ON THE VIOLATION OF OHM'S LAW FOR BOUNDED INTERACTIONS

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ABSTRACT. We study the motion of a charged particle in a constant electric field interacting via Newton's law with an infinitely extended system of particles. We discuss conditions on the particle/medium interaction which are necessary for the charged particle to reach a finite limit velocity. For small electric field intensity, the linear response theory states that the drift velocity should be directly proportional to the field (Ohm's law). In the framework of fully *Hamiltonian* systems, we support the thesis that Ohm's law validity needs the interaction particle/medium to be *unbounded*. We prove this statement in a one-dimensional model, by showing that for bounded interactions the velocity of the charged particle increases linearly in time.

1. INTRODUCTION

The task of obtaining laws of non equilibrium statistical mechanics by an Hamiltonian model is appealing and very difficult. In this paper we consider a charged particle moving under the action of a constant electric field and interacting with a system of infinitely many classical particles. This problem has been considered in a previous paper where the case of large electric fields has been rigorously studied [9]. There we discussed a necessary condition on the particle/medium interaction for the charged particle to reach a finite limit velocity (depending on the intensity of the electric field). More precisely, we investigated a system confined either in an infinite tube or in one dimension. We proved that, for each initial state of thermodynamic relevance, there exists a threshold on the electric field intensity above which the charged particle escapes to infinity close to a uniformly accelerated motion. In particular the drift velocity becomes infinite. In that paper the case of small electric fields was discussed only at an heuristic level, while we give here a rigorous analysis of this important case.

If the electric field intensity is small, the linear response theory, usually accepted in Physics, suggests that the drift is directly proportional to the electric field (Ohm's law). Here we support the following thesis: for Ohm's law to hold (asymptotically in time) the particle/medium interaction needs to be *unbounded*. It is too hard to prove this statement in general. We restrict ourselves to study a particular one-dimensional model, which is simple enough to be mathematically investigated but significant to give suggestions on the general case.

Now let us give a short (not exhaustive) comment on the efforts to rigorously obtain Ohm's law for an Hamiltonian system. There are investigations on the motion of a charged particle elastically interacting with an infinite hard core gas. When the hard core particles are fixed (Lorentz gas) a nonphysical phenomenon of

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heating arises: the average kinetic energy of the charged particle is increased by the electric field, while it cannot be dissipated by the elastic collisions with the fixed obstacles. At the same time there is not an asymptotic drift velocity [22]. The more realistic case of moving hard core particles with a finite mass seems to be too difficult. Another situation which has been considered is the motion along one axis of a charged particle with an hard core and elastically interacting with a gas of free particles. In this case it is possible to prove the existence of a limit drift velocity and to investigate some properties of the limit state. However the technical assumption that the free gas is in a Gibbs state with a cutoff on the slow transversal velocities is necessary [5, 6]. (On related topics see also [2, 3, 4, 13, 21, 25, 26]). In other works, different media are considered, where, in general, the goal is to describe a microscopic model for linear friction. We only recall the recent paper [7] on this argument and address the reader to the references quoted there in.

We finally remark that many physical problems only concern the behavior of the system on a long time scale rather than its genuine steady state. Mathematically it corresponds to study a transient situation during a long but finite time. In this case for bounded interactions also Ohm's law could approximatively hold until the time in which the escape effects become relevant. Concerning these effects we recall for instance the phenomenon of the so called *runaway* electrons discussed in [16]. In this paper we actually present an Hamiltonian model which exhibits such a behavior.

In the next section we describe the model and we state the main result of the paper. The proofs will be given in Sects. 3 and 4.

2. The model and statement of the results

We consider the following Hamiltonian system of infinitely many particles in one dimension. A charged particle of mass M and charge q is subject to a constant electric field $E > 0$, and it is coupled with an infinite system of neutral particles of unit mass by means of a non-negative, symmetric, twice differentiable, short-range, two-body potential Ψ. Without loss of generality we assume that $q/M = 1$ (it is just a redefinition of the electric field).

The neutral particles interact among themselves by means of a non-negative, symmetric, twice differentiable, short-range, two-body potential Φ. We also require $\Phi(0) > 0$, which implies that the interaction Φ is superstable [24]. Without loss of generality we further assume that both Ψ and Φ have range not greater than one, i.e.

$$\Phi(s) = 0, \quad \Psi(s) = 0 \qquad \text{if} \quad |s| > 1. \tag{2.1}$$

We denote by $(x, v) \in \mathbb{R}^2$ [resp. $(x_i, v_i) \in \mathbb{R}^2$] the position and velocity of the charged particle [resp. neutral particles]. The state $X = \{(x_i, v_i)\}_{i\in\mathbb{N}}$ is assumed to have a locally finite density and energy. In particular, for any $\mu \in \mathbb{R}$ and $R > 0$, it is well defined the quantity:

$$Q(X; \mu, R) \doteq \sum_i \chi_i(\mu, R) \left\{ \frac{v_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \Phi(x_i - x_j) + 1 \right\}, \tag{2.2}$$

where $\chi_i(\mu, R) = \chi(|x_i - \mu| \le R)$ and $\chi(A)$ denotes the characteristic function of the set A. In order to consider configurations which are typical for the thermodynamic states, we allow initial data with logarithmic divergences in the velocities and local densities. More precisely, by defining

$$Q(X) \doteq \sup_{\mu} \sup_{R:R>\log(e+|\mu|)} \frac{Q(X;\mu,R)}{2R}, \tag{2.3}$$

the set $\mathcal{X} \doteq \{X : Q(X) < +\infty\}$ has a full measure w.r.t. any Gibbs state [14].

The time evolution is defined by the following system of infinite equations:

$$\begin{cases} \ddot{x}(t) = -M^{-1} \sum_j \nabla\Psi(x(t) - x_j(t)) + E, \\ \ddot{x}_i(t) = -\nabla\Psi(x_i(t) - x(t)) - \sum_{j:j\neq i} \nabla\Phi(x_i(t) - x_j(t)), \quad i \in \mathbb{N}, \\ (x(0), v(0)) = (0,0), \quad X(0) = X. \end{cases} \tag{2.4}$$

Without loss of generality we assumed that the charged particle is initially at rest in the position $x = 0$; we also used that $q/M = 1$.

The Cauchy problem (2.4) is well posed if the initial condition X is chosen in the set $\mathcal{X}$, see e.g. [10, 9] and the pioneering papers by Lanford [17, 18]. More precisely, the solution is obtained by means of the following limiting procedure.

Given $X \in \mathcal{X}$ and $n \in \mathbb{N}$, let $I_n \doteq \{i \in \mathbb{N} : |x_i| \le n\}$. The n-partial dynamics $t \mapsto \{(x^{(n)}(t), v^{(n)}(t)); X^{(n)}(t)\}$, $X^{(n)}(t) = \{(x_i^{(n)}(t), v_i^{(n)}(t))\}_{i\in I_n}$, is defined by the solution of the differential system:

$$\begin{cases} \ddot{x}^{(n)}(t) = -M^{-1} \sum_{j\in I_n} \nabla\Psi(x^{(n)}(t) - x_j^{(n)}(t)) + E, \\ \ddot{x}_i^{(n)}(t) = -\nabla\Psi(x_i^{(n)}(t) - x^{(n)}(t)) - \sum_{j\in I_n:j\neq i} \nabla\Phi(x_i^{(n)}(t) - x_j^{(n)}(t)), \quad i \in I_n, \\ (x^{(n)}(0), v^{(n)}(0)) = (0,0), \quad X^{(n)}(0) = \{(x_i, v_i)\}_{i\in I_n}. \end{cases} \tag{2.5}$$

Then, for any $t \ge 0$, the following limits exist:

$$\begin{aligned} &\lim_{n\to+\infty} x^{(n)}(t) = x(t), \qquad \lim_{n\to+\infty} v^{(n)}(t) = v(t), \\ &\lim_{n\to+\infty} x_i^{(n)}(t) = x_i(t), \qquad \lim_{n\to+\infty} v_i^{(n)}(t) = v_i(t), \qquad i \in \mathbb{N}. \end{aligned} \tag{2.6}$$

Moreover, the flow $t \mapsto \{(x(t), v(t)); X(t)\}$, $X(t) = \{(x_i(t), v_i(t))\}_{i\in\mathbb{N}}$, is the unique (global) solution to Eqs. (2.4) such that $X(t) \in \mathcal{X}$.

We are interested in the asymptotic behavior of the charged particle when the background particle system is initially in a thermodynamic state. The full system [charged particle] + [background particles] is not in equilibrium, and its time evolution is described by the statistical solutions of the Newton equations (2.4) relative to the corresponding random initial data X. Our main result is the content of the following theorems which will be proved in Sect. 4.

Theorem 2.1. *Given $Q > 0$ and $L > 0$ let:*

$$\mathcal{B}_{Q,L} \doteq \{X \in \mathcal{X} : Q(X) \le Q, \quad |x_i| \ge L \quad \forall\, i \in \mathbb{N}\}. \tag{2.7}$$

Then, for each $E > 0$ and $Q > 0$ there exists $L_0 > 0$ such that, for any $L \ge L_0$ and $X \in \mathcal{B}_{Q,L}$,

$$\lim_{t\to+\infty} \frac{v(t)}{t} = E. \tag{2.8}$$

Theorem 2.2. *For any $E \geq 0$ and $X \in \mathcal{X}$,*

$$\liminf_{t\to+\infty} \frac{v(t)}{t} \geq 0. \tag{2.9}$$

An easy consequence of these theorems is:

Theorem 2.3. *For any Gibbs state $\langle\cdot\rangle$ of the background particle system and for any positive electric field E,*

$$\liminf_{t\to+\infty} \frac{\langle v(t)\rangle}{t} > 0. \tag{2.10}$$

Proof. By well known properties of the DLR states, see e.g. [14], the subset $\mathcal{B}_{Q,L}$ has positive measure w.r.t the state $\langle\cdot\rangle$ for Q large enough (depending on $\langle\cdot\rangle$) and for any $L \geq 0$. Then, by fixing Q large enough and then L as in Theorem 2.1,

$$\begin{aligned} \liminf_{t\to+\infty} \frac{\langle v(t)\rangle}{t} &\geq \left\langle \liminf_{t\to+\infty} \frac{v(t)}{t} \right\rangle \\ &= \left\langle \chi(\mathcal{B}_{Q,L}) \lim_{t\to+\infty} \frac{v(t)}{t} \right\rangle + \left\langle \chi(\mathcal{B}^c_{Q,L}) \liminf_{t\to+\infty} \frac{v(t)}{t} \right\rangle \\ &\geq \langle \chi(\mathcal{B}_{Q,L})\rangle\, E > 0, \end{aligned}$$

where in the first inequality we used Fatou's lemma. Eq. (2.10) is thus proved. □

We remark that this result is in contrast with the classical diffusion theory in which the motion of the charged particle, in the diffusive space-time scaling, is described as the sum of a drift term $d(E)$ plus a diffusion of diffusivity $\sigma(E)$. Moreover both $d(E)$ and $\sigma(E)$ are continuous functions of E and the Einstein relation holds, i.e.

$$\lim_{E\to 0} \frac{d(E)}{qE} = \frac{\sigma^2(0)}{2KT}, \tag{2.11}$$

where T is the absolute temperature and K the Boltzmann constant. On the contrary, from (2.10) we obtain here that:

$$d(E) = \lim_{t\to+\infty} \langle v(t)\rangle = +\infty \qquad \forall\, E > 0.$$

Remark 2.1. By exploiting the proof of Theorem 2.1 it is easy to check that we actually prove also the following statement. Assume the charged particle is initially in the position $x = 0$ with a positive velocity v_0. Then for each $X \in \mathcal{X}$ and $E > 0$ there exists a threshold $\bar{v}_0$ such that Eq. (2.8) holds for any $v_0 \geq \bar{v}_0$. This Hamiltonian model is thus an example of *runaway* charged particle.

Remark 2.2. By symmetry, the limit on the left-hand side of (2.9) is non-positive in the case $E \leq 0$. In particular the growth of the velocity is sub-linear in time for $E = 0$ (i.e. $v(t)/t \to 0$ as $t \to +\infty$). As we shall see this sub-linear growth will be the key point in the proof.

Remark 2.3. It is clear that the limit (2.10) holds not only for Gibbs states but for any reasonable equilibrium or non-equilibrium thermodynamic state.

Remark 2.4. Until now we have discussed a one-dimensional model. It would be interesting to investigate also models in higher dimensions, in which the background is composed by infinitely many interacting particles in $\mathbb{R}^2$ or in $\mathbb{R}^3$, but this seems to be a pretty hard task. In fact in this case we cannot mime the proof of the present paper, essentially because we cannot control the growth of the background particles velocity in a sub-linear way. Actually there are many papers on the time evolution of infinite systems [1, 11, 12, 14, 15, 17, 18, 19, 20, 23] but the estimates become quite bad for very long times (as we need in the present paper) when the space dimension is larger than one.

It is not hopeless to extend some results of this paper to two different models. Firstly, we could investigate the very simple case in which the background is composed by free particles in $\mathbb{R}^3$ distributed according to a Gibbs measure. The charged particle is initially at rest in the origin and interacts with a constant electric field and with the background particles via a potential Ψ. As in this paper we assume Ψ to be smooth, non-negative, and with finite range. It is reasonable to obtain an average drift velocity which increases linearly in time. Of course the time evolution of the background is trivial but the set of initial data considered in the present paper is too wide to exclude big densities close to the charged particle. We hope to overcome this difficulty by using the explicit properties of the free particles Gibbs measure.

Secondly, we could try to relax the strict one-dimensionality of the present model. In fact, here we heavily rely on the fact that the motion takes place on a straight line and so the effect of the collision between two particles is very small, because of a compensation before and after the shock. Just this property allows us to extend the result of [9] to the case of small electric fields. By using a slightly different strategy, we could investigate a model in which the infinitely many particles of the background are confined in $\mathbb{R}^3$ close to the x-axis by a non-negative twice differentiable external potential $V(r)$, which is defined in the following way: assuming cylindrical coordinates (x, r, θ), there exist $r_0 > 0$ and $0 < \alpha \leq 1$ such that $V(r) = \text{Const.}\, r^\alpha$ for any $r \geq r_0$. We have to show that the velocities of the background particles may grow at most sub-linearly in time.

We conclude the section with a notation warning: in the sequel, if not further specified, we shall denote by C a generic positive constant whose numerical value may change from line to line and it may possibly depend only on the interactions Φ and Ψ.

3. New bounds on the growth of the particles velocity

If E is large enough (w.r.t. the Gibbs state) the limit (2.10) has been proved in [9]. The proof is based on a result, firstly proved in [10], on the growth in time of the particle velocity. More precisely, it can be proved that in one dimension the particle velocity of an infinite system may grow at most linearly in time. To prove Theorem 2.3 we need to improve this result, by showing that the growth of the background particle velocity is in fact sub-linear in time.

