Perturbation theory of W^*-dynamics, Liouvilleans and KMS-states

J. Dereziński[1], V. Jakšić[2] and C.-A. Pillet[3,4]

[1]Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warszawa, Poland

[2]Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, QC, H3A 2K6, Canada

[3]PHYMAT
Université de Toulon, B.P. 132
F-83957 La Garde Cedex, France

[4] CPT-CNRS Luminy, Case 907
F-13288 Marseille Cedex 9, France

March 4, 2003

Abstract

Given a W^*-algebra $\mathfrak{M}$ with a W^*-dynamics τ, we prove the existence of the perturbed W^*-dynamics for a large class of unbounded perturbations. We compute its Liouvillean. If τ has a β-KMS state, and the perturbation satisfies some mild assumptions related to the Golden-Thompson inequality, we prove the existence of a β-KMS state for the perturbed W^*-dynamics. These results extend the well known constructions due to Araki valid for bounded perturbations.

1 Introduction

1.1 W^*-dynamics and KMS states

Let $\mathfrak{M}$ be a W^*-algebra equipped with a W^*-dynamics (a 1-parameter pointwise σ-weakly continuous group of $*$-automorphisms) $\mathbb{R} \ni t \mapsto \tau^t$. The pair $(\mathfrak{M}, \tau)$ is often called a W^*-dynamical system. Let Q be a self-adjoint element of $\mathfrak{M}$. A well known convergent power series expansion, that can be traced back at least to Schwinger and Dyson, can be used to define the perturbed W^*-dynamics which we denote by $\mathbb{R} \ni t \mapsto \tau_Q^t$. The difference of the generators of τ_Q and τ equals $\mathrm{i}[Q, \cdot]$—in fact, the W^*-dynamics τ_Q is uniquely characterized by this property.

Suppose in addition that $\beta > 0$ and that τ possesses a β-KMS state ω. Araki proved that in this case the dynamics τ_Q also possesses a canonical β-KMS state ω_Q. More precisely, if $\omega(A) = (\Omega|A\Omega)$, where Ω is the vector representative of the state ω in the standard positive cone, and L is the so-called Liouvillean of τ, then the vector $\Omega_Q := \mathrm{e}^{-\beta(L+Q)/2}\Omega$ is well defined and the state $\omega_Q(A) := (\Omega_Q|A\Omega_Q)/\|\Omega_Q\|^2$ is β-KMS for the W^*-dynamics τ_Q.

The above two constructions play an important role in applications of operator algebras to quantum statistical physics. Whereas the construction of the perturbed W^*-dynamics τ_Q is relatively easy and not very surprising, the construction of the perturbed KMS state ω_Q is more subtle and has a far-reaching physical importance. The both constructions, however, have one technical weakness which restricts the range of their applications: the perturbation Q is assumed to be bounded. In many physical applications the operator Q is unbounded and is only affiliated to $\mathfrak{M}$.

In this paper we extend the construction of the perturbed W^*-dynamics τ_Q and the (τ_Q, β)-KMS state ω_Q to a large class of unbounded perturbations Q affiliated to $\mathfrak{M}$. An application of these results is discussed in [DJ2] and concerns spectral and ergodic theory of Pauli-Fierz systems.

The proof of the first result—the construction of τ_Q—is again relatively simple and does not involve much more than an application of the Trotter product formula. The proof of the second result—the construction of ω_Q—is more involved. Its main idea is the use of the so-called Golden-Thompson inequality. The Golden-Thompson inequality in its original form says that if A and B are self-adjoint matrices, then

$$\mathrm{Tr}\, \mathrm{e}^{A+B} \leq \mathrm{Tr}\, \mathrm{e}^{A}\mathrm{e}^{B}.$$

Translated into the language of W^*-algebras and KMS states, the Golden-Thompson inequality can be put into the form

$$\|\Omega_Q\| \leq \|\mathrm{e}^{-\beta Q}\Omega\|. \tag{1.1}$$

In our approach, the Golden-Thompson inequality is used to control the perturbed KMS-states and gives an upper bound, which combined with a weak convergence argument enables us to construct Ω_Q for a large class of unbounded Q.

In the literature there exists a different approach to the construction of the perturbed KMS states for unbounded perturbations, which is restricted to perturbations bounded from below. One of its versions has been developed by Sakai [Sa2]; another version (applicable to generalized positive operators which may not have a dense domain) is due to Donald [Don] (his method is also discussed in monograph [OP]). The Sakai-Donald theory does not cover perturbations which are unbounded from both sides, and in particular is not applicable to Pauli-Fierz systems.

The W^*-algebraic form (1.1) of the Golden-Thompson inequality was first proven by Araki [Ar2]. A different proof, based on an application of Uhlmann's monotonicity theorem for the relative entropy [Uh], was given in [Don].

1.2 Liouvilleans

The term Liouvillean has become quite popular in the recent literature on algebraic quantum statistical physics. The meaning of this term can vary depending on the author. Therefore, we would like to devote some space to a discussion of possible meanings of the term Liouvillean in the context of W^*-dynamical systems.

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system. It is often important to construct a representation of $\mathfrak{M}$ equipped with a unitary implementation of the W^*-dynamics τ. There are two natural approaches to such construction.

The first approach presupposes that τ has an invariant normal state ω. In the corresponding GNS representation this state is represented by a cyclic vector Ω. Then it is easy to see that there exists a unique self-adjoint operator L such that

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \qquad L\Omega = 0.$$

The operator L defined this way can be called the Ω-Liouvillean of τ.

In the second approach one chooses a standard representation of $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$. One of the objects that go together with the standard representation is the positive cone $\mathcal{H}^+$. A general theory of standard representations implies that there exists a unique self-adjoint operator L such that

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \qquad \mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ \subset \mathcal{H}^+.$$

The operator L defined in this way can be called the standard Liouvillean of τ, or simply the Liouvillean of τ.

The two setups overlap if the invariant state ω is faithful and $\Omega \in \mathcal{H}^+$. In this case the Ω-Liouvillean of τ coincides with the standard Liouvillean of τ. This fact is important for applications of W^*-algebras to quantum statical physics.

If one is interested in the case of equilibrium, then the first approach to Liouvillean suffices. In non-equilibrium situations one needs the second approach.

The (standard) Liouvillean encodes in a particularly convenient way the properties of the dynamics. This has been demonstrated in many places in the recent literature

[BFS, DJ2, JP1, JP2, M]. The Liouvillean is also one of the main technical tools of our paper.

If L is the Liouvillean for the W^*-dynamics τ, then one may ask what is the Liouvillean for τ_Q. If Q is bounded, then the answer is $L_Q = L + Q - JQJ$, where J is the modular conjugation. We will establish the same result for unbounded Q under some mild technical assumptions.

1.3 Organization of the paper

We start our paper with a concise review of some aspects of the theory of W^*-algebras. The choice of topics is motivated by some recent applications of W^*-algebras to quantum statistical mechanics [BFS, DJ1, DJ2, JP1, JP2, M]. Among other things, we will discuss the two possible definitions of the Liouvillean. For most of the proofs in Section 2 the reader is referred to the literature, especially [BR1, BR2].

In Section 3 we describe the perturbation theory of W^*-dynamics and Liouvilleans. We describe in particular the case of unbounded perturbations, which goes beyond what we could find in the literature.

To make our paper more accessible, we have included in Section 4 the proof of the Uhlmann's monotonicity theorem [Uh] and Donald's proof of the Golden-Thompson inequality [Don]. A somewhat different presentation of this topic can be found in [OP].

Section 5 contains the perturbation theory of KMS states. The subject naturally splits into three levels. The most restrictive level concerns analytic perturbations. In this case the proofs are essentially algebraic and relatively simple. The next level concerns bounded Q. This is the case considered by Araki [Ar1], see also [KL, BR2, Sa1, Si]. Finally, we develop perturbation theory for a class of unbounded Q. In all the cases we prove a number of properties of Ω_Q, including the Peierls-Bogoliubov and the Golden-Thompson inequalities. We stress that the Golden-Thompson inequality is at the same time an important ingredient of our proof of the existence of Ω_Q. We also prove a number of estimates that can be used to compare the vectors Ω and Ω_Q. Some of these estimates appear to be new.

We have attempted to make the paper reasonably self-contained so that it can serve as a brief introduction to some recent works on algebraic quantum statistical physics. Our presentation is in some respects complementary to the presentation in the standard literature such as [BR1, BR2, OP]. In particular, we tried to emphasize the use of the standard representation and the Liouvillean.

In Appendix B we give a concise description of the Pauli-Fierz systems at positive densities. The material of this appendix is based on [DJ2]. We include this material at the request of referee to briefly explain the main physical motivation and application of the results of our paper.

Acknowledgments. The research of the first author was partly supported by the Postdoctoral Training Program HPRN-CT-2002-00277 and by the grant SPUB127 financed by Komitet Badań Naukowych. A part of this work was done during a visit of the first author at the Aarhus University supported by MaPhySto funded by the Danish National Research Foundation and during a visit to University of Montreal. The research of the second author was partly supported by NSERC. We wish to thank the referees for the careful reading of the manuscript and for numerous helpful remarks.

2 General facts about W^*-algebras

In this section we recall some basic definitions and facts about W^*-algebras which will play a role in our paper. For additional information and proofs we refer the reader to [BR1, BR2, Sa3, StZs, St].

There are two approaches to the theory of W^*-algebras: the concrete and the abstract approach. In the concrete approach one starts with the notion of a concrete W^*-algebra (called also a von Neumann algebra), defined as a $*$-algebra of bounded operators on a Hilbert space which equals its double commutant. This is in fact the original definition that dates back to the works of von Neumann. In the abstract approach, due to Sakai [Sa3], one defines an abstract W^*-algebra as a C^*-algebra that possesses a predual.

These approaches are essentially equivalent: every abstract W^*-algebra can be represented as a concrete W^*-algebra and every concrete W^*-algebra is an abstract W^*-algebra.

The concrete approach is historically the first and is used in most monographs, e.g. [BR1, BR2, StZs]. The abstract approach has been developed in [Sa3]. In some respects the abstract approach is more difficult from the pedagogical point of view—many basic properties of W^*-algebras are more difficult to show starting from Sakai's definition than starting from von Neumann's definition. Nevertheless, one can argue that Sakai's approach is conceptually superior: it helps to distinguish the notions that are intrinsic from the notions that are representation dependent. In our presentation we will stress the abstract approach.

2.1 Abstract W^*-algebras

If $\mathcal{X}$ is a Banach space, then a Banach space $\mathcal{Y}$ is called a predual of $\mathcal{X}$ iff $\mathcal{X}$ is isomorphic to the dual of $\mathcal{Y}$.

$\mathfrak{M}$ is an (abstract) W^*-algebra if it is a C^*-algebra which possesses a predual. It can be shown that every W^*-algebra $\mathfrak{M}$ possesses a unique predual (up to isomorphism). It will be denoted by $\mathfrak{M}_*$. Elements of $\mathfrak{M}_*$ will be called normal functionals on $\mathfrak{M}$.

The topology on $\mathfrak{M}$ generated by the seminorms $|\omega(A)|$, $\omega \in \mathfrak{M}_*$, is called the σ-weak topology. The topology on $\mathfrak{M}$ generated by the seminorms $|\omega(A^*A)|^{1/2}$, $\omega \in \mathfrak{M}_*$, is called the σ-strong topology.

$\mathfrak{M}_*^+$ denotes the set of positive elements of $\mathfrak{M}_*$. Elements of $\mathfrak{M}_*^+$ satisfying $\omega(1) = 1$ are called normal states. The set of normal states is denoted $\mathfrak{M}_*^{+,1}$.

Let $\omega \in \mathfrak{M}_*^+$ and let $\mathfrak{N}$ be a W^*-subalgebra of $\mathfrak{M}$. The support of ω with respect to $\mathfrak{N}$ is defined as

$$\mathrm{s}_\omega^{\mathfrak{N}} := \inf\{P \in \mathfrak{N} \ : P \text{ is an orthogonal projection and } \omega(1-P)=0\}.$$

In particular, the support with respect to $\mathfrak{M}$ will be called just the support of ω and denoted s_ω. The support of ω wrt the center of $\mathfrak{M}$ will be called the central support of ω and denoted z_ω.

$\omega \in \mathfrak{M}_*^+$ is called faithful iff $\mathrm{s}_\omega = 1$. A W^*-algebra is called σ-finite if it possesses a faithful state.

Let $\mathfrak{M}$, $\mathfrak{N}$ be W^*-algebras and $\pi : \mathfrak{M} \to \mathfrak{N}$ a homomorphism. We say that π is normal iff π is σ-weakly continuous.

2.2 Concrete W^*-algebras

Let $\mathcal{H}$ be a Hilbert space. $(\Psi|\Phi)$ will denote the scalar product of the vectors $\Psi, \Phi \in \mathcal{H}$. We adopt "physicist's convention" and our scalar product is antilinear with respect to the first argument.

If $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$, then the commutant of $\mathcal{C}$ will be denoted by $\mathcal{C}'$.

We will say that $\mathfrak{M}$ is a concrete W^*-algebra (or a von Neumann algebra) iff $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $\mathfrak{M}'' = \mathfrak{M}$. A concrete W^*-algebra in $\mathcal{B}(\mathcal{H})$ is a W^*-algebra inside $\mathcal{B}(\mathcal{H})$ containing the identity of $\mathcal{B}(\mathcal{H})$. Every abstract W^*-algebra is $*$-isomorphic to a concrete W^*-algebra.

Let $\mathfrak{M}$ be an abstract W^*-algebra and $\pi : \mathfrak{M} \mapsto \mathcal{B}(\mathcal{H})$ a representation. Then $\pi(\mathfrak{M})$ is a concrete W^*-algebra iff π is unital and normal.

Given an injective unital normal representation $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$, we will often identify $\mathfrak{M}$ with $\pi(\mathfrak{M})$.

2.3 Concrete affiliations

In the following two subsections we recall the concept of operators affiliated to a W^*-algebra. This concept is well-known in the case of concrete W^*-algebras, see e.g. [BR1].

Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a concrete W^*-algebra. Let A be a closed densely defined operator on $\mathcal{H}$ and $\mathcal{D}(A)$ its domain. We say that A is affiliated to $\mathfrak{M}$ iff for all $A' \in \mathfrak{M}'$, $A'\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AA' = A'A$, on $\mathcal{D}(A)$. Let $\mathfrak{M}^{(\eta)}$ be the set of operators affiliated to $\mathfrak{M}$.

Theorem 2.1 (1) *If A is self-adjoint on $\mathcal{H}$, then A is affiliated to $\mathfrak{M}$ iff all bounded Borel functions of A belong to $\mathfrak{M}$.*

(2) *If A is a closed operator, then A is affiliated to $\mathfrak{M}$ iff $A(1 + A^*A)^{-1/2} \in \mathfrak{M}$.*

2.4 Abstract affiliations

The concept of affiliation can be introduced for abstract W^*-algebras in a fashion independent of representations. Our definition of an operator affiliated to an abstract W^*-algebra is directly inspired by the definition of the affiliation in the context of C^*-algebras due originally to Baaj and Jungl [BaJu] and elaborated by Woronowicz [Wo]. We are grateful to S. L. Woronowicz for a discussion of this issue.

Let $\mathfrak{M}$ be an abstract W^*-algebra. In this subsection we will consider linear operators acting on $\mathfrak{M}$. The domain of an operator A on $\mathfrak{M}$ will be denoted by $\mathcal{D}om(A)$. (We reserve the notation $\mathcal{D}(A)$ to denote the domain of an operator A acting on a Hilbert space.)

Let A be a linear mapping acting on $\mathfrak{M}$. We say that A is affiliated to $\mathfrak{M}$ and write $A \in \mathfrak{M}^\eta$, iff there exists $B \in \mathfrak{M}$ such that $\|B\| \leq 1$, $(1 - BB^*)\mathfrak{M}$ is σ-weakly dense in $\mathfrak{M}$ and, for any $C, D \in \mathfrak{M}$,

$$C \in \mathcal{D}om(A) \ \text{ and } \ AC = D \iff BC = (1 - BB^*)^{1/2} D.$$

If such B exists, then it is unique. We set $z(A) := B$. In [Wo], $z(A)$ is called the z-transform of A.

One can show that if $A \in \mathfrak{M}^\eta$, then $\mathcal{D}om(A)$ is σ-weakly dense and A is closed, both in the norm topology and in the σ-weak topology.

Note that every $A \in \mathfrak{M}$ may be identified with a linear map on $\mathfrak{M}$ with $\mathcal{D}om(A) = \mathfrak{M}$ (given by $A(C) = AC$) and thus it is an element of $\mathfrak{M}^\eta$. The z-transform of $A \in \mathfrak{M}$ equals

$$z(A) = (1 + AA^*)^{-1/2} A.$$

The following theorem describes the relationship between abstract and concrete affiliations. It shows that in the case of an injective normal representation we can identify abstract and concrete affiliated operators.

Theorem 2.2 *Let $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ be a normal representation preserving the identity. Then there exists a unique extension of π to a surjective map $\pi : \mathfrak{M}^\eta \to \pi(\mathfrak{M})^{(\eta)}$ satisfying*

$$(1 + \pi(A)\pi(A)^*)^{-1/2} \pi(A) = \pi(z(A)).$$

If π is injective on $\mathfrak{M}$, then its extension on $\mathfrak{M}^\eta$ is injective as well.

2.5 Vector representatives of states

Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a concrete W^*-algebra and Ω a vector in $\mathcal{H}$. Then

$$\omega_\Omega(A) := (\Omega | A\Omega), \quad A \in \mathfrak{M},$$

defines a normal positive functional on $\mathfrak{M}$. We say that Ω is a vector representative of ω_Ω. ω_Ω is a state iff Ω is normalized.

The support and the central support of ω_Ω are also called the support and the central support of Ω and denoted s_Ω and z_Ω respectively. We thus have

$$\mathrm{s}_{\omega_\Omega} = \mathrm{s}_\Omega, \qquad \mathrm{z}_{\omega_\Omega} = \mathrm{z}_\Omega.$$

The support of Ω wrt the W^*-algebra $\mathfrak{M}'$ will be denoted s'_Ω. One shows that

$$\mathrm{Ran}\,\mathrm{s}_\Omega = (\mathfrak{M}'\Omega)^{\mathrm{cl}}, \qquad \mathrm{Ran}\,\mathrm{s}'_\Omega = (\mathfrak{M}\Omega)^{\mathrm{cl}},$$

where cl stands for the closure.

