Spectral Theory of Massless Pauli-Fierz Models

V. Georgescu
CNRS, Département de Mathématiques
Université de Cergy-Pontoise
2 avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex France
C. Gérard
Département de Mathématiques
Université de Paris Sud
91405 Orsay Cedex France
J. S. Møller*
FB Mathematik (17)
Johannes Gutenberg Universität
D-55099 Mainz Germany

April 2003

Abstract

We study the spectral theory of massless Pauli-Fierz models using an extension of the Mourre method. We prove the local finiteness of point spectrum and a limiting absorption principle away from the eigenvalues for an arbitrary coupling constant. In addition we show that the expectation value of the number operator is finite on all eigenvectors.

1 Introduction

We consider in this paper a class of QFT models describing a quantum system linearly coupled with a massless scalar photon field. The models are described on a Hilbert space $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$, where $\mathcal{K}$ is a separable Hilbert space describing the quantum system and $\mathcal{G}(\mathfrak{h})$ is the bosonic Fock space over $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, describing a field of massless scalar bosons.

The Hamiltonian H is given by $H = K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) + g\phi(v)$, where K is a bounded below Hamiltonian on $\mathcal{K}$ describing the dynamics of the quantum system, $\omega(k) = |k|$ is the boson dispersion relation, $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ is an operator valued form factor describing the coupling of the small system with the boson field and g is a coupling constant.

The most important examples are the *spin-boson model*, describing a single spin coupled to a boson field, and the *Nelson model*, describing a non-relativistic atom coupled to a boson field.

A lot of effort was devoted in recent years to the study of these models and their generalization (for example the non-relativistic model of electrons minimally coupled to the Maxwell field), see e.g. [Ar, AH1, AH2, BFS, BFSS, DG1, DG2, DJ, FGS1, FGS2, G1, G2, LMS, Sk, Sp].

---

*Supported by Carlsbergfondet

One way to study the spectral properties of a Hamiltonian H is the *Mourre commutator method*, which relies on the construction of a *conjugate operator* A such that the commutator $[H, \mathrm{i}A]$ is locally positive

$$\mathbb{1}_{\Delta}(H)[H, \mathrm{i}A]\mathbb{1}_{\Delta}(H) \geq c_0 \mathbb{1}_{\Delta}(H), \ \ c_0 > 0$$

on some energy interval Δ. The weaker estimate in which the preceding inequality is required to hold modulo a compact operator is called a *Mourre estimate*. Typically one deduces from a Mourre estimate the local finiteness of point spectrum and a limiting absorption principle away from thresholds and eigenvalues of H, which implies the absence of singular continuous spectrum.

Moreover one can deduce from a Mourre estimate propagation estimates on the unitary group $\mathrm{e}^{-\mathrm{i}tH}$ for large times which are often a key ingredient in the study of the scattering theory of H, for example in proofs of the asymptotic completeness.

In this paper we use the Mourre method to obtain results on the structure of the spectrum for massless Pauli-Fierz Hamiltonians.

1.1 Outline of the paper

To put our work in perspective, it is helpful to make a quick review of the applications of the Mourre method to various Hamiltonians arising in Quantum Mechanics, like the $N-$particle Schrödinger Hamiltonian, or the Pauli-Fierz Hamiltonian and its generalizations.

Typically the Hamiltonian H can be written as the sum $H = H_0 + V$ of a 'free' part H_0 and an 'interacting' part V. Quite often a conjugate operator for H can be guessed by choosing a conjugate operator A for H_0 and then proving that it is also a conjugate operator for H. However, except in simple situations, the proof that A is a conjugate operator for H does not follow from a perturbation argument, but relies on the following ingredients:

- A geometric decomposition of the Hilbert space (corresponding for example to the various cluster decompositions of the N-particle Hamiltonian).
- An induction step allowing to deduce a Mourre estimate for H from a Mourre estimate for subsystems.

Note also that in these proofs, *compact operators* play the role of error terms, which can be neglected by proving the Mourre estimate on a small enough energy interval.

For massive Pauli-Fierz models [DG1], and space-cutoff $P(\varphi)_2$ models [DG2], the same strategy can be applied, yielding a Mourre estimate for arbitrary coupling constant, away from the eigenvalues and thresholds of H. The *threshold set* of H is $\tau(H) = \sigma_{\mathrm{pp}}(H) + m\mathbb{N}^*$, where m is the boson mass. It corresponds to the energy levels where bosons can propagate away to infinity with zero asymptotic velocity.

Quite a number of papers have been devoted recently to the proof of a Mourre estimate for massless Pauli-Fierz models or some of their extensions, see e.g. [BFS], [BFSS], [Sk], [DJ], [FGS3]. However these papers did not follow the standard scheme outlined above. Instead the Mourre estimate for H is typically deduced from a Mourre estimate for H_0 (or a more sophisticated free Hamiltonian approximating more closely H as in [BFSS], [DJ]) and by assuming that the coupling constant g is small enough to control the commutator $[V, \mathrm{i}A]$ with the interaction.

As a consequence, in [BFS] the Mourre estimate for H is shown only outside some g^{α}-neighborhoods of the eigenvalues of H_0, or in [BFSS] and [DJ] outside some g^{α}-neighborhood of the lowest eigenvalue of H_0, assuming in addition that the *Fermi golden rule* holds at all eigenvalues of H_0 embedded in the continuous spectrum. The only exception is [Sk], where the coupling constant is small but the Mourre estimate holds on all the spectrum of H.

These results are not surprising, since one expects that a Mourre estimate should hold away from the *eigenvalues of* H, which by a formal perturbation argument can exist only in g^{α}-neighborhoods of the eigenvalues of H_0. (Note that since massless bosons propagate with speed 1, massless Pauli-Fierz models should have no thresholds.)

In our paper we prove a Mourre estimate for massless Pauli-Fierz Hamiltonians H for *arbitrary coupling constant* at all energies away from the eigenvalues of H, thereby obtaining the correct non-perturbative result one naturally expects from considering the corresponding massive case.

Let us now briefly discuss the ideas of our proof.

Instead of using just one conjugate operator, we consider a *family* A^{δ} of conjugate operators, which are of the form $A^{\delta} = \mathrm{d}\Gamma(a^{\delta})$, where a^{δ} is the generator of a semigroup of isometries on $\mathfrak{h}$. More precisely a^{δ} is the symmetric operator associated to the vector field $m^{\delta}(r)\partial_r$, where $r = |k|$ and m^{δ} is a smooth function equal to 1 in $r \geq 1$, and equal to $d(\delta)$ in $0 \leq r \leq \delta$, where $d(\delta) \to +\infty$ when $\delta \to 0$. To prove a Mourre estimate up to an energy level E we have to choose the parameter δ sufficiently small. Therefore our conjugate operators are modifications of the generator of *radial translations*, used in [DJ] and [Sk] (in [BFS] and [BFSS] the generator of *dilations* was used instead).

The method of proof is inspired by that in [DG1]. The first step is as usual to perform a geometric decomposition of the Hilbert space allowing to treat separately the bosons close to the atom and the bosons close to infinity. This decomposition alone is no more sufficient to set up an inductive proof of the Mourre estimate, because taking a boson near infinity does not decrease the energy of the remaining system, since the rest mass of the boson vanishes. To set up the induction proof, we separate again the bosons near infinity between bosons of momentum less than δ and bosons of momentum greater than δ. If there exists at least a boson near infinity of momentum greater than δ, then the energy of the remaining system is lower than the total energy by an amount at least equal to δ, which allows to start an inductive proof of the Mourre estimate. If all the bosons near infinity have momentum less than δ, then we use a different argument: namely the commutator $[H_0, \mathrm{i}A^{\delta}]$ is larger than $d(\delta)$, which suffices to get positivity of $[H, \mathrm{i}A^{\delta}]$, by controlling the error term $[V, \mathrm{i}A^{\delta}]$ in norm and choosing δ small enough.

Once a Mourre estimate is obtained, additional work is required to deduce from it consequences like a limiting absorption principle or absence of eigenvalues. In our case the commutator $[H, \mathrm{i}A^{\delta}]$ is a perturbation of the number operator, and hence is not bounded as a quadratic form on the domain of H. In [GGM] an extension of the Mourre method, as developed in [Sk], was given. We rely here on this version of the Mourre method, which is formulated in terms of C_0-semigroups in the spirit of [ABG]. Finally using an extension of the virial theorem, we can show that the expectation value of the number operator N is finite on each eigenvector of H.

1.2 Plan of the paper

Let us now describe the plan of the paper.

In Section 2 we describe the class of abstract Pauli-Fierz models considered in this paper.

We describe the hypotheses and give the two main applications, namely the confined Nelson model and the confined Nelson model after a dressing transformation. The results of the paper are formulated in Subsection 2.5.

In Section 3, we recall the definition of various operators on Fock spaces and we prove some estimates on creation/annihilation operators and on second quantized operators that will be needed later. Most of the results here are standard, except for Props. 3.4, 3.7 and 3.9.

In Section 4, we study the smoothness of abstract Pauli-Fierz Hamiltonians under a second quantized C_0-semigroup of isometries. We furthermore prove a HVZ-type theorem,

In Section 5 we recall some terminology and results of [GGM], where an extension of the Mourre method is developed.

In Section 6, we introduce the conjugate operator A that will be used to prove a Mourre estimate and we verify the abstract conditions given in Section 5, using the results of Section 4.

In Section 7, we prove the Mourre estimate for Pauli-Fierz Hamiltonians, using geometric decompositions in position and momentum space. Finally the proofs of the results of Subsection 2.5 are given in Section 8.

2 Hypotheses and results

2.1 Massless Pauli-Fierz models

Let us first describe the class of Hamiltonians that we will consider in this paper. These Hamiltonians describe a quantum system, typically a non-relativistic atom, interacting with a field of massless scalar bosons.

We refer the reader to Section 4 where abstract Pauli-Fierz models are studied in details.

The quantum system is described with a separable Hilbert space $\mathcal{K}$ and a bounded below selfadjoint operator K. Without loss of generality we will assume that K is positive.

The one-particle space is $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, where k is the boson momentum. The one-particle kinetic energy is the operator of multiplication by

$$\omega(k) = |k|.$$

The boson field is described by the Hilbert space $\Gamma(\mathfrak{h})$, and the interacting system by:

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

The free Hamiltonian is

$$H_0 = K \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega).$$

The interacting Hamiltonian is

$$H = H_0 + \phi(v),$$

for a coupling function (also called *form factor* in the physics literature) $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$. Since $\mathcal{K}$ is separable, v can be identified with a strongly measurable function:

$$\mathbb{R}^d \ni k \mapsto v(k) \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K})$$

uniquely defined almost everywhere, such that

$$\|v(\cdot)\| = \left[\sup_{\psi \in \mathcal{K},\, \|\psi\|=1} \int_{\mathbb{R}^d} \|v(k)(K+1)^{-\frac{1}{2}}\psi\|_{\mathcal{K}}^2 \mathrm{d}k \right]^{\frac{1}{2}} < \infty.$$

We assume the following hypothesis:

$$\textbf{(I1)}\ \left\{\begin{array}{l} v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h}),\ v \text{ extends as } v \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathfrak{h}) \text{ and} \\ \lim_{r\to+\infty} \left(\| \mathbb{1}_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}} v (K+r)^{-\frac{1}{2}} \|_{\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} + \| (K+r)^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}} v \|_{\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} \right) = 0. \end{array}\right.$$

2.2 Additional hypotheses

We will now collect the additional hypotheses that we will impose on K and v to prove the results of this paper. The first one concerns the system coupled to the boson field:

(H0) $(K + \mathrm{i})^{-1}$ is compact on $\mathcal{K}$.

Physically this condition means that the small system is confined.

To formulate the hypotheses on the coupling function v, we fix a function $d \in C^\infty(]0, +\infty[)$ such that:

$$d'(t) < 0,\ |d'(t)| \leq C t^{-1} d(t),\ d(t) \equiv 1 \text{ in } \{t \geq 1\},\ \lim_{t\to 0} d(t) = +\infty. \tag{2.1}$$

Remark 2.1 Let $\chi \in C_0^\infty(\mathbb{R})$, with $\chi \equiv 1$ near 0. Then a function of the form

$$d(t) = \chi(t) t^{-\epsilon} + 1 - \chi(t)$$

for $\epsilon > 0$ satisfies (2.1). Moreover if d satisfies (2.1) then d^α for $\alpha > 0$ and $\ln(d) + 1$ satisfy also (2.1).

Let us introduce polar coordinates on $\mathbb{R}^d$ using the unitary map:

$$\left\{\begin{array}{l} T : L^2(\mathbb{R}^d, \mathrm{d}k) \to L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^{d-1}) =: \tilde{\mathfrak{h}}, \\ \\ Tu(r, \theta) := r^{(d-1)/2} u(r\theta). \end{array}\right. \tag{2.2}$$

Let also

$$\tilde{v} := (\mathbb{1}_{\mathcal{K}} \otimes T) v.$$

Then we will impose

$$\textbf{(I2)}\ \left\{\begin{array}{l} (1 + r^{-\frac{1}{2}}) r^{-1} d(r) \tilde{v} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \tilde{\mathfrak{h}}) \cap \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \tilde{\mathfrak{h}}), \\ \\ (1 + r^{-\frac{1}{2}}) d(r) \partial_r \tilde{v} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \tilde{\mathfrak{h}}) \cap \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \tilde{\mathfrak{h}}), \end{array}\right.$$

and finally

(I3) $\partial_r^2 \tilde{v} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \tilde{\mathfrak{h}})$.

2.3 The massless Nelson model

The main example of a massless Pauli-Fierz model is the *Nelson model* (see [Ne], [Ca] , [A], [Ar] and [LMS]). It was originally introduced in [Ne] as a phenomenological model of non-relativistic particles interacting with a quantized scalar field.

The atom is described with the Hilbert space

$$\mathcal{K} := L^2(\mathbb{R}^{3P}, \mathrm{dx}),$$

where $\mathrm{x} = (\mathrm{x}_1, \dots, \mathrm{x}_P)$, x_i is the position of particle i, and the Hamiltonian:

$$K := \sum_{i=1}^{P} -\frac{1}{2m_i}\Delta_i + \sum_{i<j} V_{ij}(\mathrm{x}_i - \mathrm{x}_j) + W(\mathrm{x}_1, \dots, \mathrm{x}_P),$$

where m_i is the mass of particle i, V_{ij} is the interaction potential between particles i and j and W is an external confining potential.

We will assume

$$(\mathbf{H0'}) \left\{ \begin{array}{l} V_{ij} \text{ is } \Delta - \text{ bounded with relative bound } 0, \\ W \in L^2_{\mathrm{loc}}(\mathbb{R}^{3N}),\ W(\mathrm{x}) \geq c_0|\mathrm{x}|^{2\alpha} - c_1,\ c_0 > 0,\ \alpha > 0. \end{array} \right.$$

It follows from **(H0′)** that K is symmetric and bounded below on $C_0^\infty(\mathbb{R}^{3P})$. We still denote by K its Friedrichs extension. Moreover we have $\mathcal{D}((K+b)^{\frac{1}{2}}) \subset H^1(\mathbb{R}^{3P}) \cap \mathcal{D}(|\mathrm{x}|^\alpha)$, which implies:

$$(2.3) \qquad |\mathrm{x}|^\alpha (K+b)^{-\frac{1}{2}} \text{ is bounded.}$$

Note also that **(H0′)** implies that K has compact resolvent on $L^2(\mathbb{R}^{3P})$, so hypothesis **(H0)** in Subsection 2.2 is satisfied.

The one-particle space for bosons is

$$\mathfrak{h} := L^2(\mathbb{R}^3, \mathrm{d}k),$$

and the bosonic field is described with the Fock space $\Gamma(\mathfrak{h})$ and the Hamiltonian $\mathrm{d}\Gamma(|k|)$.

We assume that the interaction is of the form

$$(2.4) \qquad V := \sum_{j=1}^{N} \phi(\check{\rho}(\mathrm{x}_j)),$$

for

$$\phi(\check{\rho}(\mathrm{x})) = \frac{1}{\sqrt{2}} \int \rho(k)\mathrm{e}^{-\mathrm{i}k\cdot\mathrm{x}} \otimes a^*(k) + \bar{\rho}(k)\mathrm{e}^{\mathrm{i}k\cdot\mathrm{x}} \otimes a(k)\mathrm{d}k,$$

where $\check{\rho}$ denotes the inverse Fourier transform of $\rho \in L^2(\mathbb{R}^3)$. The Hamiltonian describing the interacting system is now:

$$H := H_0 + V.$$

Note that the interaction is translation invariant (although the full Hamiltonian H is not because of the confining potential W). Note also that using the notation introduced in Subsection 4.1 we can write $V = \phi(v)$, where $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ is defined by

$$(2.5) \qquad v(k)\psi(\mathrm{x}_1, \dots, \mathrm{x}_P) = \sum_{j=1}^{P} \mathrm{e}^{-\mathrm{i}k\cdot\mathrm{x}_j}\rho(k)\psi(\mathrm{x}_1, \dots, \mathrm{x}_P).$$

If the function ρ satisfies:

(I1′) $\int (1 + |k|^{-1})|\rho(k)|^2 \mathrm{d}k < \infty$, hypothesis **(I1)** in Subsection 2.2 is satisfied. Going to polar coordinates we have:

$$\tilde{v}(r,\theta)=\sum_{j=1}^{P}\mathrm{e}^{-\mathrm{i}r\mathrm{x}_j\cdot\theta}\tilde{\rho}(r,\theta),\ \text{ for }\tilde{\rho}(r,\theta)=r\rho(r\theta).$$

Using the identity $\partial_r\mathrm{e}^{-\mathrm{i}r\mathrm{x}\cdot\theta}\tilde{\rho}=\mathrm{e}^{-\mathrm{i}r\mathrm{x}\cdot\theta}(\partial_r\tilde{\rho}-\mathrm{i}\mathrm{x}\cdot\theta\tilde{\rho})$ and (2.3) to control the powers of x we see that if:

$$\textbf{(I2}'\textbf{)}\ \left\{\begin{array}{l}(1+r^{-\frac{1}{2}})r^{-1}d(r)\tilde{\rho}\in L^2(\mathbb{R}^+,\mathrm{d}r)\otimes L^2(S^2),\\[2mm] (1+r^{-\frac{1}{2}})d(r)\partial_r\tilde{\rho}\in L^2(\mathbb{R}^+,\mathrm{d}r)\otimes L^2(S^2),\end{array}\right.$$

(I3′) $\partial_r^2\tilde{\rho}\in L^2(\mathbb{R}^+,\mathrm{d}r)\otimes L^2(S^2)$,

and **(I1′)** are satisfied and $\alpha\geq 2$, then hypotheses **(I1)**, **(I2)** and **(I3)** of Subsection 2.2 are satisfied.