The argument leading to this new estimate is not really affected by the presence of the charged particle. Then, to simplify notation and to make more clear the global strategy, we give a detailed proof of this bound in the case when the charged particle is absent. The equations of motion then reads:

$$\begin{cases} \ddot{x}_i(t) = -\sum_{j:j\neq i} \nabla\Phi(x_i(t) - x_j(t)), \quad i \in \mathbb{N}, \\ X(0) = X. \end{cases} \tag{3.1}$$

We recall two basic estimates on the infinite dynamics. Let $t \mapsto X(t)$, $X(t) = \{(x_i(t), v_i(t))\}_{i\in\mathbb{N}}$, be the solution to Eqs. (3.1). For any initial data $X \in \mathcal{X}$ we have:

$$|v_i(t)| \le C\left[\sqrt{Q(X)\log(e+|x_i|+Q(X))}+Q(X)t\right] \qquad \forall\, i\in\mathbb{N} \quad \forall\, t\ge 0, \tag{3.2}$$

and, for any $\mu\in\mathbb{R}$ and $R>\log(e+|\mu|)$,

$$Q(X(t);\mu,R) \le CQ(X)\left[R+\log(e+Q(X))+(1+Q(X))t^2\right] \qquad \forall\, t\ge 0. \tag{3.3}$$

We briefly discuss these estimates in the Appendix. The inequality (3.2) has been firstly proved in [10]. The bound (3.3) is actually an easy consequence of the same proof of the existence of the dynamics, but it has not explicitly proved in the quoted papers. We now state the new result.

Theorem 3.1. *There exist $K\ge 1$ and $a_0\in(0,1)$ such that for any $a\in(0,a_0]$ and $X\in\mathcal{X}$ the following holds. Let $t\mapsto X(t)=\{(x_i(t),v_i(t)\}_{i\in\mathbb{N}}$ be the solution to Eqs.* (3.1)*. Then, for any $i\in\mathbb{N}$ and $t\ge 0$,*

$$|v_i(t)| \le a^{-K[1+Q(X)^3]/a^2}\sqrt{\log(e+|x_i|)}+at. \tag{3.4}$$

The proof is rather technical so let us firstly give an outline of it. We demonstrate the bound (3.4) by contradiction: we assume that at time T large enough the velocity absolute value of a particle is greater than aT and we then show this particle does not change very much its velocity during the backward motion, so that the latter is initially larger than $aT/2$. For T large enough this fact contradicts the assumptions on the initial data. To understand the general idea let us start with a particular situation: at a large time T there is only one fast particle with velocity aT, while the other particles have velocities smaller than $aT/4$ during the whole time interval $[0,T]$. Then, the average force acting on the fast particle is very small for two reasons: firstly, the fast particle interacts with a slow one for a very short time (inversely proportional to the velocity gap); secondly, there is a compensation effect in the action of the forces during a collision of two particle (later on this fact is proved by a "perturbative argument"). The fast particle thus returns to the initial time essentially with the same velocity aT and this fact gives an absurd for large T. Of course the assumption that the fast particle is alone and the background does not increase its velocity is too drastic. We now make an essential observation: Eq. (3.3) implies that, for each time $t\in[0,T]$, the total number of fast particles which are initially in a region of order T^2 (i.e. which can interact with the tagged one) does not depend on T (it is proportional to a^{-2}). For T large enough this fact imposes that there exists a velocity gap between the fast and slow particles. Then we can find an ε small enough such that in the interval $[(1-\varepsilon)T,T]$ the background does not increase very much its velocity and the fast particles remain such (as we will see in Lemma 3.1 below, the control on the background is not trivial). So the effect of the background on the fast particles is small. We must now control the mutual interactions among the fast particles: it is possible to show that each fast particle after some collisions either remains alone (and so it does not change its velocity) or it remains in a small cluster (in momentum space), whose center of mass is unchanged. In conclusion, the velocity of each fast particle in the interval $[(1-\varepsilon)T,T]$ is almost unchanged. Repeating ε^{-1} times this estimate, we arrive to an absurd.

We now give the rigorous proof of the above heuristic argument.

Proof of Theorem 3.1. By (3.2) the proof of (3.4) reduces to prove the following statement: there are $K\ge 1$ and $a_0\in(0,1)$ such that, for any $a\in(0,a_0]$, $X\in\mathcal{X}$

and $i \in \mathbb{N}$,

$$|v_i(t)| \le at \qquad \forall\, t \ge T_a(X, x_i), \tag{3.5}$$

where, for $w \in \mathbb{R}$,

$$T_a(X, w) \doteq a^{-K[1+Q(X)^3]/a^2} \sqrt{\log(e + |w|)}. \tag{3.6}$$

Let us prove (3.5). In the sequel we always assume $a \in (0, a_0]$ and the statement a_0 *small enough* means that it can be fixed small enough independently of $X \in \mathcal{X}$ and $i \in \mathbb{N}$. Analogously, the parameter $K \ge 1$ will be chosen large enough independently of $X \in \mathcal{X}$ and $i \in \mathbb{N}$.

We first assume a_0 small enough that, for all $K \ge 1$,

$$\sqrt{Q(X) \log(e + |w|)} < \frac{aT_a(X, w)}{8} \qquad \forall\, w \in \mathbb{R} \quad \forall\, X \in \mathcal{X}.$$

This implies, by the definition (2.3),

$$|v_i| < \frac{aT_a(X, x_i)}{4} \qquad \forall\, i \in \mathbb{N} \quad \forall\, X \in \mathcal{X}. \tag{3.7}$$

We next proceed by contradiction. We assume there are $i \in \mathbb{N}$ and $T > T_a(X, x_i)$ such that $|v_i(T)| > aT$. We shall prove that, for a_0 small enough and $K \ge 1$ large enough, this implies $|v_i(0)| > aT/4$ for all $a \in (0, a_0]$. By (3.7) we thus get an absurd since $v_i(0) = v_i$ and $T > T_a(X, x_i)$.

By (3.2), the j-th particle may enter into the interval $[x_i - L, x_i + L]$ during the time interval $[0, t]$ only if:

$$|x_j - x_i| \le L + Ct \left[\sqrt{Q(X) \log(e + |x_j| + Q(X))} + Q(X)t \right], \tag{3.8}$$

which implies:

$$|x_j| - \log(e + |x_j|) \le C \left[L + |x_i| + \log(e + Q(X)) + Q(X)t^2 \right],$$

whence:

$$|x_j| \le C \left[L + |x_i| + \log(e + Q(X)) + Q(X)t^2 \right].$$

Inserting the last bound in (3.8) we conclude that, for any $L \ge 1$, the j-th particle may enter into the interval $[x_i - L, x_i + L]$ during the time interval $[0, t]$ only if

$$|x_j - x_i| \le L + \log(e + L) + \log(e + |x_i|) + C \left[\log(e + Q(X)) + Q(X)t^2 \right],$$

so that, recalling the definition (3.6),

$$\inf_{s \in [0,T]} |x_j(s) - x_i| \le L \quad \Longrightarrow \quad |x_j - x_i| \le L + \log(e + L) + C[1 + Q(X)]T^2. \tag{3.9}$$

Moreover, by (3.2),

$$\sup_{s \in [0,T]} |x_i(s) - x_i| \le C[1 + Q(X)]T^2.$$

By (3.9) and the last inequality, setting $L_T = C_*[1 + Q(X)]T^2$ and defining:

$$\mathcal{P} \doteq \{ j \in \mathbb{N} : |x_j - x_i| \le 4L_T \}, \qquad \mathcal{P}_1 \doteq \{ j \in \mathbb{N} : |x_j - x_i| \le 6L_T \},$$

we can choose $C_* \geq 1$ so large that the following holds:

$$\sup_{s\in[0,T]} |x_i(s) - x_i| \leq L_T, \tag{3.10}$$

$$\inf_{s\in[0,T]} |x_j(s) - x_i| \leq 3L_T \quad \Longrightarrow \quad j \in \mathcal{P}, \tag{3.11}$$

$$\inf_{s\in[0,T]} \min_{r\in\mathcal{P}} |x_j(s) - x_r(s)| \leq 1 \quad \Longrightarrow \quad j \in \mathcal{P}_1. \tag{3.12}$$

(to get (3.12) first observe that $|x_r(s) - x_i| \leq 4L_T + \log(e + L_T) + C[1 + Q(X)]T^2$ for all $s \in [0,T]$ and $r \in \mathcal{P}$, and then use (3.9)). In particular $\mathcal{P}$ contains all the particles which can interact with the i-th particle during the time interval $[0,T]$, and $\mathcal{P}_1$ contains all the particles which can interact with those in $\mathcal{P}$ during the same time interval. Moreover, by (3.2) and the definition (3.6), there is $\rho \geq 1$ such that:

$$\max_{j\in\mathcal{P}_1} \sup_{s\in[0,T]} |v_j(s)| \leq \rho[1 + Q(X)]T. \tag{3.13}$$

We next define the subset of the *fast* particles,

$$\mathcal{J}_t \doteq \left\{ j \in \mathcal{P} \, : \, |v_j(t)| > \frac{aT}{4} \right\} \tag{3.14}$$

(clearly $i \in \mathcal{J}_T$). By the energy conservation these particles are very rare for large T. In fact, by (2.2), for any $t \in [0,T]$,

$$|\mathcal{J}_t| \leq \frac{32}{a^2T^2} \sum_{j\in\mathcal{P}} \frac{v_j(t)^2}{2} \leq \frac{32}{a^2T^2} Q(X(t); x_i, 4L_T + \rho[1 + Q(X)]T^2),$$

where we also used (3.13). Then, by (3.3) and the definition (3.6), there exists $\rho_* \geq 1$ such that:

$$|\mathcal{J}_t| < \rho_* \frac{1 + Q(X)^2}{a^2}. \tag{3.15}$$

From (3.15) it follows the existence of gaps in the spatial and momentum distribution of fast particles. More precisely, there exist $y_t^- \in [x_i - 2L_T, x_i - L_T]$ and $y_t^+ \in [x_i + L_T, x_i + 2L_T]$, such that there are no particles in $\mathcal{J}_t$ whose positions (at time t) lie in the set:

$$\left[y_t^- - \frac{C_* a^2T^2}{\rho_*[1 + Q(X)]}, y_t^- \right] \cup \left[y_t^+, y_t^+ + \frac{C_* a^2T^2}{\rho_*[1 + Q(X)]} \right].$$

We point out that, by (3.10), $x_i(t) \in (y_t^-, y_t^+)$ for all $t \in [0,T]$. Analogously there is $\gamma_t \in (1/4, 1/2)$ such that there are no particles in $\mathcal{P}$ with velocities whose absolute value (at time t) lies in the interval:

$$\left[\gamma_t aT, \gamma_t aT + \frac{a^3T}{4\rho_*[1 + Q(X)^2]} \right]. \tag{3.16}$$

Now let ε^{-1} be a positive integer to be fixed later, and set $T_k \doteq k\varepsilon T$, $k = 0, \ldots, \varepsilon^{-1}$. We next define, for all $k = 0, \ldots, \varepsilon^{-1}$,

$$\mathcal{V}_k \doteq \left\{ j \in \mathcal{P} \, : \, |v_j(T_k)| > \gamma_k aT, \quad y_k^- \leq x_j(T_k) \leq y_k^+ \right\}, \tag{3.17}$$

$$\mathcal{M}_k \doteq \left\{ j \in \mathcal{P} \, : \, |v_j(T_k)| \leq \gamma_k aT \right\}, \tag{3.18}$$

where we shorthanded $y^{\pm}_{T_k} = y^{\pm}_k$ and $\gamma_{T_k} = \gamma_k$ (observe also that $i \in \mathcal{V}_{\varepsilon^{-1}}$). We note that, by (3.13), choosing e.g.

$$\varepsilon^{-1} \geq \frac{10\rho\rho_*[1+Q(X)]^2}{C_* a^2}, \tag{3.19}$$

the particles in $\mathcal{J}_{T_k}$ can fill up during the time interval $[T_{k-1}, T_k]$ no more than one third of the spatial gaps around $[y^-_k, y^+_k]$. Then, during this time interval, the particles in $\mathcal{V}_k$ do not interact with those in $\mathcal{J}_{T_k} \setminus \mathcal{V}_k$ and, by (3.11), they may interact only with particles in $\mathcal{P}$. Thus, at each time T_k, we have isolated a finite set $\mathcal{V}_k$ (whose cardinality is independent of T) of fast particles, which can only interact among themselves and with the background $\mathcal{M}_k$ of slow particles during the time interval $[T_{k-1}, T_k]$. We next study their evolution during this time interval, by getting sufficiently sharp bounds on the velocity variations to conclude that the (very fast) i-th particle stays in $\mathcal{V}_k$ also for any $k < \varepsilon^{-1}$. Then $|v_i(0)| > \gamma_0 aT > aT/4$ and the theorem is proved.

Recalling (3.16), if a_0 is small enough, then:

$$\nexists\, j \in \mathcal{P} \,:\, \gamma_k aT \leq |v_j(T_k)| \leq (\gamma_k + 3\beta) aT, \tag{3.20}$$

where

$$\beta \doteq \frac{a^3}{1+Q(X)^2}. \tag{3.21}$$

First of all we control the evolution of the background $\mathcal{M}_k$, by showing that in the time interval $[T_{k-1}, T_k]$ the gap in momentum space which separates $\mathcal{M}_k$ from $\mathcal{V}_k$ cannot be saturated by the slow particles. This is the content of the following lemma.