A vector $\Omega \in \mathcal{H}$ is called cyclic if $\mathrm{s}'_\Omega = 1$. A vector Ω is called separating if $\mathrm{s}_\Omega = 1$, or equivalently, if it is a vector representative of a faithful state.

The following construction, called after Gelfand, Naimark and Segal, associates to every normal state a normal representation equipped with a cyclic vector.

Theorem 2.3 (The GNS construction) *Let ω be a normal state. Then there exist a (unique up to a unitary equivalence) Hilbert space $\mathcal{H}$, a normal unital representation $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ and a cyclic vector $\Omega \in \mathcal{H}$, such that*

$$\omega(A) = (\Omega|\pi(A)\Omega).$$

The representation π is injective on $\mathrm{z}_\omega\mathfrak{M}$ and zero on $(1 - \mathrm{z}_\omega)\mathfrak{M}$.

2.6 Automorphisms of W^*-algebras

Let $\mathrm{Aut}(\mathfrak{M})$ denote the group of $*$-automorphisms of a W^*-algebra $\mathfrak{M}$. We equip $\mathrm{Aut}(\mathfrak{M})$ with the following topology: if ρ_α is a net in $\mathrm{Aut}(\mathfrak{M})$ and $\rho \in \mathrm{Aut}(\mathfrak{M})$, then $\rho_\alpha \to \rho$ iff for all $A \in \mathfrak{M}$, $\rho_\alpha(A) \to \rho(A)$ σ-weakly. This topology is called the pointwise σ-weak topology.

A one parameter pointwise σ-weakly continuous group $\mathbb{R} \ni t \mapsto \tau^t \in \mathrm{Aut}(\mathfrak{M})$ is called W^*-dynamics on $\mathfrak{M}$. The pair $(\mathfrak{M}, \tau)$ is called a W^*-dynamical system.

Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a concrete W^*-algebra and $\rho \in \mathrm{Aut}(\mathfrak{M})$. We say that ρ is implemented by $U \in \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the set of unitary operators on $\mathcal{H}$, iff

$$\rho(A) = UAU^*. \tag{2.2}$$

Let $t \mapsto \tau^t$ be a W^*-dynamics on $\mathfrak{M}$ and $t \mapsto U(t) \in \mathcal{U}(\mathcal{H})$ a strongly continuous group. We say that τ^t is implemented by $U(t)$ iff

$$\tau^t(A) = U(t)AU(t)^*. \tag{2.3}$$

In general, neither $*$-automorphisms nor W^*-dynamics need be implementable. If they are, the implementation is not unique. In the next subsections we will describe two situations where there exist distinguished implementations.

2.7 Automorphisms with a fixed invariant state

Let $\omega \in \mathfrak{M}_*^+$ and $\rho \in \mathrm{Aut}(\mathfrak{M})$. We define $\rho^*\omega \in \mathfrak{M}_*^+$ by $\rho^*\omega(A) = \omega(\rho(A))$. We say that ω is ρ-invariant if $\omega = \rho^*\omega$. The automorphisms that leave ω invariant form a group denoted $\mathrm{Aut}_\omega(\mathfrak{M})$.

If $\rho \in \mathrm{Aut}_\omega(\mathfrak{M})$, then $\rho(\mathrm{z}_\omega) = \mathrm{z}_\omega$ and $\rho(\mathrm{s}_\omega) = \mathrm{s}_\omega$. Thus ρ maps $\mathrm{z}_\omega\mathfrak{M}$ and $(1-\mathrm{z}_\omega)\mathfrak{M}$ into itself, and without loss of generality we may assume that $\mathrm{z}_\omega = 1$. By passing to the GNS-representation we may assume that $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ and that Ω is a cyclic vector representative of ω.

Proposition 2.4 *There exists a unique representation*

$$\mathrm{Aut}_\omega(\mathfrak{M}) \ni \rho \mapsto U^\Omega(\rho) \in \mathcal{U}(\mathcal{H})$$

such that

$$U^\Omega(\rho)\Omega = \Omega, \qquad U^\Omega(\rho)AU^\Omega(\rho)^* = \rho(A).$$

It is continuous if we equip $\mathrm{Aut}_\omega(\mathfrak{M})$ *with the pointwise* σ*-weak topology and* $\mathcal{U}(\mathcal{H})$ *with the strong operator topology.*

Proof. One just sets

$$U^\Omega(\rho)A\Omega = \rho(A)\Omega, \quad A \in \mathfrak{M}.$$

□

$U^\Omega(\rho)$ will be called the Ω-implementation of ρ.

Suppose now that $t \mapsto \tau^t$ is a W^*-dynamics that leaves ω invariant. Then, by Theorem 2.4, τ is implemented by a strongly continuous unitary group $\mathbb{R} \ni t \mapsto U^\Omega(\tau^t) \in \mathcal{U}(\mathcal{H})$. The self-adjoint generator of $U^\Omega(\tau^t)$ will be denoted L^Ω and called the Ω-Liouvillean of τ^t. (Thus $U^\Omega(\tau^t) = \mathrm{e}^{\mathrm{i}tL^\Omega}$).

The following fact is a corollary of Proposition 2.4:

Proposition 2.5 *The operator* L^Ω *is the unique self-adjoint operator such that*

$$L^\Omega\Omega = 0, \quad \mathrm{e}^{\mathrm{i}tL^\Omega}A\mathrm{e}^{-\mathrm{i}tL^\Omega} = \tau^t(A), \quad A \in \mathfrak{M}.$$

2.8 The Tomita-Takesaki theory

Let ω be a faithful state on $\mathfrak{M}$. By passing to the GNS representation we may assume that $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ and that ω has a vector representative Ω which is cyclic and separating.

The following theorem summarizes the results of the well known Tomita-Takesaki theory.

Theorem 2.6 (1) *Define the operator S_Ω with the domain $\mathfrak{M}\Omega$ by*

$$S_\Omega A\Omega = A^*\Omega.$$

Then S_Ω is antilinear, closable, has a zero kernel and cokernel. Its closure will be denoted also S_Ω. Let $S_\Omega = J\Delta_\Omega^{1/2}$ be its polar decomposition;

(2) *J is an antiunitary involution;*

(3) *Δ_Ω is a positive operator satisfying $J\Delta_\Omega J = \Delta_\Omega^{-1}$ and $\Delta_\Omega\Omega = \Omega$;*

(4) *The map*

$$\tau_\omega^t(A) := \Delta^{-\mathrm{i}t} A \Delta^{\mathrm{i}t} \in \mathfrak{M}, \quad A \in \mathfrak{M},$$

is a W^*-dynamics on $\mathfrak{M}$ and $-\log\Delta_\Omega$ is its Ω-Liouvillean.

The W^*-dynamics $\mathbb{R} \ni t \mapsto \tau_\omega^{-t}$ is called the modular dynamics and Δ_Ω is called the modular operator.

2.9 Standard form

One of the central notions of the theory of W^*-algebras is the so-called standard form. It has been introduced by Haagerup [Haa], following the work of Araki [Ar3] and Connes [Co].

A W^*-algebra in a standard form is a quadruple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$, where $\mathcal{H}$ is a Hilbert space, $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a concrete W^*-algebra, J is an antiunitary involution on $\mathcal{H}$ (that is, J is antilinear, $J^2 = 1$, $J^* = J$) and $\mathcal{H}^+$ is a self-dual cone in $\mathcal{H}$ such that:
(1) $J\mathfrak{M}J = \mathfrak{M}'$;
(2) $JAJ = A^*$ for A in the center of $\mathfrak{M}$;
(3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}^+$;
(4) $AJA\mathcal{H}^+ \subset \mathcal{H}^+$ for $A \in \mathfrak{M}$.

If $\mathfrak{M}$ is an abstract W^*-algebra, then we will say that $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is its standard representation if $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ is an injective unital representation and $(\pi(\mathfrak{M}), \mathcal{H}, J, \mathcal{H}^+)$ is a standard form.

Theorem 2.7 *Let $\mathfrak{M}$ be a W^*-algebra with a faithful state ω. Let $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ be the corresponding GNS representation with the cyclic vector Ω. Let J be the modular conjugation obtained by the Tomita-Takesaki theory and $\mathcal{H}^+ := \{\pi(A)J\pi(A)\Omega : A \in \mathfrak{M}\}^{\mathrm{cl}}$. Then $\mathcal{H}^+$ is a self-dual cone and $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is a standard representation of $\mathfrak{M}$. If $(\pi, \mathcal{H}, J_1, \mathcal{H}_1^+)$ is another standard representation of $\mathfrak{M}$ and $\Omega \in \mathcal{H}_1^+$, then $\mathcal{H}_1^+ = \mathcal{H}_2^+$ and $J_1 = J_2$.*

Theorem 2.8 *Every W^*-algebra $\mathfrak{M}$ possesses a standard representation. Moreover, if $(\pi_1, \mathcal{H}_1, J_1, \mathcal{H}_1^+)$ and $(\pi_2, \mathcal{H}_2, J_2, \mathcal{H}_2^+)$ are two standard representations of $\mathfrak{M}$, then there* *exists a unique unitary operator* $W' : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$W'\pi_1(A) = \pi_2(A)W',$$

$$W'\mathcal{H}_1^+ = \mathcal{H}_2^+.$$

We then automatically have $W'J_1 = J_2W'$.

If $\mathfrak{M}$ is σ-finite, then Theorem 2.8 is proven e.g. in [BR1]. In this case the existence part follows from Theorem 2.7.

If $\mathfrak{M}$ is not σ-finite, the theorem is proven using weights instead of states. The details can be found in [Haa, St].

2.10 States and automorphisms in the standard representation

In this subsection we fix a W^*-algebra in the standard form $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$.

Theorem 2.9 (1)

$$\mathcal{H}^+ \ni \Omega \mapsto \omega_\Omega \in \mathfrak{M}_*^+$$

is a bijection. Its inverse will be denoted

$$\mathfrak{M}_*^+ \ni \omega \mapsto \Omega_\omega \in \mathcal{H}^+.$$

(2) *If* $\Psi, \Phi \in \mathcal{H}^+$, *then*

$$\|\Psi - \Phi\|^2 \le \|\omega_\Psi - \omega_\Phi\| \le \|\Psi - \Phi\|\|\Psi + \Phi\|.$$

(3) *If* $\Omega \in \mathcal{H}^+$, *then* Ω *is cyclic* $\Leftrightarrow$ Ω *is separating* $\Leftrightarrow$ ω_Ω *is faithful.*

(4) *For* $\Omega \in \mathcal{H}^+$, $\mathrm{s}'_\Omega = J\mathrm{s}_\Omega J$.

The vector $\Omega_\omega \in \mathcal{H}^+$ will be called the *standard vector representative of* ω.

A unitary operator U on $\mathcal{H}$ is called a standard unitary operator iff
(1) $U\mathcal{H}^+ = \mathcal{H}^+$,
(2) $U\mathfrak{M}U^* = \mathfrak{M}$.

Theorem 2.10 (1) *If* U *is a standard unitary operator, then* $JU = UJ$ *and* $U\mathfrak{M}'U^* = \mathfrak{M}'$.

(2) *There exists a unique unitary representation*

$$\mathrm{Aut}(\mathfrak{M}) \ni \rho \mapsto U(\rho) \in \mathcal{U}(\mathcal{H}) \tag{2.4}$$

satisfying the following conditions:
(a) $U(\rho)AU(\rho)^* = \rho(A)$, $A \in \mathfrak{M}$;
(b) $U(\rho)\mathcal{H}^+ \subset \mathcal{H}^+$.

(3) *The image of (2.4) is the group of all standard unitary operators.*

(4) *(2.4) is continuous if* $\mathrm{Aut}(\mathfrak{M})$ *is equipped with the pointwise* σ*-weak topology and* $\mathcal{U}(\mathcal{H})$ *with the strong operator topology.*

(5) $U(\rho)\Omega_\omega = \Omega_{\rho^{-1*}\omega}$ *for all* $\omega \in \mathfrak{M}_*^+$.

$U(\rho)$ will be called the standard implementation of ρ.

Suppose that $t \to \tau^t$ is a W^*-dynamics on $\mathfrak{M}$ and let $U(\tau^t)$ be as in Theorem 2.10. Then there exists a unique self-adjoint L such that

$$U(\tau^t) = \mathrm{e}^{\mathrm{i}tL}.$$

The operator L will be called the standard Liouvillean of the W^*-dynamics τ, or simply the Liouvillean of τ.

Theorem 2.11 *The Liouvillean of* τ *is the unique self-adjoint operator* L *satisfying*

$$\mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ \subset \mathcal{H}^+, \qquad \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} = \tau^t(A), \quad A \in \mathfrak{M},$$

for all $t \in \mathbb{R}$.

The final result we wish to mention follows easily from Theorems 2.9 and 2.10. It has been a key tool in recent investigations of invariant states of a certain class of W^*-dynamical systems called Pauli-Fierz systems [BFS, DJ2, JP1, JP2, M].

Theorem 2.12 *Let* τ *be a* W^**-dynamics and* L *the corresponding Liouvillean. Then*

$$\{\omega_\Phi : \Phi \in \mathcal{H}^+ \cap \mathrm{Ker}L\} = \{\omega \in \mathfrak{M}_*^+ : \omega \;\; \textit{is } \tau^t \textit{ invariant}\}.$$

Consequently,

(1) $\dim \mathrm{Ker}L = 0 \Leftrightarrow$ *there are no normal* τ*-invariant states.*

(2) $\dim \mathrm{Ker}L = 1 \Leftrightarrow$ *there exists exactly one normal* τ*-invariant state.*

We will not make use of this result in our paper.

2.11 Comparison

In some circumstances the setups of Subsections 2.7 and 2.10 overlap. Recall that in Subsection 2.7 we have a W^*-algebra $\mathfrak{M}$ with a faithful state ω. We can assume that $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ and that ω has a cyclic vector representative Ω.

By Theorem 2.7, we can construct J and $\mathcal{H}^+$ so that $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a standard form and $\Omega \in \mathcal{H}^+$.

Proposition 2.13 *Let* $\rho \in \mathrm{Aut}_\omega(\mathfrak{M})$. *Suppose that* $U \in \mathcal{U}(\mathcal{H})$ *implements* ρ, *that is* $\rho(A) = UAU^*$, $A \in \mathfrak{M}$. *Then the following conditions are equivalent:*

(1) $U\Omega = \Omega$ ($U = U^\Omega(\rho)$ *is the* Ω*-implementation of* ρ*);*

(2) $U\mathcal{H}^+ = \mathcal{H}^+$ ($U = U(\rho)$ *is the standard implementation of* ρ*).*

Proof. We know from Theorem 2.4 that the Ω-implementation of ρ exists and is unique. We also know from Theorem 2.10 that the standard implementation of ρ exists and is unique. Hence, it is sufficient to show the implication in one direction.

(2)⇒(1). The vector $U\Omega$ determines the state $\rho^*\omega = \omega$. Hence the vectors $U\Omega$, Ω belong to the cone $\mathcal{H}^+$ and determine the same state. This implies $U\Omega = \Omega$. □

As a corollary, if the invariant state ω is faithful, then the concepts of the Ω-Liouvillean and the standard Liouvillean coincide.

Proposition 2.14 *Let* $t \mapsto \tau^t$ *be a* W^**-dynamics on* $\mathfrak{M}$ *that leaves invariant a faithful state* ω. *Suppose that* L *is a self-adjoint operator such that* $\tau^t(A) = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}$. *Then the following conditions are equivalent:*

(1) $L\Omega = 0$ ($L = L^\Omega$ *is the* Ω*-Liouvillean of* τ*);*

(2) *For* $t \in \mathbb{R}$, $\mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ \subset \mathcal{H}^+$ (L *is the standard Liouvillean of* τ*).*

2.12 KMS states

In this subsection we recall basic properties of KMS states. Let $(\mathfrak{M}, \tau^t)$ be a W^*-dynamical system.

Definition 2.15 *Let* $\beta > 0$. $\omega \in \mathfrak{M}_*^{+,1}$ *is called a* (τ, β)*-KMS state if for any* $A, B \in \mathfrak{M}$ *there exists a function* $F_{A,B}(z)$, *analytic in the strip* $\{z : 0 < \mathrm{Im} z < \beta\}$, *continuous on its closure, and satisfying the KMS boundary conditions for* $t \in \mathbb{R}$*:*

$$F_{A,B}(t) = \omega(A\tau^t(B)),$$

$$F_{A,B}(t + \mathrm{i}\beta) = \omega(\tau^t(B)A).$$

Theorem 2.16 *Let* ω *be a* (τ, β)*-KMS state and* $\beta > 0$. *Then*

(1) ω *is* τ*-invariant.*

(2) $\mathrm{s}_\omega = \mathrm{z}_\omega$. (*In particular,* ω *is faithful on* $\mathrm{z}_\omega\mathfrak{M}$).

(3) *If* $B \in \mathrm{z}_\omega\mathfrak{Z}$, *where* $\mathfrak{Z}$ *is the center of* $\mathfrak{M}$, *then* $\tau^t(B) = B$.

(4) *Let* τ_ω *be the dynamics on* $\mathrm{z}_\omega\mathfrak{M}$ *generated by* ω. *Then*

$$\tau^t\big|_{\mathrm{z}_\omega\mathfrak{M}} = \tau_\omega^{\beta t}.$$

Theorem 2.17 *Let ω be a faithful state on $\mathfrak{M}$ and τ_ω the corresponding dynamics. Then ω is a $(\tau_\omega, 1)$-KMS state.*

Let $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form. We say that Ω is a standard (τ, β)-KMS vector iff it is a standard vector representative of a (τ, β)-KMS state.

Suppose that L is the Liouvillean of τ. The following theorem gives a criterium for the KMS property expressed in terms of Hilbert spaces.