Let us consider a particular choice of ρ of the form

$$\rho(k)=|k|^{\beta}\chi(|k|),\ \beta\in\mathbb{R},\tag{2.6}$$

where $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ near 0 is an ultraviolet cutoff, and recall that the physical case corresponds to $\beta=-\frac{1}{2}$. We see that if $\beta>\frac{1}{2}$, conditions **(I1′)**, **(I2′)** and **(I3′)** are satisfied for a function $d(r)$ equal to $r^{-\epsilon}$ near 0 and $0<\epsilon\ll 1$. In the next subsection, we will show that we can actually handle coupling functions ρ of the form (2.6) for all $\beta>-\frac{1}{2}$.

2.4 The massless Nelson model after a dressing transformation

Let us assume in addition to **(I1′)** that:

(I4′) $\int|k|^{-2}|\rho(k)|^2\mathrm{d}k<\infty$,

and set:

$$v_0(k)=P\rho(k)1\!\!1_{\mathcal{K}},\ v_1(k)=v(k)-v_0(k)=\sum_{j=1}^{P}(\mathrm{e}^{-\mathrm{i}k\cdot\mathrm{x}_j}-1)\rho(k).$$

Then it is easy to verify that:

$$H_1:=\mathrm{e}^{\mathrm{i}\phi(\frac{v_0}{\omega})}H\mathrm{e}^{-\mathrm{i}\phi(\frac{v_0}{\omega})}=K_1\otimes 1\!\!1+1\!\!1\otimes\mathrm{d}\Gamma(|k|)+\phi(v_1)+E_1,$$

for:

$$K_1=K-P\sum_{j=1}^{P}\int\omega^{-1}(k)|\rho(k)|^2(1-\cos(k\cdot\mathrm{x}_j))\mathrm{d}k,\ E_1=\frac{1}{2}\mathrm{Re}(v_0,v_0/\omega)_{\mathfrak{h}}.$$

We see that H_1 is a Pauli-Fierz Hamiltonian similar to H with v replaced by v_1, K by K_1+E_1.

It is clear that K_1 satisfies **(H0)**, since $K-K_1$ is bounded. To control the interaction v_1, we use the bound:

$$|\mathrm{e}^{-\mathrm{i}\mathrm{x}\cdot\theta}-1|\leq\hat{r}\langle\mathrm{x}\rangle,$$

for $\hat{r} := r\langle r\rangle^{-1}$. This yields if $\tilde{v}_1 = 1\!\!1_{\mathcal{K}} \otimes T v_1$:

$$\begin{aligned} |\tilde{v}_1| &\leq C|\tilde{\rho}|\hat{r}\langle \mathrm{x}\rangle, \\ |\partial_r \tilde{v}_1| &\leq C|\partial_r \tilde{\rho}|\hat{r}\langle \mathrm{x}\rangle + C\langle \mathrm{x}\rangle|\tilde{\rho}|, \\ |\partial_r^2 \tilde{v}_1| &\leq C|\partial_r^2 \tilde{\rho}|\hat{r}\langle \mathrm{x}\rangle + C\langle \mathrm{x}\rangle|\partial_r \tilde{\rho}| + C\langle \mathrm{x}\rangle^2|\tilde{\rho}|. \end{aligned} \tag{2.7}$$

It is easy to verify that if the hypotheses:

(I1″) $(1 + r^{-1})\tilde{\rho} \in L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^2)$,

(I2″) $\begin{cases} (1 + r^{-\frac{1}{2}})d(r)\tilde{\rho} \in L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^2), \\ (1 + r^{\frac{1}{2}}d(r)\langle r\rangle^{-\frac{1}{2}})\partial_r \tilde{\rho} \in L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^2), \end{cases}$

(I3″) $\hat{r}\partial_r^2 \tilde{\rho} \in L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^2)$,

are fulfilled and $\alpha > 2$, then condition **(I4′)** is satisfied and the renormalized Hamiltonian H_1 satisfies **(I1)**, **(I2)** and **(I3)**. For a coupling function of the form (2.6), these hypotheses hold for a function $d(r)$ equal to $r^{-\epsilon}$ near 0 and $0 < \epsilon \ll 1$, if $\beta > -\frac{1}{2}$.

2.5 Results

In this subsection we state the main results of this paper. The proofs will be given in Section 8.

The following notations are needed to formulate the limiting absorption principle. Let $-\frac{\partial^2}{\partial r^2}$ be the Laplacian on $L^2(\mathbb{R}^+, \mathrm{d}r)$ with Dirichlet condition at 0, and $\tilde{b} := (-\frac{\partial^2}{\partial r^2})^{\frac{1}{2}}$. We set $b := 1\!\!1_{\mathcal{K}} \otimes T^{-1}\tilde{b}T$, where $T : \mathfrak{h} \to \tilde{\mathfrak{h}}$ is defined in (2.2).

We begin with a preliminary result which describes the basic spectral properties of H. Proposition 4.8 contains more general results.

Proposition 2.2 *Assume hypotheses* **(H0)** *and* **(I1)***. Then H is selfadjoint and bounded below on $\mathcal{D}(H) = \mathcal{D}(H_0)$ and*

$$\sigma(H) = [\inf \sigma(H), +\infty[.$$

Properties of eigenvectors.

Theorem 2.3 *Assume hypotheses* **(I1)** *and* **(I2)***. Let $N = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(1\!\!1)$ be the number operator on $\mathcal{H}$. Then if u is an eigenvector of H, u belongs to $\mathcal{D}(N^{\frac{1}{2}})$.*

Theorem 2.4 *Assume hypotheses* **(H0)***,* **(I1)** *and* **(I2)***. Then for each bounded interval $I \subset \mathbb{R}$ $\mathrm{Tr}1\!\!1_I^{\mathrm{pp}}(H) < \infty$, i.e. the point spectrum of H is locally finite (counting multiplicity).*

Limiting absorption principle.

Theorem 2.5 *Assume hypotheses* **(H0)**, **(I1)** *and* **(I2)**. *Let* $I \subset \mathbb{R}\backslash\sigma_{\rm pp}(H)$ *be a compact interval. Then for* $\frac{1}{2} < s \leq 1$ *the limits:*

$$\begin{array}{l}(N+1)^{\frac{1}{2}}(\mathrm{d}\Gamma(b)+1)^{-s}R(\lambda\pm \mathrm{i}0)(\mathrm{d}\Gamma(b)+1)^{-s}(N+1)^{\frac{1}{2}}\\ :=\lim_{\mu\to 0^+}(N+1)^{\frac{1}{2}}(\mathrm{d}\Gamma(b)+1)^{-s}(H-\lambda\mp \mathrm{i}\mu)^{-1}(\mathrm{d}\Gamma(b)+1)^{-s}(N+1)^{\frac{1}{2}}\end{array}$$

exist in norm uniformly in $\lambda \in I$. *Moreover, the maps:*

$$I \ni \lambda \mapsto (N+1)^{\frac{1}{2}}(\mathrm{d}\Gamma(b)+1)^{-s}R(\lambda\pm \mathrm{i}0)(\mathrm{d}\Gamma(b)+1)^{-s}(N+1)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$$

are Hölder continuous of order $s-\frac{1}{2}$ *for the norm topology of* $\mathcal{B}(\mathcal{H})$.

Remarks 2.6 (1) Stronger forms of the limiting absorption principle can be obtained by applying Theorem 5.15 for the space $\mathcal{G} = \mathcal{D}(B^{\frac{1}{2}})$, where $B = K\otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}}\otimes \mathrm{d}\Gamma((k^2+1)^{\frac{1}{2}})$ and the conjugate operator $A = A^{\delta}$, where A^{δ} is defined in Section 6 for the parameter δ depending on the energy interval I.

(2) A weaker but more explicit form of the limiting absorption principle can be obtained by replacing in Theorem 2.5 the observable b by $|x|$, where $x := i\partial_k$ is the boson position observable.

3 Operators on Fock spaces

In this section we first recall some standard definitions on Fock spaces. Then we prove some bounds on second quantized and creation/annihilation operators which will be useful in the sequel.

3.1 Notations

General notations

Let $\mathbb{R} \ni t \mapsto \Phi(t)$ be a map with values in linear operators on a Hilbert space $\mathcal{H}$ and N is a positive selfadjoint operator on $\mathcal{H}$. For $\alpha, \beta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ we will say that

$$\Phi(t) = N^{\alpha}O(t^{\mu})N^{\beta} \text{ for } \alpha,\beta\in\mathbb{R}^+,\ \mu\in\mathbb{R}$$

if $(N+1)^{-\alpha}\Phi(t)(N+1)^{-\beta} \in \mathcal{B}(\mathcal{H})$ for $|t| \gg 1$ and $\|(N+1)^{-\alpha}\Phi(t)(N+1)^{-\beta}\| = O(t^{\mu})$. We say that $\Phi(t) = N^{\alpha}O(t^{\mu})$ if $\Phi(t) \in N^{\alpha}O(t^{\mu})N^{0}$. The notations $\Phi(t) = N^{\alpha}o(t^{\mu})N^{\beta}$ and $\Phi(t) = o(N^{\alpha})t^{\mu}$ are defined similarly.

The symbol $A^{(*)}$ in a statement means that the statement holds both for the linear operator A and its adjoint A^*.

Quadratic forms

We now fix some terminology related to quadratic forms on Hilbert spaces. All quadratic forms considered in the sequel will be assumed to be symmetric and bounded below. If q is a quadratic form with domain $\mathcal{D}(q)$ on a Hilbert space $\mathfrak{h}$ we will extend q to the whole Hilbert space by setting $q(u) = +\infty$ if $u \notin \mathcal{D}(q)$. If $U \in \mathcal{B}(\mathfrak{h})$, we denote by U^*qU the quadratic form $q(Uu)$.

If q_1, q_2 are two quadratic forms, we write $q_1 \leq q_2$ if $q_1(u) \leq q_2(u)$ for all $u \in \mathfrak{h}$. Note that with the above convention this implies that $\mathcal{D}(q_2) \subset \mathcal{D}(q_1)$.

To a bounded below selfadjoint operator a we associate the quadratic form $a(u) = (u, au)$ with domain $\mathcal{D}(|a|^{\frac{1}{2}})$. If a_1, a_2 are two bounded below selfadjoint operators, we will write $a_1 \leq a_2$ if the same relation holds for the associated quadratic forms.

3.2 Fock spaces

Let $\mathfrak{h}$ be a Hilbert space, which we will call the *one-particle space*. Let $\Gamma_n(\mathfrak{h}) := \otimes_{\rm s}^n \mathfrak{h}$ be the symmetric nth tensor power of $\mathfrak{h}$. Let $\mathcal{S}_n$ be the orthogonal projection of $\otimes^n \mathfrak{h}$ onto $\Gamma_n(\mathfrak{h})$. The Fock space over $\mathfrak{h}$ is the direct Hilbert sum

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Gamma_n(\mathfrak{h}).$$

Ω will denote the vacuum vector $(1, 0\dots) \in \Gamma(\mathfrak{h})$. The number operator N is defined as

$$N\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}} = n\mathbb{1}.$$

For $h \in \mathfrak{h}$ we denote by $a^*(h)$ and $a(h)$ the creation and annihilation operators, by $\phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h))$ the field operators and by $W(h) = {\rm e}^{{\rm i}\phi(h)}$ the Weyl operators (see e.g. [DG1, Section 2]).

If $\mathfrak{g} \subset \mathfrak{h}$ is a vector space, we denote by $\Gamma_{\rm fin}(\mathfrak{g}) \subset \Gamma(\mathfrak{h})$ the space $\oplus_0^\infty \otimes_{\rm s}^n \mathfrak{g}$ where direct sums and tensor products are taken in the algebraic sense. If $\mathfrak{g} = \mathfrak{h}$ the space $\Gamma_{\rm fin}(\mathfrak{h})$ will be the space of finite particle vectors, for which $\mathbb{1}_{[n,+\infty]}(N)u = 0$ for some $n \in \mathbb{N}$.

Let now $\mathcal{K}$ be a Hilbert space describing a quantum system.

The Hilbert space describing the quantum system interacting with a field of bosons of one-particle space $\mathfrak{h}$ is:

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

We shall identify the adjoint spaces $\mathcal{K}^* = \mathcal{K}$, $\mathfrak{h}^* = \mathfrak{h}$ and $\mathcal{H}^* = \mathcal{H}$ with the help of the Riesz isomorphism as usual. If not explicitly stated, the other Hilbert spaces that appear below are not identified with their adjoints. The space $\mathcal{K} \otimes \Gamma_{\rm fin}(\mathfrak{h})$ will be denoted by $\mathcal{H}_{\rm fin}$.

Creation/annihilation operators

We now define creation/annihilation operators associated to operator valued symbols. We recall that a densely defined operator A is closeable iff its adjoint A^* is densely defined.

Let $\mathcal{L}_1, \mathcal{L}_2$ be Hilbert spaces and $v \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathfrak{h})$, so that $v^* \in \mathcal{B}(\mathcal{L}_2^* \otimes \mathfrak{h}, \mathcal{L}_1^*)$. Then the *creation operator*

$$a^*(v) : \mathcal{D}(a^*(v)) \subset \mathcal{L}_1 \otimes \Gamma(\mathfrak{h}) \to \mathcal{L}_2 \otimes \Gamma(\mathfrak{h})$$

and the *annihilation operator*

$$a(v) : \mathcal{D}(a(v)) \subset \mathcal{L}_2^* \otimes \Gamma(\mathfrak{h}) \to \mathcal{L}_1^* \otimes \Gamma(\mathfrak{h})$$

are defined as follows:

for $n \in \mathbb{N}$ we denote by $a_n^*(v) : \mathcal{L}_1 \otimes \Gamma_n(\mathfrak{h}) \to \mathcal{L}_2 \otimes \Gamma_{n+1}(\mathfrak{h})$ the operators defined by:

$$a_n^*(v) := \sqrt{n+1}\,(1\!\!1_{\mathcal{L}_2} \otimes S_{n+1}) \circ \left(v \otimes 1\!\!1_{\Gamma_n(\mathfrak{h})}\right). \tag{3.1}$$

Then we set:

$$a_{\mathrm{fin}}^*(v) := \oplus_{n=0}^{\infty} a_n^*(v),$$

as an operator from $\mathcal{L}_1 \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ into $\mathcal{L}_2 \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$.

Similarly for $n \in \mathbb{N}$ we denote by $a_n(v) : \mathcal{L}_2^* \otimes \Gamma_{n+1}(\mathfrak{h}) \to \mathcal{L}_1^* \otimes \Gamma_n(\mathfrak{h})$ the operators defined by:

$$a_n(v) = \sqrt{n+1} v^* \otimes 1\!\!1_{\Gamma_n(\mathfrak{h})}, \tag{3.2}$$

and set:

$$a_{\mathrm{fin}}(v) := \oplus_{n=0}^{\infty} a_n(v),$$

as an operator from $\mathcal{L}_2^* \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ into $\mathcal{L}_1^* \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$. Clearly $a_{\mathrm{fin}}(v) \subset (a_{\mathrm{fin}}^*(v))^*$ hence $a_{\mathrm{fin}}^*(v)$ is closeable. We will denote by $a^*(v)$ its closure and by $a(v)$ the operator $(a^*(v))^*$, which coincides with the closure of $a_{\mathrm{fin}}(v)$.

$\mathrm{d}\Gamma(a)$ operators

If $\mathfrak{h}_1, \mathfrak{h}_2$ are Hilbert spaces and b is a closeable densely defined operator from $\mathfrak{h}_1$ to $\mathfrak{h}_2$, one first defines the linear operator $\mathrm{d}\Gamma_{\mathrm{fin}}(b)$ with domain $\Gamma_{\mathrm{fin}}(\mathcal{D}(b))$ by:

$$\begin{aligned} &\mathrm{d}\Gamma_{\mathrm{fin}}(b) : \Gamma_{\mathrm{fin}}(\mathcal{D}(b)) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}_2), \\ &\mathrm{d}\Gamma_{\mathrm{fin}}(b)\Big|_{\bigotimes_{\mathrm{s}}^n \mathcal{D}(b)} := \sum_{j=1}^{n} \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{j-1} \otimes b \otimes \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{n-j}. \end{aligned}$$

Since b is closeable, b^* is densely defined. Moreover, it is easy to see that $\mathrm{d}\Gamma_{\mathrm{fin}}(b^*) \subset \mathrm{d}\Gamma_{\mathrm{fin}}(b)^*$ which implies that $\mathrm{d}\Gamma_{\mathrm{fin}}(b)$ is closeable and we will denote by $\mathrm{d}\Gamma(b)$ its closure.