Lemma 3.1. *If a_0 is small enough and*

$$\varepsilon^{-1} = \mathrm{Int}\left[\beta^{-6}\right], \tag{3.22}$$

then, for any $k = 0, \dots, \varepsilon^{-1}$,

$$\max_{j \in \mathcal{M}_k} \sup_{s \in [T_{k-1}, T_k]} |v_j(s)| \leq (\gamma_k + 2\beta) aT. \tag{3.23}$$

Proof. Fix $k \in \{0; \dots; \varepsilon^{-1}\}$ and let:

$$t_0 \doteq \sup\left\{ t \in [T_{k-1}, T_k] \,:\, \max_{j \in \mathcal{M}_k} |v_j(t)| > (\gamma_k + 2\beta) aT \right\},$$

setting $t_0 = T_{k-1}$ if the above set is empty. We have to show that $t_0 = T_{k-1}$. We argue by contradiction and assume $t_0 > T_{k-1}$. Then there exists $q \in \mathcal{M}_k$ such that $|v_q(t_0)| = (\gamma_k + 2\beta) aT$. Setting

$$t_1 \doteq \inf\left\{ t \in [t_0, T_k] \,:\, |v_q(t)| < (\gamma_k + \beta) aT \right\},$$

we get an absurd if we show that $t_1 = T_k$. From the equations of motion and (3.12),

$$\big||v_q(t_1)| - (\gamma_k + 2\beta) aT\big| \leq \|\nabla\Phi\|_\infty \int_{t_0}^{t_1} ds \sum_{j \in \mathcal{P}_1 : j \neq q} \chi_{q,j}(s), \tag{3.24}$$

where $\chi_{q,j}(s) \doteq \chi(|x_q(s) - x_j(s)| \leq 1)$, i.e. the characteristic function of the set $\{s \in [t_0, t_1] : |x_q(s) - x_j(s)| \leq 1\}$. We look for an upper bound to the right-hand side of (3.24): if we show it can be done smaller than e.g. $\beta a T/2$, then $t_1 = T_k$. We decompose:

$$\{s \in [t_0, t_1] : |x_q(s) - x_j(s)| \le 1\} = \Delta^+_{q,j} \cup \Delta^-_{q,j},$$

with

$$\Delta^+_{q,j} \doteq \{s \in [t_0,t_1] \,:\, |v_j(s)| > \gamma_k a T, \quad |x_q(s) - x_j(s)| \le 1\},$$
$$\Delta^-_{q,j} \doteq \{s \in [t_0,t_1] \,:\, |v_j(s)| \le \gamma_k a T, \quad |x_q(s) - x_j(s)| \le 1\},$$

so that $\chi_{q,j}(s) = \chi(\Delta^+_{q,j})(s) + \chi(\Delta^-_{q,j})(s)$. We observe that:

$$|v_q(s)| - |v_j(s)| \ge \beta a T \qquad \forall\, s \in \Delta^-_{q,j} \quad \forall\, j \in \mathcal{P}_1$$

and, by (3.13),

$$\big||v_q(s)| - |v_j(s)|\big| \le 2\rho T[1 + Q(X)] \qquad \forall\, s \in \Delta^+_{q,j} \quad \forall\, j \in \mathcal{P}_1.$$

Then, letting $\theta_0 \doteq \inf\{\Delta^+_{q,j} \cup \Delta^-_{q,j}\}$, $\theta_1 \doteq \sup\{\Delta^+_{q,j} \cup \Delta^-_{q,j}\}$, and $Dv(s) \doteq v_q(s) - v_j(s)$, we get:

$$\begin{aligned}
2 \;&\ge\; \big|x_q(\theta_1) - x_j(\theta_1) - x_q(\theta_0) + x_j(\theta_0)\big| = \left|\int_{\theta_0}^{\theta_1} ds\, Dv(s)\right| \\
&=\; \left|\int_{\Delta^-_{q,j}} ds\, Dv(s) + \int_{\Delta^+_{q,j}} ds\, Dv(s)\right| \\
&\ge\; \int_{\Delta^-_{q,j}} ds\, |Dv(s)| - \int_{\Delta^+_{q,j}} ds\, |Dv(s)| \\
&\ge\; \beta a T |\Delta^-_{q,j}| - 2\rho T[1 + Q(X)]|\Delta^+_{q,j}|,
\end{aligned}$$

where we used that Dv has zero average on $[\theta_0, \theta_1] \setminus (\Delta^+_{q,j} \cup \Delta^-_{q,j})$ and that its sign is constant on $\Delta^-_{q,j}$. We thus obtain:

$$|\Delta^-_{q,j}| \le \frac{2}{\beta a T} + \frac{2\rho[1 + Q(X)]}{\beta a}|\Delta^+_{q,j}|.$$

It follows there is a decomposition $\Delta^-_{q,j} = \Delta^*_{q,j} \cup \Delta^\dagger_{q,j}$ with:

$$\Delta^*_{q,j} \cap \Delta^\dagger_{q,j} = \emptyset, \qquad |\Delta^*_{q,j}| \le \frac{2}{\beta a T}, \qquad |\Delta^\dagger_{q,j}| \le \frac{2\rho[1 + Q(X)]}{\beta a}|\Delta^+_{q,j}|.$$

Then $\chi(\Delta^-_{q,j})(s) = \chi(\Delta^*_{q,j})(s) + \chi(\Delta^\dagger_{q,j})(s)$ and therefore:

$$\begin{aligned}
\int_{t_0}^{t_1} ds\, \chi_{q,j}(s) \;&\le\; \int_{t_0}^{t_1} ds\, \chi(\Delta^*_{q,j})(s) + \left[1 + \frac{2\rho[1 + Q(X)]}{\beta a}\right]|\Delta^+_{q,j}| \\
&=\; \int_{t_0}^{t_1} ds\, \left[\chi(\Delta^*_{q,j})(s) + \frac{3\rho[1 + Q(X)]}{\beta a}\chi(\Delta^+_{q,j})(s)\right]. \qquad (3.25)
\end{aligned}$$

From (3.24) and (3.25) we obtain the following estimate:

$$\big||v_q(t_1)| - (\gamma_k + 2\beta) a T\big| \le \|\nabla\Phi\|_\infty (A_1 + A_2), \qquad (3.26)$$

with

$$A_1 = \int_{t_0}^{t_1} ds \sum_{j \in \mathcal{P}_1 : j \ne q} \chi(\Delta^*_{q,j})(s) \le \frac{2}{\beta a T}\bar{N}_k, \qquad (3.27)$$

where $\bar{N}_k$ denotes the number of particles which can interact with the q-th particle during the time $[T_{k-1}, T_k]$, and (recall $t_1 - t_0 \le \varepsilon T$)

$$A_2 = \frac{3\rho[1+Q(X)]}{\beta a} \int_{t_0}^{t_1} ds \sum_{j \in \mathcal{P}_1 : j \neq q} \chi(\Delta^+_{q,j})(s) \tag{3.28}$$

$$\le \frac{3\rho[1+Q(X)]}{\beta a} \varepsilon T \sup_{s \in [T_{k-1}, T_k]} N_k^>(s), \tag{3.29}$$

where $N_k^>(s)$ denotes the number of particles of $X(s)$ which are contained in the interval $\big[x_q(s)-1, x_q(s)+1\big]$ and whose velocity is bigger than $\gamma_k a T$. We need now to bound the quantities $\bar{N}_k$ and $N_k^>(s)$. By (3.13) we have:

$$\bar{N}_k \le N\big(X(T_{k-1}); x_q(T_{k-1}), 1 + 2\varepsilon\rho T^2[1+Q(X)]\big), \tag{3.30}$$

where $N(X;\mu,R)$ denotes the number of particles of the configuration X which are contained in the interval $[\mu - R, \mu + R]$. We next observe that, since the interaction is superstable with range bounded by 1,

$$N(X;\mu,R)^2 \le CRQ(X;\mu,R) \qquad \forall\, X \in \mathcal{X} \quad \forall\, \mu \in \mathbb{R} \quad \forall\, R \ge 1. \tag{3.31}$$

By (3.21) and (3.22) it follows $\varepsilon T^2 \ge 1$ for all $T \ge T_a(X, x_i)$, so that we can use (3.31) to bound the right-hand side of (3.30). Since $|x_q(T_{k-1})| \le |x_i| + C[1 + Q(X)]T^2$, by (3.3) and the definitions (3.6), (3.21), and (3.22) we conclude that:

$$\bar{N}_k \le C\sqrt{\varepsilon}\, T^2[1+Q(X)]. \tag{3.32}$$

Recalling that $\gamma_k > 1/4$, the quantity $N_k^>(s)$ can be bounded as $|\mathcal{J}_t|$ in (3.15):

$$\sup_{s \in [0,T]} N_k^>(s) \le C \frac{1+Q(X)^2}{a^2}. \tag{3.33}$$

From (3.27), (3.32), and recalling the definition (3.21) of β, we thus get:

$$A_1 \le C\sqrt{\varepsilon}\, \frac{1+Q(X)^5}{a^8}\, \beta a T. \tag{3.34}$$

Analogously, from (3.28) and (3.33),

$$A_2 \le C\varepsilon\, \frac{1+Q(X)^7}{a^{10}}\, \beta a T. \tag{3.35}$$

From (3.26), (3.34), (3.35), and (3.22), by choosing a_0 small enough we obtain $\big||v_q(t_1)| - (\gamma_k + 2\beta)aT\big| \le \beta a T/2$, hence $t_1 = T_k$ and the lemma is proved. $\square$

Let us go back to the proof of Theorem 3.1. To analyze the time evolution of the particles in $\mathcal{V}_k$ during the time interval $[T_{k-1}, T_k]$, we shall use a perturbative argument, based on the existence of gaps in the velocity distribution of the fast particles. To implement this, for each $t \in [T_{k-1}, T_k]$ we divide $\mathcal{V}_k$ into *clusters* of particles, according to the following rule. Given a parameter $\delta > 0$, two particles j and j' belong to the same (δ, t)-*cluster* if there is a positive integer p and a sequence of particles $j_0, \ldots, j_p \in \mathcal{V}_k$, $j_0 = j$, $j_p = j'$, such that:

$$|v_{j_{s-1}}(t) - v_{j_s}(t)| < \delta\varepsilon T \qquad \forall\, s = 1, \ldots, p.$$

We denote by $\Pi(\delta, t)$ the partition of $\mathcal{V}_k$ into (δ, t)-clusters and by $(x_{\mathcal{C}}, v_{\mathcal{C}})$ the phase point of the center of mass of the cluster $\mathcal{C}$:

$$x_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{j \in \mathcal{C}} x_j, \qquad v_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{j \in \mathcal{C}} v_j. \tag{3.36}$$

We next introduce a sequence of decompositions of $\mathcal{V}_k$ into (β^ℓ, τ_ℓ)-clusters, $\ell = 1, \dots, \ell^*$, β as in (3.21), where the times $\tau_1 \ge \tau_2 \ge \dots \ge \tau_{\ell^*}$ and the integer ℓ^* are defined by:

$$\begin{cases} \tau_1 \doteq T_k, \\ \tau_{\ell+1} \doteq \inf\left\{t \in [T_{k-1}, \tau_\ell] \,:\, \xi_\ell(t) < \dfrac{\beta^\ell \varepsilon T}{8},\ \eta_\ell(t) < \dfrac{\beta^\ell \varepsilon T}{8}\right\}, \qquad \ell \ge 1, \\ \ell^* = \min\{\ell \in \mathbb{N} \,:\, \tau_{\ell+1} = T_{k-1}\}, \end{cases} \tag{3.37}$$

where:

$$\begin{aligned} \xi_\ell(t) &\doteq \sup_{s \in [t, \tau_\ell]} \max\left\{|v_{\mathcal{C}}(s) - v_{\mathcal{C}}(\tau_\ell)| \,:\, \mathcal{C} \in \Pi(\beta^\ell, \tau_\ell)\right\}, \\ \eta_\ell(t) &\doteq \sup_{s \in [t, \tau_\ell]} \max\left\{|v_j(s) - v_{\mathcal{C}}(s)| \,:\, j \in \mathcal{C},\ \mathcal{C} \in \Pi(\beta^\ell, \tau_\ell)\right\}, \end{aligned}$$

and setting $\tau_{\ell+1} = \tau_\ell$ if $\eta_\ell(\tau_\ell) \ge \beta^\ell \varepsilon T/8$. Our goal is now to show that, by choosing $K \ge 1$ large enough in the definition (3.6) and a_0 small enough, we have:

$$\ell^* \le |\mathcal{V}_k| < \rho_* \frac{1 + Q(X)^2}{a^2}, \tag{3.38}$$

(the second inequality follows from (3.15) since $\mathcal{V}_k \subset \mathcal{J}_{T_k}$). We first note that if a_0 is small enough then, for any $t \in [\tau_{\ell+1}, \tau_\ell]$,

$$\min\left\{|v_j(t) - v_q(t)| \,:\, \mathcal{C} \in \Pi(\beta^\ell, \tau_\ell),\ j \in \mathcal{C},\ q \in \mathcal{V}_k \setminus \mathcal{C}\right\} \ge \frac{\beta^\ell \varepsilon T}{2} \tag{3.39}$$

and

$$\min\left\{|v_j(t) - v_q(t)| \,:\, \mathcal{C} \in \Pi(\beta^\ell, \tau_\ell),\ j \in \mathcal{C},\ q \in \mathcal{M}_k\right\} \ge \frac{\beta a T}{2}. \tag{3.40}$$

To obtain (3.40) we have also used Lemma 3.1. In fact, by (3.20) and (3.23), if a_0 is small enough, for any $\mathcal{C} \in \Pi(\beta^\ell, \tau_\ell)$ and $t \in [\tau_{\ell+1}, \tau_\ell]$,

$$\min_{j \in \mathcal{C}} |v_j(t)| - \max_{q \in \mathcal{M}_k} |v_q(t)| \ge \beta a T - \sum_{\ell=1}^{\ell^*-1} \frac{\beta^\ell \varepsilon T}{4} \ge \beta a T - \frac{\beta \varepsilon T}{4(1-\beta)} > \frac{\beta a T}{2}$$

(recall that $\varepsilon \ll \beta$ for small a's). Using (3.39) and (3.40) we next prove that, for K large enough and a_0 small enough,

$$\xi_\ell(\tau_{\ell+1}) < \frac{\beta^\ell \varepsilon T}{16} \qquad \forall\, \ell \le \min\{\ell^*, 2|\mathcal{V}_k|\}, \tag{3.41}$$

from which, by the definition (3.37),

$$\tau_{\ell+1} > T_{k-1} \quad \Longrightarrow \quad \eta_\ell(\tau_{\ell+1}) = \frac{\beta^\ell \varepsilon T}{8} \qquad \forall\, \ell \le \min\{\ell^*, 2|\mathcal{V}_k|\}. \tag{3.42}$$

We choose $a_0 < 1/(8\rho_*)$ with ρ_* as in (3.15) so that, by the second inequality in (3.38),

$$\frac{\beta^\ell \varepsilon T}{8|\mathcal{V}_k|} \ge \frac{\beta^\ell \varepsilon T}{8} \frac{\rho_* a^2}{1 + Q(X)^2} > \beta^{\ell+1} \varepsilon T.$$

Then, from (3.42) it follows that:

$$|\Pi(\beta^{\ell+1}, \tau_{\ell+1})| \ge |\Pi(\beta^\ell, \tau_\ell)| + 1 \qquad \forall\, \ell \le \min\{\ell^*, 2|\mathcal{V}_k|\},$$

so that there is only one particle in each cluster of $\Pi(\beta^\ell, \tau_\ell)$ if $\ell \ge |\mathcal{V}_k|$. Thus $\eta_\ell \equiv 0$ for $\ell \ge |\mathcal{V}_k|$, which implies, by (3.41) and the definition (3.37), that $\tau_{\ell+1} = T_{k-1}$ for $\ell \ge |\mathcal{V}_k|$, whence the first inequality in (3.38) follows.