Theorem 2.18 *Let $\Omega \in \mathcal{H}^+$ be a unit vector. Then*

(1) *Ω is a standard (τ, β)-KMS vector iff $\mathfrak{M}\Omega \subset \mathcal{D}(\mathrm{e}^{-\beta L/2})$ and*

$$\mathrm{e}^{-\beta L/2} A\Omega = J A^* \Omega, \quad A \in \mathfrak{M}.$$

(2) *If in addition Ω is cyclic and Δ_Ω is the corresponding modular operator, then*

$$\Delta_\Omega = \mathrm{e}^{-\beta L}.$$

2.13 Convergence

It is often convenient to reduce the study of W^*-dynamics and normal states to the study of corresponding Liouvilleans and standard vector representatives. In this subsection we apply this point of view to the convergence properties of W^*-dynamics, invariant states and KMS states.

Theorem 2.19 *Assume that $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a W^*-algebra in the standard form.*

(1) *Suppose that τ_n is a sequence of W^*-dynamics with Liouvilleans L_n, L is a self-adjoint operator, and $L_n \to L$ in the strong resolvent sense. Then*

$$\tau^t(A) := \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}$$

is a W^*-dynamics on $\mathfrak{M}$ and L is its Liouvillean.

(2) *Assume in addition that $\omega_n \in \mathfrak{M}_*^+$ are τ_n-invariant and Ω_n are their standard vector representatives. Suppose also that $\mathrm{w}-\lim_n \Omega_n = \Omega$. Then $\Omega \in \mathcal{H}^+$ and the functional ω_Ω is τ-invariant.*

(3) *Assume in addition that ω_n are (τ_n, β)-KMS states and that $\Omega \neq 0$. Then $\omega_{\Omega/\|\Omega\|}$ is a (τ, β)-KMS state.*

Proof. (1) Let $A \in \mathfrak{M}$. We have $\mathrm{s}-\lim_{n\to\infty} \mathrm{e}^{\pm\mathrm{i}tL_n} = \mathrm{e}^{\pm\mathrm{i}tL}$, hence

$$\mathrm{s}-\lim_{n\to\infty} \mathrm{e}^{\mathrm{i}tL_n} A \mathrm{e}^{-\mathrm{i}tL_n} = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \in \mathfrak{M}.$$

Therefore τ is a W^*-dynamics.

Since $\mathcal{H}^+$ is closed and $\mathrm{e}^{\mathrm{i}tL_n}$ preserve $\mathcal{H}^+$, $\mathrm{e}^{\mathrm{i}tL}$ preserves $\mathcal{H}^+$. Hence L is the Liouvillean of τ.

(2) Since $\mathcal{H}^+$ is weakly closed, $\Omega \in \mathcal{H}^+$. Moreover, since $\Omega_n \in \mathcal{D}(L_n)$ and $L_n\Omega_n = 0$, by Proposition A.6, $\Omega \in \mathcal{D}(L)$ and $L\Omega = 0$.

(3) Let $A \in \mathfrak{M}$. Ω_n are (τ_n, β)-KMS vectors, hence

$$\exp(-\beta L_n/2)A\Omega_n = JA^*\Omega_n.$$

Since $\exp(-\beta L_n/2) \to \exp(-\beta L/2)$ in the strong resolvent sense, $JA^*\Omega_n \to JA^*\Omega$ weakly, and $A\Omega_n \to A\Omega$ weakly, it follows from Proposition A.6 that $A\Omega \in \mathcal{D}(\mathrm{e}^{-\beta L/2})$ and

$$\mathrm{e}^{-\beta L/2}A\Omega = JA^*\Omega. \tag{2.5}$$

Hence $\Omega/\|\Omega\|$ is a (τ, β)-KMS vector. □

2.14 Analytic elements

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system. An element $A \in \mathfrak{M}$ is called τ-analytic if there exists a strip $I(r) = \{z : |\mathrm{Im} z| < r\}$ and a function $f : I(r) \mapsto \mathfrak{M}$ such that:
(1) $f(t) = \tau^t(A)$ for $t \in \mathbb{R}$;
(2) $I(r) \ni z \mapsto \phi(f(z))$ is analytic for all $\phi \in \mathfrak{M}_*$.

Under these conditions we write $f(z) = \tau^z(A)$. A standard argument based on the uniform boundedness theorem shows that $f(z)$ is actually analytic in the norm of $\mathfrak{M}$.

If $r = \infty$, then we say that A is τ-entire.

For $A \in \mathfrak{M}$ and $n \in \mathbb{N}$ let

$$A_n = \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-nt^2}\tau^t(A)\mathrm{d}t.$$

Theorem 2.20 *A_n is τ-entire and $A_n \nearrow A$ in the σ-strong topology. Thus the τ-entire elements form a σ-strongly dense subspace of $\mathfrak{M}$. This subspace is denoted by $\mathfrak{M}_\tau$.*

For additional discussion of analytic elements we refer the reader to [BR1].

3 The perturbation theory of W^*-dynamics

In this section, given a W^*-dynamics τ and a perturbation Q, we construct a perturbed W^*-dynamics τ_Q. We also construct the so-called Araki-Dyson expansionals $\mathrm{E}_Q^\tau(t)$ which intertwine these two dynamics. We describe these objects in three cases: for analytic perturbations, bounded perturbations, and for a large class of unbounded perturbations. The constructions in the first two cases are well known, see [Ar6, BR2].

3.1 Bounded perturbations

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system and Q a self-adjoint element of $\mathfrak{M}$. The following formula defines the W^*-dynamics τ_Q on $\mathfrak{M}$:

$$\tau_Q^t(A) = \sum_{n\geq 0} \mathrm{i}^n \int_{0\leq t_n\leq \ldots t_1\leq t} [\tau^{t_n}(Q), [\cdots, [\tau^{t_1}(Q), \tau^t(A)]\cdots]]\mathrm{d}t_1\cdots \mathrm{d}t_n. \tag{3.6}$$

If δ is the generator of τ, then the generator of τ_Q^t has the same domain as δ and equals

$$\delta_Q(A) = \delta(A) + \mathrm{i}[Q, A].$$

Let $\mathrm{E}_Q^\tau(t)$ be a one-parameter family of elements of $\mathfrak{M}$ given by

$$\mathrm{E}_Q^\tau(t) = \sum_{n\geq 0} \mathrm{i}^n \int_{0\leq t_n\leq \ldots t_1\leq t} \tau^{t_n}(Q)\cdots \tau^{t_1}(Q)\mathrm{d}t_1\cdots \mathrm{d}t_n. \tag{3.7}$$

We will call $\mathrm{E}_Q^\tau(t)$ the Araki-Dyson expansionals. Whenever there is no danger of confusion we will write $\mathrm{E}_Q(t)$ for $\mathrm{E}_Q^\tau(t)$.

We remark that integrals in (3.6) and (3.7) converge in σ-weak topology and define a norm-convergent series of bounded operators.

The expansions (3.6) and (3.7) played an important role in the works of Schwinger, Tomonaga and Dyson on QED. The operators $\mathrm{E}_Q^\tau(t)$ are closely related to the so-called Connes cocycles [Co].

Let us list some properties of Araki-Dyson expansionals:

Theorem 3.1 *Let $t, t_1, t_2 \in \mathbb{R}$. Then*

(1) $\mathrm{E}_Q(t)$ *are unitary elements of* $\mathfrak{M}$;

(2) $\tau_Q^t(A) = \mathrm{E}_Q^\tau(t)\tau^t(A)\mathrm{E}_Q^\tau(t)^{-1}$;

(3) $\mathrm{E}_Q(t)^{-1} = \mathrm{E}_Q(t)^* = \tau^t(\mathrm{E}_Q(-t))$;

(4) $\mathrm{E}_Q(t_1 + t_2) = \mathrm{E}_Q(t_1)\tau^{t_1}(\mathrm{E}_Q(t_2))$;

Assume in addition that $\mathfrak{M}$ is a concrete W^*-algebra in $\mathcal{B}(\mathcal{H})$ and that L is a self-adjoint operator on $\mathcal{H}$ such that $\tau^t(A) = \mathrm{e}^{\mathrm{i}tL}A\mathrm{e}^{-\mathrm{i}tL}$ for $A \in \mathfrak{M}$. Then

(5) $\tau_Q^t(A) = \mathrm{e}^{\mathrm{i}t(L+Q)}A\mathrm{e}^{-\mathrm{i}t(L+Q)}$ *for* $A \in \mathfrak{M}$;

(6) $\mathrm{E}_Q(t) = \mathrm{e}^{\mathrm{i}t(L+Q)}\mathrm{e}^{-\mathrm{i}tL}$.

3.2 Analytic perturbations

In this subsection we assume that Q is τ-entire. Then τ_Q extends to $\mathbb{C}$ by the formula

$$\tau_Q^z(A) = \sum_{n\geq 0} (\mathrm{i}z)^n \int_{0\leq s_n\leq \ldots s_1\leq 1} [\tau^{s_n z}(Q), [\cdots, [\tau^{s_1 z}(Q), \tau^z(A)]\cdots]]\mathrm{d}s_1\cdots \mathrm{d}s_n, \tag{3.8}$$

valid for $A \in \mathfrak{M}_\tau$. Thus $\mathfrak{M}_\tau = \mathfrak{M}_{\tau_Q}$.

For τ-analytic Q, the Araki-Dyson expansionals can be defined for all complex z by

$$\mathrm{E}_Q^\tau(z) = \sum_{n\geq 0} (\mathrm{i}z)^n \int_{0\leq s_n \leq \ldots s_1 \leq 1} \tau^{s_n z}(Q) \cdots \tau^{s_1 z}(Q) \mathrm{d}s_1 \cdots \mathrm{d}s_n. \tag{3.9}$$

The series (3.8) and (3.9) converge in norm uniformly for z in compact sets and define analytic functions with values in $\mathfrak{M}$.

Theorem 3.2 *Let $z, z_1, z_2 \in \mathbb{C}$. Then*

(1) $\mathrm{E}_Q(z) \in \mathfrak{M}_\tau$;

(2) $\tau_Q^z(A) = \mathrm{E}_Q^\tau(z)\tau^z(A)\mathrm{E}_Q^\tau(z)^{-1}$;

(3) $\mathrm{E}_Q(z)^{-1} = \mathrm{E}_Q(\overline{z})^* = \tau^z(\mathrm{E}_Q(-z))$;

(4) $\mathrm{E}_Q(z_1 + z_2) = \mathrm{E}_Q(z_1)\tau^{z_1}(\mathrm{E}_Q(z_2))$;

Assume in addition that $\mathfrak{M}$ is a concrete W^*-algebra in $\mathcal{B}(\mathcal{H})$ and that L is a self-adjoint operator on $\mathcal{H}$ such that $\tau^t(A) = \mathrm{e}^{\mathrm{i}tL}A\mathrm{e}^{-\mathrm{i}tL}$ for $A \in \mathfrak{M}$. Then

(5) $\tau_Q^z(A)\mathrm{e}^{\mathrm{i}z(L+Q)} = \mathrm{e}^{\mathrm{i}z(L+Q)}A$ *for* $A \in \mathfrak{M}_\tau$;

(6) $\mathrm{E}_Q(z)\mathrm{e}^{\mathrm{i}zL} = \mathrm{e}^{\mathrm{i}z(L+Q)}$.

3.3 Unbounded perturbations

In this subsection we consider a concrete W^*-algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ with a W^*-dynamics τ implemented by a self-adjoint operator L and assume that Q is a selfadjoint operator affiliated to $\mathfrak{M}$. We formulate the following assumption on Q:

Assumption 3.A *$L + Q$ is essentially self-adjoint on $\mathcal{D}(L) \cap \mathcal{D}(Q)$.*

Theorem 3.3 *Suppose that Assumption 3.A holds and let*

$$\tau_Q^t(A) = \mathrm{e}^{\mathrm{i}t(L+Q)}A\mathrm{e}^{-\mathrm{i}t(L+Q)}. \tag{3.10}$$

Then

(1) *τ_Q is a W^*-dynamics on $\mathfrak{M}$;*

(2) *If Q is bounded, then τ_Q defined by (3.10) coincides with τ_Q defined by (3.6).*

Proof. Let $A \in \mathfrak{M}$. The Trotter product formula (Theorem A.1) yields that

$$\tau_Q^t(A) = \mathrm{s} - \lim_{n\to\infty} \left(\mathrm{e}^{\mathrm{i}tL/n}\mathrm{e}^{\mathrm{i}tQ/n}\right)^n A \left(\mathrm{e}^{-\mathrm{i}tQ/n}\mathrm{e}^{-\mathrm{i}tL/n}\right)^n.$$

Since $\exp(\pm \mathrm{i}tQ/n) \in \mathfrak{M}$, $\tau_Q^t(A) \in \mathfrak{M}$. Therefore, τ_Q is a W^*-dynamics and (1) is proven. (2) follows from Theorem 3.1 (5). □

Under Assumption 3.A we set

$$\mathrm{E}_Q^\tau(t) := \mathrm{e}^{\mathrm{i}t(L+Q)}\mathrm{e}^{-\mathrm{i}tL}. \tag{3.11}$$

Again, for simplicity we will often write $\mathrm{E}_Q(t)$ for $\mathrm{E}_Q^\tau(t)$. By the Trotter product formula

$$\mathrm{E}_Q(t) = \mathrm{s-}\lim_{n\to\infty} \exp(\mathrm{i}tQ/n)\exp(\mathrm{i}t\tau^{t/n}(Q)/n)\cdots\exp(\mathrm{i}t\tau^{t(n-1)/n}(Q)/n),$$

hence $\mathrm{E}_Q(t) \in \mathfrak{M}$.

Theorem 3.4 *Suppose that Assumption 3.A holds. Then all the statements of Theorem 3.1 hold.*

3.4 Perturbations of Liouvilleans

We continue with the setup of the previous subsection. In addition, we suppose that $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a standard form and that L is the Liouvillean of τ.

Define

$$L_Q := L + Q - JQJ. \tag{3.12}$$

We set an additional hypothesis:

Assumption 3.B *The operator L_Q is essentially self-adjoint on $\mathcal{D}(L)\cap\mathcal{D}(Q)\cap\mathcal{D}(JQJ)$.*

The main result of this section is:

Theorem 3.5 *Assume that Assumptions 3.A and 3.B hold. Then L_Q is the Liouvillian for τ_Q.*

Proof. We have to show that for $t \in \mathbb{R}$:
(1) $\tau_Q^t(A) = \mathrm{e}^{\mathrm{i}tL_Q}A\mathrm{e}^{-\mathrm{i}tL_Q}$, $A \in \mathfrak{M}$;
(2) $\mathrm{e}^{\mathrm{i}tL_Q}\mathcal{H}^+ \subset \mathcal{H}^+$.

Clearly,

$$\mathrm{e}^{\mathrm{i}tJQJ} = J\mathrm{e}^{-\mathrm{i}tQ}J \in \mathfrak{M}'. \tag{3.13}$$

By definition, $\mathcal{D}(L+Q) \supset \mathcal{D}(L)\cap\mathcal{D}(Q)$. Therefore, $\mathcal{D}(L+Q)\cap\mathcal{D}(JQJ) \supset \mathcal{D}(L)\cap\mathcal{D}(Q)\cap\mathcal{D}(JQJ)$. Hence, by Hypothesis 3.B, L_Q is essentially self-adjoint on $\mathcal{D}(L+Q)\cap\mathcal{D}(JQJ)$, and we can use the Trotter formula (Theorem A.1) to write

$$\mathrm{e}^{\mathrm{i}tL_Q} = \mathrm{s} - \lim_{n\to\infty}\left(\mathrm{e}^{(\mathrm{i}tL+Q)/n}e^{-\mathrm{i}tJQJ/n}\right)^n.$$

Therefore, for all $A \in \mathfrak{M}$,

$$\begin{aligned}\tau_Q^t(A) &= \mathrm{e}^{\mathrm{i}t(L+Q)} A \mathrm{e}^{-\mathrm{i}t(L+Q)} \\ &= \mathrm{s}-\lim_{n\to\infty} \left(\mathrm{e}^{\mathrm{i}t(L+Q)/n} e^{-\mathrm{i}tJQJ/n}\right)^n A \left(\mathrm{e}^{\mathrm{i}tJQJ/n} \mathrm{e}^{-\mathrm{i}t(L+Q)/n}\right)^n \\ &= \mathrm{e}^{\mathrm{i}tL_Q} A \mathrm{e}^{-\mathrm{i}tL_Q}.\end{aligned} \tag{3.14}$$

This yields (1).

To establish (2), note that since $\mathrm{e}^{\mathrm{i}tQ}$ and $\mathrm{e}^{\mathrm{i}tJQJ}$ commute

$$\mathrm{e}^{\mathrm{i}t(Q-JQJ)} = \mathrm{e}^{\mathrm{i}tQ} J \mathrm{e}^{\mathrm{i}tQ} J.$$

Hence

$$\mathrm{e}^{\mathrm{i}t(Q-JQJ)} \mathcal{H}^+ \subset \mathcal{H}^+.$$

Moreover,

$$\mathrm{e}^{\mathrm{i}tL} \mathcal{H}^+ \subset \mathcal{H}^+.$$

By definition, $\mathcal{D}(Q) \cap \mathcal{D}(JQJ) \subset \mathcal{D}(Q + JQJ)$. Therefore, $\mathcal{D}(L) \cap \mathcal{D}(Q - JQJ) \supset \mathcal{D}(L) \cap \mathcal{D}(Q) \cap \mathcal{D}(JQJ)$. Hence L_Q is essentially self-adjoint on $\mathcal{D}(L) \cap \mathcal{D}(Q - JQJ)$ and it follows from Theorem A.1 that

$$\mathrm{e}^{\mathrm{i}tL_Q} = \mathrm{s} - \lim_{n\to\infty} \left(\mathrm{e}^{\mathrm{i}tL/n} \mathrm{e}^{\mathrm{i}t(Q-JQJ)/n}\right)^n.$$

This and the fact that $\mathcal{H}^+$ is a closed set imply (2). □

The following formulas are sometimes useful:

Theorem 3.6 (1) *Assume that Assumptions 3.A and 3.B hold. Then for $t \in \mathbb{R}$,*

$$\mathrm{E}_Q(t) = \mathrm{e}^{\mathrm{i}tL_Q} \mathrm{e}^{-\mathrm{i}t(L-JQJ)},$$

$$\mathrm{e}^{\mathrm{i}tL_Q} = J\mathrm{E}_Q(t) J \mathrm{e}^{\mathrm{i}tL} \mathrm{E}_Q(-t)^{-1}.$$

(2) *Assume that Q is τ-analytic. Then for $z \in \mathbb{C}$,*

$$\mathrm{E}_Q(z) = \mathrm{e}^{\mathrm{i}zL_Q} \mathrm{e}^{-\mathrm{i}z(L-JQJ)},$$

$$\mathrm{e}^{\mathrm{i}zL_Q} = J\mathrm{E}_Q(\overline{z}) J \mathrm{e}^{\mathrm{i}zL} \mathrm{E}_Q(-z)^{-1}.$$

4 Relative modular theory and relative entropy

One of the main tools used in our paper is the relative modular theory and relative entropy. We devote this section to a concise introduction to this subject. Our presentation follows partly [Ar4, Ar5, Don, Uh, OP].