For later use we extend the meaning of the operation $\mathrm{d}\Gamma$ as follows. Let $S \in \mathcal{B}(\mathcal{L}_1 \otimes \mathfrak{h}, \mathcal{L}_2 \otimes \mathfrak{h})$ (unbounded operators can be considered as well). For each $n \in \mathbb{N}$ define $\mathrm{d}\Gamma_n(S) \in \mathcal{B}(\mathcal{L}_1 \otimes \Gamma_n(\mathfrak{h}), \mathcal{L}_2 \otimes \Gamma_n(\mathfrak{h}))$ by

$$\mathrm{d}\Gamma_n(S) = \sum_{i=1}^{n} 1\!\!1_{\mathcal{K}} \otimes \tau_i^{(n)*} \circ S \otimes 1\!\!1_{\Gamma_{n-1}(\mathfrak{h})} \circ 1\!\!1_{\mathcal{K}} \otimes \tau_i^{(n)}, \tag{3.3}$$

where $\tau_i^{(n)}$ is the unitary operator on $\otimes^n \mathfrak{h}$ determined by the condition:

$$\tau_i^{(n)} h_1 \otimes \cdots \otimes h_n = h_i \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_n.$$

Then we set

$$\mathrm{d}\Gamma(S) := \oplus_{n=0}^{\infty} \mathrm{d}\Gamma_n(S).$$

This is a closed densely defined operator from $\mathcal{L}_1 \otimes \Gamma(\mathfrak{h})$ into $\mathcal{L}_2 \otimes \Gamma(\mathfrak{h})$. For example, if $S = S^\circ \otimes T$ with $S^\circ \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ and $T \in \mathcal{B}(\mathfrak{h})$, then $\mathrm{d}\Gamma(S) = S^\circ \otimes \mathrm{d}\Gamma(T)$.

$\Gamma(q)$ operators

If $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ is a bounded linear operator, one defines

$$\begin{aligned} &\Gamma_{\rm fin}(q) : \Gamma_{\rm fin}(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2) \\ &\Gamma_{\rm fin}(q)\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}_1} := q \otimes \cdots \otimes q. \end{aligned}$$

Again using that $\Gamma(q^*) \subset \Gamma(q)^*$, we see that $\Gamma_{\rm fin}(q)$ is closeable, and we denote by $\Gamma(q)$ its closure. Note that $\Gamma(q)$ is bounded iff $\|q\| \leq 1$.

Lemma 3.1 *Let* $\mathbb{R}^+ \ni t \mapsto w_t \in \mathcal{B}(\mathfrak{h})$ *be a* C_0*-semigroup of contractions, with generator* a*. Then* $\mathbb{R}^+ \ni t \mapsto \Gamma(w_t) \in \mathcal{B}(\Gamma(\mathfrak{h}))$ *is a* C_0*-semigroup of contractions whose generator is* ${\rm d}\Gamma(a)$*.*

Proof. We first recall the following standard fact on C_0-semigroups, which is a generalization of an essential selfadjointness criterion due to Nelson:

Let $\{W_t\}$ be a C_0-semigroup on a Banach space F, and let F_1 be the domain of the generator of $\{W_t\}$. Then if $E \subset F_1$ is a vector space invariant under $\{W_t\}$, E is dense in F_1 if E is dense in F.

In fact let $R_s = s^{-1}\int_0^s W_t {\rm d}t$. Then $R_s \in \mathcal{B}(F, F_1)$ and ${\rm s}\text{-}\lim_{s\to 0} R_s = \mathbb{1}$ in F and F_1. Let $\overline{E}$ be the closure of E in F_1. Since E is dense in F, and $R_s \in \mathcal{B}(F, F_1)$, we obtain that $R_s F \subset \overline{E}$. Then the statement follows from the fact that ${\rm s}\text{-}\lim_{s\to 0} R_s = \mathbb{1}$ in F_1.

Let us now prove the lemma. Clearly $\{W_t\} = \{\Gamma(w_t)\}$ is a C_0-semigroup of isometries, and ${\rm d}\Gamma_{\rm fin}(a) \subset A$, if A is the generator of $\{W_t\}$.

To show that $A = {\rm d}\Gamma(a)$, we apply the above result to $F = \Gamma(\mathfrak{h})$, $\{W_t\} = \Gamma(w_t)$, and $E = \Gamma_{\rm fin}(\mathcal{D}(a))$, which is dense in $\mathcal{H}$ and invariant under $\{W_t\}$, since $\mathcal{D}(a)$ is invariant under $\{w_t\}$. □

${\rm d}\Gamma(q, r)$ operators

If $q \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$ with $\|q\| \leq 1$, r is a closeable densely defined operator from $\mathfrak{h}_1$ to $\mathfrak{h}_2$ one defines

$$\begin{aligned} &{\rm d}\Gamma_{\rm fin}(q, r) : \Gamma_{\rm fin}(\mathcal{D}(r)) \to \Gamma(\mathfrak{h}_2), \\ &{\rm d}\Gamma_{\rm fin}(q, r)\Big|_{\bigotimes_{\rm s}^n \mathcal{D}(b)} := \sum_{j=1}^{n} \underbrace{q \otimes \cdots \otimes q}_{j-1} \otimes r \otimes \underbrace{q \otimes \cdots \otimes q}_{n-j}. \end{aligned}$$

Again using that ${\rm d}\Gamma_{\rm fin}(q^*, r^*) \subset {\rm d}\Gamma_{\rm fin}(q, r)^*$, we see that ${\rm d}\Gamma_{\rm fin}(q, r)$ is closeable and we denote by ${\rm d}\Gamma(q, r)$ its closure. We note the following identity:

$$[{\rm d}\Gamma(b), {\rm i}\Gamma(q)] = {\rm d}\Gamma(q, [b, {\rm i}q]). \tag{3.4}$$

We note the following lemma, which is an extension of [DG1, Lemma 2.8] and is proved similarly. Note that we use the convention explained above for quadratic forms and the right hand side of (3.5) can take the value $+\infty$.

Lemma 3.2 *Assume that* $\|q\| \leq 1$ *and that there exist closed densely defined operators* r_i *on* $\mathfrak{h}_i$ *such that* $|(h_2, r h_1)| \leq \|r_1 h_1\| \|r_2 h_2\|$ *for* $h_i \in \mathcal{D}(r_i)$*. Then:*

$$|(u_2, {\rm d}\Gamma(q, r) u_1)| \leq \|{\rm d}\Gamma(r_1^* r_1)^{\frac{1}{2}} u_1\| \|{\rm d}\Gamma(r_2^* r_2)^{\frac{1}{2}} u_2\|, \ u_i \in \Gamma(\mathfrak{h}_i). \tag{3.5}$$

Canonical map

Let $\mathfrak{h}_i$, $i = 1, 2$ be Hilbert spaces. Let p_i be the projection of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ onto $\mathfrak{h}_i$, $i = 1, 2$. We define

$$U : \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2),$$

by

$$\begin{aligned} &U\Omega = \Omega \otimes \Omega, \\ &Ua^{(*)}(h) = \left(a^{(*)}(p_1 h) \otimes 1\!\!1_{\Gamma(\mathfrak{h}_2)} + 1\!\!1_{\Gamma(\mathfrak{h}_1)} \otimes a^{(*)}(p_2 h)\right) U,\ h \in \mathfrak{h}_1 \oplus \mathfrak{h}_2. \end{aligned} \tag{3.6}$$

Since the vectors $a^*(h_1) \cdots a^*(h_n)\Omega$ form a total family in $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$, and since U preserves the canonical commutation relations, U extends as a unitary operator from $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ to $\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$.

Operators $\check{\Gamma}(j)$ and $\mathrm{d}\check{\Gamma}(j, k)$

Let $j_0, j_\infty \in \mathcal{B}(\mathfrak{h})$. Set $j = (j_0, j_\infty)$. We identify j with the operator

$$\begin{aligned} &j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}, \\ &jh := (j_0 h, j_\infty h). \end{aligned}$$

We have

$$\begin{aligned} &j^* : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h}, \\ &j^*(h_0, h_\infty) = j_0^* h_0 + j_\infty^* h_\infty, \end{aligned}$$

and

$$j^* j = j_0^* j_0 + j_\infty^* j_\infty.$$

By second quantization, we obtain the map

$$\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \oplus \mathfrak{h}).$$

Let U denote the canonical map between $\Gamma(\mathfrak{h} \oplus \mathfrak{h})$ and $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ introduced above . We define

$$\begin{aligned} &\check{\Gamma}(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}), \\ &\check{\Gamma}(j) := U\Gamma(j). \end{aligned}$$

Another formula defining $\check{\Gamma}(j)$ is

$$\check{\Gamma}(j)\Pi_{i=1}^n a^*(h_i)\Omega := \Pi_{i=1}^n \left(a^*(j_0 h_i) \otimes 1\!\!1 + 1\!\!1 \otimes a^*(j_\infty h_i)\right) \Omega \otimes \Omega,\ h_i \in \mathfrak{h}. \tag{3.7}$$

Let $N_0 = N \otimes 1\!\!1$, $N_\infty = 1\!\!1 \otimes N$ acting on $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$. Then if we denote by I_k the natural isometry between $\bigotimes^n \mathfrak{h}$ and $\bigotimes^{n-k} \mathfrak{h} \otimes \bigotimes^k \mathfrak{h}$, we have:

$$1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j)\Big|_{\Gamma_n(\mathfrak{h})} = I_k \sqrt{\tfrac{n!}{(n-k)!k!}}\, \underbrace{j_0 \otimes \cdots \otimes j_0}_{n-k} \otimes \underbrace{j_\infty \otimes \cdots \otimes j_\infty}_{k}.$$

Finally we set

$$\check{\Gamma}_k(j) := 1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j).$$

Let $j = (j_0, j_\infty)$, $k = (k_0, k_\infty)$ be bounded operators from $\mathfrak{h}$ to $\mathfrak{h} \oplus \mathfrak{h}$. We set

$$\begin{aligned} &\mathrm{d}\check{\Gamma}(j, k) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}), \\ &\mathrm{d}\check{\Gamma}(j, k) := U \mathrm{d}\Gamma(j, k). \end{aligned}$$

The operator $\mathrm{d}\check{\Gamma}(1, k) = U\mathrm{d}\Gamma(k)$ will be denoted simply by $\mathrm{d}\check{\Gamma}(k)$.

3.3 Bounds on second quantized operators

In this subsection we prove some bounds allowing to dominate $\mathrm{d}\Gamma(a)$ by $\mathrm{d}\Gamma(b)$ for a, b two linear operators on $\mathfrak{h}$. We start with an easy estimate whose proof is left to the reader.

Lemma 3.3 *Let $\mathcal{L}$ be a Hilbert space and let $a, b \in \mathcal{B}(\mathcal{L} \otimes \mathfrak{h}, \mathcal{L}^* \otimes \mathfrak{h})$ be selfadjoint operators. Then $\mathrm{d}\Gamma(a)$ and $\mathrm{d}\Gamma(b)$ are self-adjoint operators from $\mathcal{L} \otimes \Gamma(\mathfrak{h})$ into $\mathcal{L}^* \otimes \Gamma(\mathfrak{h})$ and*

$$0 \leq a \leq b \Rightarrow 0 \leq \mathrm{d}\Gamma(a) \leq \mathrm{d}\Gamma(b). \tag{3.8}$$

Proposition 3.4 *i) Let a be a closed, symmetric, densely defined operator on $\mathfrak{h}$. Then:*

$$\mathrm{d}\Gamma(a)^* \mathrm{d}\Gamma(a) \leq \mathrm{d}\Gamma(|a|)^2.$$

ii) Let a, b be two selfadjoint operators on $\mathfrak{h}$ with $b \geq 0$ and $a^2 \leq b^2$. Then:

$$\mathrm{d}\Gamma(a)^2 \leq \mathrm{d}\Gamma(b)^2.$$

To prove Proposition 3.4 we will use the following lemma.

Lemma 3.5 *i) Let a be a closed densely defined operator on $\mathfrak{h}$. Then:*

$$a^* \otimes a + a \otimes a^* \leq |a^*| \otimes |a| + |a| \otimes |a^*|.$$

If a is symmetric, we also have:

$$a^* \otimes a + a \otimes a^* \leq 2|a| \otimes |a|.$$

ii) Let a, b be two selfadjoint operators on $\mathfrak{h}$ with $a^2 \leq b^2$ and $b \geq 0$. Then:

$$a \otimes a \leq b \otimes b.$$

Proof. We recall the following well-known facts on the polar decomposition of a (see [Ka, Chap. VI.7]):

$$\mathcal{D}(a) = \mathcal{D}(|a|) = \{u | r^* u \in \mathcal{D}(|a^*|)\},\ a = r|a| = |a^*| r, \tag{3.9}$$

where $|a| = (a^* a)^{\frac{1}{2}}$, $|a^*| = (a a^*)^{\frac{1}{2}}$ and r is a partial isometry from $\mathrm{Im}|a|$ into $\mathrm{Im} a$. For $\epsilon > 0$ we have

$$\|a(\epsilon + |a|)^{-1}\| = \|r|a|(\epsilon + |a|)^{-1}\| \leq 1,\ \|(\epsilon + |a^*|)^{-1} a\| = \|(\epsilon + |a^*|)^{-1} |a^*| r\| \leq 1.$$

By complex interpolation we obtain that $\|(\epsilon + |a^*|)^{-\frac{1}{2}} a (\epsilon + |a|)^{-\frac{1}{2}}\| \leq 1$ and taking adjoints that $\|(\epsilon + |a|)^{-\frac{1}{2}} a^* (\epsilon + |a^*|)^{-\frac{1}{2}}\| \leq 1$. Write:

$$\begin{array}{rl} a^* \otimes a = & (\epsilon + |a|)^{\frac{1}{2}} \otimes (\epsilon + |a^*|)^{\frac{1}{2}} \\ \times & \Big((\epsilon + |a|)^{-\frac{1}{2}} a^* (\epsilon + |a^*|^{-\frac{1}{2}}) \otimes (\epsilon + |a^*|)^{-\frac{1}{2}} a (\epsilon + |a|)^{-\frac{1}{2}}\Big) \times (\epsilon + |a^*|)^{\frac{1}{2}} \otimes (\epsilon + |a|)^{\frac{1}{2}}. \end{array}$$

This yields:

$$\begin{array}{rl} 2|\mathrm{Re}(u, a^* \otimes a u)| \leq & 2\|(\epsilon + |a|)^{\frac{1}{2}} \otimes (\epsilon + |a^*|)^{\frac{1}{2}} u\| \|(\epsilon + |a^*|)^{\frac{1}{2}} \otimes (\epsilon + |a|)^{\frac{1}{2}} u\| \\ \leq & (u, \Big((\epsilon + |a^*|) \otimes (\epsilon + |a|) + (\epsilon + |a|) \otimes (\epsilon + |a^*|\Big) u). \end{array}$$

Letting $\epsilon \to 0$ we obtain *i)*. If a is symmetric then $aa^* \leq a^*a$ and hence $|a^*| \leq |a|$, since the function $\lambda \to \lambda^{\frac{1}{2}}$ is matrix monotone. Next

$$\begin{array}{rl} a^* \otimes a + a \otimes a^* \leq & |a^*| \otimes |a| + |a| \otimes |a^*| \\ = & \mathbb{1} \otimes |a|^{\frac{1}{2}} \times |a^*| \otimes \mathbb{1} \times \mathbb{1} \otimes |a|^{\frac{1}{2}} + |a|^{\frac{1}{2}} \otimes \mathbb{1} \times \mathbb{1} \otimes |a^*| \times |a|^{\frac{1}{2}} \otimes \mathbb{1} \\ \leq & 2|a| \otimes |a|. \end{array}$$

To prove *ii)*, we note that $|a|^s \leq b^s$ for $0 \leq s \leq 2$ since $a^2 \leq b^2$ and $b \geq 0$. Then, using i) in the first step:

$$\begin{array}{rl} a \otimes a \leq & |a| \otimes |a| = |a|^{\frac{1}{2}} \otimes \mathbb{1} \times \mathbb{1} \otimes |a| \times |a|^{\frac{1}{2}} \otimes \mathbb{1} \\ \leq & |a|^{\frac{1}{2}} \otimes \mathbb{1} \times \mathbb{1} \otimes b \times |a|^{\frac{1}{2}} \otimes \mathbb{1} = \mathbb{1} \otimes b^{\frac{1}{2}} \times |a| \otimes \mathbb{1} \times \mathbb{1} \otimes b^{\frac{1}{2}} \\ \leq & \mathbb{1} \otimes b^{\frac{1}{2}} \times b \otimes \mathbb{1} \times \mathbb{1} \otimes b^{\frac{1}{2}} = b \otimes b. \ \Box \end{array}$$

Proof of Proposition 3.4. We first prove *i)*. Using the fact that the closure of the operator $\mathrm{d}\Gamma_{\mathrm{fin}}(a)$ is $\mathrm{d}\Gamma(a)$ it suffices to prove the inequality as forms on $\Gamma_{\mathrm{fin}}(\mathcal{D}(a)) = \Gamma_{\mathrm{fin}}(\mathcal{D}(|a|))$. Let

$$a_i = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes a \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{n-j},$$

acting on $\otimes_{\mathrm{s}}^n \mathfrak{h}$. Then it suffices to prove

$$(a_1^* + \ldots + a_n^*)(a_1 + \ldots + a_n) \leq (|a|_1 + \ldots + |a|_n)^2$$

as forms on $\otimes_s^n \mathfrak{h}$. But

$$(a_1^* + \ldots + a_n^*)(a_1 + \ldots + a_n) = \sum_i a_i^* a_i + \sum_{i<j} 2\mathrm{Re} a_i^* a_i = \sum_i |a_i|^2 + \sum_{i<j} 2\mathrm{Re} a_i^* a_i.$$

We have $|a_i| = |a|_i$ and $2\mathrm{Re} a_i^* a_i \leq 2|a|_i |a|_j$ by Lemma 3.5 *i)*. This completes the proof of *i)*. The proof of *ii)* is similar, using Lemma 3.5 *ii)*. $\Box$

3.4 Bounds on creation/annihilation operators

This subsection is devoted to some bounds on creation/annihilation operators with operator valued symbols.

We fix an auxiliary Hilbert space $\mathcal{L}$ and consider $u, v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^* \otimes \mathfrak{h})$. Then $a(v)$ is a map from $\mathcal{L} \otimes \Gamma(\mathfrak{h})$ to $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ and $a^*(u)$ is a map from $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ to $\mathcal{L}^* \otimes \Gamma(\mathfrak{h})$, so that they can be composed. On the other hand $v^* \in \mathcal{B}(\mathcal{L} \otimes \mathfrak{h}, \mathcal{K})$ so that $uv^* \in \mathcal{B}(\mathcal{L} \otimes \mathfrak{h}, \mathcal{L}^* \otimes \mathfrak{h})$. A straightforward computation involving (3.1)–(3.3) gives

$$a^*(u)a(v)f = \mathrm{d}\Gamma(uv^*)f \text{ for all } f \in \mathcal{L} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}). \tag{3.10}$$

One can simplify the computations by using the following preliminary argument. For fixed f both members of (3.10) are bilinear strongly continuous functions of (u, v^*). A simple approximation procedure shows that it suffices to prove (3.10) in the particular case $u = q' \otimes h', v = q'' \otimes h''$ for some $q', q'' \in \mathcal{B}(\mathcal{K}, \mathcal{L}^*)$ and $h', h'' \in \mathfrak{h}$, which is easy.