To prove (3.41) we apply the perturbative argument firstly used in [10]. Let $\mathcal{C} \in \Pi(\beta^\ell, \tau_\ell)$ and define:

$$\begin{aligned} p_{\mathcal{C}}(t) \quad &\doteq \quad v_{\mathcal{C}}(t) + \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \sum_{q\in\mathcal{V}_k\setminus\mathcal{C}} \int_0^t ds\, \nabla\Phi(x_j(s) - x_q(s)) \\ &\quad + \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \sum_{q\in\mathcal{M}_k} \frac{\Phi(x_j(t) - x_q(t))}{v_j(t) - v_q(t)}. \end{aligned} \tag{3.43}$$

From the equations of motion and recalling that Φ is symmetric we have:

$$\begin{aligned} \dot{p}_{\mathcal{C}}(t) \quad &= \quad \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \sum_{q\in\mathcal{M}_k} \sum_{s:s\neq j} \frac{\Phi(x_j(t) - x_q(t))\nabla\Phi(x_j(t) - x_s(t))}{(v_j(t) - v_q(t))^2} \\ &\quad - \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \sum_{q\in\mathcal{M}_k} \sum_{s:s\neq q} \frac{\Phi(x_j(t) - x_q(t))\nabla\Phi(x_q(t) - x_s(t))}{(v_j(t) - v_q(t))^2}. \end{aligned}$$

Then, recalling (3.40),

$$|\dot{p}_{\mathcal{C}}(t)| \leq \|\Phi\|_\infty \|\nabla\Phi\|_\infty \frac{4}{(\beta a T)^2} \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} N_j(t)^2 \qquad \forall\, t \in [\tau_{\ell+1}, \tau_\ell]. \tag{3.44}$$

where $N_j(t)$ is the number of particles which are contained at time t in the interval $[x_j(t) - 2, x_j(t) + 2]$. Then, for any $t \in [\tau_{\ell+1}, \tau_\ell]$,

$$|p_{\mathcal{C}}(t) - p_{\mathcal{C}}(\tau_\ell)| \leq \frac{C}{(\beta a T)^2} \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \sup_{s\in[T_{k-1}, T_k]} N_j(s) \int_{\tau_{\ell+1}}^{\tau_\ell} dt'\, N_j(t').$$

By (3.31):

$$N_j(t)^2 \leq C Q(X(t); x_j(t), 2) \leq C[1 + Q(X)^2] T^2,$$

where in the last inequality we used (3.3) with $|x_j(t)| \leq |x_i| + C[1 + Q(X)]T^2$ and the definition (3.6). We next observe that, during the time $[\tau_{\ell+1}, \tau_\ell]$, the distance between the particle $j \in \mathcal{C}$ and a particle $q \in \mathcal{M}_k$ can be smaller than 2 only for a time not longer than $4/(\beta a T)$. Moreover, the total number $\tilde{N}_k$ of particles $q \in \mathcal{M}_k$ such that $|x_j(t) - x_q(t)| \leq 2$ for some $t \in [T_{k-1}, T_k]$ can be bounded similarly to $\bar{N}_k$ in (3.32). Then:

$$\begin{aligned} \int_{\tau_{\ell+1}}^{\tau_\ell} ds\, N_j(s) \quad &= \quad \sum_{q\in\mathcal{V}_k} \int_{\tau_{\ell+1}}^{\tau_\ell} ds\, \tilde{\chi}_{j,q}(s) + \sum_{q\in\mathcal{M}_k} \int_{\tau_{\ell+1}}^{\tau_\ell} ds\, \tilde{\chi}_{j,q}(s) \leq |\mathcal{V}_k| \varepsilon T + \frac{4\tilde{N}_k}{\beta a T} \\ &\leq \quad C \frac{1 + Q(X)^2}{a^2} \varepsilon T + C\sqrt{\varepsilon} T \frac{1 + Q(X)}{\beta a} \leq C\sqrt{\varepsilon} T \frac{1 + Q(X)}{\beta a}, \end{aligned}$$

where $\tilde{\chi}_{j,q}(s) = \chi(|x_j(s) - x_q(s)| \leq 2)$ and we used the second inequality in (3.38). In conclusion:

$$|p_{\mathcal{C}}(t) - p_{\mathcal{C}}(\tau_\ell)| \leq \frac{C}{\beta^2},$$

where we used the definitions (3.21) and (3.22). On the other hand, by the definition (3.43) and using (3.39) and (3.40),

$$\begin{aligned} |v_{\mathcal{C}}(t) - p_{\mathcal{C}}(t)| \quad &\leq \quad \frac{1}{|\mathcal{C}|} \sum_{j\in\mathcal{C}} \left[\frac{2\|\nabla\Phi\|_\infty}{\beta^\ell \varepsilon T} |\mathcal{V}_k| + \frac{2\|\Phi\|_\infty}{\beta a T} N_j(t) \right] \\ &\leq \quad C \left[\frac{1}{\beta^{\ell+1} \varepsilon T} + \frac{1}{\beta^2} \right] \qquad \forall\, t \in [T_{k-1}, T_k], \end{aligned} \tag{3.45}$$

where we again used (3.21) and (3.22). Then, for any $t \in [\tau_{\ell+1}, \tau_\ell]$,

$$\begin{aligned} |v_{\mathcal{C}}(t) - v_{\mathcal{C}}(\tau_\ell)| &\leq |v_{\mathcal{C}}(t) - p_{\mathcal{C}}(t)| + |p_{\mathcal{C}}(t) - p_{\mathcal{C}}(\tau_\ell)| + |p_{\mathcal{C}}(\tau_\ell) - v_{\mathcal{C}}(\tau_\ell)| \\ &\leq C\left[\frac{1}{\beta^{\ell+1}\varepsilon T} + \frac{1}{\beta^2}\right]. \end{aligned} \tag{3.46}$$

Recalling the definitions (3.6), (3.21), (3.22), and the second inequality in (3.38), if $K \geq 1$ is large enough and a_0 is small enough the inequality (3.46) implies the bound (3.41).

We can now conclude the proof of the theorem. From the previous analysis the velocity variation of a particle $j \in \mathcal{V}_k$ in the time interval $[T_{k-1}, T_k]$ is bounded by:

$$\begin{aligned} |v_j(T_{k-1}) - v_j(T_k)| &\leq \sum_\ell{}^{*} |v_j(\tau_{\ell+1}) - v_j(\tau_\ell)| \\ &\leq \sum_\ell{}^{*} \left(|v_j(\tau_{\ell+1}) - v_{\mathcal{C}_\ell}(\tau_{\ell+1})| + |v_{\mathcal{C}_\ell}(\tau_{\ell+1}) - v_{\mathcal{C}_\ell}(\tau_\ell)| + |v_{\mathcal{C}_\ell}(\tau_\ell) - v_j(\tau_\ell)|\right) \\ &\leq \sum_{\ell=1}^{\ell^*-1} \frac{3\beta^\ell \varepsilon T}{8} \leq \frac{3\beta\varepsilon T}{8(1-\beta)} \leq \varepsilon\frac{aT}{2}. \end{aligned} \tag{3.47}$$

where $\mathcal{C}_\ell$ is the cluster of $\Pi(\beta^\ell, \tau_\ell)$ which contains the j-th particle and $\sum_\ell{}^*$ means the sum over those $\ell \in \{1; \dots; \ell^* - 1\}$ for which $\tau_{\ell+1} < \tau_\ell$ (for the last inequality recall that $a_0 < 1/8$ whence $\beta < a/8$). In particular, the previous bound guarantees that the i-th particle, whose velocity at time T is assumed bigger than aT, remains *fast*, i.e. it belongs to $\mathcal{V}_k$ for all $k = 0, \dots, \varepsilon^{-1}$ (in fact we obtained $|v_i(0)| > aT/2$). The theorem is thus proved. □

Remark 3.1. The dependence of $T_a(x_i, X)$ on the small parameter a is very bad: we have to wait a super-exponentially large (w.r.t. a^{-1}) time to catch the asymptotic estimate on the i-th particle velocity. On the other hand, in our strategy, this choice is a useful mathematical device to control the effect of the mutual interaction among the fast particles. In fact it guarantees an *apriori* bound on the maximal number of refinements into small clusters are needed to follow the evolution of the fast particles during the time interval $[T_{k-1}, T_k]$.

Remark 3.2. It is worthwhile to emphasize here the following fact. For any integer n, by defining

$$T_a(X, w) = a^{-K[1+Q(X)^{2n+1}]/a^{2n}} \sqrt{\log(e + |w|)},$$

instead of (3.6) (which corresponds to the case $n = 1$), and decomposing $\mathcal{V}_k$ into (δ^ℓ, τ_ℓ)-clusters with $\delta = \zeta\beta^n$ ($\zeta > 0$ a fixed parameter), we analogously prove that, provided a_0 small enough and K large enough, for any $j \in \mathcal{V}_k$ and $t \in [T_k, T_{k-1}]$,

$$|v_j(t) - v_j(T_k)| \leq \frac{3\delta\varepsilon T}{8(1-\delta)} \leq \frac{\delta\varepsilon T}{2}.$$

4. Proof of Theorems 2.1 and 2.2

We start by recalling the estimates analogous to Eqs. (3.2)–(3.3) for the system [charged particle] + [background particles], whose proof is discussed in the Appendix. Let $t \mapsto \{(x(t), v(t)); X(t)\}$, $X(t) = \{(x_i(t), v_i(t))\}_{i\in\mathbb{N}}$, be the solution to Eqs. (2.4) and define

$$Q_E(X) \doteq Q(X) + E. \tag{4.1}$$

For any $X \in \mathcal{X}$, $i \in \mathbb{N}$, and $t \geq 0$,

$$|v(t)| \leq C\left[\sqrt{Q_E(X)\log(e+Q_E(X))} + Q_E(X)t\right], \tag{4.2}$$

$$|v_i(t)| \leq C\left[\sqrt{Q_E(X)\log(e+|x_i|+Q_E(X))} + Q_E(X)t\right], \tag{4.3}$$

and, for any $\mu \in \mathbb{R}$, $R > \log(e+|\mu|)$, and $t \geq 0$,

$$Q(X(t);\mu,R) \leq CQ_E(X)\left[R + \log(e+Q_E(X)) + (1+Q_E(X))t^2\right]. \tag{4.4}$$

Proof of Theorem 2.1. We fix $E, Q > 0$ and define $Q_E \doteq Q+E$ so that $Q_E(X) \leq Q_E$ for any $X \in \mathcal{B}_{Q,L}$. By (4.2) and (4.3), a necessary condition for the i-th particle to satisfy

$$\inf_{s\in[0,t]} |x(s) - x_i(s)| \leq 2$$

is that:

$$|x_i| - Ct\left[\sqrt{Q_E\log(e+|x_i|+Q_E)} + Q_Et\right] \leq 2 + Ct\left[\sqrt{Q_E\log(e+Q_E)} + Q_Et\right],$$

which implies:

$$|x_i| - \log(e+|x_i|) \leq C\left[\log(e+Q_E) + Q_Et^2\right].$$

Then there exists a constant $\bar{C} \geq 1$ such that:

$$\inf_{s\in[0,t]} |x(s) - x_i(s)| \leq 2 \quad \Longrightarrow \quad |x_i| \leq Y_t \doteq \bar{C}\left[\log(e+Q_E) + Q_Et^2\right], \tag{4.5}$$

from which it follows that if $|x_i| \geq 2\bar{C}\log(e+Q_E)$ then:

$$\inf_{s\in[0,t]} |x(s) - x_i(s)| \leq 2 \quad \Longrightarrow \quad t \geq \sqrt{\frac{|x_i|}{2\bar{C}Q_E}}. \tag{4.6}$$

The parameter $L_0 = L_0(E,Q)$ for which we shall prove the theorem is chosen in the following way. Let

$$0 < a_0 \leq \frac{\min\{1;E\}}{8}, \qquad K \geq 1,$$

be two parameters to be fixed later. Then $L_0 \geq 2\bar{C}\log(e+Q_E)$ is chosen large enough that, for any $a \in (0,a_0]$, $K \geq 1$, and $L \geq L_0$,

$$T \geq a^{-K(1+Q_E^3)/a^2}\sqrt{\log\left[e+Y_T+1\right]} \qquad \forall\, T \geq T_L, \tag{4.7}$$

where

$$T_L \doteq \sqrt{\frac{L}{2\bar{C}Q_E}}. \tag{4.8}$$

Now let:

$$U(t) \doteq \max_i G\left(|x_i| - Y_t\right) \sup_{s\in[0,t]} |v_i(s)|, \tag{4.9}$$

with Y_t as in (4.5) and $G \in C(\mathbb{R})$ a not increasing function satisfying: $G(x) = 1$ for $x \leq 0$, $G(x) = 0$ for $x \geq 1$. By (4.5), the continuous and non decreasing function $U(\cdot)$ is an upper bound for the maximal velocity of any particle which may interact with the charged one during the time $[0,t]$. We next define:

$$T^* \doteq \sup\left\{t > 0 \,:\, \max\{U(s); |v(s) - Es|\} \leq \frac{Es}{4} \quad \forall\, s \in [0,t]\right\}, \tag{4.10}$$

setting $T^* = 0$ if the above set is empty.

By (4.6), (4.8), and recalling that $|x_i| \ge L \ge 2\bar{C}\log(e + Q_E)$, we have $U(t) = 0$ and $v(t) = Et$ for $t \le T_L$, whence $T^* > T_L$. We next prove that if a_0 is small enough and K is large enough then:

$$\max\{U(t); |v(t) - Et|\} \le \frac{Et}{8} \qquad \forall\, t \in [T_L, T^*), \tag{4.11}$$

which implies, by continuity, $T^* = +\infty$. Moreover, since we actually prove that $|v(t) - Et| = \mathcal{O}(\log t)$ for $t < T^*$, the limit (2.8) follows.