4.1 Relative modular operator

Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a W^*-algebra. Let $\Phi, \Psi \in \mathcal{H}$. Following Araki [Ar5], we define the operator $S_{\Phi,\Psi}$ on domain $\mathfrak{M}\Psi + (1 - \mathrm{s}'_\Psi)\mathcal{H}$ by

$$S_{\Phi,\Psi}(A\Psi + \Theta) = \mathrm{s}_\Psi A^*\Phi,$$

where $A \in \mathfrak{M}$ and $\Theta \in (1-\mathrm{s}'_\Psi)\mathcal{H} = (\mathfrak{M}\Psi)^\perp$. It is easy to check that $S_{\Phi,\Psi}$ is a well defined antilinear closable operator. Its closure will be denoted by the same symbol.

It is useful to note that

$$\mathfrak{M}\Psi = \{A\Psi : A \in \mathfrak{M},\ A\mathrm{s}_\Psi = A\},$$

and that for $A \in \mathfrak{M}$ satisfying $A\mathrm{s}_\Psi = A$ and Θ as above we have

$$S_{\Phi,\Psi}(A\Psi + \Theta) = A^*\Phi. \tag{4.15}$$

The positive operator

$$\Delta_{\Phi,\Psi} = S^*_{\Phi,\Psi} S_{\Phi,\Psi}$$

will be called the relative modular operator. The following facts are proven in [Ar5]:

Theorem 4.1 (1) $\mathrm{Ker}\,\Delta_{\Phi,\Psi} = \mathrm{Ker}\,\mathrm{s}'_\Psi \mathrm{s}_\Phi$;

(2) $\Delta_{\lambda\Phi,\mu\Psi} = \frac{\lambda^2}{\mu^2}\Delta_{\Phi,\Psi}$, $\lambda, \mu \in \mathbb{R}$;

(3) *if B belongs to the center of $\mathfrak{M}$, then B commutes with $\Delta_{\Phi,\Psi}$.*

In the remaining part of the theorem we assume that $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a standard form and $\Phi, \Psi \in \mathcal{H}^+$. Then

(4) $S_{\Phi,\Psi} = J\Delta^{1/2}_{\Phi,\Psi}$;

(5) $\Delta^{1/2}_{\Phi,\Psi}\Psi = \Delta^{1/2}_{\Phi,\Psi}\mathrm{s}_\Phi\Psi = \mathrm{s}'_\Psi\Phi$;

(6) $J\Delta_{\Psi,\Phi}J\Delta_{\Phi,\Psi} = \Delta_{\Phi,\Psi}J\Delta_{\Psi,\Phi}J = \mathrm{s}'_\Psi \mathrm{s}_\Phi$.

The following convergence property of relative modular operators will be useful.

Theorem 4.2 *Let $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form. Suppose that $\Psi_n, \Phi_n \in \mathcal{H}^+$, that $\Delta_{\Phi_n,\Psi_n} \to M$ in the strong resolvent sense, and that* $\mathrm{w}-\lim_n \Psi_n = \Psi$, $\mathrm{s}-\lim_n \mathrm{s}_{\Psi_n} = \mathrm{s}_\Psi$ *and* $\mathrm{w}-\lim_n \Phi_n = \Phi$. *Then $M = \Delta_{\Phi,\Psi}$.*

Proof. For $A \in \mathfrak{M}$,

$$\Delta^{1/2}_{\Phi_n,\Psi_n} A\Psi_n = J\mathrm{s}_{\Psi_n}A^*\Phi_n.$$

Note that $A\Psi_n \to A\Psi$ weakly and $J\mathrm{s}_{\Psi_n}A^*\Phi_n \to J\mathrm{s}_\Psi A^*\Phi$ weakly. Hence, by Proposition A.6 and remark after it, $A\Psi \in \mathcal{D}(M)$ and

$$MA\Psi = J\mathrm{s}_\Psi A^*\Phi.$$

Now let $\Theta \in (1 - \mathrm{s}'_\Psi)\mathcal{H}$ and $\Theta_n := (1 - \mathrm{s}'_{\Psi_n})\Theta$. Since $\mathrm{s}'_{\Psi_n} \to \mathrm{s}'_\Psi$ strongly, $\Theta_n \to \Theta$ strongly. Since $\Delta_{\Phi_n,\Psi_n}\Theta_n = 0$, $\Theta \in \mathcal{D}(M)$ and $M\Theta = 0$. This yields $M = \Delta_{\Phi,\Psi}$. □

4.2 Relative entropy

Let $\mathfrak{M}$ be a W^*-algebra. The relative entropy of two functionals $\psi, \phi \in \mathfrak{M}_*^+$, denoted $\mathrm{Ent}(\psi|\phi)$, is defined as follows. Choose a standard form $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ of $\mathfrak{M}$ and let Ψ, Φ, be the standard vector representatives of ψ, ϕ. Then

$$\mathrm{Ent}(\psi|\phi) = \begin{cases} (\Psi|\log \Delta_{\Phi,\Psi}\Psi) & \text{if } \mathrm{s}_\psi \le s_\phi, \\ \\ -\infty & \text{otherwise.} \end{cases}$$

The relative entropy was introduced by Araki in fundamental papers [Ar4, Ar5]. In the above definition we used the sign and ordering convention of [BR2]. The relative entropy is discussed in detail in the monograph [OP].

We will need the following well-known facts [Ar4, Ar5, OP, Don].

Theorem 4.3 (1)

$$\mathrm{Ent}(\psi|\phi) = \lim_{t\downarrow 0} t^{-1}\left(\|\Delta_{\Phi,\Psi}^{t/2}\Psi\|^2 - \|\Psi\|^2\right); \tag{4.16}$$

(2) *for* $\mu, \lambda \in \mathbb{R}^+$,

$$\mathrm{Ent}(\lambda\psi|\mu\phi) = \lambda\mathrm{Ent}(\psi|\phi) + \lambda\psi(1)(\log\mu - \log\lambda);$$

(3)

$$\mathrm{Ent}(\psi|\phi) \le \psi(1)(\log\phi(\mathrm{s}_\psi) - \log\psi(1)),$$

in particular, if $\phi(\mathrm{s}_\psi) = \psi(1)$ *then*

$$\mathrm{Ent}(\psi|\phi) \le 0;$$

(4) *if* Q *is a self-adjoint element in the center of* $\mathfrak{M}$ *and* $\psi(1) = 1$*, then*

$$\mathrm{Ent}(\psi|\phi) + \psi(Q) \le \log\phi(\mathrm{e}^Q).$$

Proof. (1) Assume first that $\mathrm{s}_\Psi \le \mathrm{s}_\Phi$. Then the statement follows from the spectral theorem, monotone convergence theorem and the fact that

$$\lim_{t\downarrow 0} t^{-1}(x^t - 1) = \log x,$$

monotonically on the intervals $0 \le x \le 1$, $x \ge 1$. If $\mathrm{s}_\Phi\Psi \ne \Psi$, then $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 \ne 0$, $\Psi_1 \perp \Psi_2$ and $\Psi_1 \in \mathrm{Ker}\Delta_{\Phi,\Psi}$, and one easily shows that the limit in (4.16) is $-\infty$.

Scaling property of Theorem 4.1 yields (2).

We first prove the part (3) under the assumption $\phi(\mathrm{s}_\psi) = \psi(1) = 1$. Using

$$\log x \leq x - 1, \quad x > 0, \tag{4.17}$$

we get

$$\log \Delta_{\Phi,\Psi} \leq \Delta_{\Phi,\Psi} - 1.$$

Thus

$$\mathrm{Ent}(\psi|\phi) \leq \|\Delta_{\Phi,\Psi}^{1/2}\Psi\|^2 - \|\Psi\|^2 = \phi(\mathrm{s}_\psi) - \psi(1) = 0.$$

(We used $\Delta_{\Phi,\Psi}^{1/2}\Psi = \mathrm{s}'_\Psi\Phi = J\mathrm{s}_\Psi\Phi$).

To extend (3) to arbitrary ϕ, ψ, use (2).

To prove (4), note that since e^Q commutes with $\Delta_{\Phi,\Psi}$

$$\log \Delta_{\Phi,\Psi} + Q - \log \phi(\mathrm{e}^Q\mathrm{s}_\psi) = \log\left(\Delta_{\Phi,\Psi}\mathrm{e}^Q/\phi(\mathrm{e}^Q\mathrm{s}_\psi)\right)$$

The inequality (4.17) yields

$$\log\left(\Delta_{\Phi,\Psi}\mathrm{e}^Q/\phi(\mathrm{e}^Q\mathrm{s}_\psi)\right) \leq \Delta_{\Phi,\Psi}\mathrm{e}^Q/\phi(\mathrm{e}^Q\mathrm{s}_\psi) - 1.$$

Hence

$$\begin{aligned}\mathrm{Ent}(\psi|\phi) + \psi(Q) - \log\phi(\mathrm{e}^Q\mathrm{s}_\psi) &\leq \|\Delta_{\Phi,\Psi}^{1/2}\mathrm{e}^{Q/2}\Psi\|^2/\phi(\mathrm{e}^Q\mathrm{s}_\psi) - 1 \\ &= \|\mathrm{e}^{Q/2}\mathrm{s}'_\psi\Phi\|^2/\phi(\mathrm{e}^Q\mathrm{s}_\psi) - 1 = 0,\end{aligned}$$

where we used $\|\mathrm{e}^{Q/2}\mathrm{s}'_\psi\Phi\| = \|\mathrm{e}^{Q/2}J\mathrm{s}_\psi J\Phi\| = \|J\mathrm{e}^{Q/2}s_\psi\Phi\| = \|\mathrm{e}^{Q/2}s_\psi\Phi\|$. □

4.3 Uhlmann's monotonicity theorem

In this subsection we prove a relative entropy inequality due to Uhlmann [Uh]. Our proof follows the steps of an argument in [OP] and is based on an interpolation theorem for self-adjoint operators (Theorem A.7 in the appendix). A different proof can be found in [PuWo].

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be W^*-algebras. A map $\gamma : \mathfrak{M}_1 \mapsto \mathfrak{M}_2$ is called a Schwartz map iff $\gamma(1) = 1$ and $\gamma(A^*A) \geq \gamma(A)^*\gamma(A)$.

Theorem 4.4 (Uhlmann's monotonicity theorem) *Let ψ_i, ϕ_i be normal states on $\mathfrak{M}_i$, $i = 1, 2$, and let $\gamma : \mathfrak{M}_1 \mapsto \mathfrak{M}_2$ be a Schwartz map such that*

$$\psi_2 \circ \gamma = \psi_1, \tag{4.18}$$

$$\phi_2 \circ \gamma = \phi_1. \tag{4.19}$$

Then

$$\mathrm{Ent}(\psi_2|\phi_2) \leq \mathrm{Ent}(\psi_1|\phi_1).$$

The following inequality is a consequence of Uhlmann's theorem:

Corollary 4.5 *Let $\mathfrak{N} \subset \mathfrak{M}$ be W^*-algebras with common identity and $\psi, \phi \in \mathfrak{M}_*^{+,1}$. Then*

$$\mathrm{Ent}_{\mathfrak{M}}(\psi|\phi) \leq \mathrm{Ent}(\psi\big|_{\mathfrak{N}} \,|\, \phi\big|_{\mathfrak{N}}).$$

Proof. The inclusion map $\gamma : \mathfrak{N} \to \mathfrak{M}$ is Schwartz and satisfies the conditions of Theorem 4.4 with respect to ψ, ϕ and the restricted states $\psi\big|_{\mathfrak{N}}, \phi\big|_{\mathfrak{N}}$. $\Box$

To prove Uhlmann's theorem it is convenient to work in the standard representation and to translate the problem into the language of operators on Hilbert spaces. Hence we assume that $\mathfrak{M}_i \subset \mathcal{B}(\mathcal{H}_i)$ and that $(\mathfrak{M}_i, \mathcal{H}_i, J_i, \mathcal{H}_i^+)$ is a standard form. Let $\gamma : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a Schwartz map. Let $\psi_i \in \mathfrak{M}_{i,*}^+$ satisfy (4.18) and let Ψ_i be the standard vector representatives of ψ_i. Set $\mathcal{D}_1 := \mathfrak{M}_1\Psi_1 + (\mathfrak{M}_1\Psi_1)^\perp$. We define a linear map $T : \mathcal{D}_1 \mapsto \mathcal{H}_2$ by

$$T(A\Psi_1 + \Theta_1) := \gamma(A)\Psi_2$$

for $A \in \mathfrak{M}_1$ and $\Theta_1 \in (\mathfrak{M}_1\Psi_1)^\perp$. Since $\gamma(1) = 1$, $T\Psi_1 = \Psi_2$.

Lemma 4.6 *The map T is well defined and extends to a contraction from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof.

$$\begin{aligned} \|\gamma(A)\Psi_2\|^2 &= \psi_2(\gamma(A)^*\gamma(A)) \\ &\leq \psi_2(\gamma(A^*A)) \\ &= \psi_1(A^*A) = \|A\Psi_1\|^2. \end{aligned} \tag{4.20}$$

Hence if $(A - B)\Psi_1 = 0$, then $(\gamma(A) - \gamma(B))\Psi_2 = 0$. Therefore, T is well defined. By (4.20), T is a contraction. $\Box$

Let Φ_i be the standard vector representative of ϕ_i. The main step of the proof of Theorem 4.4 is the following interpolation estimate for the relative modular operator:

Lemma 4.7 *For $0 \leq t \leq 1$,*

$$\|\Delta_{\Phi_2,\Psi_2}^{t/2}\Psi_2\| \leq \|\Delta_{\Phi_1,\Psi_1}^{t/2}\Psi_1\|.$$

Proof. The space $\mathcal{D}_1$, defined above, is a core for $\Delta^{1/2}_{\Phi_1,\Psi_1}$. Let $A \in \mathfrak{M}$ with $A = A\mathrm{s}_{\Psi_1}$. For $\Omega_1 = A\Psi_1 + \Theta_1 \in \mathcal{D}_1$ we get

$$\Delta^{1/2}_{\Phi_2,\Psi_2} T\Omega_1 = \Delta^{1/2}_{\Phi_2,\Psi_2}\gamma(A)\Psi_2 = J\mathrm{s}_{\Psi_2}\gamma(A)^*\Phi_2,$$

$$\Delta^{1/2}_{\Phi_1,\Psi_1}\Omega_1 = \Delta^{1/2}_{\Phi_1,\Psi_1}A\Psi_1 = JA^*\Phi_1.$$

By (4.19)

$$\|J\mathrm{s}_{\Psi_2}\gamma(A)^*\Phi_2\|^2 \le \phi_2(\gamma(A)\gamma(A)^*) \le \phi_2(\gamma(AA^*)) = \phi_1(AA^*) = \|JA^*\Phi_1\|^2.$$

Hence

$$\|\Delta^{1/2}_{\Phi_2,\Psi_2} T\Omega_1\| = \|\Delta^{1/2}_{\Phi_1,\Psi_1}\Omega_1\|.$$

By Lemma 4.6, T is a contraction. Hence, by Theorem A.7, for $t \in [0,1]$

$$\|\Delta^{t/2}_{\Phi_2,\Psi_2} T\Omega_1\| \le \|\Delta^{t/2}_{\Phi_1,\Psi_1}\Omega_1\|.$$

Setting $\Omega_1 = \Psi_1$ we derive the statement. □

Proof of Theorem 4.4. Using Theorem 4.3 (1), Lemma 4.7 and $1 = \|\Psi_1\|^2 = \|\Psi_2\|^2$, we obtain

$$\begin{aligned}
\mathrm{Ent}(\psi_2|\phi_2) &= \lim_{t\downarrow 0} t^{-1}\left(\|\Delta^{t/2}_{\Phi_2,\Psi_2}\Psi_2\|^2 - \|\Psi_2\|^2\right) \\
&\le \lim_{t\downarrow 0} t^{-1}\left(\|\Delta^{t/2}_{\Phi_1,\Psi_1}\Psi_1\|^2 - \|\Psi_1\|^2\right) \\
&= \mathrm{Ent}(\psi_1|\phi_1).
\end{aligned}$$

□

5 Perturbation theory of KMS states

Let $\beta > 0$. In this section, given a (τ,β)-KMS state ω and a perturbation Q, we describe the construction of the perturbed β-KMS state ω_Q. We also prove various properties of this state, including the Peierls-Bogoliubov and the Golden-Thompson inequalities. The Golden-Thompson inequality plays an important role in our construction.

The construction is performed on three levels: for analytic perturbations, bounded perturbations and a class of unbounded perturbations. Although the results on the first two levels are well known, the method of the proof on the second level (bounded perturbations) is new. The results concerning unbounded perturbations are new and they are the main results of our paper.

5.1 Bounded perturbations

Let $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a W^*-algebra in the standard form. Let τ be the W^*-dynamics on $\mathfrak{M}$ with the standard Liouvillian L. Let ω be a faithful (τ, β)-KMS state with the standard vector representative Ω.

Let $Q \in \mathfrak{M}$ be self-adjoint and τ_Q the perturbed W^*-dynamics defined by (3.6). By Theorem 3.5, $L_Q = L + Q - JQJ$ is the standard Liouvillean of τ_Q.

The following two theorems summarize the (bounded) perturbation theory of KMS states developed by Araki.

Theorem 5.1 (1) $\Omega \in \mathcal{D}(\mathrm{e}^{-\beta(L+Q)/2})$. *Set*

$$\Omega_Q := \mathrm{e}^{-\beta(L+Q)/2}\Omega, \qquad \omega_Q(A) = (\Omega_Q|A\Omega_Q)/\|\Omega_Q\|^2.$$

(2) $\Omega_Q \in \mathcal{H}^+$.