We would like to have a similar identity for the operator $a(v)a^*(u)$. First we note that this time we need $u, v \in \mathcal{B}(\mathcal{L}, \mathcal{K} \otimes \mathfrak{h})$ in order to be able to define $a(v)a^*(u)$ as an operator from

$\mathcal{L}\otimes\Gamma(\mathfrak{h})$ into $\mathcal{L}^*\otimes\Gamma(\mathfrak{h})$. Observe then that $v^*\in\mathcal{B}(\mathcal{K}\otimes\mathfrak{h},\mathcal{L}^*)$ so that v^*u is a well defined element of $\mathcal{B}(\mathcal{L},\mathcal{L}^*)$. Moreover, we have two linear continuous maps $u\otimes\mathbb{1}_{\mathfrak{h}}:\mathcal{L}\otimes\mathfrak{h}\to\mathcal{K}\otimes\mathfrak{h}\otimes\mathfrak{h}$ and $v^*\otimes\mathbb{1}_{\mathfrak{h}}:\mathcal{K}\otimes\mathfrak{h}\otimes\mathfrak{h}\to\mathcal{L}^*\otimes\mathfrak{h}$. Thus we can define a new operator $v\tilde{\otimes}u\in\mathcal{B}(\mathcal{L}\otimes\mathfrak{h},\mathcal{L}^*\otimes\mathfrak{h})$ by the following relation

$$v\tilde{\otimes}u := v^*\otimes\mathbb{1}_{\mathfrak{h}}\circ\mathbb{1}_{\mathcal{K}}\otimes\sigma\circ u\otimes\mathbb{1}_{\mathfrak{h}}, \tag{3.11}$$

where σ is the unitary operator in $\mathfrak{h}\otimes\mathfrak{h}$ defined by the condition $\sigma[h\otimes g]=g\otimes h$. Clearly

$$\|v\tilde{\otimes}u\|\leq\|u\|\|v\|.$$

Note that $v\tilde{\otimes}u$ is uniquely characterized by the relation $v\tilde{\otimes}u[\psi\otimes h]=(v^*(h)\otimes\mathbb{1}_{\mathfrak{h}})[u(\psi)]$ for all $\psi\in\mathcal{L}$ and $h\in\mathfrak{h}$. Here $v^*(h)\in\mathcal{B}(\mathcal{K},\mathcal{L}^*)$ is given by $\psi\mapsto v^*(\psi\otimes h)$. Finally a new straightforward computation involving (3.1)–(3.3) gives

$$a(v)a^*(u)=v^*u\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}+\mathrm{d}\Gamma(v\tilde{\otimes}u)\text{ on }\mathcal{L}\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}). \tag{3.12}$$

The computation can be simplified by the same argument as in the case of (3.10). Thus it suffices to consider the particular case $u=q'\otimes h', v=q''\otimes h''$ for some $q',q''\in\mathcal{B}(\mathcal{L},\mathcal{K})$ and $h',h''\in\mathfrak{h}$. Then $v\tilde{\otimes}u=(q''^*q')\otimes(h'h''^*)$, where $h'h''^*:h\mapsto h'(h'',h)$, and the proof is trivial.

Lemma 3.6 *Let $u,v\in\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$. If $S_1,S_2\in\mathcal{B}(\mathcal{K},\mathcal{L})$ and $T_1,T_2\in\mathcal{B}(\mathfrak{h})$ then*

$$(S_1^*\otimes T_1)\circ v\tilde{\otimes}u\circ(S_2\otimes T_2^*)=[(\mathbb{1}_{\mathcal{K}}\otimes T_2)vS_1]^\dagger\,[(\mathbb{1}_{\mathcal{K}}\otimes T_1)uS_2]\,. \tag{3.13}$$

Proof. This follows from $\mathbb{1}_{\mathfrak{h}}\otimes T_1\circ\sigma\circ\mathbb{1}_{\mathfrak{h}}\otimes T_2^*=T_2^*\otimes\mathbb{1}_{\mathfrak{h}}\circ\sigma\circ T_1\otimes\mathbb{1}_{\mathfrak{h}}$. □

We shall use the preceding formalism in order to prove some estimates involving the creation and annihilation operators. The inequalities (3.15) and (3.18) below are proved in [DJ, Proposition 4.1] in the case $K=0$. The general case is treated in [G1, Appendix A] with conditions on v slightly stronger than here. In particular, our constants are better (see the comment at the end of this subsection).

Assume that $\mathcal{L}\subset\mathcal{K}$ continuously and densely and $v\in\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$. We defined $a^*(v)$ as a closed operator with dense domain $\mathcal{D}(a^*(v))\subset\mathcal{L}\otimes\Gamma(\mathfrak{h})$ and with values in $\mathcal{K}\otimes\Gamma(\mathfrak{h})$. But now we have $\mathcal{L}\otimes\Gamma(\mathfrak{h})\subset\mathcal{K}\otimes\Gamma(\mathfrak{h})$ continuously and densely, hence $a^*(v)$ can also be viewed as a densely defined operator acting in $\mathcal{H}=\mathcal{K}\otimes\mathfrak{h}$. If this operator is closeable we denote its closure by the same symbol $a^*(v)$. Similarly for $a(v)$ if $v\in\mathcal{B}(\mathcal{K},\mathcal{L}^*\otimes\mathfrak{h})$. We stress that the right hand side in the inequalities (3.15) and (3.18) is allowed to have the value $+\infty$.

Proposition 3.7 *Let K and ω be positive self-adjoint operators on $\mathcal{K}$ and $\mathfrak{h}$ respectively.*
i) Let $v\in\mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}),\mathcal{K}\otimes\mathfrak{h})$. For $r>0$ let

$$C_1(r,v)=\|(\mathbb{1}_{\mathcal{K}}\otimes\omega^{-\frac{1}{2}})v(K+r)^{-\frac{1}{2}}\|^2:=\lim_{\varepsilon\downarrow 0}\|(\mathbb{1}_{\mathcal{K}}\otimes(\omega+\varepsilon)^{-\frac{1}{2}})v(K+r)^{-\frac{1}{2}}\|^2. \tag{3.14}$$

Then for all $f\in\mathcal{D}(a^*(v))$ one has:

$$\|a^*(v)f\|^2\leq(f,v^*v\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}f)+C_1(r,v)(f,(K+r)\otimes\mathrm{d}\Gamma(\omega)f). \tag{3.15}$$

Moreover, if we set $C_0(r,v)=\|v(K+r)^{-\frac{1}{2}}\|^2$, then

$$\|a^*(v)f\|^2\leq C_0(r,v)(f,(K+r)\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}f)+C_1(r,v)(f,(K+r)\otimes\mathrm{d}\Gamma(\omega)f). \tag{3.16}$$

ii) Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathfrak{h})$ *and for* $r > 0$ *let:*

$$C_2(r,v) = \|((K+r)^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}})v\|^2 := \lim_{\varepsilon \downarrow 0} \|((K+r)^{-\frac{1}{2}} \otimes (\omega+\varepsilon)^{-\frac{1}{2}})v\|^2. \tag{3.17}$$

Then for all $f \in \mathcal{D}(a(v))$ *one has:*

$$\|a(v)f\|^2 \le C_2(r,v)(f, (K+r) \otimes \mathrm{d}\Gamma(\omega) f). \tag{3.18}$$

Proof. We will set $\mathcal{L} = \mathcal{D}(K^{\frac{1}{2}})$. Let us first prove (3.15). It suffices to prove (3.15) for $f \in \mathcal{L} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$. Indeed, the projection f_N of any $f \in \mathcal{D}(a^*(v))$ onto $\oplus_{n=0}^N \mathcal{K} \otimes \Gamma_n(\mathfrak{h})$ belongs again to $\mathcal{D}(a^*(v))$, one has $f_N \to f$ in the graph topology of $\mathcal{D}(a^*(v))$, and the right hand side of (3.15) with f replaced by f_N is an increasing function of N; moreover, one can regularize, if needed, f_N with the help of K to get an element of $\mathcal{L} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$. We shall further simplify the problem, although this is not strictly necessary. First, it suffices to prove (3.15) (f being fixed in $\mathcal{L} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$) with ω replaced by $\omega + \varepsilon$; we let $\varepsilon \to 0$ at the end of the proof. Then, we can replace ω by $\inf(\omega, M)$ with $M > 0$ real and let $M \to \infty$ at the end of the proof. We thus see that it suffices to assume that ω is a bounded self-adjoint operator with $\omega \ge c > 0$. Finally, to simplify notations, we can include r in K. Thus it suffices to prove

$$a(v)a^*(v) \le v^*v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + \|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})vK^{-\frac{1}{2}}\|^2 K \otimes \mathrm{d}\Gamma(\omega) \tag{3.19}$$

as forms on $\mathcal{L} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$. The identity (3.12) gives $a(v)a^*(v) = v^*v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + \mathrm{d}\Gamma(v\tilde{\otimes}v)$. Then, by using Lemma 3.6 with $u = v$, $S_1 = S_2 = K^{-\frac{1}{2}}$ and $T_1 = T_2 = \omega^{-\frac{1}{2}}$, we get

$$\begin{aligned} v\tilde{\otimes}v = {} & K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}}[K^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}} v\tilde{\otimes}v K^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}}]K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}} \\ = {} & K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}}[1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}} vK^{-\frac{1}{2}}]\tilde{\otimes}[1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}} vK^{-\frac{1}{2}}]K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}} \\ \le {} & \|1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}} vK^{-\frac{1}{2}}\|^2 K \otimes \omega. \end{aligned}$$

Now using (3.8) we get

$$\mathrm{d}\Gamma(v\tilde{\otimes}v) \le \|1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}} vK^{-\frac{1}{2}}\|^2 \mathrm{d}\Gamma(K \otimes \omega).$$

This is the last term in (3.19) because $\mathrm{d}\Gamma(K \otimes \omega) = K \otimes \mathrm{d}\Gamma(\omega)$ as maps $\mathcal{L} \otimes \Gamma(\mathfrak{h}) \to \mathcal{L}^* \otimes \Gamma(\mathfrak{h})$. Thus (3.15) is proved and (3.16) is an immediate consequence of the bound

$$v^*v = K^{\frac{1}{2}} K^{-\frac{1}{2}} v^*v K^{-\frac{1}{2}} K^{\frac{1}{2}} \le \|K^{-\frac{1}{2}} v^*v K^{-\frac{1}{2}}\| K = \|vK^{-\frac{1}{2}}\|^2 K.$$

To prove (3.18) we use (3.10), (3.8) and the fact that:

$$\begin{aligned} vv^* = {} & K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}} \Big(K^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}} vv^* K^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}} \Big) K^{\frac{1}{2}} \otimes \omega^{\frac{1}{2}} \\ \le {} & \|K^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}} v\|^2 K \otimes \omega. \ \Box \end{aligned}$$

Remark 3.8 $\omega^{-\frac{1}{2}}$ is naturally realized as a selfadjoint, not densely defined in general, operator in $\mathfrak{h}$, and so are the tensor products $1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}}$ and $(K+r)^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}}$ in $\mathcal{H}$. From (an abstract version of) Fatou Lemma it follows that the condition $C_1(r,v) < \infty$ is equivalent to $v\mathcal{D}(K^{\frac{1}{2}}) \subset \mathcal{D}(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})$ while $C_2(r,v) < \infty$ means $v\mathcal{K} \subset \mathcal{D}(1\!\!1_{\mathcal{D}(K^{\frac{1}{2}})^*} \otimes \omega^{-\frac{1}{2}})$.

The next proposition is also a slightly improved version of a result from [G1, Appendix A].

Proposition 3.9 *Let $\mathcal{L}$ be a Hilbert space such that $\mathcal{L} \subset \mathcal{K}$ continuously and densely and let $v \in \mathcal{B}(\mathcal{L}, \mathcal{K} \otimes \mathfrak{h})$. If $\omega \geq 0$ is a self-adjoint operator on $\mathfrak{h}$ and $f \in \mathcal{D}(a^*(v))$ then:*

$$|(f, a^*(v)f)| \leq \|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\| \|1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega)^{\frac{1}{2}} f\| \tag{3.20}$$

where

$$\|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\| := \lim_{\varepsilon \downarrow 0} \|(1\!\!1_{\mathcal{K}} \otimes (\omega + \varepsilon)^{-\frac{1}{2}})v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\|$$

and the value $+\infty$ is allowed.

Proof. It is easily seen, as in the proof of Proposition 3.7, that it suffices to assume that $f \in \mathcal{L} \otimes \Gamma_{\text{fin}}(\mathfrak{h})$ and that ω is a bounded self-adjoint operator with $\omega \geq c > 0$. A further simplification of the problem is obtained as follows. Let $0 < a < b < \infty$ such that the spectrum of ω is included in the interval $]a, b]$ and let E be the spectral measure of ω, so that $\omega = \int_{]a,b]} \lambda E(\mathrm{d}\lambda)$. Set $I_n^k =]a + (k-1)(b-a)/n, a + k(b-a)/n]$ for $1 \leq k \leq n \in \mathbb{N}$ and let $\omega_n = \sum_{k=1}^n (a + k(b-a)/n) E(I_n^k)$. Then ω_n is a self-adjoint operator with finite spectrum and $\|\omega_n - \omega\| \leq 1/n$. Hence if (3.20) holds with ω replaced by ω_n, then after letting $n \to \infty$ we get it for ω. Thus, it suffices to prove (3.20) for operators ω having the following property: there is an orthonormal basis $\{e_i\}$ of $\mathfrak{h}$ and there is a family $\{\lambda_i\}$ of strictly positive numbers which takes only a finite number of distinct values, such that $\omega e_i = \lambda_i e_i$ for all i.

It is easy to see that for each i there is a unique operator $v_i \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ such that $v(h) = \sum_i v_i(h) \otimes e_i$ for $h \in \mathcal{L}$. Then $a^*(v) = \mathrm{s} - \sum_i v_i \otimes a^*(e_i)$ as operators with domain $\mathcal{L} \otimes \Gamma_{\text{fin}}(\mathfrak{h})$ and

$$(f, a^*(v)f) = \sum_i (f, v_i \otimes a^*(e_i)f) = \sum_i (\lambda_i^{\frac{1}{2}} 1\!\!1_{\mathcal{K}} \otimes a(e_i)f, \lambda_i^{-\frac{1}{2}} v_i \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f).$$

Hence by the Cauchy-Schwarz inequality we get:

$$|(f, a^*(v)f)|^2 \leq \sum_i \|\lambda_i^{\frac{1}{2}} 1\!\!1_{\mathcal{K}} \otimes a(e_i)f\|^2 \sum_i \|\lambda_i^{-\frac{1}{2}} v_i \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\|^2.$$

The first factor on the right hand side is equal to

$$\sum_i (f, 1\!\!1_{\mathcal{K}} \otimes \lambda_i a^*(e_i) a(e_i) f) = (f, 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) f).$$

The second factor can be written as

$$\sum_i \|(v_i \otimes 1\!\!1_{\Gamma(\mathfrak{h})}) f \otimes \omega^{-\frac{1}{2}} e_i\|^2 = \|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})v \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\|^2.$$

This finishes the proof of the proposition. □

Corollary 3.10 *Let K and ω be positive self-adjoint operators on $\mathcal{K}$ and $\mathfrak{h}$ respectively and let $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$. For $r > 0$ let $C_1(r, v)$ be defined by (3.14). Then for all $f \in \mathcal{D}(a^*(v))$ one has*

$$|(f, a^*(v)f)| \leq C_1(r, v) \|(K + r)^{\frac{1}{2}} \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f\| \|1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega)^{\frac{1}{2}} f\|. \tag{3.21}$$

3.5 Additional remarks on the spaces $\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$ and $\mathcal{B}(\mathcal{K},\mathcal{L}^*\otimes\mathfrak{h})$

We first give an alternative description of the spaces $\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2\otimes\mathfrak{h})$ in the important particular case where $\mathfrak{h}=L^2(\mathbb{R}^d,\mathrm{d}k)$ and $\mathcal{L}_1,\mathcal{L}_2$ are separable Hilbert spaces. Let $L^2_w(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))$ be the space of (equivalence classes of) strongly measurable maps $v(\cdot):\mathbb{R}^d\to\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)$ such that the map $k\mapsto\|v(k)\psi\|^2$ is integrable for all $\psi\in\mathcal{L}_1$, and let us equip it with the norm

$$\|v(\cdot)\|=\left[\sup_{\psi\in\mathcal{L}_1\|\psi\|=1}\int_{\mathbb{R}^d}\|v(k)\psi\|^2\mathrm{d}k\right]^{\frac{1}{2}}. \tag{3.22}$$

We get a Banach space such that the natural map $L^2_w(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))\to\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2\otimes L^2(\mathbb{R}^d))$ is bijective and isometric. Note that $\mathcal{L}_2\otimes L^2(\mathbb{R}^d)=L^2(\mathbb{R}^d;\mathcal{L}_2)$. Observe that the subspace $L^2(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))$ defined by the condition $\int_{\mathbb{R}^d}\|v(k)\|^2\mathrm{d}k<\infty$ is a strict subspace of $L^2_w(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))$ if $\mathcal{L}_1$ and $\mathcal{L}_2$ are infinite dimensional. For example, $L^2_w(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))$ is stable by Fourier transformation, but $L^2(\mathbb{R}^d;\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2))$ is not. Also, if $\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}$ is infinite dimensional and if $v(\cdot)$ satisfies (3.22), the function $k\mapsto v(k)^*$ does not satisfy it in general. We shall further discuss this question below in a context of interest for us.

