big(1 - \frac{N_2}{N}\Big)\Big\} \tag{5.12}$$

The above identities can be derived from the probabilistic interpretation of k_1, k_2 as

$$k_i = |\{ \text{ jumps performed before time } t \text{ having starting point in } W_i\}|$$

and from the following simple identities:

$$\mathbb{P}\big(T_1 + T_2 + \cdots + T_n \in [z, z+dz)\big) = e^{-\kappa z} \kappa^n \frac{z^{n-1}}{(n-1)!} dz$$

$$\mathbb{P}\big(T_1 + T_2 + \cdots + T_n \leq z \text{ and } T_1 + T_2 + \cdots + T_n + T_{n+1} > z\big) = e^{-\kappa z} \kappa^n \frac{z^n}{n!}$$

where $z \geq 0$ and $T_1, T_2, \ldots$ are independent exponential variables with parameter κ.

Finally, we only need to prove that, for suitable positive constant $c, p > 0$,

$$\limsup_{E \downarrow -\infty} \Phi_i \leq c e^{-pt}, \qquad \forall i = 1,2,3. \tag{5.13}$$

We give the proof in the case $i = 1$, the case $i = 2$ is completely similar while the case $i = 3$ follows directly from (5.7).

We fix $\gamma : \alpha < \gamma < 1$, set $k_0 := A^\gamma$ and write

$$\Phi_1 = \Phi_1^{\leq k_0} + \Phi_1^{> k_0}$$

where $\Phi_1^{\leq k_0}$ is the contribution to Φ_1 of addenda in the r.h.s. of (5.10) with $1 \leq k_1 \leq k_0$ and $k_2 \geq 0$.

If $k_1 > k_0$ then

$$\Big(\frac{N_1}{N}\Big)^{k_1} = \Big(\frac{N-N_2}{N}\Big)^{k_1}\Big(1 - \frac{\tilde{n}}{N-N_2}\Big)^{k_1} \leq \Big(\frac{N-N_2}{N}\Big)^{k_1}\Big(1-\frac{1}{N}\Big)^{A^\gamma}$$

thus implying that

$$\Phi_1^{>k_0} \leq \Phi_1\Big(\frac{N-N_2}{N}, N_2\Big)\Big(1-\frac{1}{N}\Big)^{A^\gamma} \leq \Big(1-\frac{1}{N}\Big)^{A^\gamma} \downarrow 0 \qquad \text{as } E \downarrow -\infty.$$

where $\Phi_1\Big(\frac{N-N_2}{N}, N_2\Big)$ is defined as in the r.h.s. of (5.10) with N_1 replaced by $N - N_1$. Note that is does not exceed 1 since it corresponds to the probability of a certain event.

Let us now consider the term $\Phi_1^{\leq k_0}$. To this aim, since $\binom{k_1+k_2}{k_1} \leq 2^{k_1+k_2}$,

$$\Phi_1^{\leq k_0} \leq \sum_{k_1=1}^{k_0}\sum_{k_2=0}^{\infty}\Big(\frac{2AN_1}{N}\Big)^{k_1}\Big(\frac{2tN_2}{N}\Big)^{k_2}\frac{I(0,t)}{(k_1-1)!k_2!}$$

where

$$I(w_1,w_2) := \int_{w_1}^{w_2} e^{-(t-u)-Au}u^{k_1-1}du.$$

Fix $m > 0$ with $2p_2 + 2mp_1 < 1$ (recall that $2p_2 < 1$). Then trivially

$$I(0,\frac{mt}{A}) \leq \frac{1}{k_1}e^{-t\left(1-\frac{m}{A}\right)}\Big(\frac{mt}{A}\Big)^{k_1} \tag{5.14}$$

From such a bound, one gets immediately

$$\sum_{k_1=1}^{k_0}\sum_{k_2=0}^{\infty}\Big(\frac{2AN_1}{N}\Big)^{k_1}\Big(\frac{2tN_2}{N}\Big)^{k_2}\frac{1}{(k_1-1)!k_2!}I(0,\frac{mt}{A}) \leq c\,\exp\Big\{-t\Big(1-\frac{m}{A}-2\frac{N_1}{N}m-2\frac{N_2}{N}\Big)\Big\} \tag{5.15}$$

The last expression, when $E \downarrow -\infty$, converges to $c\exp\big(-t(1-2p_1m-2p_2)\big)$, in agreement with (5.13).