(3) Ω_Q *is a cyclic and separating vector for* $\mathfrak{M}$.

(4) *The state* ω_Q *is a* (τ_Q, β)*-KMS state.*

(5) $\log \Delta_{\Omega_Q} = -\beta L_Q$.

(6) *For all self-adjoint* $Q_1, Q_2 \in \mathfrak{M}$,

$$(\Omega_{Q_1})_{Q_2} = \Omega_{Q_1+Q_2}, \qquad (\omega_{Q_1})_{Q_2} = \omega_{Q_1+Q_2}.$$

(7) $\log \Delta_{\Omega_Q,\Omega} = \log \Delta_\Omega - \beta Q$.

(8) $\log \Delta_{\Omega,\Omega_Q} = \log \Delta_{\Omega_Q} + \beta Q$.

(9) $\mathrm{Ent}(\omega|\omega_Q) + \beta\omega(Q) = -\log \|\Omega_Q\|^2$.

(10) $\mathrm{Ent}(\omega_Q|\omega) - \beta\omega_Q(Q) = \log \|\Omega_Q\|^2$.

(11) *The Peierls-Bogoliubov inequality holds:*

$$\mathrm{e}^{-\beta(\Omega|Q\Omega)/2} \le \|\Omega_Q\|.$$

(12) *The Golden-Thompson inequality holds:*

$$\|\Omega_Q\| \le \|\mathrm{e}^{-\beta Q/2}\Omega\|.$$

(13) *Assume that* $Q_n \in \mathfrak{M}$ *are self-adjoint and* $Q_n \to Q$ *strongly. Then* $\Omega_{Q_n} \to \Omega_Q$ *and* $\omega_{Q_n} \to \omega_Q$ *in norm.*

Theorem 5.2 *Let*

$$T_{\beta,n} = \{(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n \ : \ \beta_i \geq 0, \ i = 1, \ldots, n, \ \beta_1 + \cdots + \beta_n \leq \beta/2\}.$$

Then $\Omega \in \mathcal{D}(\mathrm{e}^{-\beta_1 L}Q \cdots \mathrm{e}^{-\beta_n L}Q)$ *for* $(\beta_1, \ldots, \beta_n) \in T_{\beta,n}$*, the function*

$$T_{\beta,n} \ni (\beta_1, \ldots, \beta_n) \mapsto \mathrm{e}^{-\beta_1 L}Q \cdots \mathrm{e}^{-\beta_n L}Q\,\Omega$$

is norm continuous,

$$\sup_{(\beta_1,\cdots,\beta_n)\in T_{\beta,n}} \|\mathrm{e}^{-\beta_1 L}Q \cdots \mathrm{e}^{-\beta_n L}Q\,\Omega\| \leq \|Q\|^n, \tag{5.21}$$

and

$$\Omega_Q = \sum_{n=0}^{\infty}(-1)^n \int \cdots \int_{T_{\beta,n}} \mathrm{e}^{-\beta_1 L}Q \cdots \mathrm{e}^{-\beta_n L}Q\,\Omega\,\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n. \tag{5.22}$$

We have separated Theorem 5.2 from the other results of Araki's theory for several reasons.

Theorem 5.2 contains the main idea of Araki's original proof of Theorem 5.1. In fact, his proof was centered around the expansion (5.22). Our methods are in a certain sense orthogonal to Araki's and we do not need Theorem 5.2 to prove Theorem 5.1.

The expansion (5.22) is an additional information about Ω_Q which, strictly speaking, cannot be derived by our methods alone. Hence, for bounded perturbations our method yields a slightly weaker result than the Araki method. On the other hand, our method is simpler and easily extends to a large class of unbounded perturbations Q.

Both Araki and our methods start with analytic perturbations. In this case, the proofs of Theorems 5.1 and 5.2 are essentially algebraic and relatively easy. For a general bounded Q one picks a sequence of analytic Q_n with $Q_n \to Q$ and uses various limit arguments to establish the theorems. The key difference between the two methods concerns these limit arguments—we use weak limits while Araki uses strong limits. The use of weak limits leads to some technical simplifications and the method naturally extends to unbounded perturbations.

Finally, we mention some additional estimates which can be used to compare Ω with Ω_Q.

Theorem 5.3 (1) $\|\Omega_Q - \Omega\| \leq (\mathrm{e}^{\beta\|Q\|/2} - 1)$.

(2)

$$\begin{aligned}\beta(\Omega|Q\Omega)/2 \ &\geq \|\Omega\|^2 - (\Omega|\Omega_Q) \geq \beta(\Omega|Q\Omega_Q)/2 \\ &\geq (\Omega|\Omega_Q) - \|\Omega_Q\|^2 \geq \beta(\Omega_Q|Q\Omega_Q)/2.\end{aligned}$$

(3)
$$\beta(\Omega|Q\Omega) \geq \|\Omega\|^2 - \|\Omega_Q\|^2 \geq \beta(\Omega_Q|Q\Omega_Q).$$

(4)
$$\|\Omega_Q - \Omega\|^2 \leq \beta(\Omega|Q\Omega)/2 - \beta(\Omega_Q|Q\Omega_Q)/2.$$

(5)
$$\|\Omega_Q - \Omega\| \leq \beta f(\|Q\Omega\|, \|Q\Omega_Q\|)/2,$$

where, for $x, y > 0$, we set

$$f(x,y) := \begin{cases} \frac{x-y}{\log x - \log y}, & x \neq y; \\ x & x = y. \end{cases}$$

The estimate (1) follows immediately from (5.21) and is of course well-known. The estimates (2)–(5) appear to be new.

5.2 Analytic perturbations—proofs

In this section we prove Theorem 5.1 for analytic self-adjoint perturbations $Q \in \mathfrak{M}_\tau$. The proofs are based on the algebraic arguments and are relatively easy.
Proof of Theorem 5.1 in the analytic case (1) For t real,

$$\mathrm{E}_Q(t)\Omega = \mathrm{e}^{\mathrm{i}t(L+Q)}\mathrm{e}^{-\mathrm{i}tL}\Omega = \mathrm{e}^{\mathrm{i}t(L+Q)}\Omega.$$

Since $\mathrm{E}_Q(t)$ has an analytic continuation to an entire function $z \mapsto \mathrm{E}_Q(z)$, $\Omega \in \mathcal{D}(\mathrm{e}^{\mathrm{i}z(L+Q)})$ for all $z \in \mathbb{C}$ and $\mathrm{E}_Q(z)\Omega = \mathrm{e}^{\mathrm{i}z(L+Q)}\Omega$. In particular,

$$\Omega_Q = \mathrm{E}_Q(\mathrm{i}\beta/2)\Omega. \tag{5.23}$$

(2) We have

$$\begin{aligned} \mathrm{E}_Q(\mathrm{i}\beta/2) &= \mathrm{E}_Q(\mathrm{i}\beta/4)\tau^{\mathrm{i}\beta/4}(\mathrm{E}_Q(\mathrm{i}\beta/4)) \\ &= \mathrm{E}_Q(\mathrm{i}\beta/4)\tau^{\mathrm{i}\beta/2}(\mathrm{E}_Q(\mathrm{i}\beta/4)^*). \end{aligned}$$

Hence, by (5.23),

$$\begin{aligned} \Omega_Q &= \mathrm{E}_Q(\mathrm{i}\beta/4)\mathrm{e}^{-\beta L/2}\mathrm{E}_Q(\mathrm{i}\beta/4)^*\Omega \\ &= \mathrm{E}_Q(\mathrm{i}\beta/4)J\mathrm{E}_Q(\mathrm{i}\beta/4)\Omega. \end{aligned}$$

Therefore, $\Omega_Q \in \mathcal{H}^+$.

(3) Since $\mathrm{E}_Q(\mathrm{i}\beta/2)$ is an invertible element of $\mathfrak{M}$, Ω_Q is obviously a cyclic and separating vector for $\mathfrak{M}$.

(4) Theorem 3.6 yields

$$\mathrm{e}^{-\beta L_Q/2} = J\mathrm{E}_Q(-\mathrm{i}\beta/2)J\mathrm{e}^{-\beta L/2}\mathrm{E}_Q(-\mathrm{i}\beta/2)^{-1},$$

and $\mathfrak{M}\Omega_Q = \mathfrak{M}\Omega \subset \mathcal{D}(\mathrm{e}^{-\beta L_Q/2})$. Moreover, for $A \in \mathfrak{M}$,

$$\begin{aligned}
\mathrm{e}^{-\beta L_Q/2}A\Omega_Q &= J\mathrm{E}_Q(-\mathrm{i}\beta/2)J\mathrm{e}^{-\beta L/2}\mathrm{E}_Q(-\mathrm{i}\beta/2)^{-1}A\mathrm{E}_Q(\mathrm{i}\beta/2)\Omega \\
&= J\mathrm{E}_Q(-\mathrm{i}\beta/2)\mathrm{E}_Q(\mathrm{i}\beta/2)^*A^*\mathrm{E}_Q(-\mathrm{i}\beta/2)^{-1*}\Omega \\
&= J\mathrm{E}_Q(-\mathrm{i}\beta/2)\mathrm{E}_Q(-\mathrm{i}\beta/2)^{-1}A^*\mathrm{E}_Q(\mathrm{i}\beta/2)\Omega = JA^*\Omega_Q.
\end{aligned}$$

(5) By Theorem 3.5, we know that $L_Q := L + Q - JQJ$ is the Liouvillean of τ_Q. By Theorem 2.18 we know that $\Delta_{\Omega_Q} = \mathrm{e}^{-\beta L_Q}$.

(6) follows from

$$\mathrm{E}_{Q_2}^{\tau_{Q_1}}(\mathrm{i}\beta/2)\mathrm{E}_{Q_1}^{\tau}(\mathrm{i}\beta/2) = \mathrm{E}_{Q_1+Q_2}^{\tau}(\mathrm{i}\beta/2),$$

which is an immediate consequence of Theorem 3.1 (6), where $L + Q_1$ is to be used for L in the expression for $\mathrm{E}_{Q_2}^{\tau_{Q_1}}(t)$.

(7) The relation

$$S_\Omega\mathrm{E}_Q(\mathrm{i}\beta/2)^*A\Omega = A^*\Omega_Q = S_{\Omega_Q,\Omega}A\Omega$$

implies that

$$S_{\Omega_Q,\Omega} = S_\Omega E_Q(\mathrm{i}\beta/2)^*.$$

Hence

$$\begin{aligned}
\Delta_{\Omega,\Omega_Q} &= S_{\Omega_Q,\Omega}^*S_{\Omega_Q,\Omega} \\
&= \mathrm{E}_Q(\mathrm{i}\beta/2)\Delta_\Omega\mathrm{E}_Q^*(\mathrm{i}\beta/2) \\
&= \left(\mathrm{E}_Q(\mathrm{i}\beta/2)\mathrm{e}^{-\beta L/2}\right)\left(\mathrm{e}^{-\beta L/2}\mathrm{E}_Q(\mathrm{i}\beta/2)^*\right) \\
&= \mathrm{e}^{-\beta(L+Q)},
\end{aligned}$$

where we used $\Delta_\Omega = \mathrm{e}^{-\beta L}$.

(8) follows from (7) if we note that, by (6), $(\Omega_Q)_{-Q} = \Omega$.

(9) Set $\tilde{Q} := Q + \beta^{-1}\log\|\Omega_Q\|^2$. Then $\omega_Q = \omega_{\tilde{Q}}$ and $\Omega_{\tilde{Q}} := \Omega_Q/\|\Omega_Q\|$. Using (7) we get

$$\log\Delta_{\Omega_{\tilde{Q}},\Omega} = \log\Delta_\Omega - \beta\tilde{Q},$$

which implies

$$\mathrm{Ent}(\omega|\omega_Q) = -\beta\omega(\tilde{Q}).$$

(10) Similarly, using (8) we get

$$\log \Delta_{\Omega,\tilde{\Omega}_Q} = \log \Delta_{\Omega_{\tilde{Q}}} + \beta \tilde{Q},$$

which implies

$$\mathrm{Ent}(\omega_Q|\omega) = \beta \omega_Q(\tilde{Q}).$$

(11) Since $\mathrm{Ent}(\omega|\omega_Q) \leq 0$, (9) yields that

$$\mathrm{e}^{-\beta(\Omega|Q\Omega)/2} \leq \|\Omega_Q\|.$$

This is the Peierls-Bogoliubov inequality.

(12) Let $\mathfrak{N}$ be the Abelian von Neumann subalgebra of $\mathfrak{M}$ generated by Q. Then,

$$\begin{aligned}
\log \|\Omega_Q\|^2 &= \mathrm{Ent}(\omega_Q|\omega) - \beta\omega_Q(Q) \\
&\leq \mathrm{Ent}\big(\omega_Q\big|_{\mathfrak{N}}\,|\,\omega\big|_{\mathfrak{N}}\big) - \beta\omega_Q(Q) \\
&\leq \log \omega(\mathrm{e}^{-\beta Q}) \\
&= \log \|\mathrm{e}^{-\beta Q/2}\Omega\|^2,
\end{aligned} \tag{5.24}$$

and so

$$\|\Omega_Q\| \leq \|\mathrm{e}^{-\beta Q/2}\Omega\|.$$

This is the Golden-Thompson inequality. In the first step of (5.24) we used (10), in the second—Uhlmann's estimate of Corollary 4.5 and in the third—Theorem 4.3 (4) with Q replaced by $-\beta Q$.

(13) is a general fact which has the same proof for analytic and bounded perturbations. Its proof is given in the next section. □

We remark that the Golden-Thompson inequality was first proven by Araki [Ar2]. The proof described in (12) is due to Donald [Don].

5.3 Bounded perturbations—proofs

In this subsection we prove Theorem 5.1. We assume that Q is an arbitrary self-adjoint element of $\mathfrak{M}$. By Theorem 2.20, we can find a sequence Q_n of self-adjoint τ-analytic elements such that $Q_n \to Q$ σ-strongly. This implies that $Q_n \to Q$ strongly and the following lemma holds:

Lemma 5.4 (1) *$L + Q_n \to L + Q$ in the strong resolvent sense.*

(2) *$L_{Q_n} \to L_Q$ in the strong resolvent sense.*

Proof of Theorem 5.1. (1) Clearly, $\lim_n e^{-\beta Q_n/2}\Omega = e^{-\beta Q/2}\Omega$. Hence there exists C such that for all n

$$\|e^{-\beta Q_n/2}\Omega\| \leq C.$$

By the Golden-Thompson inequality for analytic perturbations,

$$\|\Omega_{Q_n}\| \leq \|e^{-\beta Q_n/2}\Omega\|.$$

Hence $\|\Omega_{Q_n}\| \leq C$. Now by Proposition A.6, $\Omega \in \mathcal{D}(e^{-\beta(L+Q)/2})$ and

$$\mathrm{w}-\lim_{n\to\infty} e^{-\beta(L+Q_n)/2}\Omega = e^{-\beta(L+Q)/2}\Omega.$$

(2) follows from the analytic case of (2) and the fact that $\mathcal{H}^+$ is weakly closed.

(3) Let $P := 1 - \mathrm{s}_{\Omega_Q}$. Clearly, $P \in \mathfrak{M}$, $\tau_Q^t(P) = P$ and $P\Omega_Q = 0$. Set

$$\Omega(z) = e^{-z(L+Q)}\Omega.$$

By Proposition A.2, the vector-valued function $\Omega(z)$ is analytic inside the strip $0 < \mathrm{Re} z < \beta/2$ and norm continuous on its closure. Moreover, $\Omega(\beta/2) = \Omega_Q$ and

$$\begin{aligned} e^{\mathrm{i}t(L+Q)}P\Omega(\mathrm{i}t + \beta/2) &= e^{\mathrm{i}t(L+Q)}Pe^{-\mathrm{i}t(L+Q)}\Omega(\beta/2) \\ &= \tau_Q^t(P)\Omega_Q \\ &= P\Omega_Q = 0. \end{aligned}$$

Thus, for all real t, $P\Omega(\mathrm{i}t + \beta/2) = 0$. This implies that $P\Omega(z) = 0$ for all z in the strip $0 \leq \mathrm{Re} z \leq \beta/2$. In particular, $P\Omega(0) = P\Omega = 0$. Since Ω is a separating vector for $\mathfrak{M}$, $P = 0$. Hence $\mathrm{s}_{\Omega_Q} = 1$ and Ω_Q is a separating vector for $\mathfrak{M}$. Since Ω_Q is separating, (2) and Theorem 2.9 (3) imply that Ω_Q is also cyclic.

(4) follows from the analytic case of (4) and Theorem 2.19.

(5), (7) and (8) follow from their analytic versions and Theorem 4.2.

(6) Let now Q_1, Q_2 be two self-adjoint elements and $Q_{1,n}$, $Q_{2,n}$ the sequences of the corresponding analytic approximations. Then, by the analytic case of (6)

$$(\Omega_{Q_{1,m}})_{Q_{2,n}} = \Omega_{Q_{1,m}+Q_{2,n}}.$$

As $n \to \infty$, $(\Omega_{Q_{1,m}})_{Q_{2,n}} \to (\Omega_{Q_{1,m}})_{Q_2}$ weakly, $\Omega_{Q_{1,m}+Q_{2,n}} \to \Omega_{Q_{1,m}+Q_2}$ weakly, and so

$$(\Omega_{Q_{1,m}})_{Q_2} = \Omega_{Q_{1,m}+Q_2}. \tag{5.25}$$

By the arguments of the proof of (1), as $m \to \infty$, $\Omega_{Q_{1,m}+Q_2} \to \Omega_{Q_1+Q_2}$ weakly. Moreover,

$$(\Omega_{Q_{1,m}})_{Q_2} = e^{-\beta(L+Q_{1,m}-JQ_{1,m}J+Q_2)/2}\Omega_{Q_{1,m}},$$

$\Omega_{Q_{1,m}} \to \Omega_{Q_1}$ weakly and $L + Q_{1,m} - JQ_{1,m}J + Q_2 \to L + Q_1 - JQ_1J + Q_2$ in the strong resolvent sense. Hence by Proposition A.6, $\Omega_{Q_1} \in \mathcal{D}(\mathrm{e}^{-\beta(L+Q_1-JQ_1J+Q_2)/2}$ and

$$(\Omega_{Q_1})_{Q_2} = e^{-\beta(L+Q_1-JQ_1J+Q_2)/2}\Omega_{Q_1} = \Omega_{Q_1+Q_2}.$$

(9) and (10) follow from (7) and (8) precisely as in the analytic case.