We now discuss certain peculiarities of the space $\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$ when $\mathcal{K}$, $\mathcal{L}$ and $\mathfrak{h}$ are infinite dimensional Hilbert spaces. We will assume that $\mathfrak{h}$ is equipped with an isometric conjugation $h\mapsto\overline{h}$. This allows us to use the canonical identification of $\mathcal{K}\otimes\mathfrak{h}$ with the space $\mathcal{B}_2(\mathfrak{h},\mathcal{K})$ of Hilbert-Schmidt operators $\mathfrak{h}\to\mathcal{K}$, obtained by identifying $\psi\otimes h$ with the map $f\mapsto\psi(\overline{h},f)\psi$. Thus

$$\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})\equiv\mathcal{B}(\mathcal{L},\mathcal{B}_2(\mathfrak{h},\mathcal{K}))\subset\mathcal{B}(\mathcal{L},\mathcal{B}_\infty(\mathfrak{h},\mathcal{K}))\subset\mathcal{B}(\mathcal{L},\mathcal{B}(\mathfrak{h},\mathcal{K})) \tag{3.23}$$

where $\mathcal{B}_\infty(\mathfrak{h},\mathcal{K})$ is the space of compact operators $\mathfrak{h}\to\mathcal{K}$. Thus if $v\in\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$ then for each $\psi\in\mathcal{L}$ we have a linear map $v(\psi):\mathfrak{h}\to\mathcal{K}$ and this map is Hilbert-Schmidt.

In Subsection 4.2, we will need to consider the operator $v^\dagger\in\mathcal{B}(\mathcal{K},\mathcal{B}(\mathfrak{h},\mathcal{L}^*))$ defined by

$$v^\dagger(\psi)(h):=v^*(\psi\otimes h),\ \psi\in\mathcal{K},\ h\in\mathfrak{h}.$$

Note that since $v^*\in\mathcal{B}(\mathcal{K}\otimes\mathfrak{h},\mathcal{L}^*)$, $v^\dagger$ belongs indeed to $\mathcal{B}(\mathcal{K},\mathcal{B}(\mathfrak{h},\mathcal{L}^*))$.

Assume now additionally that $\mathcal{L}\subset\mathcal{K}$ densely and that $v\in\mathcal{B}(\mathcal{K},\mathcal{L}^*\otimes\mathfrak{h})$. Then $v^*\in\mathcal{B}(\mathcal{L}\otimes\mathfrak{h},\mathcal{K})$ so the operator $v^\dagger$ belongs also to $\mathcal{B}(\mathcal{L},\mathcal{B}(\mathfrak{h},\mathcal{K}))$. Thus $v^\dagger$ belongs to the last space in (3.23) and in fact it does not, in general, belong to the other ones, as the following example shows. Choose $\varphi\in\mathcal{K}$ and $J\in\mathcal{B}(\mathcal{K},\mathfrak{h})$ and set $v(u)=\varphi\otimes J(u)$ for $u\in\mathcal{K}$. Then $v\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$ and a straightforward computation gives $v^\dagger(\psi)=(\varphi,\psi)J^*\in\mathcal{B}(\mathfrak{h},\mathcal{K})$ for $\psi\in\mathcal{K}$.

To summarize, if $v\in\mathcal{B}(\mathcal{K},\mathcal{L}^*\otimes\mathfrak{h})$ then we have a well defined element $v^\dagger\in\mathcal{B}(\mathcal{L},\mathcal{B}(\mathfrak{h},\mathcal{K}))$ and, according to (3.23), we can impose as further restrictions $v^\dagger\in\mathcal{B}(\mathcal{L},\mathcal{B}_\infty(\mathfrak{h},\mathcal{K}))$ or $v^\dagger\in\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$. The intermediate assumption $v^\dagger\in\mathcal{B}(\mathcal{L},\mathcal{B}_\infty(\mathfrak{h},\mathcal{K}))$ means that for each $\psi\in\mathcal{L}$ the map $h\mapsto v^*(\psi\otimes h)$ is a compact operator $\mathfrak{h}\to\mathcal{K}$, while the strongest condition $v^\dagger\in\mathcal{B}(\mathcal{L},\mathcal{K}\otimes\mathfrak{h})$ means that this is a Hilbert-Schmidt operator.

Let us now restate the main conditions on v from Proposition 3.7 in the case $\mathfrak{h}=L^2(\mathbb{R}^d,\mathrm{d}k)$ assuming that ω is the operator of multiplication by a positive measurable function $\omega(\cdot)$ on $\mathbb{R}^d$ and that $\mathcal{K}$ is separable. Then the operator v from part *i)* of the proposition is identified with a strongly measurable map $v(\cdot):\mathbb{R}^d\to\mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}),\mathcal{K})$ and

$$C_1(r,v)=\sup_{\psi\in\mathcal{L}_1\|\psi\|=1}\int_{\mathbb{R}^d}\|v(k)(K+r)^{-\frac{1}{2}}\psi\|^2\frac{\mathrm{d}k}{\omega(k)}.$$

The operator v from part *ii)* of the proposition is identified with a strongly measurable map $v(\cdot) : \mathbb{R}^d \to \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^*)$ and

$$C_2(r, v) = \sup_{\psi \in \mathcal{L}_1 \|\psi\|=1} \int_{\mathbb{R}^d} \|(K+r)^{-\frac{1}{2}} v(k)\psi\|^2 \frac{\mathrm{d}k}{\omega(k)}.$$

We now describe $v^\dagger$ in the case when $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ (equipped with the usual conjugation) and $\mathcal{K}, \mathcal{L}$ are separable. Assume $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^* \otimes \mathfrak{h})$ and let $v(\cdot) : \mathbb{R}^d \to \mathcal{B}(\mathcal{K}, \mathcal{L}^*)$ be the map defining it. Then $k \mapsto v(k)^* \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ is weakly measurable and hence strongly measurable since $\mathcal{L}, \mathcal{K}$ are separable, and we clearly have $v^*(\psi \otimes h) = \int v(k)^* \psi h(k) \mathrm{d}k$ for $\psi \in \mathcal{L}$ and $h \in \mathfrak{h}$ (the integral exists in the weak sense). Hence $v^\dagger(\psi) = v(\cdot)^*\psi$ but this function *does not* belong to $L^2(\mathbb{R}^d; \mathcal{K})$ in general, being only weakly of class L^2, i.e. we only have $\int |(v(k)^*\psi, u)|^2 \mathrm{d}k < \infty$ for each $u \in \mathcal{K}$. Thus we see that $v^\dagger \in \mathcal{B}(\mathcal{L}, \mathcal{K} \otimes \mathfrak{h})$ if and only if $v(\cdot)^*\psi \in L^2(\mathbb{R}^d; \mathcal{K})$ for all $\psi \in \mathcal{L}$, i.e. if and only if $v(\cdot)^* \in L^2_w(\mathbb{R}^d; \mathcal{B}(\mathcal{L}, \mathcal{K}))$.

4 Abstract Pauli-Fierz Hamiltonians

In this section we consider a class of Hamiltonians H called *Pauli-Fierz Hamiltonians* describing a quantum system interacting with a boson field. This class of Hamiltonians has been introduced and studied in various degrees of generality in [DG1, DJ, G1]. Pauli-Fierz Hamiltonians are defined in Subsection 4.1. In Subsection 4.3 we study the smoothness of Pauli-Fierz Hamiltonians with respect to some semigroups of isometries. The results of this subsection will be used later to check the conditions **(M1)**, **(M3)**, **(M4)** and **(M5)** introduced in Subsection 5.3.

4.1 Abstract Pauli-Fierz Hamiltonians

We describe now an abstract framework introduced in [DG1] which describes a small system interacting with a bosonic field.

The small system is described by a Hilbert space $\mathcal{K}$ and a bounded below selfadjoint operator K on $\mathcal{K}$. Without loss of generality we will assume that K is positive.

The bosonic field is described with a one-particle space $\mathfrak{h}$ and the one-particle energy by a positive self-adjoint operator ω on $\mathfrak{h}$.

The Hilbert space of the interacting system is $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$, introduced in Subsection 3.2.

The free Hamiltonian is

$$H_0 := K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) \text{ acting on } \mathcal{H}. \tag{4.1}$$

The interaction term of the Hamiltonian is the field operator $\phi(v)$ associated to a coupling function $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$. We recall that

$$\phi(v) = \frac{1}{\sqrt{2}}(a^*(v) + a(v)). \tag{4.2}$$

Under the stated condition on v one can not, in general, realize $\phi(v)$ as a densely defined operator on $\mathcal{H}$. However, one can realize it as a symmetric densely defined form by setting

$$(f, \phi(v)f) := \sqrt{2}\mathrm{Re}(f, a^*(v)f), \ \ f \in \mathcal{D}(a^*(v)). \tag{4.3}$$

We first state two direct consequences of Proposition 3.7 and Corollary 3.10.

Proposition 4.1 *i) Assume that* $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$. *Set*

$$C_1(r,v) := \|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}}) v (K+r)^{-\frac{1}{2}}\|^2,$$

defined as in (3.14). Then:

$$\pm\phi(v) \leq \sqrt{2} C_1(r,v)(H_0 + r).$$

ii) Assume that $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$ *and that* v *extends as* $v \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathfrak{h})$. *Set:*

$$C_0(r,v) := \|v(K+r)^{-\frac{1}{2}}\|^2, \ \ C_2(r,v) := \|((K+r)^{-\frac{1}{2}} \otimes \omega^{-\frac{1}{2}}) v\|^2,$$

defined as in (3.17). Then:

$$\|\phi(v)u\|^2 \leq C_0(r,v)(u, (H_0+r)u) + \frac{1}{2}(C_1(r,v) + C_2(r,v))\|(H_0+r)u\|^2.$$

Proof. We apply Proposition 3.7 and Corollary 3.10 and use the inequalities:

$$(K+r) \otimes 1\!\!1_{\Gamma(\mathfrak{h})} \leq H_0 + r, \ \ 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) \leq H_0 + r, \ \ (K+r) \otimes \mathrm{d}\Gamma(\omega) \leq (H_0+r)^2/2. \ \Box$$

The following essentially optimal condition under which the form $\phi(v)$ is small with respect to H_0 has been isolated in [G1]. It follows immediately from Proposition 4.1.

Corollary 4.2 *Assume that*

(Ia1) $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$ and $\lim_{r \to +\infty} C_1(r,v) = 0$.

Then the form $\phi(v)$ *is* H_0*–form bounded with relative bound zero.*

Definition 4.3 *Let* K, ω *and* v *be such that* **(Ia1)** *holds. Then the self-adjoint operator* $H = H_0 + \phi(v)$*, the sum being interpreted in form sense, is the* Pauli-Fierz Hamiltonian *associated to* (K, ω, v).

A Pauli-Fierz Hamiltonian is bounded from below and its form domain is explicitly known:

$$\mathcal{D}(|H|^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}}) = \mathcal{D}(K^{\frac{1}{2}}) \otimes \Gamma(\mathfrak{h}) \cap \mathcal{K} \otimes \mathcal{D}(\mathrm{d}\Gamma(\omega)^{\frac{1}{2}}). \tag{4.4}$$

Applying Proposition 4.1 we obtain conditions under which $\phi(v)$ is a densely defined symmetric operator on $\mathcal{H}$, small with respect to H_0 in operator sense.

Corollary 4.4 *Assume that:*

$$\textbf{(Ia2)} \begin{cases} v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h}), \ v \text{ extends as } v \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathfrak{h}), \\ \text{and } \lim_{r \to +\infty}(C_1(r,v) + C_2(r,v)) = 0. \end{cases}$$

Then $\phi(v)$ *is a symmetric operator on* $\mathcal{D}(H_0)$ *and is* H_0-*bounded with relative bound* 0. *In particular:*

$$\mathcal{D}(H) = \mathcal{D}(H_0) = \mathcal{D}(K) \otimes \Gamma(\mathfrak{h}) \cap \mathcal{K} \otimes \mathcal{D}(\mathrm{d}\Gamma(\omega)). \tag{4.5}$$

Proof. From Proposition 4.1 *ii)* we get

$$\|\phi(v)f\|^2 \leq C_0(r,v)(f, (H_0+r)f) + (C_1(r,v) + C_2(r,v))(f, (H_0+r)^2 f)/2.$$

We have $C_0(r,v) \leq C_0(1,v)$ if $r \geq 1$ and $H_0 \leq \nu H_0^2 + 1/(4\nu)$ for all $\nu > 0$. Thus, by taking r sufficiently large, for each $\varepsilon > 0$ we find a real number $c(\varepsilon)$ such that $\phi(v)^2 \leq \varepsilon H_0^2 + c(\varepsilon)$ as forms on $\mathcal{D}(H_0) \cap \mathcal{H}_{\mathrm{fin}}$. Finally, use the fact that $\mathcal{D}(H_0) \cap \mathcal{H}_{\mathrm{fin}}$ is a core for H_0. $\Box$

4.2 Essential spectrum of abstract Pauli-Fierz Hamiltonians

Our next purpose is to get a description of the essential spectrum of H under general conditions. For this we need two technical lemmas. The first one contains an alternative description of the condition **(Ia1)** (condition **(Ia2)** can be expressed similarly).

Lemma 4.5 *Let* $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$*. Then* $\lim_{r \to +\infty} C_1(r, v) = 0$ *if and only if*

$$\lim_{r \to \infty} \|(1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}}) v (K+1)^{-\frac{1}{2}} 1\!\!1_{[r,\infty[}(K)\| = 0. \tag{4.6}$$

Proof. In this proof we abbreviate $1\!\!1_r = 1\!\!1_{[r,\infty[}(K)$ and $\omega^{-\frac{1}{2}} = 1\!\!1_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}}$. We recall that all computations have to be done with $\omega^{-\frac{1}{2}}$ replaced by $(\omega + \varepsilon)^{-\frac{1}{2}}$ and then one has to take sup over ε in the final expressions. We have

$$\|\omega^{-\frac{1}{2}} v (K+1)^{-\frac{1}{2}} 1\!\!1_r\|^2 = \|\omega^{-\frac{1}{2}} v (K+1)^{-1} 1\!\!1_r v^* \omega^{-\frac{1}{2}}\| \leq \|\omega^{-\frac{1}{2}} v 2 (K+r)^{-1} v^* \omega^{-\frac{1}{2}}\|$$

hence (4.6) follows from $\lim_{r \to +\infty} C_1(r, v) = 0$. Reciprocally, if $r, s \geq 1$ then

$$(K+r)^{-1} = (K+r)^{-1} 1\!\!1_s^{\perp} + (K+r)^{-1} 1\!\!1_s \leq \frac{s+1}{s+r}(K+1)^{-1} + (K+1)^{-1} 1\!\!1_s,$$

hence

$$\begin{aligned}
\|\omega^{-\frac{1}{2}} v (K+r)^{-\frac{1}{2}}\|^2 &= \|\omega^{-\frac{1}{2}} v (K+r)^{-1} v^* \omega^{-\frac{1}{2}}\| \\
&\leq \frac{s+1}{s+r}\|\omega^{-\frac{1}{2}} v (K+1)^{-\frac{1}{2}}\|^2 + \|\omega^{-\frac{1}{2}} v (K+1)^{-\frac{1}{2}} 1\!\!1_s\|^2
\end{aligned}$$

from which the needed result follows easily. □

The next result concerns the so called "pull-through formula". For $f \in \mathfrak{h}$ we shall still denote by $a^{(*)}(f)$ the operator $1\!\!1_{\mathcal{K}} \otimes a^{(*)}(f)$ acting on $\mathcal{H}$. If $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$ then $v^* \in \mathcal{B}(\mathcal{K} \otimes \mathfrak{h}, \mathcal{D}(K^{\frac{1}{2}})^*)$ and for $f \in \mathfrak{h}$ we denote $v^*(f) \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^*)$ the operator defined by $v^*(f)\psi = v^*(\psi \otimes f)$ for $\psi \in \mathcal{K}$. We write $f \in \mathcal{D}(\omega^{-\frac{1}{2}})$ if $\sup_{\varepsilon > 0} \|(\omega + \varepsilon)^{-\frac{1}{2}} f\| < \infty$.

Lemma 4.6 *Assume that condition* **(Ia1)** *is fulfilled and let c be a number such that* $H + c \geq 1$*. If* $f \in \mathcal{D}(\omega^{-\frac{1}{2}})$ *then* $a^{(*)}(f)$ *is a bounded operator* $\mathcal{D}(|H|^{\frac{1}{2}}) \to \mathcal{H}$ *and there is a constant C depending only on H such that*

$$\|a^{(*)}(f) u\| \leq C \|(1 + \omega^{-\frac{1}{2}}) f\| \|(H+c)^{\frac{1}{2}} u\|. \tag{4.7}$$

If $f \in \mathcal{D}(\omega) \cap \mathcal{D}(\omega^{-\frac{1}{2}})$ *and* $z \in \mathbb{C} \setminus \sigma(H)$ *then the closure* $[a^*(f), (H-z)^{-1}]^\circ$ *of the form* $[a^*(f), (H-z)^{-1}]$ *is a bounded operator and we have*

$$[a^*(f), (H-z)^{-1}]^\circ = (H-z)^{-1} \left(a^*(\omega f) + \frac{1}{\sqrt{2}} v^*(f) \otimes 1\!\!1_{\Gamma(\mathfrak{h})} \right) (H-z)^{-1}. \tag{4.8}$$

Proof. Note that the operator $a^{(*)}(f)$ is just $a^{(*)}(w)$, where $w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ acts as $w(\psi) = \psi \otimes f$. Then (4.7) follows from (3.16) and (3.18) for $K = 0$, $r = 1$, using that $\mathcal{D}(|H|^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$.