In order to estimate the integral $I(\frac{mt}{A},t) = e^{-t}\int_{\frac{mt}{A}}^{t} e^{-(A-1)u}u^{k_1-1}$, we observe that

$$\int_s^w e^{-zu}u^n du = (-1)^n\frac{d}{dz^n}\frac{e^{-zs}-e^{-zw}}{z} \leq \frac{e^{-zs}}{z}(s+\frac{1}{z})^n + \frac{e^{-zw}}{z}(w+\frac{1}{z})^n, \qquad \forall s,w,z \geq 0,$$

thus implying the bound

$$I(\frac{mt}{A},t) \leq ce^{-t}A^{-k_1}(mt+1)^{k_1-1} + e^{-At}(A-1)^{-1}(t+\frac{1}{A-1})^{k_1-1}$$

The contribution of $ce^{-t}A^{-k_1}(mt+1)^{k_1-1}$ to $\Phi_1^{>k_0}$ can be treated by means of estimates similar to the ones leading to (5.15).

In order to conclude we only need to show that

$$e^{-At}\sum_{k_1=1}^{A^\gamma}\sum_{k_2=1}^{\infty}\Big(\frac{2AtN_1}{N}\Big)^{k_1-1}\Big(\frac{2tN_2}{N}\Big)^{k_2}\frac{1}{(k_1-1)!k_2!} \downarrow 0, \qquad \text{as } E \downarrow -\infty. \tag{5.16}$$

To this aim observe that

$$\begin{aligned}\text{r.h.s. of } (5.16) &\le e^{-At+2t\frac{N_2}{N}}\sum_{k_1=0}^{A^\gamma}\Big(\frac{2AtN_1}{N}\Big)^{k_1}\\ &\le c(t)e^{-At}A^\gamma(4tA)^{A^\gamma} = c(t)\exp\Big\{-At+\gamma\ln A + A^\gamma\ln(2tA)\Big\}\end{aligned}$$

Since $0<\gamma<1$, we get (5.16), thus concluding the proof. □

Appendix A. Laplace Transform

Proposition A.1. *Let $G(t)$ be a bounded measurable function on $(0,\infty)$ and let us consider the Laplace transform*

$$\hat G(\omega) = \int_0^\infty G(t)e^{-t\omega}dt$$

well defined if $\Re(\omega)>0$. Let us define

$$\mathcal{A} := \{re^{i\theta}\ :\ 0<r<\infty,\ |\theta|\le \frac{3}{4}\pi\} \tag{A.1}$$

Suppose that $\hat G$ can be analytically continued to $\mathbb{C}\setminus(-\infty,0]$ and that there are positive constants γ,β,α,c and $B\in\mathbb{R}$ such that

$$|\,\hat G(\omega)\,| \le c\,|\omega|^{-\gamma} \qquad \forall \omega\in\mathcal{A},\ |\omega|\ge 1, \tag{A.2}$$

$$|\omega^\beta \hat G(\omega) - B\,| \le c\,|\omega|^\alpha \quad \forall\omega\in\mathcal{A},\ |\omega|\le 1. \tag{A.3}$$

Then,

$$\lim_{s\uparrow\infty} s^{1-\beta}G(s) = \frac{B}{c(\beta)} \text{ where } c(\beta) := \int_0^\infty y^{\beta-1}e^{-y}dy.$$

Proof. If we set $H(s) := \frac{B}{c(\beta)}s^{\beta-1}$ with $s>0$, then the Laplace transform $\hat H(\omega)$ is well defined for $\Re(\omega)>0$, $\hat H(\omega) = B\omega^{-\beta}$ and trivially $\hat H(\omega)$ can be analytically continued to $\mathbb{C}\setminus(-\infty,0]$.