To bound $U(t)$ we can apply in the present context the analysis developed in Sect. 3. We first observe that by (4.7) and the definition (3.6),

$$T \ge T_L \qquad \Longrightarrow \qquad T \ge T_a(X, x_i) \quad \forall\, i\, :\, |x_i| \le Y_T + 1.$$

Moreover, since for $t < T^*$ the charged particle is much faster than the particles it meets, the interaction with this particle does not affect too much the velocity of each background particle up to this time. More precisely, for any $0 \le \tau_1 \le \tau_2 < T^*$ and $i \in \mathbb{N}$,

$$\left| \int_{\tau_1}^{\tau_2} ds\, \nabla\Psi(x_i(s) - x(s)) \right| \le \frac{2\|\nabla\Psi\|_\infty}{ET_L}. \tag{4.12}$$

Note in fact that the i-th particle may interact with the charged one only after the time T_L, and hence for a time not bigger than $2/(E\tau_1) \le 2/(ET_L)$. From the previous estimate the strategy used for proving (3.5) applies in this case almost unchanged, getting:

$$|v_i(T)| \le aT \qquad \forall\, T \in [T_L, T^*) \quad \forall\, i\, :\, |x_i| \le Y_T + 1, \tag{4.13}$$

which in particular implies, by the definition of $U(\cdot)$,

$$U(t) \le \max\{|v_i(t)|\, :\, |x_i| \le Y_t + 1\} \le \frac{Et}{8} \tag{4.14}$$

(recall we assume $a_0 \le E/8$). To prove (4.13) only two minor modifications are needed with respect to the proof of (3.5). The first one occurs in the proof of the analogous of Lemma 3.1. More precisely, by using (4.12), Eq. (3.24) now reads:

$$\big| |v_q(t_1)| - (\gamma_k + 2\beta)aT \big| \le \|\nabla\Phi\|_\infty \int_{t_0}^{t_1} ds \sum_{j \in \mathcal{P}_1 : j \ne q} \chi_{q,j}(s) + \frac{2\|\nabla\Psi\|_\infty}{ET_L}$$

(where the parameter β is now defined as in (3.21) but with $Q(X)$ replaced by $Q_E(X)$) so that Eq. (3.26) becomes:

$$\big| |v_q(t_1)| - (\gamma_k + 2\beta)aT \big| \le \|\nabla\Phi\|_\infty (A_1 + A_2) + \frac{2\|\nabla\Psi\|_\infty}{ET_L}.$$

Since for a_0 small enough the extra-term on the right-hand side can be done much smaller than βaT, its presence does not affect the conclusion of the proof of the lemma. The second modification occurs in the proof of (3.41). We modify the definition (3.43) by setting:

$$\begin{aligned} p_{\mathcal{C}}(t) &\doteq v_{\mathcal{C}}(t) + \frac{1}{|\mathcal{C}|}\sum_{j\in\mathcal{C}}\sum_{q\in\mathcal{M}_k} \frac{\Phi(x_j(t)-x_q(t))}{v_j(t)-v_q(t)} \\ &\quad + \frac{1}{|\mathcal{C}|}\sum_{j\in\mathcal{C}}\sum_{q\in\mathcal{V}_k\setminus\mathcal{C}} \int_0^t ds\, \nabla\Phi(x_j(s)-x_q(s)) \\ &\quad + \frac{1}{|\mathcal{C}|}\sum_{j\in\mathcal{C}} \int_0^t ds\, \nabla\Psi(x_j(s)-x(s)), \end{aligned} \tag{4.15}$$

from which we compute:

$$\begin{aligned} \dot{p}_{\mathcal{C}}(t) &= \frac{1}{|\mathcal{C}|}\sum_{j\in\mathcal{C}}\sum_{q\in\mathcal{M}_k} \frac{\Phi(x_j(t)-x_q(t))}{(v_j(t)-v_q(t))^2} \Big[\nabla\Psi(x_j(t)-x(t)) - \nabla\Psi(x_q(t)-x(t)) \\ &\quad + \sum_{s:s\neq j}\nabla\Phi(x_j(t)-x_s(t)) - \sum_{s:s\neq q}\nabla\Phi(x_q(t)-x_s(t))\Big], \end{aligned}$$

which can be bounded analogously to (3.44). On the other hand, by (4.12), the extra-term in the definition of $p_{\mathcal{C}}(t)$ does not change the last estimate in Eq. (3.45), which now reads:

$$\begin{aligned} |v_{\mathcal{C}}(t) - p_{\mathcal{C}}(t)| &\leq \frac{1}{|\mathcal{C}|}\sum_{j\in\mathcal{C}} \left[\frac{2\|\nabla\Phi\|_\infty}{\beta^\ell\varepsilon T}|\mathcal{V}_k| + \frac{2\|\Phi\|_\infty}{\beta a T}N_j(t) + \frac{2\|\nabla\Psi\|_\infty}{ET_L}\right] \\ &\leq C\left[\frac{1}{\beta^{\ell+1}\varepsilon T} + \frac{1}{\beta^2}\right] \qquad \forall\, t\in[T_{k-1},T_k]. \end{aligned}$$

We are left with an upper bound for $|v(t)-Et|$ when $T_L < t < T^*$ (recall in fact that $|v(t)-Et| = 0$ for $t \leq T_L$). We use again the perturbative argument, by defining:

$$p(t) \doteq v(t) - Et + \sum_i \frac{\Psi(x(t)-x_i(t))}{M(v(t)-v_i(t))}.$$

By the equations of motion,

$$\begin{aligned} \dot{p}(t) &= \sum_{i,j} \frac{\Psi(x(t)-x_i(t))\nabla\Psi(x(t)-x_j(t))}{M^2[v(t)-v_i(t)]^2} \\ &\quad - \sum_{i\neq j} \frac{\Psi(x(t)-x_i(t))\nabla\Phi(x_i(t)-x_j(t))}{M[v(t)-v_i(t)]^2} \\ &\quad - \sum_i \frac{\Psi(x(t)-x_i(t))[E+\nabla\Psi(x_i(t)-x(t))]}{M[v(t)-v_i(t)]^2}, \end{aligned}$$

and therefore, for $T_L \leq t < T^*$,

$$|p(t)-p(T_L)| \leq C\int_{T_L}^t ds\, \frac{N(s)}{Es}\left[\frac{1}{s} + \frac{1+N(s)}{Es}\right], \tag{4.16}$$

where $N(s) = N(X(s);x(s),2)$. By (3.31) and (4.4) we get:

$$N(s)^2 \leq CQ(X(s);x(s),2) \leq CQ_E\left[\log(e+Q_E) + (1+Q_E)s^2\right], \tag{4.17}$$

where we used that $|x(s)| \le Cs[\sqrt{Q_E \log(e+Q_E)} + Q_E s]$, which follows by (4.2). The term $[1+N(s)]/(Es)$ in (4.16) can be bounded using (4.17); by the definitions of L and T_L we thus obtain, for any $t \ge T_L$,

$$\begin{aligned} |p(t)-p(T_L)| &\le C\frac{1+Q_E}{E}\int_{T_L}^t ds\, \frac{N(s)}{Es} \\ &= C\frac{1+Q_E}{E}\sum_j \int_{T_L}^t ds\, \frac{1}{Es}\chi(|x(s)-x_j(s)| \le 2). \end{aligned}$$

An upper bound for the right-hand side of the above inequality can be obtained in the following way. We first observe that by (4.5) and the definition (2.7) only the particles which are initially in $[-Y_t,-L]\cup[L,Y_t]$ may contribute to the above integral. We next define:

$$N_q = \{j \in \mathbb{N} \,:\, 2^q < |x_j| \le 2^{q+1}\}.$$

Since $t < T^*$ we have:

$$\sum_j \int_{T_L}^t ds\, \frac{1}{Es}\chi(|x_j(s)-x(s)| \le 2) \le \sum_{q=q_L}^{q_t+1} \frac{4|N_q|}{E^2 t_q^2}, \tag{4.18}$$

where q_L [resp. q_t] is the integer such that $2^{q_L} < L \le 2^{q_L+1}$ [resp. $2^{q_t} < Y_t \le 2^{q_t+1}$] and

$$t_q \doteq \min_{j\in N_q} \inf\{s \in [T_L,T^*) \,:\, |x(s)-x_j(s)| \le 2\},$$

setting $t_q = +\infty$ if the above set is empty for all $j \in N_q$. We may assume L so large (i.e. a_0 small enough) that $2^{q-1} > \log(e+2^{q+1})$ for any $q \ge q_L$. Then, since $2^{q+1}-2^q = 2^q$, by (2.2) and (2.3) we have $|N_q| \le 2Q(X)2^q \le Q_E 2^{q+1}$. On the other hand, from (4.6) it follows that $t_q > \sqrt{2^{q-1}/(\bar{C}Q_E)}$. Inserting the previous bounds in (4.18) we obtain, for any $t \in [T_L,T^*)$,

$$\sum_j \int_{T_L}^t ds\, \frac{1}{Es}\chi(|x_j(s)-x(s)| \le 2) \le C\frac{Q_E^2}{E^2}\log\frac{Y_t}{L} \le C\frac{Q_E^2}{E^2}\log t,$$

so that:

$$|p(t)-p(T_L)| \le C\frac{1+Q_E^3}{E^3}\log t \qquad \forall\, t \in [T_L,T^*). \tag{4.19}$$

Since $v(T_L)-ET_L = 0$ we have:

$$|v(t)-Et| \le |v(t)-Et-p(t)| + |p(t)-p(T_L)| + |v(T_L)-ET_L-p(T_L)|.$$

By the definition of $p(t)$, the first [resp. third] term on the right-hand side is smaller than a constant multiple of $N(t)/(Et)$ [resp. $N(T_L)/(ET_L)$], which we have already shown to be bounded by $C(1+Q_E)/E$ for any $t \ge T_L$. Finally, the second term is bounded in (4.19). In conclusion:

$$|v(t)-Et| \le C\frac{1+Q_E}{E}\left(1+\frac{Q_E^2}{E^2}\log t\right) \qquad \forall\, t \in [T_L,T^*), \tag{4.20}$$

which in particular implies, if a_0 is small enough,

$$|v(t)-Et| \le \frac{Et}{8} \qquad \forall\, t \in [T_L,T^*).$$

By (4.14) and the last bound Eq. (4.11) is thus proved, whence $T^* = +\infty$. The limit (2.8) thus follows from (4.20) and the theorem is proved. □

Proof of Theorem 2.2. We prove there exist $a_0 \in (0,1)$ and $T_a = T_a(X,E) > 0$ such that, for any $a \in (0, a_0]$,

$$v(t) \geq -at \qquad \forall\, t \geq T_a. \tag{4.21}$$

This cannot be achieved by the same strategy used in the proof of Theorem 2.3. Let us firstly review the proof of the case without electric field. We divide the time interval $[0,T]$ in many subintervals $[T_{k-1}, T_k]$ and the fast particles into many disjoint clusters. By using the equations of motion and some tricks we prove that the velocity of the center of mass of each cluster remains almost constant. Of course a cluster may increase its size due to the internal forces, thus approaching an adjacent one. But the first time τ_1 (in the backward evolution) when this happens, we decompose the set of fast particles into smaller clusters which remain disjoint until a time τ_2 and so on. The important point is that the number of clusters increases at each step, so that the number of steps is not bigger than the cardinality of the set of the fast particles. This procedure holds in each time interval and we go back to time zero with some fast particles, thus getting a contradiction because of the initial data we have chosen.

The above strategy fails in the present context. In fact, due to the presence of the electric field, an estimate like (3.46) (which implies (3.41) and hence (3.42)) does not hold for the cluster containing the charged particle: the velocity of its center of mass decreases during the backward evolution. Therefore this cluster could approach the adjacent one without modifying its size. For this reason, a refinement into smaller clusters does not anymore guarantee that the number of clusters increases. We then need a non trivial modification of that part of the proof.

Assume that the charged particle velocity is negative and very large in absolute value at time T_k. Consider the first time $\sigma_0 \in [T_{k-1}, T_k)$ in the backward evolution when the particle has lost some amount of its (negative) velocity. We now focus our attention on the cluster which contains the charged particle (for the other ones the procedure remains unchanged). During the forward evolution from time σ_0 to the time, say $\sigma_1 \in (\sigma_0, T_k]$, when the charged particle has gained some amount of its (negative) velocity, the electric field acts in the good direction and the center of mass cannot decrease too much its velocity. So a refinement at time σ_1 allows to decrease the cardinality of this cluster. We then start from time σ_1 and consider the backward evolution up to the time $\sigma_2 \in [\sigma_0, \sigma_1)$ in which the charged particle has lost a smaller amount of its (negative) velocity. Since in this case the center of mass cannot increase its velocity, a further refinement decreases the cardinality of the cluster. So on, alternating backward and forward evolutions, we are able to prove that if the charged particle has a large negative velocity at time T then the latter cannot increase too much during the time intervals $[T_{k-1}, T_k]$. It then arrives at time zero too fast, thus getting a contradiction.

Let us now give the rigorous demonstration. Eq. (4.21) will be proved with

$$T_a(X,E) \doteq a^{-K[1+Q_E(X)^7]/a^6}, \tag{4.22}$$

where the parameter $K \geq 1$ will be fixed later. In the sequel we modify the previous notation and denote by (x_0, v_0) the position and velocity of the charged particle. We proceed by contradiction. We assume there exists $T > T_a(X,E)$ for which $v_0(T) < -aT$ and we show that, for a_0 small enough and $K \geq 1$ large enough, this implies $v_0(0) < -aT/2$ for any $a \in (0, a_0]$, thus getting an absurd since $v_0(0) = 0$.

Using (4.2), (4.3), and (4.4), we can repeat the same reasoning from Eq. (3.8) to Eq. (3.16), with the charged particle playing the role of the i-th particle. We then define in the present context the sets $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{J}_t$, and the functions $y_t^\pm$, γ_t, by replacing everywhere $x_i = x_i(0)$ [resp. $Q(X)$] with $x_0(0) = 0$ [resp. $Q_E(X)$]. We analogously introduce the times $T_k \doteq k\varepsilon T$, $k = 0, \ldots, \varepsilon^{-1}$, with ε as in (3.19) (where $Q(X)$ is again replaced by $Q_E(X)$).

We shall prove that, if a_0 is small enough and K large enough then, for any $k = 1, \ldots, \varepsilon^{-1}$,

$$v_0(T_k) < -\frac{a(T+T_k)}{2} \qquad \Longrightarrow \qquad v_0(T_{k-1}) < -\frac{a(T+T_{k-1})}{2}. \tag{4.23}$$

By (4.23) and the assumption $v_0(T) < -aT$ it follows that $v_0(0) < -aT/2$, and the theorem is proved.