Thus, if $f \in \mathcal{D}(\omega^{-\frac{1}{2}})$ the operator $a^*(f)$ extends to a continuous operator $\mathcal{D}(|H|^{\frac{1}{2}}) \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{D}(|H|^{\frac{1}{2}})^*$ (use the adjoint of the continuous operator $a(f) : \mathcal{D}(|H|^{\frac{1}{2}}) \to \mathcal{H}$). Now it is easy to show that the form $[a^*(f), (H-z)^{-1}]$ extends to a bounded operator on $\mathcal{H}$ and

$$(4.9) \qquad [a^*(f), (H-z)^{-1}]^\circ = (H-z)^{-1}[H, a^*(f)](H-z)^{-1}$$

where $[H, a^*(f)]$ is a well defined continuous operator $\mathcal{D}(H) \to \mathcal{D}(H)^*$.

On the other hand, if $f \in \mathcal{D}(\omega)$ then $a^*(f)$ maps $\mathcal{D}(K^{\frac{1}{2}}) \otimes \Gamma_{\text{fin}}(\mathcal{D}(\omega))$ into itself and a straightforward computation gives the following pull-through formula (see [G1]):

$$(4.10) \qquad Ha^*(f) - a^*(f)H = a^*(\omega f) + \frac{1}{\sqrt{2}} v^*(f) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})},$$

as forms on $\mathcal{D}(K^{\frac{1}{2}}) \otimes \Gamma_{\text{fin}}(\mathcal{D}(\omega))$. However, this does not prove yet the relation (4.8) because we do not have sufficient information on the domain of H if only condition **(Ia1)** is fulfilled. In order to avoid this technical difficulty we proceed as follows.

Let $f \in \mathcal{D}(\omega) \cap \mathcal{D}(\omega^{-\frac{1}{2}})$. Assume for a moment that (4.5) is satisfied. Then the subspace $\mathcal{D}(K) \otimes \Gamma_{\text{fin}}(\mathcal{D}(\omega))$ is dense in $\mathcal{D}(H)$ hence (4.10) remains valid in the sense of forms on $\mathcal{D}(H)$. Combining with (4.9) we see that (4.8) is true if (4.5) is satisfied.

We reduce the general case to this one by an approximation procedure. Let ν be a strictly positive number and $v_\nu = v(1+\nu K)^{-1} \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. We have $C_i(r, v_\nu) \le r^{-1} \|(\mathbb{1}_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}}) v_\nu\|^2$ for $i = 1, 2$, so v_ν satisfies condition **(Ia2)** and one can apply Corollary 4.4 to the operator $H^\nu = H_0 + \phi(v_\nu)$. By the preceding remark, the relation (4.8) holds if H, v are replaced by H^ν, v_ν and $z \notin \sigma(H^\nu)$. In particular, if $u \in \mathcal{D}(H_0^{\frac{1}{2}})$ then:

$$\begin{aligned}(a(f)u, (H^\nu - z)^{-1}u) - ((H^\nu - \bar{z})^{-1}u, a^*(f)u) &= ((H^\nu - \bar{z})^{-1}u, a^*(\omega f)(H^\nu - z)^{-1}u) \\ &+ \frac{1}{\sqrt{2}}((H^\nu - \bar{z})^{-1}u, v_\nu^*(f) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})}(H^\nu - z)^{-1}u).\end{aligned}$$

We have $v_\nu^*(f) = (1+\nu K)^{-1} v^*(f) \to v^*(f)$ strongly as operators $\mathcal{K} \to \mathcal{D}(K^{\frac{1}{2}})^*$ when $\nu \to 0$. From (4.4) we get $\mathcal{D}(H_0^{\frac{1}{2}}) \subset \mathcal{D}(K^{\frac{1}{2}}) \otimes \Gamma(\mathfrak{h})$. Thus, if we show that $(H^\nu - z)^{-1} \to (H-z)^{-1}$ strongly in $\mathcal{B}(\mathcal{H}, \mathcal{D}(H_0^{\frac{1}{2}}))$ when $\nu \to 0$, then by taking the limit as $\nu \to 0$ in the preceding formula we obtain (4.8) and the proof of the lemma will be finished.

We shall prove a stronger assertion, namely

$$(4.11) \qquad \lim_{\nu \to 0} R^\nu(z) = R(z) \text{ in norm in } \mathcal{B}(\mathcal{D}(H_0^{\frac{1}{2}})^*, \mathcal{D}(H_0^{\frac{1}{2}}))$$

if $z \notin \sigma(H)$. Here $R^\nu(z) = (H^\nu - z)^{-1}$ and $R(z) = (H-z)^{-1}$ and below we also make the convention $H^\nu = H$ and $R^\nu(z) = R(z)$ if $\nu = 0$. It suffices in fact to prove this for one point z_0 with $\text{Im} z_0 \neq 0$. Indeed, then we use

$$\begin{aligned} R^\nu(z) &= R^\nu(z_0)(1 - (z - z_0)R^\nu(z_0))^{-1} \text{ for } |z - z_0| \text{ small,} \\ R^\nu(z) &= R^\nu(z_0) + (z - z_0)R^\nu(z_0)^2 + (z - z_0)^2 R^\nu(z_0)R^\nu(z)R^\nu(z_0). \end{aligned}$$

If (4.11) holds for $z = z_0$ then the first relation above and a connexity argument allows us to prove norm convergence in $\mathcal{B}(\mathcal{H})$ for all $z \notin \sigma(H)$ and then the second relation gives norm convergence in $\mathcal{B}(\mathcal{D}(H_0^{\frac{1}{2}})^*, \mathcal{D}(H_0^{\frac{1}{2}}))$.

Proposition 4.1 gives

$$\pm\phi(v_\nu) \le \sqrt{2}C_1(r, v_\nu)(H_0 + r) \le \sqrt{2}C_1(r, v)(H_0 + r).$$

We choose r conveniently and find a number b such that $\pm\phi(v_\nu) \le \frac{1}{2}H_0 + b$ for all ν. It follows easily that one can choose a number a such that $H^\nu + a \ge H_0 + 1$ for all ν. We shall take $z_0 = -a$. The operator H^ν has $\mathcal{D}(H_0^{\frac{1}{2}})$ as form domain so $H^\nu + a$ extends to an isomorphism $\mathcal{D}(H_0^{\frac{1}{2}}) \to \mathcal{D}(H_0^{\frac{1}{2}})^*$. Also $H - H^\nu = \phi(v - v_\nu)$ holds in $\mathcal{B}(\mathcal{D}(H_0^{\frac{1}{2}}), \mathcal{D}(H_0^{\frac{1}{2}})^*)$. Thus, if we set $R^\nu = (H^\nu + a)^{-1}$ and $R = (H + a)^{-1}$, we have:

$$R^\nu - R = R^\nu \phi(v - v_\nu) R \quad \text{in} \quad \mathcal{B}(\mathcal{D}(H_0^{\frac{1}{2}})^*, \mathcal{D}(H_0^{\frac{1}{2}})).$$

Let $S = (H_0 + 1)^{\frac{1}{2}}$. We get

$$\|S(R^\nu - R)S\| \le \|SR^\nu S\| \|S^{-1}\phi(v - v_\nu)S^{-1}\| \|SRS\| \le \|S^{-1}\phi(v - v_\nu)S^{-1}\|$$

where we used $R_\nu \le (H_0 + 1)^{-1} = S^{-2}$ hence $0 \le SR^\nu S \le 1$. Now observe that we have $\|S^{-1}\phi(v - v_\nu)S^{-1}\| \le \theta_\nu$ if $\pm\phi(v - v_\nu) \le \theta_\nu(H_0 + 1)$. From Proposition 4.1 we get $\theta_\nu \le \sqrt{2}C_1(1, v - v_\nu)$ hence the proof of the lemma is finished if we show that

$$C_1(1, v - v_\nu) = \|(\mathbb{1}_{\mathcal{K}} \otimes \omega^{-\frac{1}{2}})(v - v_\nu)(K + 1)^{-\frac{1}{2}}\|^2 \to 0 \text{ when } \nu \to 0. \tag{4.12}$$

We shall use the notations introduced in the proof of Lemma 4.5. For $r > 0$ we have

$$\begin{aligned} & \|\omega^{-\frac{1}{2}}(v - v_\nu)(K + 1)^{-\frac{1}{2}}\| = \|\omega^{-\frac{1}{2}} v \nu K(1 + \nu K)^{-1}(K + 1)^{-\frac{1}{2}}\| \\ \le \; & \|\omega^{-\frac{1}{2}} v (K + 1)^{-\frac{1}{2}} \mathbb{1}_r^\perp \nu K(1 + \nu K)^{-1}\| + \|\omega^{-\frac{1}{2}} v (K + 1)^{-\frac{1}{2}} \mathbb{1}_r \nu K(1 + \nu K)^{-1}\| \\ \le \; & \|\omega^{-\frac{1}{2}} v (K + 1)^{-\frac{1}{2}}\| \nu r(1 + \nu r)^{-1} + \|\omega^{-\frac{1}{2}} v (K + 1)^{-\frac{1}{2}} \mathbb{1}_r\|. \end{aligned}$$

Thus

$$\limsup_{\nu \to 0} \|\omega^{-\frac{1}{2}}(v - v_\nu)(K + 1)^{-\frac{1}{2}}\| \le \|\omega^{-\frac{1}{2}} v (K + 1)^{-\frac{1}{2}} \mathbb{1}_r\|$$

and now (4.12) follows from Lemma 4.5. □

Remark 4.7 We mention the following consequence of (4.7): *if $\{f_n\}$ is a sequence in $\mathcal{D}(\omega^{-\frac{1}{2}})$ such that $\|\omega^{-\frac{1}{2}} f_n\| \le$ const and $f_n \to 0$ weakly in $\mathfrak{h}$, and if $u \in \mathcal{D}(|H|^{\frac{1}{2}})$, then $\|a(f_n)u\| \to 0$.* Indeed, let $\mathbb{1}_k = \mathbb{1}_{[0,k]}(N)$ and $\mathbb{1}_k^\perp = \mathbb{1} - \mathbb{1}_k$. Since $\mathcal{D}(|H|^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$ is stable under $\mathbb{1}_k$ and $\mathbb{1}_k$ commutes with H_0, we have

$$\|a(f_n)u\| \le \|a(f_n)\mathbb{1}_k u\| + \|a(f_n)\mathbb{1}_k^\perp u\| \le \|a(f_n)\mathbb{1}_k u\| + \|(1 + \omega^{-\frac{1}{2}})f_n\| \|\mathbb{1}_k^\perp (H_0^{\frac{1}{2}} + 1)u\|.$$

The last term tends to zero when $k \to \infty$ uniformly in n and clearly $\|a(f_n)\mathbb{1}_k u\| \to 0$ for each k.

In the next proposition we describe the essential spectrum of abstract Pauli-Fierz Hamiltonians.

Proposition 4.8 *Assume that v satisfies hypothesis* **(Ia1)** *and that*

$$\mathfrak{h} \ni f \mapsto (K + 1)^{-1} v^*(\psi \otimes (\omega + 1)^{-1} f) \in \mathcal{K} \text{ is compact for each } \psi \in \mathcal{D}(K^{\frac{1}{2}}). \tag{4.13}$$

Let $m \ge 0$ and assume $[m, +\infty[\subset \sigma(\omega)$. Then $[\inf \sigma(H) + m, +\infty[\subset \sigma_{\text{ess}}(H)$.

Remark 4.9 Let us first note that using the notation in Subsection 3.5, the map in (4.13) is equal to $(K+1)^{-1}v^{\dagger}(\psi)(1+\omega)^{-1}$. Let us describe two situations in which condition (4.13) in Proposition 4.8 is satisfied for $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})$. First if we assume that $(K+1)^{-1}$ is compact, then $(K+1)^{-1}v^{\dagger}(\psi) \in \mathcal{B}(\mathfrak{h}, \mathcal{D}(K^{\frac{1}{2}}))$ and hence is compact for each $\psi \in \mathcal{K}$.

Let us now assume that $v \in \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^{*}\otimes\mathfrak{h})$. From the discussion in Subsection 3.5, we see that if $v^{\dagger} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})$ then $v^{\dagger}(\psi)$ is Hilbert-Schmidt and hence compact for $\psi \in \mathcal{D}(K^{\frac{1}{2}})$. In particular if $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ and v is associated to the map $v(\cdot) \in L^2_w(\mathbb{R}^d; \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}))$, then $v^{\dagger} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})$ iff $v(\cdot)^{*} \in L^2_w(\mathbb{R}^d; \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}))$.

More generally if ω is the operator of multiplication by a positive measurable function $\omega(k)$ and if $(1+\omega(\cdot))^{-1}(K+1)^{-1}v^{*}(\cdot) \in L^2_w(\mathbb{R}^d; \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}))$ then the operator $(K+1)^{-1}v^{\dagger}(\psi)(1+\omega)^{-1}$ is compact for $\psi \in \mathcal{D}(K^{\frac{1}{2}})$. This condition is satisfied in particular if

$$\int_{\mathbb{R}^d} \|v(k)(K+1)^{-1}\|^2_{\mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^{*})}(1+\omega(k))^{-1}\mathrm{d}k < \infty.$$

Proof of Proposition 4.8. We shall use the following fact: let H be an arbitrary selfadjoint operator on a Hilbert space $\mathcal{H}$. Let $\mu \in \mathbb{R}$ and assume that there is a sequence of vectors $u_n \in \mathcal{H}$ such that $\|u_n\| \to 1$ and $\|(H+\mathrm{i})^{-1}(H-\mu)u_n\| \to 0$. Then $\mu \in \sigma(H)$.

Let $E = \inf\sigma(H)$ and $\lambda > m$. In the rest of the proof we shall construct a sequence $\{u_n\}$ as above for $\mu = E+\lambda$. Thus $[\inf\sigma(H)+m, +\infty[\subset \sigma(H)$, which implies the assertion of the proposition. It follows easily from (4.4) and from the fact that N commutes with H_0 that the space $\mathcal{E} := \mathcal{D}(K)\otimes\Gamma_{\mathrm{fin}}(\mathcal{D}(\omega))$ is a form core for H_0 hence for H (we recall that all tensor products in the definition of $\mathcal{E}$ are algebraic). Thus, for any $\varepsilon > 0$ there is $u_\varepsilon \in \mathcal{E}$ such that $\|u_\varepsilon\| = 1$ and $\|(H+c)^{-\frac{1}{2}}(H-E)u_\varepsilon\| \le \varepsilon$, where c is a fixed number such that $H+c \ge H_0+1$. Then for each integer $n > 2/\lambda$ let us choose $f_n \in \mathfrak{h}$ such that $\|f_n\| = 1$, $\mathbb{1}_{[\lambda-1/n, \lambda+1/n]}(\omega)f_n = f_n$ $f_n \to 0$ weakly in $\mathfrak{h}$. Then $\|\omega^{-\frac{1}{2}}f_n\| \le \sqrt{2/\lambda}$ and $\|(\omega-\lambda)f_n\| \le 1/n$. The vectors u_n will be of the form $a^{*}(f_n)u_\varepsilon$ for some conveniently chosen ε.

From (4.10) we get

$$(H-E-\lambda)a^{*}(f_n)u_\varepsilon = a^{*}(f_n)(H-E)u_\varepsilon + a^{*}((\omega-\lambda)f_n)u_\varepsilon + \frac{1}{\sqrt{2}}v^{*}(f_n)\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}u_\varepsilon. \tag{4.14}$$

We apply $(H+c)^{-1}$ to (4.14) and estimate each term on the right hand side as follows. For the first term we use (4.8) and obtain:

$$\begin{aligned} (H+c)^{-1}a^{*}(f_n)(H-E)u_\varepsilon &= (H+c)^{-1}a^{*}(\omega f_n)(H+c)^{-1}(H-E)u_\varepsilon \\ &+ \frac{1}{\sqrt{2}}(H+c)^{-1}(v^{*}(f_n)\otimes\mathbb{1}_{\Gamma(\mathfrak{h})})(H+c)^{-1}(H-E)u_\varepsilon \\ &+ a^{*}(f_n)(H+c)^{-1}(H-E)u_\varepsilon. \end{aligned} \tag{4.15}$$

In the sequel $C_1, C_2, \ldots$, are constants independent of n and ε. We have $\|(1+\omega^{-\frac{1}{2}})\omega f_n\| \le C_1$ hence from (4.7) we get

$$\|(H+c)^{-1}a^{*}(\omega f_n)(H+c)^{-1}(H-E)u_\varepsilon\| \le C_2\|(H+c)^{-\frac{1}{2}}(H-E)u_\varepsilon\| \le C_2\varepsilon.$$

The same argument gives $\|a^{*}(f_n)(H+c)^{-1}(H-E)u_\varepsilon\| \le C_3\varepsilon$. Finally the second term on the right hand side of (4.15) is bounded by

$$\|(H_0+1)^{-\frac{1}{2}}(v^{*}(f_n)\otimes\mathbb{1}_{\Gamma(\mathfrak{h})})(H+c)^{-1}(H-E)u_\varepsilon\|$$

which in turn is smaller than $\|v(K+1)^{-\frac{1}{2}}\|\|(H+c)^{-1}(H-E)u_\varepsilon\| \leq C_4\varepsilon$. Thus we have:

$$\|(H+c)^{-1}a^*(f_n)(H-E)u_\varepsilon\| \leq C_5\varepsilon. \tag{4.16}$$

Using (4.7) again we get:

$$\|a^*((\omega-\lambda)f_n)u_\varepsilon\| \leq C\|(1+\omega^{-\frac{1}{2}})(\omega-\lambda)f_n)\|\|(H+c)^{\frac{1}{2}}u_\varepsilon\| \leq C_6/n. \tag{4.17}$$

From (4.14), (4.16) and (4.17) we obtain

$$\|(H+c)^{-1}(H-E-\lambda)a^*(f_n)u_\varepsilon\| \leq C_5\varepsilon + \frac{C_6}{n} + \frac{1}{\sqrt{2}}\|(H+c)^{-1}v^*(f_n)\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}u_\varepsilon\|. \tag{4.18}$$

We now show that the last term above converges to zero when $n \to \infty$. Since $u_\varepsilon \in \mathcal{E}$ it suffices to prove that $\|(H+c)^{-\frac{1}{2}}(v^*(f_n)\otimes \mathbb{1}_{\Gamma(\mathfrak{h})})(\psi\otimes g)\| \to 0$ if $\psi \in \mathcal{D}(K^{\frac{1}{2}})$ and $g \in \Gamma(\mathfrak{h})$. But $(K+1)\otimes \mathbb{1}_{\Gamma(\mathfrak{h})} \leq H_0+1 \leq H+c$, hence it suffices to show that $\|(K+1)^{-\frac{1}{2}}v^*(\psi\otimes f_n)\| \to 0$. We use Lemma 4.5 and the notations from its proof:

$$\begin{array}{rcl} \|(K+1)^{-\frac{1}{2}}v^*(\psi\otimes f_n)\| & \leq & \|(K+1)^{-\frac{1}{2}}\mathbb{1}_r^{\perp}v^*(\psi\otimes f_n)\| + \|(K+1)^{-\frac{1}{2}}\mathbb{1}_r v^*(\psi\otimes f_n)\| \\ & \leq & (r+1)^{\frac{1}{2}}\|(K+1)^{-1}v^*(\psi\otimes(\omega+1)^{-1}(\omega+1)f_n)\| \\ & + & \|(K+1)^{-\frac{1}{2}}\mathbb{1}_r v^*(\mathbb{1}_{\mathcal{K}}\otimes\omega^{-\frac{1}{2}})\|\|\psi\otimes(\omega^{\frac{1}{2}}f_n)\|. \end{array}$$

We have $\|\omega^{\frac{1}{2}}f_n\| \leq C_8$ and $(\omega+1)f_n \to 0$ weakly. From Lemma 4.5 the second term in the right hand side above tends to 0 when $r \to \infty$ uniformly in n and since by hypothesis the operator $f \mapsto (K+1)^{-1}v^*(\psi\otimes(\omega+1)^{-1}f)$ is compact, the first term in the right hand side tends to 0 when $n \to \infty$. Picking first $r \gg 1$ and then $n \gg 1$, we see that the last term in (4.18) converges to zero as $n \to \infty$.