By the inverse formula for Laplace transform (see Chapter 4, Section 4 in [16]), we have

$$G(s) = \lim_{K\to\infty}\frac{1}{2\pi i}\int_{x-iK}^{x+iK} e^{s\omega}\hat G(\omega)d\omega, \qquad \forall s>0,\ x>0, \tag{A.4}$$

where ω runs over the vertical path connecting $x-iK$ and $x+iK$. The above formula remains true if substituting G with H. Therefore,

$$s^{1-\beta}G(s) - \frac{B}{c(\beta)} = \lim_{K\to\infty}\frac{s^{1-\beta}}{2\pi i}\int_{x-iK}^{x+iK} e^{s\omega}\big(\hat G(\omega)-\hat H(\omega)\big)d\omega, \quad \forall s>0,\ x>0, \tag{A.5}$$

Let $\rho := \min(\gamma,\beta)/2$. Fix a positive number x and, given K and s, define the following paths (see figure below).

γ_K is the vertical path from $x-iK$ to $x+iK$. $\gamma_{1,+}$ is the segment from $-s^{-1}+s^{-1}i$ to $-1+i$. $\gamma_{2,+}$ is an arc from $-1+i$ to $-K^\rho+iK$ given by the parametrisation $z(t) = -t+it^{1/\rho}$ with $t\in[1,K^\rho]$. $\gamma_{3,+}$ is the horizontal segment from $-K^\rho+iK$ to $x+iK$. For $i=1,2,3$ we define the path $\gamma_{i,-}$ by considering the reflection of $\gamma_{i,+}$ w.r.t. the real axis and inverting the orientation. Let γ_0 be the positive–oriented circular arc of radius s^{-1} from $-s^{-1}-s^{-1}i$ to $-s^{-1}+s^{-1}i$ crossing the axis of positive real numbers.

Note that the above paths depend on s and/or K.

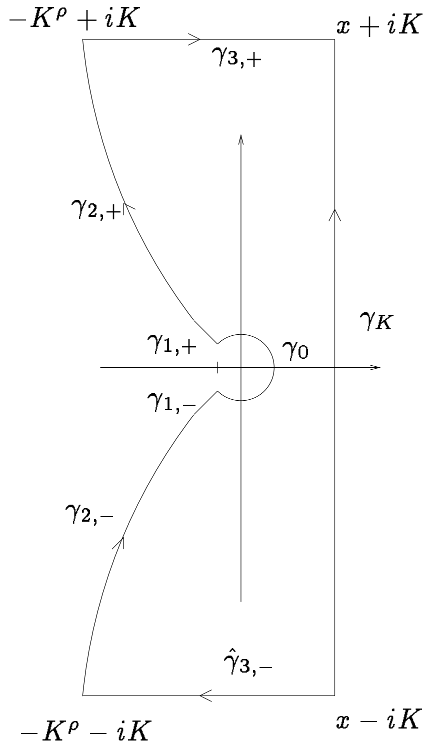


FIGURE 1. The integration paths

Because of analyticity, the integral over γ_K of $e^{s\omega}\hat{G}(\omega)$ is equal to the sum of the integrals over $\gamma_{3,-}$, $\gamma_{2,-}$, $\gamma_{1,-}$, γ_0, $\gamma_{1,+}$, $\gamma_{2,+}$, $\gamma_{3,+}$. The same is valid with $\hat{G}$ replaced with $\hat{H}$.

By (A.2) we have that

$$\int_{\gamma_{3,\pm}} |e^{s\omega}\hat{G}(\omega)|\,|d\omega| \le c\,e^{sx}K^{\rho-\gamma} \downarrow 0 \qquad \text{as } K\uparrow\infty \tag{A.6}$$

and, for a suitable rational function f,

$$\begin{aligned} s^{1-\beta}\int_{\gamma_{2,\pm}} |e^{s\omega}\hat{G}(\omega)|\,|d\omega| &\le s^{1-\beta}\int_1^{\infty}\left|e^{s(-t+it^{1/\rho})}\hat{G}(-t+it^{1/\rho})(-1+\rho^{-1}t^{\frac{1-\rho}{\rho}}i)\right|dt \\ &\le c\,s^{1-\beta}\int_1^{\infty} e^{-st}f(t)dt \le c's^{1-\beta}e^{-\frac{s}{2}}\downarrow 0, \qquad \text{as } s\uparrow\infty. \end{aligned} \tag{A.7}$$

Similarly, it can be proved that the corresponding integrals with $\hat{G}$ substituted with $\hat{H}$ go to 0 by taking the limits $K\uparrow\infty$, $s\uparrow\infty$.