Let $k \in \{1; \ldots; \varepsilon^{-1}\}$ and assume $v_0(T_k) < -a(T+T_k)/2$. We define:

$$\sigma_0 \doteq \inf\left\{ t \in [T_{k-1}, T_k) \,:\, v_0(s) < v_0(T_k) + \frac{a\varepsilon T}{2} \quad \forall\, s \in [t, T_k] \right\}. \tag{4.24}$$

Equation (4.23) follows if $\sigma_0 = T_{k-1}$. To prove this we argue by contradiction, by assuming $\sigma_0 > T_{k-1}$. Analogously to (3.17) and (3.18), we introduce the sets:

$$\begin{aligned} \mathcal{V}_{\sigma_0} &\doteq \left\{ j \in \mathcal{P} \,:\, |v_j(\sigma_0)| > \gamma_{\sigma_0} aT, \quad y_{\sigma_0}^- \le x_j(\sigma_0) \le y_{\sigma_0}^+ \right\}, && (4.25)\\ \mathcal{M}_{\sigma_0} &\doteq \left\{ j \in \mathcal{P} \,:\, |v_j(\sigma_0)| \le \gamma_{\sigma_0} aT \right\}. && (4.26) \end{aligned}$$

By (4.24) the charged particle belongs to the set of fast particles, i.e. $0 \in \mathcal{V}_{\sigma_0}$. The same arguments leading from Eq. (3.19) to Eq. (3.21) apply here to the set $\mathcal{V}_{\sigma_0}$ with β as in (3.21) (where $Q(X)$ is replaced by $Q_E(X)$). The particles of the set $\mathcal{V}_{\sigma_0}$ can thus interact only among themselves and with the background $\mathcal{M}_{\sigma_0}$ of slow particles during the time interval $[\sigma_0, T_k]$. Moreover, by [the analogous of] (3.15),

$$|\mathcal{V}_{\sigma_0}| < \rho_* \frac{1 + Q_E(X)^2}{a^2}. \tag{4.27}$$

Finally, analogously to Lemma 3.1, for a_0 small enough and ε as in (3.22),

$$\max_{j \in \mathcal{M}_{\sigma_0}} \sup_{s \in [\sigma_0, T_k]} |v_j(s)| \le (\gamma_{\sigma_0} + 2\beta) aT. \tag{4.28}$$

Observe in fact that in the proof of the lemma we have only to replace $\|\nabla\Phi\|_\infty$ by $\max\{\|\nabla\Phi\|_\infty; \|\nabla\Psi\|_\infty\}$ in the right-hand side of Eq. (3.24), thus taking into account of the interaction with the charged particle.

Let Π_0 be the partition of $\mathcal{V}_{\sigma_0}$ into $(\beta^{\ell_0}, \sigma_0)$-clusters with ℓ_0 the smallest integer such that:

$$\max_{i \in \mathcal{C}_0} |v_i(\sigma_0) - v_{\mathcal{C}_0}(\sigma_0)| < \frac{\beta^{\ell_0} \varepsilon T}{32}, \tag{4.29}$$

where $\mathcal{C}_0$ denotes the cluster of Π_0 which contains the charged particle. We define:

$$\begin{aligned} w_0 &\doteq \min_{i \in \mathcal{C}_0} v_i(\sigma_0), \quad W_0 \doteq \max_{i \in \mathcal{C}_0} v_i(\sigma_0), \\ D_0(t) &\doteq \inf_{s \in [\sigma_0, t]} \min\{ |v_i(s) - v_j(s)| \,:\, i \in \mathcal{C}_0,\ j \in \mathcal{V}_{\sigma_0} \setminus \mathcal{C}_0 \}, \end{aligned}$$

and

$$\sigma_1 \doteq \sup\left\{ t \in (\sigma_0, T_k] \,:\, \inf_{s \in [\sigma_0, t]} v_0(s) > w_0 - \frac{\beta^{\ell_0}\varepsilon T}{8},\ D_0(t) > \frac{\beta^{\ell_0}\varepsilon T}{2} \right\}. \tag{4.30}$$

We now consider the backward evolution of the cluster $\mathcal{C}_0$ during the time interval $[\sigma_0, \sigma_1]$. Let Π_1 be the partition of $\mathcal{C}_0$ into $(\beta^{\ell_1}, \sigma_1)$-clusters with ℓ_1 the smallest integer greater or equal to $\ell_0 + 1$ such that:

$$\max_{i \in \mathcal{C}_1} |v_i(\sigma_1) - v_{\mathcal{C}_1}(\sigma_1)| < \frac{\beta^{\ell_1} \varepsilon T}{32},$$

where $\mathcal{C}_1$ is the cluster of Π_1 which contains the charged particle. We next define:

$$\begin{aligned} w_1 &\doteq \min_{i \in \mathcal{C}_1} v_i(\sigma_1), \quad W_1 \doteq \max_{i \in \mathcal{C}_1} v_i(\sigma_1), \\ D_1(t) &\doteq \inf_{s \in [t, \sigma_1]} \min\{|v_i(s) - v_j(s)| \,:\, i \in \mathcal{C}_1,\ j \in \mathcal{C}_0 \setminus \mathcal{C}_1\}, \end{aligned}$$

and set:

$$\sigma_2 \doteq \inf \left\{ t \in [\sigma_0, \sigma_1) \,:\, \sup_{s \in [t, \sigma_1]} v_0(s) < W_1 + \frac{\beta^{\ell_1} \varepsilon T}{8},\ D_1(t) > \frac{\beta^{\ell_1} \varepsilon T}{2} \right\}. \tag{4.31}$$

We iterate the above procedure and define recursively a sequence of time intervals,

$$[\sigma_0, T_k] \supseteq [\sigma_0, \sigma_1] \supseteq [\sigma_2, \sigma_1] \supseteq [\sigma_2, \sigma_3] \supseteq [\sigma_4, \sigma_3] \supseteq [\sigma_4, \sigma_5] \supseteq [\sigma_6, \sigma_5] \ldots,$$

according to the following rule. For any integer $n \geq 1$ we set:

$$\begin{aligned} D_{2n}(t) &\doteq \inf_{s \in [\sigma_{2n}, t]} \min\{|v_i(s) - v_j(s)| \,:\, i \in \mathcal{C}_{2n},\ j \in \mathcal{C}_{2n-1} \setminus \mathcal{C}_{2n}\}, \\ D_{2n+1}(t) &\doteq \inf_{s \in [t, \sigma_{2n+1}]} \min\{|v_i(s) - v_j(s)| \,:\, i \in \mathcal{C}_{2n+1},\ j \in \mathcal{C}_{2n} \setminus \mathcal{C}_{2n+1}\}, \end{aligned}$$

and

$$\begin{aligned} \sigma_{2n+1} &\doteq \sup \Bigg\{ t \in (\sigma_{2n}, \sigma_{2n-1}] \,:\, \inf_{s \in [\sigma_{2n}, t]} v_0(s) > w_{2n} - \frac{\beta^{\ell_{2n}} \varepsilon T}{8} \\ &\qquad \text{and } D_{2n}(t) > \frac{\beta^{\ell_{2n}} \varepsilon T}{2} \Bigg\}, \qquad (4.32) \\ \sigma_{2n+2} &\doteq \inf \Bigg\{ t \in [\sigma_{2n}, \sigma_{2n+1}) \,:\, \sup_{s \in [t, \sigma_{2n+1}]} v_0(s) < W_{2n+1} + \frac{\beta^{\ell_{2n+1}} \varepsilon T}{8}, \\ &\qquad \text{and } D_{2n+1}(t) > \frac{\beta^{\ell_{2n+1}} \varepsilon T}{2} \Bigg\}, \qquad (4.33) \end{aligned}$$

where, for any $n \geq 2$,

$$w_n \doteq \min_{i \in \mathcal{C}_n} v_i(\sigma_n), \qquad W_n \doteq \max_{i \in \mathcal{C}_n} v_i(\sigma_n),$$

and the set $\mathcal{C}_n$ is the cluster of Π_n which contains the charged particle, Π_n denoting the partition of $\mathcal{C}_{n-1}$ into $(\beta^{\ell_n}, \sigma_n)$-clusters with ℓ_n the smallest integer greater or equal to $\ell_{n-1} + 1$ such that:

$$\max_{i \in \mathcal{C}_n} |v_i(\sigma_n) - v_{\mathcal{C}_n}(\sigma_n)| < \frac{\beta^{\ell_n} \varepsilon T}{32}. \tag{4.34}$$

We remark that the function $D_{2n}(\cdot)$ [resp. $D_{2n+1}(\cdot)$] is not increasing [resp. not decreasing].

In the sequel we shall assume $a_0 < 1/(32\rho_*)$ so that, by (4.27),

$$\frac{\beta^{\ell_{n-1}} \varepsilon T}{32 |\mathcal{V}_{\sigma_0}|} > \beta^{\ell_n} \varepsilon T.$$

Since $|\mathcal{C}_n| \le |\mathcal{C}_0| \le |\mathcal{V}_{\sigma_0}|$ then $\ell_0 \le |\mathcal{V}_{\sigma_0}|$ and $\ell_n \le 1 + \ell_{n-1} + |\mathcal{C}_{n-1}| - |\mathcal{C}_n|$ for $n \ge 1$, whence:

$$\ell_n \le 2|\mathcal{V}_{\sigma_0}| + n. \tag{4.35}$$

Lemma 4.1. *For a_0 small enough and K large enough, if $\sigma_0 > T_{k-1}$ then, for any integer $n = 0, \dots, 2|\mathcal{V}_{\sigma_0}|$,*

$$D_n(\sigma_{n+1}) \ge \frac{5\beta^{\ell_n}\varepsilon T}{8} \tag{4.36}$$

and

$$v_{\mathcal{C}_n}(t) - v_{\mathcal{C}_n}(\sigma_n) \begin{cases} \ge -\dfrac{\beta^{\ell_n}\varepsilon T}{64} & \forall\, t \in [\sigma_n, \sigma_{n+1}] \quad \text{if } n \text{ is even,} \\[2ex] \le +\dfrac{\beta^{\ell_n}\varepsilon T}{64} & \forall\, t \in [\sigma_{n+1}, \sigma_n] \quad \text{if } n \text{ is odd.} \end{cases} \tag{4.37}$$

Before proving the lemma let us conclude the proof of the theorem. Let $\bar{n} \doteq 2|\mathcal{V}_{\sigma_0}|$ and set $\sigma_{-1} \doteq T_k$. By continuity and Eq. (4.36) we have:

$$\sigma_{2n+1} < \sigma_{2n-1}, \quad 1 \le 2n+1 < \bar{n} \quad \Longrightarrow \quad v_0(\sigma_{2n+1}) = w_{2n} - \frac{\beta^{\ell_{2n}}\varepsilon T}{8}, \tag{4.38}$$

$$\sigma_{2n} > \sigma_{2n-2}, \quad 2 \le 2n \le \bar{n} \quad \Longrightarrow \quad v_0(\sigma_{2n}) = W_{2n-1} + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8}. \tag{4.39}$$

It then follows that for a_0 small enough the times $T_k, \sigma_0, \sigma_1, \dots, \sigma_{\bar{n}}$ are distinct. In fact, if $\sigma_1 = T_k$ we get an absurd, since:

$$\begin{aligned} v_0(\sigma_1) &\ge w_0 - \frac{\beta^{\ell_0}\varepsilon T}{8} > v_0(\sigma_0) - \frac{2\beta^{\ell_0}\varepsilon T}{32} - \frac{\beta^{\ell_0}\varepsilon T}{8} = v_0(T_k) + \frac{a\varepsilon T}{2} - \frac{3\beta^{\ell_0}\varepsilon T}{16} \\ &> v_0(T_k). \end{aligned}$$

where we used (4.29), definition (4.30), and that $v_0(\sigma_0) = v_0(T_k) + a\varepsilon T/2$ (recall we are assuming $\sigma_0 > T_{k-1}$). We then proceed by induction. If $n \ge 1$ and $\sigma_{2n-1} < \sigma_{2n-3}$ then, by (4.34) and (4.38),

$$\begin{aligned} v_0(\sigma_{2n}) &\le W_{2n-1} + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8} \le v_0(\sigma_{2n-1}) + \frac{2\beta^{\ell_{2n-1}}\varepsilon T}{32} + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8} \\ &= w_{2n-2} - \frac{\beta^{\ell_{2n-2}}\varepsilon T}{8} + \frac{3\beta^{\ell_{2n-1}}\varepsilon T}{16} < v_0(\sigma_{2n-2}), \end{aligned}$$

which implies $\sigma_{2n} > \sigma_{2n-2}$ (in the last inequality we used that $\ell_{2n-1} > \ell_{2n-2}$ and $\beta < 2/3$ for a_0 small). Analogously, if $n \ge 1$ and $\sigma_{2n} > \sigma_{2n-2}$ then, by (4.34) and (4.39),

$$\begin{aligned} v_0(\sigma_{2n+1}) &\ge w_{2n} - \frac{\beta^{\ell_{2n}}\varepsilon T}{8} \ge v_0(\sigma_{2n}) - \frac{2\beta^{\ell_{2n}}\varepsilon T}{32} - \frac{\beta^{\ell_{2n}}\varepsilon T}{8} \\ &= W_{2n-1} + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8} - \frac{3\beta^{\ell_{2n}}\varepsilon T}{16} > v_0(\sigma_{2n-1}), \end{aligned}$$

which implies $\sigma_{2n+1} < \sigma_{2n-1}$. We now observe that, by (4.34), (4.38), and (4.37), for any $1 \le 2n+1 < \bar{n}$,

$$\begin{aligned} v_0(\sigma_{2n+1}) - v_{\mathcal{C}_{2n}}(\sigma_{2n+1}) &= [w_{2n} - v_{\mathcal{C}_{2n}}(\sigma_{2n})] - \frac{\beta^{\ell_{2n}}\varepsilon T}{8} \\ &\quad + [v_{\mathcal{C}_{2n}}(\sigma_{2n}) - v_{\mathcal{C}_{2n}}(\sigma_{2n+1})] \\ &\le \frac{2\beta^{\ell_{2n}}\varepsilon T}{32} - \frac{\beta^{\ell_{2n}}\varepsilon T}{8} + \frac{\beta^{\ell_{2n}}\varepsilon T}{64} \\ &= -\frac{3\beta^{\ell_{2n}}\varepsilon T}{64}, \end{aligned}$$

and, by (4.34), (4.39), and (4.37), for any $2 \le 2n \le \bar{n}$,

$$\begin{aligned} v_0(\sigma_{2n}) - v_{\mathcal{C}_{2n-1}}(\sigma_{2n}) &= [W_{2n-1} - v_{\mathcal{C}_{2n-1}}(\sigma_{2n-1})] + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8} \\ &\quad + \left[v_{\mathcal{C}_{2n-1}}(\sigma_{2n-1}) - v_{\mathcal{C}_{2n-1}}(\sigma_{2n})\right] \\ &\ge -\frac{2\beta^{\ell_{2n-1}}\varepsilon T}{32} + \frac{\beta^{\ell_{2n-1}}\varepsilon T}{8} - \frac{\beta^{\ell_{2n-1}}\varepsilon T}{64} \\ &= \frac{3\beta^{\ell_{2n-1}}\varepsilon T}{64}. \end{aligned}$$