(11) (The Peierls-Bogoliubov inequality) follows from (9) just as in the analytic case.

(12) $\lim_n \mathrm{e}^{-\beta Q_n/2}\Omega = \mathrm{e}^{-\beta Q/2}\Omega$ implies

$$\lim_{n\to\infty} \|\mathrm{e}^{-\beta Q_n/2}\Omega\| = \|\mathrm{e}^{-\beta Q/2}\Omega\|. \tag{5.26}$$

Moreover, $\mathrm{w}-\lim_n \Omega_{Q_n} = \Omega_Q$ implies

$$\|\Omega_Q\| \leq \liminf_{n\to\infty} \|\Omega_{Q_n}\|. \tag{5.27}$$

By the Golden-Thompson inequality for analytic perturbations,

$$\|\Omega_{Q_n}\| \leq \|\mathrm{e}^{-\beta Q_n/2}\Omega\|. \tag{5.28}$$

Now (5.26), (5.27) and (5.28) imply the Golden-Thompson inequality:

$$\|\Omega_Q\| \leq \|\mathrm{e}^{-\beta Q/2}\Omega\|. \tag{5.29}$$

(13) Let $Q_n \in \mathfrak{M}$ be an arbitrary sequence of self-adjoint elements which converges strongly to Q. The proof of (1) yields that $\Omega_{Q_n} \to \Omega_Q$ weakly. Using first the chain rule and then the Golden-Thompson inequality we get

$$\|\Omega_{Q_n}\| = \|(\Omega_Q)_{Q_n-Q}\| \leq \|\mathrm{e}^{-\beta(Q_n-Q)/2}\Omega_Q\|.$$

Hence, $\limsup_n \|\Omega_{Q_n}\| \leq \|\Omega_Q\|$. Combining this estimate with (5.27) we get $\|\Omega_{Q_n}\| \to \|\Omega_Q\|$, and so $\Omega_{Q_n} \to \Omega_Q$ in norm. By Theorem 2.9, this implies that $\omega_{Q_n} \to \omega_Q$ in norm. □

5.4 Perturbative expansion of Ω_Q and the estimates

In this subsection we prove Theorems 5.2 and 5.3. The proof of Theorem 5.2 is based on the following technical result of Araki.

Theorem 5.5 (1) *Set*

$$S_{\beta,n} := \{(z_1, \ldots, z_n) : \mathrm{Im} z_i \geq 0, \quad i = 1, \ldots, n, \ \mathrm{Im} z_1 + \cdots + \mathrm{Im} z_n \leq \beta/2\}.$$

Then for $(z_1, \ldots, z_n) \in S_{\beta,n}$, Ω *belongs to* $\mathcal{D}(\mathrm{e}^{\mathrm{i}z_nL}Q_n \cdots \mathrm{e}^{\mathrm{i}z_1L}Q_1)$, *the function*

$$S_{\beta,n} \ni (z_n, \ldots, z_1) \mapsto \mathrm{e}^{\mathrm{i}z_nL}Q_n \cdots \mathrm{e}^{\mathrm{i}z_1L}Q_1\Omega \tag{5.30}$$

is norm continuous on $S_{\beta,n}$, *analytic on its interior, and*

$$\sup_{(z_1,\ldots,z_n)\in S_{\beta,n}} \|\mathrm{e}^{\mathrm{i}z_nL}Q_n \cdots \mathrm{e}^{\mathrm{i}z_1L}Q_1\Omega\| \leq \|Q_n\| \cdots \|Q_1\|. \tag{5.31}$$

(2) *Let* $Q_{i,m} \underset{m\to\infty}{\to} Q_i$ *strongly,* $Q_{i,m}^* \underset{m\to\infty}{\to} Q_i^*$ *strongly. Then*

$$\lim_{m\to\infty} \mathrm{e}^{\mathrm{i}z_nL}Q_{n,m}\cdots \mathrm{e}^{\mathrm{i}z_1L}Q_{1,m}\Omega = \mathrm{e}^{\mathrm{i}z_nL}Q_n\cdots \mathrm{e}^{\mathrm{i}z_1L}Q_1\Omega, \tag{5.32}$$

uniformly for $(z_1,\dots,z_n)$ *in compact subsets of* $S_{\beta,n}$.

Proof. The proof follows by induction wrt n. For $n=1$, the statement follows from the Proposition A.2 and the KMS condition (Theorem 2.18).

Suppose that the statement is true for $n-1$. Set

$$\begin{aligned}\Omega(z_1,\dots,z_{n-1}) &:= Q_n\mathrm{e}^{\mathrm{i}z_{n-1}L}Q_{n-1}\cdots \mathrm{e}^{\mathrm{i}z_1L}Q_1\Omega,\\ \Omega^*(z_1,\dots,z_{n-1}) &:= JQ_1^*\mathrm{e}^{-\mathrm{i}\overline{z}_1L}Q_2^*\cdots \mathrm{e}^{-\mathrm{i}\overline{z}_{n-1}L}Q_{n-1}^*\Omega.\end{aligned}$$

Consider $\Phi\in\mathcal{D}(\mathrm{e}^{-\beta L/2})$ and the function

$$F(z_1,\dots,z_{n-1}) := (\Phi|\Omega^*(z_1,\dots,z_{n-1}))\,.$$

By the induction assumption, the function F is continuous on $S_{\beta,n-1}$, analytic on its interior, and satisfies the estimate

$$|F(z_1,\cdots z_{n-1})| \le \|\Phi\|\|Q_1\|\cdots\|Q_n\|, \tag{5.33}$$

which gives the estimate (5.31) for $z_n=0$.

The function

$$G(z_1,\dots,z_{n-1}) := \left(\mathrm{e}^{(\mathrm{i}\overline{z}_1+\cdots+\mathrm{i}\overline{z}_{n-1}-\beta/2)L}\Phi|\Omega(z_1,\dots,z_{n-1})\right) \tag{5.34}$$

is also analytic and continuous on the same domain. (Here we used the induction assumption, the assumption $\Phi\in\mathcal{D}(\mathrm{e}^{-\beta L/2})$ and Proposition A.2).

For $z_1,\dots,z_{n-1}\in\mathbb{R}$, set $s_2=z_1$, $s_3:=z_2+z_1$, $\dots$ $s_n=z_{n-1}+\cdots+z_1$. Then

$$\begin{aligned}F(z_1,\cdots,z_{n-1}) &= (\Phi|JQ_1^*\tau^{-s_2}(Q_2^*)\cdots\tau^{-s_n}(Q_n^*)\Omega)\\ &= (\Phi|\mathrm{e}^{-\beta L/2}\tau^{-s_n}(Q_n)\cdots\tau^{-s_2}(Q_2)Q_1\Omega)\\ &= G(z_1,\dots,z_{n-1}),\end{aligned}$$

and by the edge of wedge theorem, the functions F and G coincide on their whole domains. Thus, by (5.33)

$$|G(z_1,\dots,z_{n-1})| \le \|\Phi\|\|Q_1\|\cdots\|Q_n\|. \tag{5.35}$$

For $z_n=\mathrm{i}\beta/2-z_1-\cdots-z_{n-1}$ and $(z_1,\dots,z_{n-1})\in S_{\beta,n-1}$, this implies that

$$\Omega(z_1,\dots,z_{n-1})\in\mathcal{D}(\mathrm{e}^{\mathrm{i}z_nL}),$$

and

$$\Omega^*(z_1, \dots, z_{n-1}) = \mathrm{e}^{\mathrm{i}z_n L}\Omega(z_1, \dots, z_{n-1}). \tag{5.36}$$

(5.35) gives also the estimate (5.31) for $z_n = \mathrm{i}\beta/2 - z_1 - \cdots - z_{n-1}$.

The estimate (5.31) for $0 \le \mathrm{Im}z_n \le \beta/2 - \mathrm{Im}z_1 - \cdots - \mathrm{Im}z_{n-1}$ follows from (5.33), (5.35) and Proposition A.2.

By Proposition A.2 and Hartog's theorem of holomorphy, $(\mathrm{e}^{\mathrm{i}\overline{z}_n L}\Phi|\Omega(z_1, \dots, z_n))$ is analytic on the interior of $S_{\beta,n}$, for $\Phi \in \mathcal{D}(\mathrm{e}^{-\beta L/2})$. Using the estimate (5.31) we see that it is analytic for all Φ. Hence we can conclude that the function (5.30) is weakly analytic. Since the weak analyticity is equivalent to the norm analyticity, we have proven all the statements of (1) except that (5.30) is norm continuous on the whole $S_{\beta,n}$.

Next we turn to the proof of (2) for n. Set

$$\begin{aligned}\Omega_m(z_1, \dots, z_{n-1}) &:= Q_{n,m}\mathrm{e}^{\mathrm{i}z_{n-1}L}Q_{n-1,m}\cdots \mathrm{e}^{\mathrm{i}z_1 L}Q_{1,m}\Omega,\\ \Omega^*_m(z_1, \dots, z_{n-1}) &:= JQ^*_{1,m}\mathrm{e}^{-\mathrm{i}\overline{z}_1 L}Q^*_{2,m}\cdots \mathrm{e}^{-\mathrm{i}\overline{z}_{n-1}L}Q^*_{n-1,m}\Omega.\end{aligned}$$

By the uniform boundedness principle, independently of m, we have

$$\|Q_{i,m}\| \le C, \quad i = 1, \dots, n. \tag{5.37}$$

Now

$$\begin{aligned}&\|\Omega^*_m(z_1, \dots, z_{n-1}) - \Omega^*(z_1, \dots, z_{n-1})\|\\ &\le \|Q_{1,m}\|\|\mathrm{e}^{-\mathrm{i}\overline{z}_1 L}Q^*_{2,m}\cdots \mathrm{e}^{-\mathrm{i}\overline{z}_{n-1}L}Q^*_{n-1,m}\Omega - \mathrm{e}^{-\mathrm{i}\overline{z}_1 L}Q^*_2\cdots \mathrm{e}^{-\mathrm{i}\overline{z}_{n-1}L}Q^*_{n-1}\Omega\|\\ &+\|(Q^*_{1,m} - Q^*_1)\mathrm{e}^{-\mathrm{i}\overline{z}_1 L}Q^*_2\cdots \mathrm{e}^{-\mathrm{i}\overline{z}_{n-1}L}Q^*_{n-1}\Omega\|.\end{aligned}$$

The first term on the right goes to zero uniformly on compact subsets of $S_{\beta,n-1}$ by the induction assumption and (5.37) for $i = 1$. The second term on the right goes to zero uniformly on compact subsets of $S_{\beta,n-1}$ by the induction assumption, Lemma A.3 and the strong convergence $Q^*_{1,m} \to Q^*_1$.

By the proof of (1) (see the identity (5.36)), we have for $z_1, \dots, z_{n-1} \in S_{\beta,n-1}$,

$$\Omega(z_1, \dots, z_{n-1}) - \Omega_m(z_1, \dots, z_{n-1}) \in \mathcal{D}(\mathrm{e}^{(-\mathrm{i}z_1 - \cdots - \mathrm{i}z_{n-1} - \beta/2)L}),$$

$$\begin{aligned}&\Omega^*(z_1, \dots, z_{n-1}) - \Omega^*_m(z_1, \dots, z_{n-1})\\ &\qquad = \mathrm{e}^{(-\mathrm{i}z_1 - \cdots - \mathrm{i}z_{n-1} - \beta/2)L}(\Omega(z_1, \dots, z_{n-1}) - \Omega_m(z_1, \dots, z_{n-1})).\end{aligned}$$

Hence,

$$\lim_{m\to\infty}\left\|\mathrm{e}^{(-\mathrm{i}z_1 - \cdots - \mathrm{i}z_{n-1} - \beta/2)L}(\Omega(z_1, \dots, z_{n-1}) - \Omega_m(z_1, \dots, z_{n-1}))\right\| = 0$$

uniformly on compact subsets of $S_{\beta,n-1}$. By the induction assumption,

$$\lim_{m\to\infty}\|\Omega(z_1, \dots, z_{n-1}) - \Omega_m(z_1, \dots, z_{n-1})\| = 0$$

uniformly on compact subsets of $S_{\beta,n-1}$. Hence, by Proposition A.2

$$\lim_{m\to\infty}\left\|\mathrm{e}^{\mathrm{i}z_nL}\Big(\Omega(z_1,\dots,z_{n-1})-\Omega_m(z_1,\dots,z_{n-1})\Big)\right\|=0$$

uniformly for $0\le \mathrm{Im}z_n\le\beta/2-\mathrm{Im}z_1-\cdots-\mathrm{Im}z_{n-1}$ and $(z_1,\dots,z_{n-1})$ in compact subsets of $S_{\beta,n-1}$. In particular, the convergence is uniform on compact subsets of $S_{\beta,n}$. This ends the proof of (2) for n.

It remains to prove the norm continuity part of (1). Let $Q_{i,m}\in\mathfrak{M}_\tau$ such that $Q_{i,m}\underset{m\to\infty}{\to}Q_i$ strongly and $Q_{i,m}^*\underset{m\to\infty}{\to}Q_i^*$ strongly. The function

$$\mathbb{C}^n\ni(z_1,\dots,z_n)\mapsto\mathrm{e}^{\mathrm{i}z_nL}Q_{n,m}\cdots\mathrm{e}^{\mathrm{i}z_1L}Q_{1,m}\Omega$$

is entire analytic and in particular, it is norm continuous. By the uniform convergence on compact subsets of $S_{\beta,n}$, proven in (2), and the local compactness of $S_{\beta,n}$ we conclude that (5.30) is norm continuous on $S_{\beta,n}$. □

Proof of Theorem 5.2. Let $Q_n\in\mathfrak{M}_\tau$ be such that $Q_n\to Q$ strongly. Since $\Omega_{Q_n}=\mathrm{E}_{Q_n}(\mathrm{i}\beta/2)\Omega$, the expansion (3.9) yields that Theorem 5.2 holds for Q_n. Moreover,

$$\begin{aligned}\Omega_Q&=\mathrm{w}-\lim_{n\to\infty}\Omega_{Q_n}\\&=\mathrm{w}-\lim_{n\to\infty}\sum_{m=0}^{\infty}(-1)^m\underset{T_{\beta,m}}{\int\cdots\int}\mathrm{e}^{-\beta_1L}Q_n\cdots\mathrm{e}^{-\beta_mL}Q_n\,\Omega\,\mathrm{d}\beta_1\cdots\mathrm{d}\beta_m\\&=\sum_{m=0}^{\infty}(-1)^m\underset{T_{\beta,m}}{\int\cdots\int}\mathrm{e}^{-\beta_1L}Q\cdots\mathrm{e}^{-\beta_mL}Q\,\Omega\,\mathrm{d}\beta_1\cdots\mathrm{d}\beta_m.\end{aligned}$$

The first identity follows from Theorem 5.1 (recall the proof of (1) or use (13)), the second is obvious, and the third follows from Theorem 5.5. □

Proof of Theorem 5.3. Theorem 5.2 yields (1). By Theorem 5.1 (13) it suffices to prove (2)–(5) for $Q\in\mathfrak{M}_\tau$.

(2)-(3). Our proof is motivated by [Sa2]. By Theorem 3.2, $\Omega\in\mathcal{D}(\mathrm{e}^{-z(L+Q)})$ for all z and

$$\mathrm{E}_Q(\mathrm{i}z)\Omega=\mathrm{e}^{-z(L+Q)}\Omega$$

is an entire vector-valued function. Set

$$f(z):=(\Omega|\mathrm{e}^{-z(L+Q)}\Omega)=(\Omega|\mathrm{E}_Q(\mathrm{i}z)\Omega).$$

Then f is an entire function, $f''(x) \geq 0$ for $x \in \mathbb{R}$, and

$$\begin{aligned}
&f(0) = \|\Omega\|^2 = 1, \quad f(\beta/2) = (\Omega|\Omega_Q), \quad f(\beta) = \|\Omega_Q\|^2, \\
&f'(0) = -(\Omega|(L+Q)\Omega) = -(\Omega|Q\Omega), \\
&f'(\beta/2) = -(\Omega|(L+Q)\Omega_Q) = -(\Omega|JQJ\Omega_Q) = -(\Omega|Q\Omega_Q), \\
&f'(\beta) = -(\Omega_Q|(L+Q)\Omega_Q) = -(\Omega_Q|JQJ\Omega_Q) = -(\Omega_Q|Q\Omega_Q)
\end{aligned}$$

(we used $L\Omega = 0$ and $(L + Q - JQJ)\Omega_Q = 0$). These relations combined with the mean-value theorem yield (2)–(3).

(4) follows easily from (2).

To prove (5), consider the function

$$F(z) := \tau_Q^z(Q)\mathrm{E}_Q(z)\Omega.$$

Since $\tau_Q^z(Q)$ and $\mathrm{E}_Q(z)$ are uniformly bounded on the strip $0 \leq \mathrm{Im} z \leq \beta/2$, $F(z)$ is also bounded on the this strip. Moreover,

$$\|F(z)\| \leq \begin{cases} \|Q\Omega\| & \text{if } \mathrm{Im} z = 0; \\ \|\tau_Q^{\mathrm{i}\beta/2}(Q)\Omega_Q\| & \text{if } \mathrm{Im} z = \beta/2. \end{cases}$$

Since $\tau_Q^{\mathrm{i}\beta/2}(Q)\Omega_Q = \mathrm{e}^{-\beta L_Q/2}Q\Omega_Q = JQ\Omega_Q$,

$$\|F(z)\| \leq \|Q\Omega_Q\| \qquad \text{if} \quad \mathrm{Im} z = \beta/2.$$

Hence, by the three-line theorem, for $0 \leq t \leq \beta/2$,

$$\|F(\mathrm{i}t)\| \leq \|Q\Omega_Q\|^{1-2t/\beta}\|Q\Omega\|^{2t/\beta}.$$

Since

$$\Omega_Q - \Omega = -\int_0^{\beta/2} \tau_Q^{\mathrm{i}t}(Q)\mathrm{E}_Q(\mathrm{i}t)\Omega \mathrm{d}t,$$

we derive

$$\begin{aligned}
\|\Omega_Q - \Omega\| &\leq \textstyle\int_0^{\beta/2} \|F(\mathrm{i}t)\|\mathrm{d}t \\
&\leq \tfrac{\beta}{2} \textstyle\int_0^1 \|Q\Omega\|^{1-s}\|Q\Omega_Q\|^s \mathrm{d}s = \beta f(\|Q\Omega\|, \|Q\Omega_Q\|)/2.
\end{aligned}$$

□

5.5 Unbounded perturbations

This subsection contains our main results. It extends the construction of KMS states to a large class of unbounded perturbations.