Let us now estimate $s^{1-\beta}\int_{s^{-1}}^{1} e^{-st}t^{\alpha-\beta}dt$ by dividing the path of integration in two paths. Choosing $0<\delta<1$,

$$s^{1-\beta}\int_{s^{-1}}^{s^{-1+\delta}} e^{-st}t^{\alpha-\beta}dt \le c\,s^{1-\beta}\left|s^{(-1+\delta)(1+\alpha-\beta)} - s^{-(1+\alpha-\beta)}\right| \le c's^{-\alpha} + c's^{-\alpha+\delta(1+\alpha-\beta)},$$

$$s^{1-\beta}\int_{s^{-1+\delta}}^{1} e^{-st}t^{\alpha-\beta}dt \le s^{1-\beta}e^{-s^{\delta}}g(s)$$

for a suitable rational function $g(s)$. In particular, choosing δ small enough, the above upper bounds imply

$$\lim_{s\uparrow\infty} s^{1-\beta}\int_{s^{-1}}^{1} e^{-st}t^{\alpha-\beta}dt = 0.$$

This result, together with (A.3), implies

$$\lim_{s\uparrow\infty} s^{1-\beta}\frac{1}{2\pi i}\int_{\gamma_{1,\pm}} e^{s\omega}\big(\hat G(\omega)-\hat H(\omega)\big)d\omega = 0.$$

Trivially, by (A.3),

$$s^{1-\beta}\Big|\int_{\gamma_0} e^{s\omega}\big(\hat G(\omega)-\hat H(\omega)\big)d\omega\Big| \le s^{-\alpha}\downarrow 0 \qquad \text{as } s\uparrow\infty.$$

The Proposition now follows from the estimates above and (A.5). □

APPENDIX B. PERTURBATION THEORY

In this appendix we comment on a paper by Melin and Butaud [25] where the eigenvalues and eigenfunctions of the generator of our model were computed using perturbation theory. As pointed out earlier, these results are at variance with our exact results, and it may be worthwhile to point out the flaw in their arguments. Melin and Butaud write the generator $\mathbb{L}$ defined in (2.3) as $\mathbb{L} = T + \frac{1}{N}T^{(1)}$ where

$$T := \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_N \end{pmatrix}, \qquad T^{(1)} := \begin{pmatrix} -x_1 & -x_1 & \dots & -x_1 \\ -x_2 & -x_2 & \dots & -x_2 \\ \vdots & \vdots & \ddots & \vdots \\ -x_N & -x_N & \dots & -x_N \end{pmatrix} \tag{B.1}$$

The factor $1/N$ in front of the second term encourages them to consider this term as a small perturbation. Both T and $T^{(1)}$ are symmetric operators on $L^2(\mu)$ where $\mu(i) := x_i^{-1}$. We denote $< \cdot,\cdot >$ the scalar product in $L^2(\mu)$ and assume $x_1,\dots,x_N$ to be distinct positive numbers.

Given an operator $A : L^2(\mu) \to L^2(,\mu)$ we write $||A||$ for its operator norm. Because of symmetry, $||T||$ and $||T^{(1)}||$ are given by the maximum of $|\lambda|$, with λ eigenvalue. Trivially, T has eigenvalues $x_1,\dots,x_N$ and $T(e_i) = x_ie_i$ where $e_1,\dots,e_N$ is the canonical basis of $\mathbb{R}^N$, while $T^{(1)}$ has eigenvalues $0, -(x_1+x_2+\dots+x_N)$.

Given $z \in \mathbb{C}$ we can define the holomorphic function $T(z) = T + zT^{(1)}$. A natural condition in order to apply perturbation theory to $T(z)$ (see [23], chapter II) is

$$|z| < \frac{d}{2a_0} \tag{B.2}$$

where

$$d = \inf_{i\neq j}|x_i - x_j|, \qquad a_0 := \min_{a\in\mathbb{R}} ||T^{(1)} - a|| = \frac{x_1+x_2+\dots+x_N}{2}.$$

In this case, we can conclude that $T(z) = T + zT^{(1)}$ has N eigenvalues $\lambda_1(z), \dots, \lambda_N(z)$ with $\lambda_k(z) = \sum_{n=0}^{\infty} \lambda_k^{(n)} z^n$, $|\lambda_k^{(n)}| \leq a_0^n (2/d)^{n-1}$ and

$$\lambda_k^{(1)} = \frac{< T^{(1)} e_k, e_k >}{< e_k, e_k >}$$

$$\lambda_k^{(2)} = \sum_{j \neq k} (x_k - x_j)^{-1} \frac{< T^{(1)} e_k, e_j >^2}{< e_k, e_k >< e_j, e_j >}$$

$$\dots$$

Similar series exist for the perturbed eigenvectors.