In conclusion:

$$|v_0(\sigma_n) - v_{\mathcal{C}_{n-1}}(\sigma_n)| \ge \frac{3\beta^{\ell_{n-1}}\varepsilon T}{64} \qquad \forall\, n = 1, \dots, \bar{n},$$

which in particular implies $|\mathcal{C}_{n-1}| > 1$ for all $n = 1, \dots, \bar{n}$. On the other hand, recalling (4.27), if $a_0 < 3/(64\rho_*)$ then $|\Pi_n| > 1$, i.e. $|\mathcal{C}_n| < |\mathcal{C}_{n-1}|$ for any $n = 1, \dots, \bar{n}$. But $|\mathcal{C}_0| \le |\mathcal{V}_{\sigma_0}| \le \bar{n}/2$, whence $|\mathcal{C}_{n_*}| = 1$ for some $n_* \le \bar{n}/2$, then getting an absurd. The assumption $\sigma_0 > T_{k-1}$ is thus disproved and the theorem follows. □

Proof of Lemma 4.1. We prove the result by induction. We first consider the case $n = 0$. The forward motion of the particles in $\mathcal{V}_{\sigma_0} \setminus \mathcal{C}_0$ during the time interval $[\sigma_0, \sigma_1]$ can be studied as the backward motion of the fast particles $\mathcal{V}_k$ in the proof of Theorem 3.1. More precisely, we introduce a sequence of partitions $\bar{\Pi}_\ell$ of $\mathcal{V}_{\sigma_0} \setminus \mathcal{C}_0$ into $(\bar{\beta}^\ell, \bar{\tau}_\ell)$-clusters, $\bar{\beta} = \beta^{\ell_0}/4$, $\ell = 1, \dots, \ell^*$, where the times $\bar{\tau}_1 \le \bar{\tau}_2 \le \dots \le \bar{\tau}_{\ell^*}$ and the integer ℓ^* are defined, analogously to Eq. (3.37), by:

$$\begin{cases} \bar{\tau}_1 \doteq \sigma_0, \\ \bar{\tau}_{\ell+1} \doteq \inf\left\{ t \in [\bar{\tau}_\ell, \sigma_1] \,:\, \bar{\xi}_\ell(t) < \dfrac{\bar{\beta}^\ell \varepsilon T}{8},\ \bar{\eta}_\ell(t) < \dfrac{\bar{\beta}^\ell \varepsilon T}{8} \right\}, & \ell \ge 1, \\ \ell^* = \min\{\ell \in \mathbb{N} \,:\, \bar{\tau}_{\ell+1} = \sigma_1\}, \end{cases}$$

where:

$$\begin{aligned} \bar{\xi}_\ell(t) &\doteq \sup_{s \in [\bar{\tau}_\ell, t]} \max\left\{ |v_{\mathcal{C}}(s) - v_{\mathcal{C}}(\tau_\ell)| \,:\, \mathcal{C} \in \bar{\Pi}_\ell \right\}, \\ \bar{\eta}_\ell(t) &\doteq \sup_{s \in [\bar{\tau}_\ell, t]} \max\left\{ |v_j(s) - v_{\mathcal{C}}(s)| \,:\, j \in \mathcal{C},\ \mathcal{C} \in \bar{\Pi}_\ell \right\}, \end{aligned}$$

and setting $\bar{\tau}_{\ell+1} = \bar{\tau}_\ell$ if $\bar{\eta}_\ell(\bar{\tau}_\ell) \ge \bar{\beta}^\ell \varepsilon T/8$. By (4.28) and since $D_0(t) \ge \beta^{\ell_0}\varepsilon T/2 = 2\bar{\beta}\varepsilon T$, analogously to (3.39) and (3.40), for any $t \in [\bar{\tau}_\ell, \bar{\tau}_{\ell+1}]$,

$$\min\left\{ |v_j(t) - v_q(t)| \,:\, \mathcal{C} \in \bar{\Pi}_\ell,\ j \in \mathcal{C},\ q \in \mathcal{V}_{\sigma_0} \setminus \mathcal{C} \right\} \ge \frac{\bar{\beta}^\ell \varepsilon T}{2}$$

and

$$\min\left\{|v_j(t)-v_q(t)| \,:\, \mathcal{C}\in\bar\Pi_\ell,\ j\in\mathcal{C},\ q\in\mathcal{M}_{\sigma_0}\right\} \ge \frac{\beta aT}{2}.$$

Moreover, both ℓ_0 and ℓ^* are not bigger than $|\mathcal{V}_{\sigma_0}|$. Then, recalling Remark 3.2, by (4.27) and the choice (4.22), we get:

$$\sup_{t\in[\sigma_0,\sigma_1]} |v_j(t)-v_j(\sigma_0)| \le \frac{\beta^{\ell_0}\varepsilon T}{8} \qquad \forall\, j\in\mathcal{V}_{\sigma_0}\setminus\mathcal{C}_0. \tag{4.40}$$

We omit the details. We shall prove below that, for a_0 small enough,

$$w_0 - \frac{\beta^{\ell_0}\varepsilon T}{4} \le v_i(t) \le W_0 + \frac{\beta^{\ell_0}\varepsilon T}{4} \qquad \forall\, t\in[\sigma_0,\sigma_1] \qquad \forall\, i\in\mathcal{C}_0. \tag{4.41}$$

Since $D_0(\sigma_0)\ge\beta^{\ell_0}\varepsilon T$, by (4.40) and (4.41), we have:

$$D_0(\sigma_1) \ge D_0(\sigma_0) - \frac{\beta^{\ell_0}\varepsilon T}{4} - \frac{\beta^{\ell_0}\varepsilon T}{8} \ge \frac{5\beta^{\ell_0}\varepsilon T}{8},$$

which proves (4.36) for $n=0$.

To obtain (4.41) we have to show that $\sigma_*=\sigma_1$, where:

$$\sigma_* \doteq \sup\left\{t\in(\sigma_0,\sigma_1] \,:\, w_0-\frac{\beta^{\ell_0}\varepsilon T}{4} < v_i(s) < W_0+\frac{\beta^{\ell_0}\varepsilon T}{4}\ \forall\, i\in\mathcal{C}_0,\ s\in[\sigma_0,t]\right\}.$$

We first observe that, by the assumption $\sigma_0 > T_{k-1}$ and the definition (4.24), $v_0(t)<v_0(\sigma_0)\le W_0$ for all $t\in(\sigma_0,T_k]$, so that, by (4.30),

$$w_0 - \frac{\beta^{\ell_0}\varepsilon T}{8} \le v_0(t) \le W_0 \qquad \forall\, t\in[\sigma_0,\sigma_1]. \tag{4.42}$$

Given $\delta>0$, let $\Pi(\delta,t)$ be the partition of $\mathcal{C}_0$ into (δ,t)-clusters. We denote by $\mathcal{K}^{\rm ext}_-(\delta,t)$ the set of all the particles belonging to clusters $\mathcal{C}\in\Pi(\delta,t)$ such that:

$$\max_{j\in\mathcal{C}} v_j(t) < w_0 - \left(\frac{\beta^{\ell_0}}{8}+\delta\right)\varepsilon T,$$

and by $\mathcal{K}^{\rm ext}_+(\delta,t)$ the set of all the particles belonging to clusters $\mathcal{C}\in\Pi(\delta,t)$ such that:

$$\min_{j\in\mathcal{C}} v_j(t) > W_0 + \delta\varepsilon T.$$

Finally, $\mathcal{K}^{\rm ext}(\delta,t)\doteq\mathcal{K}^{\rm ext}_-(\delta,t)\cup\mathcal{K}^{\rm ext}_+(\delta,t)$ and $\mathcal{K}^{\rm int}(\delta,t)\doteq\mathcal{C}_0\setminus\mathcal{K}^{\rm ext}(\delta,t)$. We then introduce a sequence of partitions of $\mathcal{C}_0$ into (δ^κ,t_κ)-clusters, with $\delta=\beta^{\ell_0+1}/16$, $\kappa=1,\dots,\kappa^*$, where the times $t_1\ge t_2\ge\ldots\ge t_{\kappa^*}$ and the integer κ^* are defined by:

$$\begin{cases} t_1 \doteq \sigma_*, \\ t_{\kappa+1} \doteq \inf\left\{t\in[\sigma_0,t_\kappa] \,:\, \theta_\kappa(t) > \dfrac{\delta^\kappa\varepsilon T}{4},\right\}, & \kappa\ge 1, \\ \kappa^* = \min\{\kappa\in\mathbb{N} \,:\, t_{\kappa+1}=\sigma_0\}, \end{cases}$$

where:

$$\theta_\kappa(t) \doteq \inf_{s\in[t,t_\kappa]} \min\{|v_i(s)-v_j(s)| \,:\, i\in\mathcal{K}^{\rm int}(\delta^\kappa,t_\kappa),\ j\in\mathcal{K}^{\rm ext}(\delta^\kappa,t_\kappa)\},$$

setting $\theta_\kappa(t)=+\infty$ if $\mathcal{K}^{\rm ext}(\delta^\kappa,t_\kappa)$ is empty. Observe that, by (4.42), the charged particle belongs to $\mathcal{K}^{\rm int}(\delta^\kappa,t_\kappa)$ for any κ. The backward motion of the particles in $\mathcal{K}^{\rm ext}(\delta^\kappa,t_\kappa)$ during the time interval $[t_{\kappa+1},t_\kappa]$ can be analyzed analogously to the proof of (4.40). In fact, by (4.27) and the choice (4.22), we can apply the strategy of Theorem 3.1 to the particles in $\mathcal{K}^{\text{ext}}(\delta^\kappa, t_\kappa)$, with the parameter β replaced by $\delta^\kappa/2 = [\beta^{(\ell_0+1)}/16]^\kappa/2$, for any $\kappa \le \min\{\kappa^*; 2|\mathcal{C}_0|\}$, see Remark 3.2. We thus obtain, for a_0 small enough and $\kappa \le \min\{\kappa^*; 2|\mathcal{C}_0|\}$,

$$\sup_{t\in(t_{\kappa+1},t_\kappa]} |v_j(t) - v_j(t_\kappa)| \le \frac{\delta^\kappa \varepsilon T}{4} \qquad \forall\, j \in \mathcal{K}^{\text{ext}}(\delta^\kappa, t_\kappa). \tag{4.43}$$

For $t_{\kappa+1} > \sigma_0$ and $\kappa \le \min\{\kappa^*; 2|\mathcal{C}_0|\}$, since $\theta_\kappa(t_{\kappa+1}) = \delta^\kappa\varepsilon T/4$, the inequality (4.43) implies that there exists $i \in \mathcal{K}^{\text{int}}(\delta^\kappa, t_\kappa)$ such that:

$$v_i(t_{\kappa+1}) \le w_0 - \left(\frac{\beta^{\ell_0}}{8} + \delta^\kappa\right)\varepsilon T + \frac{\delta^\kappa \varepsilon T}{2}$$

or

$$v_i(t_{\kappa+1}) \ge W_0 + \delta^\kappa \varepsilon T - \frac{\delta^\kappa \varepsilon T}{2}.$$

In both cases, by (4.27) and assuming a_0 so small that $\delta < 1/(2|\mathcal{V}_{\sigma_0}|)$, we have $i \in \mathcal{K}^{\text{ext}}(\delta^{\kappa+1}, t_{\kappa+1})$. In conclusion, for a_0 small enough, if $t_{\kappa+1} > \sigma_0$ and $\kappa \le \min\{\kappa^*; 2|\mathcal{C}_0|\}$ then:

$$|\mathcal{K}^{\text{ext}}(\delta^{\kappa+1}, t_{\kappa+1})| \ge |\mathcal{K}^{\text{ext}}(\delta^\kappa, t_\kappa)| + 1,$$

so that $t_{\kappa+1} = \sigma_0$ if $\kappa \ge |\mathcal{C}_0|$, i.e. $\kappa^* \le |\mathcal{C}_0|$.

The proof of (4.41) now follows by contradiction. For a_0 small enough, if $\sigma_* < \sigma_1$ and $\delta = \beta^{\ell_0+1}/16$, the set $\mathcal{K}^{\text{ext}}(\delta, \sigma_*)$ is non empty whence, by the previous analysis, also $\mathcal{K}^{\text{ext}}(\delta^{\kappa^*}, \sigma_0)$ is non empty, which contradicts the definitions of w_0 and W_0.

We now conclude the proof of (4.36) by induction. We assume it is true for any integer $m = 0, \dots, n$ and we prove it also holds for $m = n+1$. We first note that, by the argument after Eqs. (4.38)–(4.39), this assumption implies that the times $\sigma_0, \dots, \sigma_{n+1}$ are all distinct. On the other hand, (setting $\sigma_{-1} \doteq T_k$), for any integer $p \ge 1$,

$$\begin{aligned} \sigma_{2p-1} > \sigma_{2p-3} \implies \quad & w_{2p-1} \le v_0(t) \le W_{2p-1} + \frac{\beta^{\ell_{2p-1}}\varepsilon T}{8} \quad && \forall\, t \in [\sigma_{2p}, \sigma_{2p-1}], \\ \sigma_{2p} > \sigma_{2p-2} \implies \quad & w_{2p} - \frac{\beta^{\ell_{2p}}\varepsilon T}{8} \le v_0(t) \le W_{2p} && \forall\, t \in [\sigma_{2p}, \sigma_{2p+1}], \end{aligned}$$

which give the *apriori* bounds (analogous to (4.42)) on the velocity of the charged particle. Thus, in both cases $n+1 = 2p-1$ or $n+1 = 2p$, we can repeat the same argument used in the case $n = 0$. Recalling (4.35), if a_0 is small enough then $\ell_n \le 4|\mathcal{V}_{\sigma_0}|$ for any $n \le 2|\mathcal{V}_{\sigma_0}|$. This guarantees that the choice (4.22) is large enough for the argument of Theorem 3.1 to work with gaps of order $\beta^{\ell_n \kappa}$, $\kappa \le 2|\mathcal{C}_n|$, for any $n \le 2|\mathcal{V}_{\sigma_0}|$. We omit the details.

We are left with the proof of (4.37). We apply the perturbative argument to the cluster $\mathcal{C}_n$. In fact, for n even (resp. odd) during the time interval $[\sigma_n, \sigma_{n+1}]$ (resp. $[\sigma_{n+1}, \sigma_n]$) there is a velocity gap $\beta^{\ell_n}\varepsilon T/2$ between the particles in $\mathcal{C}_n$ and the other particles which may interact with the first ones.