To conclude, we have

$$\limsup_{n\to\infty} \|(H+c)^{-1}(H-E-\lambda)a^*(f_n)u_\varepsilon\| \leq C_5\varepsilon.$$

On the other hand, using that

$$\|a^*(f_n)u_\epsilon\|^2 = \|f_n\|^2\|u_\epsilon\|^2 + \|a(f_n)u_\epsilon\|^2,$$

Remark 4.7, and the facts that $\|\omega^{-\frac{1}{2}}f_n\| \leq \sqrt{2/\lambda}$ and $f_n \to 0$ weakly in $\mathfrak{h}$, we have:

$$\lim_{n\to\infty} \|a^*(f_n)u_\varepsilon\| = 1.$$

An easy argument finishes the proof. □

4.3 Smoothness of abstract Pauli-Fierz Hamiltonians

Let H be a Pauli-Fierz Hamiltonian as in Subsection 4.1. We assume that hypothesis **(Ia1)** from Corollary 4.2 holds.

Let $\mathbb{R}^+ \ni t \mapsto w_t \in \mathcal{B}(\mathfrak{h})$ be a C_0-semigroup of isometries with generator a. We set $W_t := \mathbb{1}_{\mathcal{K}}\otimes\Gamma(w_t)$, which defines a C_0-semigroup of isometries of $\mathcal{H}$ whose generator we denote by A. Recall that $A = \mathbb{1}_{\mathcal{K}}\otimes \mathrm{d}\Gamma(a)$, see Lemma 3.1.

We fix another selfadjoint operator $b \geq 0$ on $\mathfrak{h}$ and set:

$$B := K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(b),\ \mathcal{G} := \mathcal{D}(B^{\frac{1}{2}}).$$

We will give sufficient conditions which ensure that $\mathcal{G}$ is b-stable under $\{W_t\}$ and $\{W_t^*\}$ and that $H \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and give an expression for $[H, \mathrm{i}A]^0$. We refer to Subsection 5.2 for notation.

Throughout this subsection, if $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$ is a coupling function, we denote simply by av the operator $(1\!\!1_{\mathcal{K}} \otimes a)v$.

Proposition 4.10 *Let ω, b, K and v be as above. Then: i) If*

$$w_t^* b w_t \leq C_t b, (\text{ resp. } w_t b w_t^* \leq C_t b) \text{ with } \sup_{0<t<1} C_t < \infty, \tag{4.19}$$

then

$$\mathcal{G} \text{ is b-stable under } \{W_t\}(\text{ resp. } \{W_t^*\}). \tag{4.20}$$

ii) Assume (4.19) and

$$\omega \leq Cb,\ |(u_2, (\omega w_t - w_t \omega) u_1)| \leq Ct \|b^{\frac{1}{2}} u_1\| \|b^{\frac{1}{2}} u_2\|,\ u_i \in \mathcal{D}(b^{\frac{1}{2}}),\ 0 < t < 1. \tag{4.21}$$

Then $H_0 \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and

$$[H_0, \mathrm{i}A]^0 = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma([\omega, \mathrm{i}a]^0).$$

iii) Assume (4.19) and

$$v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(a))),\ av \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(b^{-\frac{1}{2}})). \tag{4.22}$$

Then $\phi(v) \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and

$$[\phi(v), \mathrm{i}A]^0 = -\phi(\mathrm{i}av).$$

iv) Assume (4.19), (4.21) and (4.22). Then $H \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and

$$[H, \mathrm{i}A]^0 = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma([\omega, \mathrm{i}a]^0) - \phi(\mathrm{i}av).$$

Remarks 4.11 (1) Condition (4.19) implies that $\mathcal{D}(b^{\frac{1}{2}})$ is b-stable under $\{w_t\}$ (resp. $\{w_t^*\}$) and is equivalent to it if $b \geq c > 0$.
(2) Condition (4.21) implies that $\omega \in C^1(a; \mathcal{D}(b^{\frac{1}{2}}), \mathcal{D}(b^{\frac{1}{2}})^*)$ and is equivalent to it if $b \geq c > 0$. Therefore $[\omega, \mathrm{i}a]^0$ is well defined as an element of $\mathcal{B}(\mathcal{D}(b^{\frac{1}{2}}), \mathcal{D}(b^{\frac{1}{2}})^*)$.

Corollary 4.12 *Let ω, K and v be as above and let a be a selfadjoint operator on $\mathfrak{h}$, $A = \mathrm{d}\Gamma(a)$. Assume that*

$$\pm(\mathrm{e}^{-\mathrm{i}ta} \omega \mathrm{e}^{\mathrm{i}ta} - \omega) \leq C|t|\omega,\ 0 < |t| < 1, \tag{4.23}$$

and

$$v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(a))),\ av \in \mathcal{B}(\mathcal{D}((K)^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(\omega^{-\frac{1}{2}})). \tag{4.24}$$

Then $\mathcal{G} := \mathcal{D}(|H|^{\frac{1}{2}})$ is b-stable under $\{\mathrm{e}^{\mathrm{i}tA}\}_{t \in \mathbb{R}}$ and H is of class $C^1(A; \mathcal{G}, \mathcal{G}^*)$ and hence of class $C^1(A)$. Moreover:

$$[H, \mathrm{i}A]^0 = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma([\omega, \mathrm{i}a]^0) - \phi(\mathrm{i}av).$$

Proof. We apply Proposition 4.10 for $b = \omega$. Hypothesis (4.23) implies (4.19) and hence that $\mathcal{D}(\omega^{\frac{1}{2}})$ is b-stable under $\{\mathrm{e}^{\mathrm{i}ta}\}$. We see then that it also implies (4.21). Thus we get that H is of class $C^1(A; \mathcal{G}, \mathcal{G}^*)$. The fact that H is of class $C^1(A)$ follows then from [ABG, Lemma 7.5.3]. □

Proof of Proposition 4.10. Note first that *iv)* follows from *ii)* and *iii)*, since by (4.22) and Corollary 4.2 we have $H = H_0 + \phi(v)$ as an operator sum in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$.

Let us first prove *i)*. Since W_t does not act on $\mathcal{K}$ we can without loss of generality assume that $\mathcal{K} = \mathbb{C}$ and $K = 0$. We observe that $\Gamma_{\mathrm{fin}}(\mathcal{D}(b))$ is a form core for $\mathrm{d}\Gamma(b)$. Using Lemma 3.3 and the fact that $w_t^* w_t = 1\!\!1$, we get:

$$(W_t u, \mathrm{d}\Gamma(b) W_t u) = (u, \mathrm{d}\Gamma(w_t^* b w_t) u) \leq C_t (u, \mathrm{d}\Gamma(b) u),\ u \in \Gamma_{\mathrm{fin}}(\mathcal{D}(b)).$$

By density this yields:

$$W_t^* \mathrm{d}\Gamma(b) W_t \leq C_t \mathrm{d}\Gamma(b),$$

which implies that $W_t : \mathcal{G} \to \mathcal{G}$. Moreover

$$\|W_t u\|_{\mathcal{G}}^2 = (W_t u, \mathrm{d}\Gamma(b) W_t u) + (u, u) \leq (C_t + 1)\Big((u, \mathrm{d}\Gamma(b) u) + (u, u)\Big) = (C_t + 1)\|u\|_{\mathcal{G}}^2,$$

which proves that $\mathcal{G}$ is b-stable under $\{W_t\}$.

To prove the corresponding statement for W_t^*, we estimate for $u \in \Gamma_{\mathrm{fin}}(\mathcal{D}(b))$:

$$(W_t^* u, \mathrm{d}\Gamma(b) W_t^* u) = (u, W_t \mathrm{d}\Gamma(b) W_t^* u) = (u, \mathrm{d}\Gamma(w_t w_t^*, w_t b w_t^*) u) \leq C_t (u, \mathrm{d}\Gamma(b) u),$$

using that $w_t w_t^* \leq 1\!\!1$, $w_t b w_t^* \leq C_t b$. Then we argue similarly.

Let us now prove *ii)*. As above we may assume that $\mathcal{K} = \mathbb{C}$ and $K = 0$. For $u_1, u_2 \in \Gamma_{\mathrm{fin}}(\mathcal{D}(b))$ we have by (3.4):

$$|(u_2, (H_0 W_t - W_t H_0) u_1)| = |(u_2, \mathrm{d}\Gamma(w_t, \omega w_t - w_t \omega) u_1)| \leq Ct \|\mathrm{d}\Gamma(b)^{\frac{1}{2}} u_1\| \|\mathrm{d}\Gamma(b)^{\frac{1}{2}} u_2\|,$$

using Lemma 3.2. By density this extends to $u_1, u_2 \in \mathcal{G}$ and shows that $H_0 \in C^1(A; \mathcal{G}, \mathcal{G}^*)$. By Remark 4.11 we know that $\omega \in C^1(a; \mathcal{D}(b^{\frac{1}{2}}), \mathcal{D}(b^{\frac{1}{2}})^*)$ which yields:

$$\text{s-}\lim_{t \to 0^+} t^{-1}(\omega w_t - w_t \omega) = [\omega, \mathrm{i}a]^0 \text{ in } \mathcal{B}(\mathcal{D}(b^{\frac{1}{2}}), \mathcal{D}(b^{\frac{1}{2}})^*). \tag{4.25}$$

Hence we have

$$\pm [\omega, \mathrm{i}a]^0 \leq Cb. \tag{4.26}$$

Again by Remark 4.11, $\mathcal{D}(b^{\frac{1}{2}})$ is b-stable under $\{w_t\}$ and hence $\text{s-}\lim_{t \to 0^+} w_t = 1\!\!1$ in $\mathcal{D}(b^{\frac{1}{2}})$ which implies that for $u_1, u_2 \in \Gamma_{\mathrm{fin}}(\mathcal{D}(b))$:

$$\lim_{t \to 0^+} (u_2, \mathrm{d}\Gamma(w_t, \omega w_t - w_t \omega) u_1) = (u_2, \mathrm{d}\Gamma([\omega, \mathrm{i}a]^0) u_1),$$

and hence

$$[H, \mathrm{i}A]^0 = \mathrm{d}\Gamma([\omega, \mathrm{i}a]^0).$$

It remains to prove *iii)*. To prove that $\phi(v) \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ we will apply Proposition 5.10. Note that using Corollary 4.2 and the fact that $\omega \leq Cb$, we see that $\phi(v) \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$. We consider the quadratic form on $\mathcal{D}(A_{\mathcal{G}^*}^*) \times \mathcal{D}(A_{\mathcal{G}}) \subset \mathcal{G}^* \times \mathcal{G}$:

$$(u_2, {}_2[\phi(v), \mathrm{i}A]_1 u_1) := (u_2, \mathrm{i}\phi(v) A_{\mathcal{G}} u_1)_{\mathcal{G}^*} + (\mathrm{i}A_{\mathcal{G}^*}^* u_2, \phi(v) u_1)_{\mathcal{G}^*}.$$

By Proposition 5.10, we know that:

$$(u_2, {}_2[\phi(v), \mathrm{i}A]_1 u_1) = \lim_{t\to 0^+} t^{-1}(u_2, \Big(\phi(v)W_t - W_t\phi(v)\Big)u_1)_{\mathcal{G}^*}.$$

We will show that for $u_1 \in \mathcal{D}(A_{\mathcal{G}}), u_2 \in \mathcal{D}(A^*_{\mathcal{G}^*})$:

$$\lim_{t\to 0^+} t^{-1}(u_2, \Big(\phi(v)W_t - W_t\phi(v)\Big)u_1)_{\mathcal{G}^*} = (u_2, -\phi(\mathrm{i}av)u_1)_{\mathcal{G}^*}. \tag{4.27}$$

Note that since $av \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathcal{D}(b^{-\frac{1}{2}}))$, the right hand side of (4.27) is by Corollary 3.10 a bounded quadratic form on $\mathcal{G}^*\times\mathcal{G}$. Hence (4.27) implies that $\phi(v) \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and that

$$[\phi(v), \mathrm{i}A]^0 = -\phi(\mathrm{i}av).$$

It remains to prove (4.27). By [DG1, Lemma 2.7]:

$$W_t\phi(v) = \phi(w_t v)W_t,$$

and hence:

$$\phi(v)W_t - W_t\phi(v) = \phi(v - w_t v)W_t. \tag{4.28}$$

Set $b_1 = b+1$, $B_1 = K\otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}}\otimes \mathrm{d}\Gamma(b_1)$. We note that b_1 satisfies (4.19) and hence by *i)* $\mathcal{D}(B_1^{\frac{1}{2}})$ is b-stable under $\{W_t\}$. In particular $\{W_t\}$ is uniformly bounded on $\mathcal{D}(B_1^{\frac{1}{2}})$ for $0\le t\le 1$. Next since $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathcal{D}(a))$, we obtain that

$$\|v - w_t v\|_{\mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})} \le Ct, \ \ 0\le t\le 1.$$

and hence applying Corollary 3.10, we obtain that $t^{-1}\phi(v - w_t v)$ is uniformly bounded as a quadratic form on $\mathcal{D}(B_1^{\frac{1}{2}})$.

Set $\mathcal{D}_1 := \mathcal{D}(K^{\frac{1}{2}})\otimes\Gamma_{\mathrm{fin}}(\mathcal{D}(a; \mathcal{D}(b^{\frac{1}{2}})))$ and $\mathcal{D}_2 := \mathcal{D}(K^{\frac{1}{2}})\otimes\Gamma_{\mathrm{fin}}(\mathcal{D}(a^*; \mathcal{D}(b^{\frac{1}{2}})))$. By (4.22), we have $t^{-1}(v - w_t v) \to -\mathrm{i}av$ in $\mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})$ strongly when $t\to 0^+$. By a direct computation, we obtain that

$$\lim_{t\to 0^+} t^{-1}(u_2, \phi(v - w_t v)W_t u_1) = -(u_2, \phi(\mathrm{i}av)u_1), \ \ u_1, u_2 \in \mathcal{D}_1.$$

Since $\mathcal{D}_1$ is dense in $\mathcal{D}(B_1^{\frac{1}{2}})$, we obtain that

$$\lim_{t\to 0^+} t^{-1}(u_2, \Big(\phi(v)W_t - W_t\phi(v)\Big)u_1) = -(u_2, \phi(\mathrm{i}av)u_1), \ \ u_1, u_2 \in \mathcal{D}(B_1^{\frac{1}{2}}).$$

This shows that

$$t^{-1}(u_2, \Big(\phi(v)W_t - W_t\phi(v)\Big)u_1)_{\mathcal{G}^*} = ((B+1)^{-1}u_2, \Big(\phi(v)W_t - W_t\phi(v)\Big)u_1)$$

converges to

$$-((B+1)^{-1}u_2, \phi(\mathrm{i}av)u_1) = -(u_2, \phi(\mathrm{i}av)u_1)_{\mathcal{G}^*}$$

when $t\to 0^+$ if $u_1, (B+1)^{-1}u_2 \in \mathcal{D}(B_1^{\frac{1}{2}})$. In particular this holds if $u_1 \in \mathcal{D}_1, u_2 \in \mathcal{D}_2$.

We note that by Lemma 3.1, $\mathcal{D}_1$ is dense in $\mathcal{D}(A_{\mathcal{G}})$, and $\mathcal{D}_2$ is dense in $\mathcal{D}(A^*_{\mathcal{G}})$ and hence in $\mathcal{D}(A^*_{\mathcal{G}^*})$. Then (4.27) follows by a density argument. □

5 The Mourre method

In this section, we fix some terminology and recall the main results from [GGM]. We refer the reader to [GGM] for more details and proofs.

5.1 The $C^1(A)$ class

In this subsection we recall the definition of the $C^1(A)$ class introduced in [GGM], where A is a closed and densely defined operator. This definition is an extension of the standard $C^1(A)$ class for A selfadjoint (see [ABG]).

In all this subsection A will be a closed densely defined operator on a Hilbert space $\mathcal{H}$.