However, the crucial condition (B.2) is hardly satisfied when $z = \frac{1}{N}$, since it reads

$$\mathrm{Av}_{j=1}^N x_j \leq \inf_{i \neq j} |x_i - x_j| \tag{B.3}$$

while a.s. the l.h.s. of (B.3) has non zero limit and the r.h.s. converges to 0 like $1/N$.

The fact that the conditions for the application of perturbation theory are violated explains why its predictions are incorrect. This discrepancy happens not to be too obvious as far as the eigenvalues are concerned (which are caught between the diagonal elements of the generator and thus are somewhat similar to them, but the shape of the eigenfunctions is sharply different).

Namely, by of Proposition 2.1, when $j \neq 1$ and x_{j-1}, x_j are very near each other, the eigenvector $\psi^{(j)}$ related to the eigenvalue $\lambda_j : x_{j-1} < \lambda_j < x_j$ has two main peaks of opposite sign given by $\psi_{j-1}^{(j)}$ and $\psi_j^{(j)}$, this is very different from the predictions of [25] (see their Figure 4).

Appendix C. Complex integral representation

Let $\mathbb{L}$ be a Markov generator on the state space $\mathcal{S} := \{1, 2, \dots, N\}$, reversible w.r.t. a positive measure μ. We can think of $\mathbb{L}$ as a linear operator on $\mathbb{R}^N$, symmetric w.r.t. to the scalar product $(\cdot, \cdot)_\mu$ where

$$(a, b)_\mu = \sum_{i=1}^N \mu(i) a_i b_i$$

In what follows we endow $\mathbb{R}^N$ with the scalar product $(\cdot, \cdot)_\mu$ (and not with the standard Euclidean scalar product). Since $\mathbb{L}$ is symmetric, we can orthogonally decompose $\mathbb{R}^N$ as $\mathbb{R}^N = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ such that $\mathbb{L} = \sum_{k=1}^m \lambda_k P_{W_k}$, where P_{W_k} denotes the orthogonal projection on W_k and $\lambda_i \neq \lambda_j$ if $i \neq j$. Given $\lambda \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_m\}$, we write $R(\lambda)$ for the resolvent

$$R(\lambda) := (\lambda \mathbb{I} - \mathbb{L})^{-1} = \sum_{k=1}^m \frac{1}{\lambda - \lambda_k} P_{W_k}.$$

Then, the Residue Theorem implies the integral representation

$$e^{-t\mathbb{L}} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} R(\lambda) d\lambda \tag{C.1}$$

where γ is a positive oriented loop containing in its interior $\lambda_1, \lambda_2, \dots, \lambda_m$.

Given a probability measure ν on $\mathcal{S}$, we denote by $\mathbb{P}_\nu$ the probability measure on the path space associated to the continuous–time random walk $Y(t)$ on $\mathcal{S}$ with generator $\mathbb{L}$ and initial distribution ν. Fix $j \in \mathcal{S}$ and let $v \in \mathbb{R}^N$ be such that $v_i = \delta_{i,j}$. We write

$\frac{d\nu}{d\mu}$ for the Radon derivate, i.e. $\frac{d\nu}{d\mu}(i)=\frac{\nu(i)}{\mu(i)}$. Then the symmetry of $\mathbb{L}$ w.r.t. the scalar product $(\cdot,\cdot)_\mu$ implies

$$\begin{aligned}\mathbb{P}_\nu(\,Y(t)=j\,) &= \sum_{k=1}^{N}\nu(k)\big(e^{-t\mathbb{L}}\big)_{k,j} = \mu\big(\frac{d\nu}{d\mu}, e^{-t\mathbb{L}}v\big) = \mu\big(e^{-t\mathbb{L}}\frac{d\nu}{d\mu}, v\big)\\ &= \mu(j)\sum_{k=1}^{N}\big(e^{-t\mathbb{L}}\big)_{j,k}\frac{\nu(k)}{\mu(k)}.\end{aligned} \tag{C.2}$$