Since the charged particle belongs to $\mathcal{C}_n$, we have to modify the perturbative analysis by taking into account of the electric field. The mass of the charged particle is M, so that the velocity of the center of mass of $\mathcal{C}_n$ is:

$$v_{\mathcal{C}_n}(t) = \frac{1}{M + |\mathcal{C}_n| - 1} \sum_{j\in\mathcal{C}_n} [(M-1)\delta_{j,0} + 1]\, v_j(t),$$

and the associate slowly varying function:

$$
\begin{aligned}
p_{\mathcal{C}_n}(t) \quad &\doteq \quad v_{\mathcal{C}_n}(t) - \frac{1}{M+|\mathcal{C}_n|-1}\left[MEt - \sum_{q\in\mathcal{V}_{\sigma_0}\setminus\mathcal{C}_n}\sum_{j\in\mathcal{C}_n}\int_0^t ds\,\nabla\Phi_{j,q}(s)\right] \\
&\quad + \frac{1}{M+|\mathcal{C}_n|-1}\sum_{q\in\mathcal{M}_{\sigma_0}}\sum_{j\in\mathcal{C}_n}\frac{\Phi_{j,q}(t)}{v_j(t)-v_q(t)},
\end{aligned}
$$

where $\Phi_{j,q}(t) = \Phi(x_j(t)-x_q(t))$ if $j,q\ge 1$, $\Phi_{j,q}(t) = \Psi(x_j(t)-x_q(t))$ if $j=0$ or $q=0$, and analogously for $\nabla\Phi_{j,q}(t)$. From the equations of motion we have in this case:

$$
\begin{aligned}
\dot p_{\mathcal{C}_n}(t) \quad &= \quad \frac{1}{M+|\mathcal{C}_n|-1}\sum_{q\in\mathcal{M}_{\sigma_0}}\sum_{j\in\mathcal{C}_n}\frac{\Phi_{j,q}(t)}{(v_j(t)-v_q(t))^2} \\
&\quad \times\left\{\sum_{s:s\ne j}\left[\frac{\nabla\Phi_{j,s}(t)}{(M-1)\delta_{j,0}+1} - E\delta_{j,0}\right] - \sum_{s:s\ne q}\nabla\Phi_{q,s}(t)\right\}.
\end{aligned}
$$

Then, by proceeding as done to get (3.46), we obtain, for n even (resp. odd) and any $t\in[\sigma_n,\sigma_{n+1}]$ (resp. $t\in[\sigma_{n+1},\sigma_n]$),

$$
\left|v_{\mathcal{C}_n}(t) - v_{\mathcal{C}_n}(\sigma_n) - \frac{ME(t-\sigma_n)}{M+|\mathcal{C}_n|-1}\right| \le C\left[\frac{1}{\beta^{\ell_n+1}\varepsilon T}+\frac{1}{\beta^2}\right],
$$

which implies (4.37) by (4.22), (4.35), and since $E\ge 0$.

Appendix

In this appendix we briefly discuss the derivation of Eqs. (4.2), (4.3), and (4.4). Eqs. (3.2) and (3.3) are recovered by putting $\Psi=0$, $E=0$, and neglecting the charged particle.

In the sequel we adopt the last notation, denoting by (x_0,v_0) the position and velocity of the charged particle and calling $\bar X = \{(x_0,v_0);X\}$ the state of the whole system. We also denote by $t\mapsto \bar X^{(n)}(t) = \{(x_0^{(n)}(t),v_0^{(n)}(t));X^{(n)}(t)\}$ the n-partial dynamics relative to Eqs. (2.5).

The first step is the control on the growth of the local energy and density of the whole system defined by:

$$
\bar Q(\bar X;\mu,R) \doteq \sum_i \chi_i(\mu,R)\left\{\frac{(M-1)\delta_{i,0}+1}{2}v_i^2 + \frac12\sum_{j:j\ne i}\Phi_{i,j}+1\right\}, \tag{A.1}
$$

where $\Phi_{i,j}=\Phi(x_i-x_j)$ if $i,j\ge 1$ and $\Phi_{i,j}=\Psi(x_i-x_j)$ if $i=0$ or $j=0$. Note that $\Phi_{i,j}$ is superstable and non-negative. Then, recalling the definition (4.1), for any $X\in\mathcal{X}$ and $n\in\mathbb{N}$,

$$
\sup_\mu \bar Q(\bar X^{(n)}(t);\mu,R_n(t)) \le CQ_E(X)R_n(t) \qquad \forall\, t\ge 0, \tag{A.2}
$$

where

$$
R_n(t) \doteq \log(e+n) + \int_0^t ds\,V_n(s) \tag{A.3}
$$

and

$$
V_n(t) \doteq \max_{i\in\bar I_n}\sup_{s\in[0,t]}|v_i^{(n)}(s)|, \tag{A.4}
$$

with $\bar{I}_n \doteq I_n \cup \{0\}$. As a consequence of this estimate one can show that, for any $X \in \mathcal{X}$, $n \in \mathbb{N}$, $i \in \bar{I}_n$, and $t \geq 0$,

$$|v_i^{(n)}(t)| \leq C \left[\sqrt{Q_E(X) \log(e+n)} + Q_E(X)t \right]. \tag{A.5}$$

By applying an iterative procedure one obtains from (A.5) the existence of the infinite dynamics. More precisely, defining

$$\delta_i(n,t) \doteq |x_i^{(n)}(t) - x_i^{(n-1)}(t)| + |v_i^{(n)}(t) - v_i^{(n-1)}(t)|, \tag{A.6}$$

there is $\alpha > 1$ such that, for any $i \in \mathbb{N} \cup \{0\}$, $X \in \mathcal{X}$, and $t \geq 0$, setting $n^* = n^*(i,t) = \mathrm{Int}\left[\alpha(1 + x_i^2 + Q_E(X)^2)e^t\right]$, we have:

$$\delta_i(n,t) \leq C Q_E(X) \exp\left[-\frac{n}{C[1+Q_E(X)]\log^2(e+n)} \right] \qquad \forall\, n > n^*. \tag{A.7}$$

From (A.7) the existence of the limits (2.6) follow. Moreover:

$$|v_i^{(n)}(t) - v_i^{(n^*)}(t)| \leq C \qquad \forall\, n > n^*.$$

so that, by (A.5), for any $n \in \mathbb{N}$,

$$|v_i^{(n)}(t)| \leq C \left[\sqrt{Q_E(X) \log(e + |x_i| + Q_E(X))} + Q_E(X)t \right], \qquad \forall\, i \in \bar{I}_n. \tag{A.8}$$

By taking the limit $n \to +\infty$ in (A.8) we get Eqs. (4.2) and (4.3).

The bounds (A.2), (A.5), and the iterative argument leading to Eq. (A.7) can be found in [10] in the case without electric field E, and in [9, Theorem 2.2] in the case of a (more complicated) quasi one-dimensional model. We omit the proofs. We instead discuss in more details Eq. (4.4), which is a corollary of the previous estimates but it has been never explicitly stated in the quoted papers. We shall prove the analogous result for the function in (A.1): for any $\mu \in \mathbb{R}$, $R > \log(e+|\mu|)$, and $t \geq 0$,

$$\bar{Q}(\bar{X}(t);\mu,R) \leq C Q_E(X) \left[R + \log(e + Q_E(X)) + (1 + Q_E(X))t^2 \right], \tag{A.9}$$

which clearly implies (4.4) by the positivity of the interaction Ψ. We introduce a mollified version of $\bar{Q}(\bar{X};\mu,R)$ by defining:

$$W(\bar{X};\mu,R) \doteq \sum_i f_i^{\mu,R} \left\{ \frac{(M-1)\delta_{i,0}+1}{2} v_i^2 + \frac{1}{2} \sum_{j:j\neq i} \Phi_{i,j} + 1 \right\}, \tag{A.10}$$

where

$$f_i^{\mu,R} = f\left(\frac{|x_i - \mu|}{R} \right) \tag{A.11}$$

and $f \in C^\infty(\mathbb{R}_+)$ is not increasing and satisfies: $f(x) = 1$ for $x \in [0,1]$, $f(x) = 0$ for $x \geq 2$, and $|f'(x)| \leq 2$. Obviously:

$$\bar{Q}(\bar{X};\mu,R) \leq W(\bar{X};\mu,R) \leq \bar{Q}(\bar{X};\mu,2R), \tag{A.12}$$

which in particular shows it is enough to prove Eq. (A.9) with $\bar{Q}(\bar{X}(t);\mu,R)$ replaced by $W(\bar{X}(t);\mu,R)$. Given $\alpha_0 \geq 1$, $\mu \in \mathbb{R}$, and $R > \log(e+|\mu|)$ let:

$$n_0 = n_0(\mu,R,t) = \mathrm{Int}\left[\alpha_0(e + Q_E(X)^2)e^{2(R+t)} \right] + 1.$$

Since $\log(e+n_0) > R$, by (A.2), (A.5), and (A.12),

$$\begin{aligned} W(\bar X^{(n_0)}(t);\mu,R) &\le \bar Q(\bar X^{(n_0)}(t);\mu,2R_{n_0}(t)) \le C\sup_\nu \bar Q(\bar X^{(n_0)}(t);\nu,R_{n_0}(t)) \\ &\le CQ_E(X)R_{n_0}(t) \le CQ_E(X)\left[\log(e+n_0)+Q_E(X)t^2\right] \\ &\le CQ_E(X)\left[R+\log(e+Q_E(X))+(1+Q_E(X))t^2\right], \end{aligned}$$

where in the second inequality we used the positivity of the potential, see [10, Eq. (A.12)]. On the other hand:

$$\begin{aligned} W(\bar X(t);\mu,R) &\le W(\bar X^{(n_0)}(t);\mu,R) \\ &\quad + \sum_{n>n_0} |W(\bar X^{(n)}(t);\mu,R) - W(\bar X^{(n-1)}(t);\mu,R)|. \end{aligned} \tag{A.13}$$

Let us estimate the sum on the right-hand side of (A.13). We have:

$$\begin{aligned} &|W(\bar X^{(n)}(t);\mu,R) - W(\bar X^{(n-1)}(t);\mu,R)| \\ &\quad \le \sum_i f\left(\frac{|x_i^{(n)}(t)-\mu|}{R}\right)|\varepsilon_i^{(n)} - \varepsilon_i^{(n-1)}| \\ &\qquad + \sum_i \left| f\left(\frac{|x_i^{(n)}(t)-\mu|}{R}\right) - f\left(\frac{|x_i^{(n-1)}(t)-\mu|}{R}\right)\right| \varepsilon_i^{(n-1)}, \end{aligned} \tag{A.14}$$

where

$$\varepsilon_i^{(n)} = \frac{(M-1)\delta_{i,0}+1}{2}|v_i^{(n)}|^2 + \frac12 \sum_{j:j\ne i} \Phi_{i,j}^{(n)} + 1$$

and $\Phi_{i,j}^{(n)}$ is defined as $\Phi_{i,j}$ in (A.1) with x_i, x_j replaced by $x_i^{(n)}$, $x_j^{(n)}$ respectively. By (A.8), if $|x_i^{(n)}(t)-\mu| \le 2R$ then all the particles $j \in I_n$ such that $|x_i^{(n)}(t)-x_j^{(n)}(t)| \le 1$ or $|x_i^{(n-1)}(t)-x_j^{(n-1)}(t)| \le 1$ are initially contained in the interval with center μ and radius $R(t)$, where:

$$R(t) = C[R + Q_E(X)(1+t^2)].$$

In particular, by choosing α_0 large enough, for any $n \ge n_0$ each particle i such that $|x_i^{(n)}(t)-\mu| \le 2R$ does not interact with the particles $j \in I_n \setminus I_{n-1}$, so that:

$$\left|\varepsilon_i^{(n)} - \varepsilon_i^{(n-1)}\right| \le C\left[\frac{|v_i^{(n)}(t)| + |v_i^{(n-1)}(t)|}{2}\delta_i(n,t) + \sum_{j:j\ne i}{}^{*} [\delta_i(n,t)+\delta_j(n,t)]\right],$$

(recall the definition (A.6)) where $\sum^*_{j:j\ne i}$ denotes the sum restricted to all the particles $j \in I_{n-1}$ such that $|x_i^{(n)}(t)-x_j^{(n)}(t)| \le 1$ or $|x_i^{(n-1)}(t)-x_j^{(n-1)}(t)| \le 1$. The number of these particles is therefore bounded by:

$$N(X;\mu,R(t)) \le 2Q(X)R(t),$$

where we used (2.2), (2.3), and that $R > \log(e+|\mu|)$. Then, setting

$$\Delta_n(t) \doteq \max\{\delta_i(n,t) \,:\, |x_i-\mu| \le R(t)\}$$

and using (A.8), if i is such that $|x_i-\mu| \le R(t)$, for any $n > n_0$,

$$\begin{aligned} \left|\varepsilon_i^{(n)} - \varepsilon_i^{(n-1)}\right| &\le C\left[\sqrt{Q_E(X)\log(e+|\mu|+R(t))} + Q_E(X)(t+R(t))\right]\Delta_n(t) \\ &\le C[1+Q_E(X)^2]\log^2(e+n)\,\Delta_n(t). \end{aligned} \tag{A.15}$$

On the other hand:

$$\begin{aligned}\left| f\left(\frac{|x_i^{(n)}(t)-\mu|}{R}\right) - f\left(\frac{|x_i^{(n-1)}(t)-\mu|}{R}\right)\right| \\ &\leq 2\frac{|x_i^{(n)}(t)-x_i^{(n-1)}(t)|}{R}\chi\big(|x_i^{(n-1)}(t)-\mu| \leq \delta_i(n,t)+2R\big) \\ &\leq C\chi\big(|x_i^{(n-1)}(t)-\mu| \leq \Delta_n(t)+2R\big)\,\Delta_n(t). \qquad \text{(A.16)}\end{aligned}$$

By (A.7) and the definition of n_0, if α_0 is large enough,

$$\Delta_n(t) \leq CQ_E(X)\exp\left[-\frac{n}{C[1+Q_E(X)]\log^2(e+n)}\right] \qquad \forall\, n > n_0. \qquad \text{(A.17)}$$

In particular $\Delta_n(t) \leq C$. Then, inserting the bounds (A.15) and (A.16) in (A.14),

$$\begin{aligned}|W(\bar{X}^{(n)}(t);\mu,R) - W(\bar{X}^{(n-1)}(t);\mu,R)| \\ &\leq C[1+Q_E(X)^2]\log^2(e+n)N(X^{(n)}(t);\mu,2R)\,\Delta_n(t) \\ &\qquad + W(\bar{X}^{(n-1)}(t);\mu,\Delta_n(t)+2R)\,\Delta_n(t) \\ &\leq C[1+Q_E(X)^2]\log^2(e+n)W(\bar{X}^{(n)}(t);\mu,2R_n(t))\,\Delta_n(t) \\ &\qquad + W(\bar{X}^{(n-1)}(t);\mu,C+2R_{n-1}(t))\,\Delta_n(t) \\ &\leq C[1+Q_E(X)^2]\log^2(e+n)\,[R_{n-1}(t)+R_n(t)]\;\Delta_n(t).\end{aligned}$$

where in the last inequality we used the positivity of the potential, (A.12), and (A.2). Again by (A.8) we have that $R_n(t) \leq C[1+Q_E(X)^2]\log^2(e+n)$ for $n \geq n_0$. By (A.17) we then conclude that the sum on the right-hand side of (A.13) is bounded by a constant.

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