Let Q be a self-adjoint operator affiliated to $\mathfrak{M}$ satisfying Assumptions 3.A and 3.B. Let τ_Q be the dynamics defined as in Subsection 3.3. Recall that by Theorem 3.5 its Liouvillean equals

$$L_Q = L + Q - JQJ.$$

In order to construct the perturbed KMS state we will need an additional assumption:

Assumption 5.A $\quad \|e^{-\beta Q/2}\Omega\| < \infty.$

Theorem 5.6 *Assume 3.A, 3.B and 5.A. Then*

(1) $\Omega \in \mathcal{D}(e^{-\beta(L+Q)/2})$. *Set*

$$\Omega_Q := e^{-\beta(L+Q)/2}\Omega, \qquad \omega_Q(A) := (\Omega_Q|A\Omega_Q)/\|\Omega_Q\|^2.$$

(2) $\Omega_Q \in \mathcal{H}^+$.

(3) Ω_Q *is cyclic and separating.*

(4) ω_Q *is a* (τ_Q, β)*-KMS state.*

(5) $\log \Delta_{\Omega_Q} = -\beta L_Q$.

(6) $\log \Delta_{\Omega_Q,\Omega} = -\beta L - \beta Q$.

(7) $\mathrm{Ent}(\omega|\omega_Q) = -\beta\omega(Q) - \log \|\Omega_Q\|^2$.

(8) *The Peierls-Bogoliubov inequality holds*

$$e^{-\beta(\Omega|Q\Omega)/2} \le \|\Omega_Q\|.$$

(9) *The Golden-Thompson inequality holds:*

$$\|\Omega_Q\| \le \|e^{-\beta Q/2}\Omega\|.$$

(10) *For any* $0 \le \lambda \le 1$, λQ *satisfies the assumptions of the theorem, hence* $\Omega_{\lambda Q}$ *is well defined. Moreover,* $\lim_{\lambda\downarrow 0} \|\Omega_{\lambda Q} - \Omega\| = 0$.

Remark. The formula for relative entropy of (7) requires a comment. Because of Assumption 5.A, $\omega(Q_-)$ is finite, where $Q_- = 1_{]-\infty,0]}(Q)Q$. Therefore, $\omega(Q)$ is a finite number or $+\infty$.

Set

$$Q_n := 1_{[-n,n]}(Q)Q,$$

where $1_{[-n,n]}(Q)$ is the spectral projection of Q on the interval $[-n, n]$.

Lemma 5.7 (1) *$L + Q_n \to L + Q$ in the strong resolvent sense.*

(2) *$L_{Q_n} \to L_Q$ in the strong resolvent sense.*

Proof. We prove only (2) (the proof of (1) is similar). Let $\mathcal{D}_0 = \mathcal{D}(L) \cap \mathcal{D}(Q) \cap \mathcal{D}(JQJ)$. By Assumption 3.B, L_Q is essentially self-adjoint on $\mathcal{D}_0$. Moreover, $L_{Q_n}\Psi \to L_Q\Psi$, $\Psi \in \mathcal{D}_0$. Hence the statement follows from Proposition A.5. □

Proof of Theorem 5.6. Given the approximating sequence Q_n defined above and Lemma 2, the parts (1)-(9) follow from Theorem 5.1 in the same way as the analogous parts of Theorem 5.1 followed from the analytic case of Theorem 5.1.

The only part requiring a separate argument is (10). To prove it, note that $L+\lambda Q \to L$ in the strong resolvent sense as $\lambda \downarrow 0$. This implies that $\Omega_{\lambda Q} \to \Omega$ weakly as $\lambda \downarrow 0$ and

$$\|\Omega\| \leq \liminf_{\lambda\downarrow 0} \|\Omega_{\lambda Q}\| \leq \limsup_{\lambda\downarrow 0} \|\Omega_{\lambda Q}\| \leq \lim_{\lambda\downarrow 0} \|\mathrm{e}^{-\beta\lambda Q/2}\Omega\| = \|\Omega\|.$$

Hence, $\|\Omega_{\lambda Q}\| \to \|\Omega\|$ as $\lambda \downarrow 0$, and this implies that $\Omega_{\lambda Q} \to \Omega$ as $\lambda \downarrow 0$. □

5.6 Perturbations of Liouvilleans revisited

In Theorem 3.5 we have shown that L_Q is the Liouvillean of τ_Q by invoking Theorem 2.11 and checking that

$$\tau_Q^t(A) = \mathrm{e}^{\mathrm{i}tL_Q} A \mathrm{e}^{-\mathrm{i}tL_Q}, \qquad \mathrm{e}^{\mathrm{i}tL_Q}\mathcal{H}^+ \subset \mathcal{H}^+. \tag{5.38}$$

Under the conditions of Theorem 5.6 (recall Proposition 2.14), the second relation in (5.38) is equivalent to

$$L_Q\Omega_Q = 0. \tag{5.39}$$

In this section we give an elementary direct proof of (5.39). This verifies that L_Q is the Liouvillean of τ_Q without resort to Theorem 2.11.

We consider only the case of analytic perturbations $Q \in \mathfrak{M}_\tau$. The extension to bounded Q and unbounded Q satisfying conditions of Theorem 5.6 is immediate using the strong resolvent convergence of Liouvilleans and the weak convergence of β-KMS vectors.

First, the relation

$$\mathrm{e}^{\mathrm{i}t(L+Q)}\Omega_Q = \mathrm{E}_Q(t + \mathrm{i}\beta)\Omega$$

and analytic continuation yield that $\Omega_Q \in \mathcal{D}(\exp(\mathrm{i}z(L + Q))$ for all z, and so $\Omega_Q \in \mathcal{D}(L + Q) = \mathcal{D}(L_Q)$.

Since $\mathrm{e}^{\mathrm{i}tL}\mathfrak{M}'\mathrm{e}^{-\mathrm{i}tL} = \mathfrak{M}'$, $JQJ \in \mathfrak{M}'$, and $\mathrm{e}^{\mathrm{i}tL}J = J\mathrm{e}^{\mathrm{i}tL}$, the Trotter product formula yields

$$\mathrm{e}^{\mathrm{i}t(L+Q)}JQJ\mathrm{e}^{-\mathrm{i}t(L+Q)} = \mathrm{e}^{\mathrm{i}tL}JQJ\mathrm{e}^{-\mathrm{i}tL} = J\mathrm{e}^{\mathrm{i}tL}Q\mathrm{e}^{-\mathrm{i}tL}J.$$

By analytic continuation, the relation

$$(\mathrm{e}^{\beta(L+Q)/2}\Phi|JQJ\mathrm{e}^{-\beta(L+Q)/2}\Omega) = (\Phi|J\tau^{\mathrm{i}\beta/2}(Q)J\Omega)$$

holds for all Φ in a dense domain $\tilde{\mathcal{D}} = \cup_{r>0} \mathrm{Ran} 1_{[-r,r]}(L+Q)$. Using

$$J\tau^{\mathrm{i}\beta/2}(Q)J\Omega = J\Delta^{\frac{1}{2}}Q\Omega = Q\Omega,$$

we derive

$$(\mathrm{e}^{\beta(L+Q)/2}\Phi|JQJ\Omega_Q) = (\Phi|Q\Omega).$$

This relation yields

$$\begin{aligned}(\mathrm{e}^{\beta(L+Q)/2}\Phi|(L+Q-JQJ)\Omega_Q) &= (\mathrm{e}^{\beta(L+Q)/2}\Phi|(L+Q)\Omega_Q) - (\Phi|Q\Omega) \\ &= (\Phi|(L+Q)\Omega) - (\Phi|Q\Omega) \\ &= (\Phi|L\Omega) = 0.\end{aligned}$$

Since $\mathrm{e}^{\beta(L+Q)/2}\tilde{\mathcal{D}} = \tilde{\mathcal{D}}$ is dense in $\mathcal{H}$, $L_Q\Omega_Q = 0$.

A Technical facts

In this appendix we collect some technical facts which have been used throughout the paper.

A.1 Operators and resolvent convergence

First, we recall the Trotter product formula (see [RS1], Theorem VIII.31).

Theorem A.1 *If A and B are self-adjoint operators and $A+B$ is essentially self-adjoint on $\mathcal{D}(A)\cap\mathcal{D}(B)$, then*

$$\mathrm{s}-\lim_{n\to\infty}\left(\mathrm{e}^{\mathrm{i}tA/n}\mathrm{e}^{\mathrm{i}tB/n}\right)^n = \mathrm{e}^{\mathrm{i}t(A+B)}.$$

The next proposition follows easily from the spectral theorem and the three-line theorem (see also Lemma 4 in [Ar2]).

Proposition A.2 *Let H be a self-adjoint operator and $\Omega \in \mathcal{D}(\mathrm{e}^{\delta H})$ for some $\delta > 0$. Then the vector-valued function $\mathrm{e}^{zH}\Omega$ is analytic inside the strip $0 < \mathrm{Re} z < \delta$, norm continuous on its closure and*

$$\|\mathrm{e}^{zH}\Omega\| \le \|\mathrm{e}^{\delta H}\Omega\|^{\mathrm{Re} z/\delta}\|\Omega\|^{1-\mathrm{Re} z/\delta}.$$

Lemma A.3 *Let Z be a compact metric space and $Z \ni z \mapsto \Omega(z) \in \mathcal{H}$ a norm continuous function. Let A_n be bounded operators and assume that $A_n \to A$ strongly. Then*

$$\lim_{n\to\infty}\|(A_n - A)\Omega(z)\| = 0$$

uniformly on Z.

Proof. Note first that $\{\Omega(z) : z \in Z\}$ is a compact subset of $\mathcal{H}$ and that by the uniform boundedness principle $C := \sup_n \|A_n\| < \infty$. Let $\epsilon > 0$ be given. Then there exists a finite dimensional projection P such that $\sup_{z\in Z} \|(1-P)\Omega(z)\| < \epsilon$. Since

$$\begin{aligned} \|(A_n - A)\Omega(z)\| &\leq \|(A_n - A)P\| \sup_{z\in Z} \|\Omega(z)\| \\ &\quad + \sup_n \|A_n - A\| \sup_{z\in Z} \|(1-P)\Omega(z)\|, \end{aligned}$$

we derive $\limsup_n \|(A_n - A)\Omega(z)\| < 2C\epsilon$.□

The following properties of the strong convergence of functions of self-adjoint operators are proven e.g. in [RS1]:

Proposition A.4 *Suppose that H_n, H are self-adjoint operators. Then the following conditions are equivalent:*

(1) *Let $z_0 \notin \left(\bigcup_{n=1}^{\infty} \sigma(H_n)\right)^{\rm cl}$ (for instance, $\mathrm{Im} z_0 \neq 0$). Then*

$$\mathrm{s-}\lim_{n\to\infty} (z_0 - H_n)^{-1} = (z_0 - H)^{-1}.$$

(2) *If f is a bounded continuous function on $\left(\bigcup_{n=1}^{\infty} \sigma(H_n)\right)^{\rm cl}$, then $f(H_n) \to f(H)$ strongly.*

Note that (1) in the above proposition holds for any choice of z_0 if it holds for one choice of z_0.

If the conditions of above proposition are satisfied we say that $H_n \to H$ in the strong resolvent sense.

Proposition A.5 *Suppose that H_n, H are self-adjoint operators, H is essentially self-adjoint on $\mathcal{D}$ and $\lim_n H_n\Psi = H\Psi$ for $\Psi \in \mathcal{D}$. Then $H_n \to H$ in the strong resolvent sense.*

Proof. Let $\mathrm{Im} z \neq 0$. Then $(z-H)\mathcal{D} =: \mathcal{D}_1$ is dense in $\mathcal{H}$. For $\Psi \in \mathcal{D}_1$,

$$(z-H)^{-1}\Psi - (z-H_n)^{-1}\Psi = (z-H_n)^{-1}(H-H_n)(z-H)^{-1}\Psi \to 0.$$

□

The following proposition plays an important role in several arguments in our paper.

Proposition A.6 *Suppose that H_n, H are self-adjoint operators and $H_n \to H$ in the strong resolvent sense. Suppose that $\Omega_n, \Omega \in \mathcal{H}$ such that $\Omega_n \to \Omega$ weakly and $\|H_n\Omega_n\| \leq C$. Then $\Omega \in \mathcal{D}(H)$, $\mathrm{w-}\lim_n H_n\Omega_n$ exists and $H\Omega = \mathrm{w-}\lim_n H_n\Omega_n$.*

Remark. By the uniform boundedness principle, the condition $\|H_n\Omega_n\| \leq C$ can be replaced by the existence of w$-\lim H_n\Omega_n$.

Proof. Since the ball of radius C in a Hilbert space is weakly sequentially compact, one can find a weakly convergent subsequence $H_{n_k}\Omega_{n_k}$. Set $\Psi = \text{w} - \lim_{k\to\infty} H_{n_k}\Omega_{n_k}$.

Let $\mathcal{D} := \bigcup_{r>0} \text{Ran}1_{[-r,r]}(H)$. Let $\Phi \in \mathcal{D}$ and $f \in C_0^\infty(\mathbb{R})$ such that $f(H)\Phi = \Phi$. Then

$$\Phi = f(H)\Phi = \lim_{n\to\infty} f(H_n)\Phi$$

$$H\Phi = f(H)H\Phi = \lim_{n\to\infty} f(H_n)H_n\Phi,$$

and

$$\begin{aligned}(H\Phi|\Omega) &= \lim_{k\to\infty}(H_{n_k}f(H_{n_k})\Phi|\Omega_{n_k}) \\ &= \lim_{k\to\infty}(f(H_{n_k})\Phi|H_{n_k}\Omega_{n_k}) \\ &= (\Phi|\Psi).\end{aligned} \tag{A.40}$$

Since $\mathcal{D}$ is a core for H, $\Omega \in \mathcal{D}(H)$ and $H\Omega = \Psi$

Now assume that w$-\lim_n H_n\Omega$ does not exist. Then there exists $\Phi \in \mathcal{H}$ and a subsequence $H_{n_k}\Omega$ such that

$$|(\Phi|H_{n_k}\Omega) - (\Phi|H\Omega)| \geq \epsilon > 0. \tag{A.41}$$

Using again the weak sequential compactness of the ball of radius C and passing to a subsubsequence we may assume that w$-\lim_{k\to\infty} H_{n_k}\Omega$ exists. Repeating the arguments of (A.40), we see that w$-\lim_{k\to\infty} H_{n_k}\Omega = H\Omega$. This contradicts (A.41). $\square$

A.2 An interpolation theorem

Various versions of the following interpolation theorem for linear operators can be found throughout literature, see e.g. [OP] (where a different proof is outlined) and [RS2].

Theorem A.7 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let H_i be a positive (possibly unbounded) operator on $\mathcal{H}_i$. Let $\mathcal{D}_1$ be a core of H_1. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $\|T\| = c_0$ be such that:*
(a) *$T\mathcal{D}_1 \subset \mathcal{D}(H_2)$.*
(b) *For $\Psi \in \mathcal{D}_1$, $\|H_2T\Psi\| \leq c_1\|H_1\Psi\|$.*
Then, for any $0 \leq \lambda \leq 1$, $T\mathcal{D}(H_1^\lambda) \subset \mathcal{D}(H_2^\lambda)$ and for $\Psi \in \mathcal{D}(H_1^\lambda)$,

$$\|H_2^\lambda T\Psi\| \leq c_0^{1-\lambda}c_1^\lambda\|H_1^\lambda\Psi\|. \tag{A.42}$$

Proof. Clearly, we may assume that $c_0 = c_1 = 1$.

First let us show that $T\mathcal{D}(H_1) \subset \mathcal{D}(H_2)$ and

$$\|H_2 T\Psi\| \leq \|H_1\Psi\|, \quad \Psi \in \mathcal{D}(H_1). \tag{A.43}$$

Let $\Psi \in \mathcal{D}(H_1)$. Then there exist $\Psi_n \in \mathcal{D}_1$ such that $\Psi_n \to \Psi$ and $H_1\Psi_n \to H_1\Psi$. Now

$$\|H_2(T\Psi_n - T\Psi_m)\| \leq \|H_1(\Psi_n - \Psi_m)\|.$$

Thus $H_2T\Psi_n$ is Cauchy, hence convergent. $T\Psi_n$ is obviously convergent. H_2 is closed. Hence $T\Psi \in \mathcal{D}(H_2)$. (A.43) follows by passing to the limit in $\|H_2T\Psi_n\| \leq \|H_1\Psi_n\|$.