By plugging (C.1) in the r.h.s. of (C.2) we get the integral representation

$$\mathbb{P}_\nu(\,Y(t)=j\,)=\frac{1}{2\pi i}\int_\gamma e^{-t\lambda}\{\sum_{k=1}^{N}\mu(j)R_{jk}(\lambda)\frac{\nu(k)}{\mu(k)}\}d\lambda. \tag{C.3}$$

In particular, given h function on $\mathcal{S}$

$$\mathbb{E}_{\mathbb{P}_\nu}\big(\,h(Y(t))\,\big)=\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{2\pi i}h(j)\mu(j)\frac{\nu(k)}{\mu(k)}\int_\gamma e^{-t\lambda}R_{jk}(\lambda)d\lambda$$

thus allowing to get an integral representation for $\Pi(t,t_w):=\mathbb{P}_\nu($ no jump in $[t_w,t_w+t])$. If we set $\mu(i)=\tau_i=x_i^{-1}$ and $\nu(i)=N^{-1}$ (uniform initial probability), then

$$\mathbb{P}_\nu(\,Y(t)=j\,)=\frac{1}{2\pi i}\int_\gamma e^{-t\lambda}\{\frac{1}{N}\sum_k\frac{x_k}{x_j}R_{jk}(\lambda)\}d\lambda. \tag{C.4}$$

Let us consider now the special case given by the Bouchaud's REM–like trap model where $\mathbb{L}:=\mathbb{L}_N$ is defined in (2.3) and ν is the uniform distribution on $\mathcal{S}$. Note the all the integral formulas obtained in Section 2 can be derived from the following one:

$$\mathbb{P}_\nu(\,Y(t)=j\,)=\frac{1}{2\pi i}\int_\gamma e^{-t\lambda}\frac{1}{(\lambda-x_j)\phi(\lambda)}d\lambda \tag{C.5}$$

where $\phi(\lambda)=\sum_{k=1}^{N}\frac{\lambda}{\lambda-x_k}$. In what follows we prove that (C.5) corresponds to (C.4).

We know already that $\det(\lambda\mathbb{I}-\mathbb{L})$ has distinct zeros given by the N distinct zeros of $\phi(\lambda)$. In particular, it must be

$$\det(\lambda\mathbb{I}-\mathbb{L})=\frac{1}{N}\phi(\lambda)\prod_j(\lambda-x_j)=\frac{1}{N}\lambda\sum_k\prod_{j:\,j\neq k}(\lambda-x_j) \tag{C.6}$$

Given a matrix A we write $[A]_{i,j}$ for the determinant of the matrix obtain from A by erasing the i–th row and the j– column. Since

$$R_{j,k}(\lambda)=(-1)^{j+k+1}\frac{[\lambda\mathbb{I}-\mathbb{L}]_{k,j}}{\det(\lambda\mathbb{I}-\mathbb{L})}$$

and due to (C.6), in order to derive (C.5) from (C.4) we only have to show that

$$\sum_k(-1)^{j+k+1}\frac{x_k}{x_j}[\lambda\mathbb{I}-\mathbb{L}]_{k,j}=\prod_{s:\,s\neq j}(\lambda-x_s) \tag{C.7}$$

In order to prove the above identity observe that $[\lambda\mathbb{I}-\mathbb{L}]_{k,j}$ is a polynomial of degree $N-1$ if $k=j$, otherwise it has degree $N-2$. The l.h.s. of (C.7) is a monomic polynomial of degree $N-1$. At this point we only have to verify that x_s, $s\neq j$, are zeros of the l.h.s. of (C.7). This is trivial if one observes that the l.h.s. of (C.7) is the determinant of the matrix obtain from $\lambda\mathbb{I}-\mathbb{L}$ by replacing the j–column with the vector w with $w_i = \frac{x_i}{x_j}$ for $i = 1, 2, \dots, N$. It is simple to verify that, if $\lambda = x_s$ for some $s \neq j$, the j–th row and the s–th row in such a matrix are proportional, thus implying the thesis.

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