Let $\Phi \in \mathcal{D}(H_2)$, $\Omega \in \mathcal{H}_1$ and $\epsilon > 0$. For $0 \leq \mathrm{Re} z \leq 1$ set

$$F_\epsilon(z) := (H_2^{\bar{z}}\Phi | T(H_1 + \epsilon)^{-z}\Omega).$$

$F_\epsilon(z)$ is a continuous function in the strip $0 \leq \mathrm{Re} z \leq 1$, analytic in its interior, and

$$|F_\epsilon(z)| \leq \|(H_2 + 1)\Phi\|\epsilon^{-1}\|\Omega\|.$$

For $\mathrm{Re} z = 0$

$$|F_\epsilon(z)| \leq \|\Phi\|\|\Omega\|.$$

For $\mathrm{Re} z = 1$, $(H_1 + \epsilon)^{-z}\Omega \in \mathcal{D}(H_1)$, and

$$\begin{aligned}|F_\epsilon(z)| &\leq \|\Phi\|\|H_2T(H_1 + \epsilon)^{-z}\Omega\| \\ &\leq \|\Phi\|\|H_1(H_1 + \epsilon)^{-z}\Omega\| \leq \|\Phi\|\|\Omega\|.\end{aligned}$$

These estimates and the three-line theorem yield that for $0 \leq \lambda \leq 1$

$$|F_\epsilon(\lambda)| \leq \|\Phi\|\|\Omega\|.$$

Therefore, for $\Omega \in \mathcal{H}_1$,

$$\|H_2^\lambda T(H_1 + \epsilon)^{-\lambda}\Omega\| \leq \|\Omega\|,$$

and for $\Psi \in \mathcal{D}(H_1^\lambda)$

$$\begin{aligned}\|H_2^\lambda T\Psi\| &= \lim_{\epsilon\downarrow 0} \|H_2^\lambda T(H_1 + \epsilon)^{-\lambda}(H_1 + \epsilon)^\lambda\Psi\| \\ &\leq \lim_{\epsilon\downarrow 0} \|(H_1 + \epsilon)^\lambda\Psi\| \\ &= \|H_1^\lambda\Psi\|.\end{aligned}$$

□

B Pauli-Fierz systems

B.1 Introduction

A large part of the motivation for the formalism and the results of our paper comes from quantum statistical physics. A detailed description of their application to Pauli-Fierz systems—a certain class of physically motivated W^*-dynamical systems—can be found in [DJ2]. In this appendix we briefly describe these applications.

Pauli-Fierz systems describe a small quantum system (an atom or a molecule) interacting with a large bosonic reservoir. They arise as an approximation to non-relativistic QED (see e.g. [DJ1, RZ]), and they have been widely used in physics literature as a basic paradigm of an open quantum system [LCD, We].

We are interested in the case where the radiation density of the bosonic reservoir is not zero (in particular, the reservoir is not at zero temperature). For example, the radiation density can be given by the Planck law at the inverse temperature $\beta < \infty$, see (B.46) below. This corresponds to the case of bosons in thermal equilibrium. We are also interested in situations outside thermal equilibrium. For example, the reservoir may consist of several subreservoirs at distinct temperatures.

W^*-dynamical systems provide a natural framework to describe Pauli-Fierz systems with non-zero radiation density, as it will be sketched below.

B.2 Bose gas at density ρ—Araki-Woods algebras

If $\mathcal{Z}$ is a Hilbert space, then we will write $\Gamma_{\mathrm{s}}(\mathcal{Z})$ for the bosonic Fock space over the 1-particle space $\mathcal{Z}$. Ω will denote the vacuum vector.

For definiteness, we will consider the Bose gas with the 1-particle space $L^2(\mathbb{R}^d)$. Assume that $\mathbb{R}^d \ni \xi \mapsto \rho(\xi)$ is a nonnegative real measurable function describing the density of bosons with the momentum $\xi \in \mathbb{R}^d$. To describe the Bose gas at density ρ one uses a special von Neumann algebra first described by Araki and Woods in [ArW]. It can be defined by its representation in the Hilbert space

$$\mathcal{H}^{\mathrm{AW}} := \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)).$$

We will write $a_{\mathrm{l}}(\xi)$, $a_{\mathrm{l}}^*(\xi)$, $a_{\mathrm{r}}(\xi)$, $a_{\mathrm{r}}^*(\xi)$ for the creation and annihilation operators corresponding to the left and right $L^2(\mathbb{R}^d)$ resp. We define the left/right Araki-Woods creation and annihillation opetators

$$\begin{aligned}
a_{\rho,\mathrm{l}}^*(\xi) &:= \sqrt{1+\rho(\xi)}a_{\mathrm{l}}^*(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}(\xi),\\
a_{\rho,\mathrm{l}}(\xi) &:= \sqrt{1+\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}^*(\xi) &:= \sqrt{\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{1+\rho(\xi)}a_{\mathrm{r}}^*(\xi),\\
a_{\rho,\mathrm{r}}(\xi) &:= \sqrt{\rho(\xi)}a_{\mathrm{l}}^*(\xi) + \sqrt{1+\rho(\xi)}a_{\mathrm{r}}(\xi).
\end{aligned}$$

The left Araki-Woods algebra is denoted by $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ and defined as the W^*-algebra generated by the operators

$$\exp\Big(\mathrm{i}\int(f(\xi)a_{\rho,\mathrm{l}}^*(\xi)+\overline{f}(\xi)a_{\rho,\mathrm{l}}(\xi))\mathrm{d}\xi\Big).$$

Let $J^{\mathrm{AW}}:=\Gamma(\epsilon)$, where ϵ is an antilinear involution on $L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$ given by

$$\epsilon(f_1,\overline{f}_2):=(f_2,\overline{f}_1),$$

and Γ is the second quantization functor, and let $\mathcal{H}_\rho^{\mathrm{AW},+}$ be the closure of the cone in $\mathcal{H}^{\mathrm{AW}}$ generated by

$$AJA\Omega,\quad A\in\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}.$$

Then $(\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}},\mathcal{H}^{\mathrm{AW}},J^{\mathrm{AW}},\mathcal{H}_\rho^{\mathrm{AW},+})$ is a von Neumann algebra in a standard form. It describes the Bose gas at density ρ.

B.3 Araki-Woods algebra coupled to a type I factor

We denote by $\mathcal{K}$ the Hilbert space of the small quantum system. For simplicity, we assume that $\dim\mathcal{K}<\infty$. We would like to describe the W^*-algebra of the joint system consisting of the small system with the algebra of observables $\mathcal{B}(\mathcal{K})$ and the Bose gas at density ρ.

One way to define this algebra is to identify it with the von Neumann algebra

$$\mathfrak{M}_\rho:=\mathcal{B}(\mathcal{K})\otimes\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$$

acting on the Hilbert space $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d))$. The identity representation of this algebra on $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d))$ will be called the semi-standard representation of $\mathfrak{M}_\rho$.

It is easy to describe the standard representation of $\mathfrak{M}_\rho$. Let $\overline{\mathcal{K}}$ be the Hilbert space complex conjugate to $\mathcal{K}$ (see e.g. Section 4.6 in [DJ2]). The standard representation acts on the space

$$\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)) \tag{B.44}$$

and is given by $\pi(A\otimes B):=A\otimes 1_{\overline{\mathcal{K}}}\otimes B$ for $A\in\mathcal{B}(\mathcal{K})$, $B\in\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$. The modular conjugation is given by

$$J\,\Psi_1\otimes\overline{\Psi}_2\otimes\Phi:=\Psi_2\otimes\overline{\Psi}_1\otimes J^{\mathrm{AW}}\Phi.$$

Note that it is useful to consider the two representations of $\mathfrak{M}_\rho$—the semi-standard and the standard representations in a parallel way. The semi-standard representation is simpler whereas the standard representation has special mathematical properties.

B.4 Pauli-Fierz W^*-dynamical systems

Suppose that K is a self-adjoint operator on $\mathcal{K}$ describing the Hamiltonian of the small system. Let $|\xi|$ be the energy of the boson of momentum ξ. Let $\mathbb{R}^d \ni \xi \mapsto v(\xi) \in \mathcal{B}(\mathcal{K})$ describe the coupling of the small system to the Bose gas. We assume that the Bose gas is at the density ρ. Let $\lambda \in \mathbb{R}$.

We introduce the following operators on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$:

$$L_{\mathrm{fr}}^{\mathrm{semi}} := K \otimes 1 + 1 \otimes \int |\xi| \big(a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi)\big) \mathrm{d}\xi,$$

$$Q_\rho^{\mathrm{semi}} := \int \big(v(\xi) \otimes a_{\rho,\mathrm{l}}^*(\xi) + v^*(\xi) \otimes a_{\rho,\mathrm{l}}(\xi)\big) \mathrm{d}\xi,$$

The operator $L_{\mathrm{fr}}^{\mathrm{semi}}$ will be called the free semi-Liouvillean. The full semi-Liouvillean for the density ρ equals

$$L_\rho^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}} + \lambda Q_\rho^{\mathrm{semi}}. \tag{B.45}$$

For $A \in \mathfrak{M}_\rho$ we set

$$\tau_{\mathrm{fr}}^t(A) := \mathrm{e}^{\mathrm{i}tL_{\mathrm{fr}}^{\mathrm{semi}}} A \mathrm{e}^{-\mathrm{i}tL_{\mathrm{fr}}^{\mathrm{semi}}},$$
$$\tau_\rho^t(A) := \mathrm{e}^{\mathrm{i}tL_\rho^{\mathrm{semi}}} A \mathrm{e}^{-\mathrm{i}tL_\rho^{\mathrm{semi}}}.$$

We also introduce the following operators on $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$:

$$L_{\mathrm{fr}} = K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1 + 1 \otimes 1 \otimes \int |\xi| \big(a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi)\big) \mathrm{d}\xi.$$

$$Q_\rho = \int \big(v(\xi) \otimes 1 \otimes a_{\rho,\mathrm{l}}^*(\xi) + v^*(\xi) \otimes 1 \otimes a_{\rho,\mathrm{l}}(\xi)\big) \mathrm{d}\xi.$$

It is easy to see that

$$JQ_\rho J = \int \big(1 \otimes \overline{v}(\xi) \otimes a_{\rho,\mathrm{r}}^*(\xi) + 1 \otimes \overline{v}^*(\xi) \otimes a_{\rho,\mathrm{r}}(\xi)\big) \mathrm{d}\xi.$$

Set

$$L_\rho := L_{\mathrm{fr}} + \lambda Q_\rho - \lambda JQ_\rho J.$$

We denote by $l^2(\mathcal{K})$ the vector space $\mathcal{B}(\mathcal{K})$ equipped with the inner product $(A|B) = \mathrm{Tr}(A^*B)$ (recall that $\dim \mathcal{K} < \infty$). The following theorem describes the case of the free dynamics.

Theorem B.1 (1) *For any t, τ_{fr}^t preserves the algebra $\mathfrak{M}_\rho$ and $(\mathfrak{M}_\rho, \tau_{\mathrm{fr}})$ is a W^*-dynamical system.*

(2) *L_{fr} is the Liouvillean for the dynamics τ_{fr}.*

(3) *Let* $\beta > 0$,

$$\rho(\xi) = (\mathrm{e}^{\beta|\xi|} - 1)^{-1}, \tag{B.46}$$

and $\Psi_{\mathrm{fr}} := \mathrm{e}^{-\beta K/2} \otimes \Omega$. *Using the natural identification of* $l^2(\mathcal{K})$ *with* $\mathcal{K} \otimes \overline{\mathcal{K}}$, Ψ_{fr} *can be understood as an element of the Hilbert space* $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$. *Then* Ψ_{fr} *is a* β*-KMS vector for* τ_{fr}.

The results of our paper are the main technical input in the proof of the following theorem, which describes the interacting dynamics:

Theorem B.2 (1) *Assume that*

$$\int (1 + |\xi|^2)(1 + \rho(\xi))\|v(\xi)\|^2 \mathrm{d}\xi < \infty. \tag{B.47}$$

Then for any t, τ_ρ^t *preserves the algebra* $\mathfrak{M}_\rho$ *and* $(\mathfrak{M}_\rho, \tau_\rho)$ *is a* W^**-dynamical system.*

(2) L_ρ *is the Liouvillean for the dynamics* τ_ρ.

(3) *Assume that (B.46) holds and that*

$$\int (|\xi|^{-1} + |\xi|^2)\|v(\xi)\|^2 \mathrm{d}\xi < \infty.$$

Then (B.47) holds, and there exists a β*-KMS vector for* τ_ρ.

The W^*-dynamical system $(\mathfrak{M}_\rho, \tau_\rho)$ is called the Pauli-Fierz system at density ρ. It is canonically defined given $\mathcal{K}$, K, v and ρ.

The proof of Theorem B.2 is given in [DJ2]. To prove (1) we check that Q_ρ^{semi} is affiliated to $\mathfrak{M}_\rho$ and that $L_{\mathrm{fr}}^{\mathrm{semi}} + \lambda Q_\rho^{\mathrm{semi}}$ is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}^{\mathrm{semi}}) \cap \mathcal{D}(Q_\rho^{\mathrm{semi}})$. Then we apply Theorem 3.3. To prove (2), in a similar way we apply Theorem 3.5. Finally, to show (3) we use Theorem 5.6. The details can be found in [DJ2].

We finish with several remarks.

The perturbation Q_ρ^{semi} is unbounded from above and below, and the existing results in the literature [Ar1, Don, Sa2] are not applicable to Pauli-Fierz systems.

The first result about existence of KMS-states for Pauli-Fierz systems goes back to [FNV] where the spin-boson system was considered. A result similar to Theorem B.2 was proven in [BFS] under a more restrictive infrared condition. Theorem B.2 covers the physical infrared regime of non-relativistic QED (often called the ohmic case in the context of Pauli-Fierz systems, see e.g. [DJ1, DJ2, LCD, We]).

References

[Ar1] Araki, H.: Relative Hamiltonian for a faithful normal states of a von Neumann algebra, Pub. R.I.M.S., Kyoto Univ. **9**, 165 (1973).

[Ar2] Araki, H.: Golden-Thompson and Peierls-Bogoliubov inequalities for a general von Neumann algebra, Commun. Math. Phys. **34**, 167 (1973).

[Ar3] Araki, H.: Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki theory, in *C^*-algebras and their Applications to Statistical Mechanics and Quantum Field Theory,* (D. Kastler editor), Amsterdam, North-Holand (1976).

[Ar4] Araki, H.: Relative entropy of states of von Neumann algebras, Pub. R.I.M.S., Kyoto Univ. **11**, 809 (1976).

[Ar5] Araki, H.: Relative entropy of states of von Neumann algebras II, Pub. R.I.M.S., Kyoto Univ. **13**, 173 (1977).

[Ar6] Araki, H.: Expansionals in Banach algebras, Ann. Sci. Ecole Norm. Sup. **6**, 67 (1973).

[ArW] Araki, H., Woods, E. J.: Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas, J. Math. Phys. **4**, 637 (1963).

[BaJu] Baaj, S., Jungl, P.: Théorie bivariant de Kasparow et operateurs non borne'ees dans les C^*-modules hilbertiens, C. R. Acad. Sci. Paris, Serie I 296, 875 (1983).

[BFS] Bach, V., Fröhlich, J., Sigal, I.: Return to equilibrium, J. Math. Phys. **41**, 3985 (2000).

[BR1] Brattelli, O, Robinson, D. W.: *Operator Algebras and Quantum Statistical Mechanics 1*, Springer-Verlag, Berlin, second edition (1987).

[BR2] Brattelli, O, Robinson, D. W.: *Operator Algebras and Quantum Statistical Mechanics 2*, Springer-Verlag, Berlin, second edition (1996).

[Co] Connes, A.: Une classification des facteurs de type III, Ann. Sci. Ecole Norm. Sup. **6**, 133 (1973).

[Don] Donald, M.J.: Relative Hamiltonians which are not bounded from above, J. Func. Anal. **91**, 143 (1990).

[DJ1] Derezinski, J., Jakšić, V.: Spectral theory of Pauli-Fierz operators, J. Func. Anal. **180**, 241 (2001).

[DJ2] Derezinski, J., Jakšić, V.: Return to equilibrium for Pauli-Fierz systems, preprint, Maphysto, Aarhus University (2001).

[FNV] Fannes, M., Nachtergale, B., Verbeure, A.: The equilibrium states of the spin-boson model, Comm. Math. Phys. **114**, 537 (1988).

[Haa] Haagerup, U.: The standard form of von Neumann algebras, Math. Scand. **37**, 271 (1975).

[JP1] Jakšić, V., Pillet C.-A.: On a model for quantum friction III. Ergodic properties of the spin-boson system, Commun. Math. Phys. **178**, 627 (1996).

[JP2] Jakšić, V., Pillet, C.-A.: Non-equilibrium steady states for finite quantum systems coupled to thermal reservoirs, to appear in Commun. Math. Phys.

[KL] Klein, A., Landau, L.: Stochastic processes associated with KMS states, J. Func. Anal. **42**, 368 (1981).

[LCD] Legget, A.J., Chakravarty, S., Dorsey, A.T., Fisher M.P.A., Garg, A., Zwerger, W.: Dynamics of the dissipative two-state system, Rev. Mod. Phys. **59**, 1 (1987).

[M] Merkli, M.: Positive commutators in non-equilibrium quantum statistical mechanics, preprint.

[OP] Ohya, M., Petz, D.: *Quantum Entropy and its Use*, Springer-Verlag, Berlin (1993).

[PuWo] Pusz, W., Woronowicz, S. L.: Form convex functions and the WYDL and other inequalities, Lett. Math. Phys. **2**, 505 (1978).

[RS1] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics I. Functional Analysis*, Academic Press, San Diego (1973).

[RS2] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, San Diego (1975).

[RZ] Raggio, G.A., Zivi, S.H.: Semiclassical description of N-level systems interacting with radiation fields, in "Quantum Porbability II, Heidelberg 1984", Lecture Notes in Math., Vol. 1136, Springer-Verlag, Berlin-New York, 1985.

[Sa1] Sakai, S.: *Operator Algebras in Dynamical Systems. The Theory of Unbounded Derivations in C^*-algebras.* Encyclopedia of Mathematics and its Applications **41**, Cambridge University Press, Cambridge (1991).

[Sa2] Sakai, S: Perturbations of KMS states in C^*-dynamical systems (Generalization of the absence theorem of phase transition to continuous quantum systems), Contemp. Math. **62**, 187 (1987).

[Sa3] Sakai, S: *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin (1971).

[Si] Simon, B.: *Statistical Mechanics of Lattice Gasses, Volume 1*, Princeton University Press, Princeton (1991).

[St] Stratila, S.: *Modular theory in operator algebras*, Abacus Press, Turnbridge Wells (1981).

[StZs] Stratila, S., Zsido, L.: *Lectures on von Neumann algebras*, Abacus Press, Turnbridge Wells (1979).

[Uh] Uhlmann, A.: Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory, Commun. Math. Phys. **54**, 123 (1977).

[We] Weiss, U.: *Quantum Dissipative Systems*, Series in Modern Condensed Matter Physics-Vol. 2, World Scientific, Singapore, second enlarged edition 1999.

[Wo] Woronowicz, S. L.: Unbounded elements afflliated with C^*-algebras and non-compact quantum groups, Commun. Math. Phys. **136**, 399 (1991).