We start by considering the $C^1(A)$ class of bounded operators. If $S \in \mathcal{B}(\mathcal{H})$ we denote by $[A, S]$ the sesquilinear form on $\mathcal{D}(A^*) \times \mathcal{D}(A)$ defined by:

$$(u, [A, S]v) := (A^*u, Sv) - (S^*u, Av), \ u \in \mathcal{D}(A^*), \ v \in \mathcal{D}(A).$$

Definition 5.1 *An operator $S \in \mathcal{B}(\mathcal{H})$ is* of class $C^1(A)$ *if the sesquilinear form $[A, S]$ is continuous for the topology of $\mathcal{H} \times \mathcal{H}$. If this is the case, we denote by $[A, S]^\circ$ the unique bounded operator on $\mathcal{H}$ associated to the quadratic form $[A, S]$ (note that $\mathcal{D}(A^*) \times \mathcal{D}(A)$ is dense in $\mathcal{H} \times \mathcal{H}$). We denote by $C^1(A)$ the linear space*

$$C^1(A) := \{S \in \mathcal{B}(\mathcal{H}) | S \textit{ is of class } C^1(A)\}.$$

It is then possible to extend the $C^1(A)$ property to an unbounded operator S, by considering the resolvent $(S - z)^{-1}$.

Definition 5.2 *If S is a closed and densely defined operator on $\mathcal{H}$, then the* A-regular resolvent set *of S is the set $\rho(S, A)$ of $z \in \mathbb{C} \backslash \sigma(S)$ such that $R(z) := (S - z)^{-1}$ is of class $C^1(A)$.*

Definition 5.3 *Let S be a closed and densely defined operator. We say that S is* of class $C^1(A)$ *if there is a sequence of complex numbers $z_\nu \in \rho(S, A)$ with $|z_\nu| \to \infty$ such that $\|(z_\nu - S)^{-1}\| \leq C|z_\nu|^{-1}$ for some constant C. If S is of class $C^1(A)$ and $\rho(S, A) = \mathbb{C} \backslash \sigma(S)$ then we say that S is* of full class $C^1(A)$.

Remark 5.4 If follows from [GGM] that if A is selfadjoint and S is of class $C^1(A)$ then S is of full class $C^1(A)$.

The $C^1(A)$ property has some consequences expressed in terms of the commutator $[S, A]$:

Definition 5.5 *Let A, S be two closed and densely defined linear operators on $\mathcal{H}$. We define $[A, S]$ as the sesquilinear form with domain $[\mathcal{D}(A^*) \cap \mathcal{D}(S^*)] \times [\mathcal{D}(A) \cap \mathcal{D}(S)]$ given by:*

$$(u, [A, S]v) := (A^*u, Sv) - (S^*u, Av).$$

Proposition 5.6 *Let S be an operator of class $C^1(A)$. Then $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ are cores for S and S^* respectively and the form $[A, S]$ has a unique extension to a continuous sesquilinear form $[A, S]^\circ$ on $\mathcal{D}(S^*) \times \mathcal{D}(S)$. One has:*

$$[A, R(z)]^\circ = -R(z)[A, S]^\circ R(z), \ z \in \rho(S, A) \tag{5.1}$$

where on the right hand side of (5.1) we consider $[A, S]^\circ$ as a bounded operator $\mathcal{D}(S) \to \mathcal{D}(S^*)^*$.

5.2 Smoothness with respect to C_0-semigroups

The $C^1(A)$ class can be further studied if A is the generator of a C_0-semigroup.

Definition 5.7 *A map* $\mathbb{R}^+ \ni t \mapsto W_t \in \mathcal{B}(\mathcal{H})$ *is a* C_0-semigroup *if:*
i) $W_0 = \mathbb{1}$, $W_t W_s = W_{t+s}$, $t, s \geq 0$;
ii) $\mathrm{w}-\lim_{t\to 0^+} W_t = \mathbb{1}$.

We define the generator A of $\{W_t\}$ by the rule

$$\mathcal{D}(A) := \{u \in \mathcal{H} \mid \lim_{t\to 0^+} (\mathrm{i}t)^{-1}(W_t u - u) =: Au \text{ exists}\}.$$

Thus we formally have $W_t = \mathrm{e}^{\mathrm{i}tA}$, which is not the usual convention but is natural in our context.

The map $\mathbb{R}^+ \ni t \mapsto W_t^* \in \mathcal{B}(\mathcal{H})$ is weakly continuous, hence defines a C_0-semigroup. It is easy to see that the generator of W_t^* is $-A^*$.

Let now $\mathcal{G}, \mathcal{H}$ be two Hilbert spaces with $\mathcal{G} \subset \mathcal{H}$ continuously and densely. We identify the adjoint space $\mathcal{H}^*$ with $\mathcal{H}$ by using the Riesz isomorphism. Then by taking adjoints we get a scale of Hilbert spaces $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$.

Definition 5.8 *Let* $\mathcal{G}, \mathcal{H}$ *be as above and let* $\{W_t\}$ *be a* C_0*-semigroup on* $\mathcal{H}$*. in* $\mathcal{H}$*. We say that* $\mathcal{G}$ *is* b-stable (boundedly stable) *under* $\{W_t\}$*, or that* $\{W_t\}$ b-preserves $\mathcal{G}$*, if* $W_t\mathcal{G} \subset \mathcal{G}$ *for all* $t > 0$ *and* $\sup_{0<t<1} \|W_t u\|_{\mathcal{G}} < \infty$ *for each* $u \in \mathcal{G}$*.*

It is easy to see that $\{W_t\}$ extends to a C_0-semigroup in $\mathcal{G}^*$ iff $\mathcal{G}$ is b-stable under $\{W_t^*\}$. If $\mathcal{G}$ is b-stable under $\{W_t\}$, then $\{W_t\}$ induces a C_0-semigroup on $\mathcal{G}$, whose generator we denote by $A_{\mathcal{G}}$. The domain of $A_{\mathcal{G}}$ will be denoted by $\mathcal{D}(A_{\mathcal{G}})$ or $\mathcal{D}(A; \mathcal{G})$. Similarly if $\mathcal{G}$ is b-stable under $\{W_t^*\}$, $\{W_t\}$ induces a C_0-semigroup on $\mathcal{G}^*$ and we will use the notation $A_{\mathcal{G}^*}$, $\mathcal{D}(A_{\mathcal{G}^*}) = \mathcal{D}(A; \mathcal{G}^*)$.

We recall the following definition and result from [GGM]:

Definition 5.9 *Let* $\{W_{1,t}\}, \{W_{2,t}\}$ *be two* C_0*-semigroups on Hilbert spaces* $\mathcal{H}_1, \mathcal{H}_2$ *with generators* A_1, A_2*. We say that* $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *is* of class $C^1(A_1, A_2)$ *if:*

$$\|W_{2,t}S - SW_{1,t}\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \leq Ct,\ 0 \leq t \leq 1.$$

Proposition 5.10 *S is of class $C^1(A_1, A_2)$ if and only if the sesquilinear form ${}_2[S, A]_1$ on $\mathcal{D}(A_2^*) \times \mathcal{D}(A_1)$ defined by $(u_2, {}_2[S, A]_1 u_1) = (S^* u_2, A_1 u_1) - (A_2^* u_2, S u_1)$ is bounded for the topology of $\mathcal{H}_2 \times \mathcal{H}_1$. If we denote by ${}_2[S, A]_1^0 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the associated operator we have:*

$$ {}_2[S, A]_1^0 = \text{s-}\lim_{t\to 0^+} (\mathrm{i}t)^{-1}(SW_{1,t} - W_{2,t}S). \tag{5.2}$$

Remark 5.11 It follows from Proposition 5.10 *that if* $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ *and* $W_{1,t} = W_{2,t} = W_t$*, then* $C^1(A_1, A_2) = C^1(A)$*.*

This relationship between the classes $C^1(A_1, A_2)$ and $C^1(A)$ can be extended to arbitrary closed densely defined operators:

Proposition 5.12 *Let S be a closed densely defined regular operator. Then S is of class $C^1(A)$ if and only if for each $u \in \mathcal{D}(S^*), v \in \mathcal{D}(S)$ there is $c < \infty$ such that $|(S^*u, W_t v) - (u, W_t S v)| \leq ct$ if $0 \leq t \leq 1$. If this is the case, then $\lim_{t\to 0^+} t^{-1}[(S^*u, W_t v) - (u, W_t S v)] = (u, [S, \mathrm{i}A]^\circ v)$.*

5.3 Hypotheses

We recall the abstract set of hypotheses under which a limiting absorption principle is shown in [GGM]. We consider three operators H, H' and A such that *H is self-adjoint, H' is symmetric closed and densely defined, and A is closed and densely defined.* Note that one of the conditions below says that H' is a realization of the formal commutator $[H, \mathrm{i}A]$. We set $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}(H')$ equipped with the intersection topology.

The first two assumptions concern the operators H and H':

(M1) *H is of full class $C^1(H')$, $\mathcal{D}$ is a core of H', and $\mathcal{D}(H) \cap \mathcal{D}(H'^*) = \mathcal{D}$.*

(M2) *A bounded open set $J \subset \mathbb{R}$ is given and there are numbers $a, b > 0$ such that the inequality $H' \geq \left(a1\!\!1_J(H) - b1\!\!1_J(H)^\perp\right) \langle H \rangle$ holds in the sense of forms on $\mathcal{D}$, where $1\!\!1_J(H)^\perp = 1\!\!1 - 1\!\!1_J(H)$.*

We choose a number $c > 0$ such that $H' + c\langle H \rangle \geq \langle H \rangle$ as forms on $\mathcal{D}$. Such a number exists because of hypothesis **(M2)** (e.g. let $c = b + 1$). It follows that the operator $H' + c\langle H \rangle$ is symmetric and bounded below on $\mathcal{D}$ and hence has a Friedrichs extension G satisfying $G \geq \langle H \rangle$. We set

$$\mathcal{G} := \mathcal{D}(G^{\frac{1}{2}}), \text{ equipped with the graph norm.}$$

Note that $\mathcal{G}$ can be identified with the completion of $\mathcal{D}$ for the norm $\|u\|_{\mathcal{G}} = \sqrt{(u, (H' + c\langle H \rangle)u)}$.

We shall denote by $\|\cdot\|_{\mathcal{G}^*}$ the norm dual to $\|\cdot\|_{\mathcal{G}}$. Thus for $v \in \mathcal{H}$

$$\|v\|_{\mathcal{G}^*} = \sup\{|(u, v)| \mid u \in \mathcal{D}, \|u\|_{\mathcal{G}} \leq 1\} = \|G^{-\frac{1}{2}}v\|.$$

The completion of $(\mathcal{H}, \|\cdot\|_{\mathcal{G}^*})$ is canonically identified with the adjoint space $\mathcal{G}^*$. Thus we get a scale of spaces

$$\mathcal{D} \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \subset \mathcal{D}^*, \tag{5.3}$$

with dense and continuous embeddings.

For later use we recall a lemma (see [GGM]) which can be used to verify condition **(M1)** in more concrete situations.

Lemma 5.13 *Let H, M be two selfadjoint operators such that $H \in C^1(M)$ and $\mathcal{D}(H) \cap \mathcal{D}(M)$ is a core for M. Let R be a symmetric operator with $\mathcal{D}(R) \supset \mathcal{D}(H)$ and let us denote by H' the closure of the operator $M + R$ defined on $\mathcal{D}(S) \cap \mathcal{D}(M)$. Then H is of full class $C^1(H')$, $\mathcal{D}(H) \cap \mathcal{D}(H')$ is a core for H' and $\mathcal{D}(H) \cap \mathcal{D}(H') = \mathcal{D}(H) \cap \mathcal{D}(H'*) = \mathcal{D}(H) \cap \mathcal{D}(M)$.*

The last three assumptions concern the operators H, H' and A:

(M3) *A is the generator of a C_0-semigroup $\{W_t\}_{t \geq 0}$ in $\mathcal{H}$.*

(M4) *For all $u \in \mathcal{D}$ we have: $\lim_{t \to 0^+} \frac{1}{t}\left[(u, W_t H u) - (Hu, W_t u)\right] = (u, H'u)$.*

(M5) *There is $H'' \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$ such that $\lim_{t \to 0^+} \frac{1}{t}\left[(u, W_t H' u) - (H'u, W_t u)\right] = (u, H''u), u \in \mathcal{D}$.*

Remark 5.14 Using the results recalled in Subsection 5.2 we see that if $\mathcal{G}$ is b-stable under $\{W_t\}$ and $\{W_t^*\}$ then the conditions **(M4)** and **(M5)** follow from: *$H \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ with $[H, \mathrm{i}A]^\circ = H'$ and $H' \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ with $[H', \mathrm{i}A]^\circ = H''$.*

5.4 Limiting absorption principle

The limiting absorption principle in [GGM] has its most convenient formulation if W_t are isometries and $\mathcal{G}$ is b-stable under $\{W_t^*\}$. Then $\{W_t\}$ extends to a C_0-semigroup on $\mathcal{G}^*$. whose generator is denoted by $A_{\mathcal{G}^*}$. We set for $0 < s < 1$:

$$\mathcal{G}_s^* := \mathcal{D}(|A_{\mathcal{G}^*}|^s), \quad \mathcal{G}_{-s} := (\mathcal{G}_s^*)^*.$$

We emphasize that the absolute value $|A_{\mathcal{G}^*}|$ is defined relatively to the Hilbert space structure of the space $\mathcal{G}^*$. The space $\mathcal{G}_{-s}$ can be defined directly in terms of the generator $A_{\mathcal{G}}^*$ of the C_0-semigroup induced by $\{W_t^*\}$ on $\mathcal{G}$, and both spaces $\mathcal{G}_s^*$ and $\mathcal{G}_{-s}$ can be obtained by complex interpolation.

In the sequel we set $R(z) = (H - z)^{-1}$ and

$$J_0^{\pm} = \{\lambda \pm \mathrm{i}\mu | \lambda \in J, \ \mu > 0\}, \quad J^{\pm} = \{\lambda \pm \mathrm{i}\mu | \lambda \in J, \mu \geq 0\}.$$

Theorem 5.15 *Assume that hypotheses* **(M1)**–**(M5)** *hold and that* W_t *are isometries and* $\mathcal{G}$ *is* b-*stable under* $\{W_t^*\}$. *Then if* $z \in J_0^{\pm}$ *and* $\mathrm{Im} z \neq 0$, $R(z)$ *induces a bounded operator* $R(z) : \mathcal{G}_s^* \to \mathcal{G}_{-s}$ *for all* $\frac{1}{2} < s \leq 1$. *Moreover, the limits* $R(\lambda \pm \mathrm{i}0) := \lim_{\mu \to \pm 0} R(\lambda + \mathrm{i}\mu)$ *exist in the norm topology of* $\mathcal{B}(\mathcal{G}_s^*, \mathcal{G}_{-s})$ *locally uniformly in* $\lambda \in J$, *and the maps* $J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in \mathcal{B}(\mathcal{G}_s^*, \mathcal{G}_{-s})$ *are locally Hölder continuous of order* $s - \frac{1}{2}$.

We refer the reader to [GGM] for more general versions of the limiting absorption principle formulated in terms of optimal Besov spaces.

5.5 The virial theorem

We now recall a version of the virial theorem, proved in [GGM]. To formulate it we first introduce some notation. We will use the convention for quadratic forms recalled in Subsection 3.1.

We recall the following easy fact, which can be checked using the concept of *gauges* on topological vector spaces (see e.g. [ABG, Proposition 2.1.1]):

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with $\mathcal{H}_2 \subset \mathcal{H}_1$ continuously. Then if Q is a symmetric bounded below quadratic form on $\mathcal{H}_1$, Q is closed (resp. closeable) on $\mathcal{H}_2$ if Q is closed (resp. closeable) on $\mathcal{H}_1$. Moreover if Q is closeable on $\mathcal{H}_1$, then the domain of the closure of Q on $\mathcal{H}_2$ is $\mathcal{D}(\overline{Q}) \cap \mathcal{H}_2$.

Let now take $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{D}(H)$ and $Q(u) = (u, H'u) + c(u, \langle H \rangle u)$, with domain $\mathcal{D}$. We saw in Subsection 5.3 that Q is closeable on $\mathcal{D}$ with closure (u, Gu) with domain $\mathcal{G}$. By the above remark, the quadratic form $(u, H'u)$ on $\mathcal{D}(H)$ with domain $\mathcal{D} \cap \mathcal{D}(H)$ is closeable on $\mathcal{D}(H)$. We denote its closure by $(u, \dot{H}u)$, which has domain $\mathcal{G} \cap \mathcal{D}(H)$.

The following result is shown in [GGM].

Proposition 5.16 *Assume that there is a sequence of selfadjoint operators* A_n *such that for each* n *the operator* H *is of class* $C^1(A_n)$ *and* $[H, \mathrm{i}A_n]^\circ$ *is a symmetric form on* $\mathcal{D}(H)$ *and such that*

$$\lim_{n \to \infty} (v, [H, \mathrm{i}A_n]^\circ v) = (v, \dot{H}v),$$

for all $v \in \mathcal{D}(H)$, *where in the l.h.s. we mean the limit in* $\mathrm{I\!R} \cup +\infty$. *Then if* u *is an eigenvector of* H, *we have* $u \in \mathcal{G}$ *and* $(u, \dot{H}u) = 0$.

6 The conjugate operator

In this section we define the conjugate operator A which we will use to prove the Mourre estimate in Section 7 and we verify some of the abstract hypotheses introduced in Subsection 5.3.

6.1 Construction of some vector fields

Let $d(t) \in C^{\infty}(]0, +\infty[)$ be a function as in Subsection 2.2, i.e. such that:

$$d'(t) < 0, \ |d'(t)| \leq Ct^{-1}d(t), \ d(t) \equiv 1 \text{ in } \{t \geq 1\}, \ \lim_{t \to 0} d(t) = +\infty. \tag{6.1}$$

Fix $\chi \in C_0^{\infty}(\mathbb{R})$, $\chi \equiv 1$ in $|t| \leq \frac{1}{2}$, $\chi \equiv 0$ in $|t| \geq 1$. For $0 < \delta \leq \frac{1}{2}$, we set:

$$s^{\delta}(t) := \chi(\frac{t}{\delta})d(\delta)t^{-1} + (1 - \chi)(\frac{t}{\delta})t^{-1}d(t).$$

For $0 < \delta < \frac{1}{2}$, $n \in \mathbb{N}$, we define as in [Sk] a regularized version of s^{δ}:

$$s_n^{\delta}(t) := \chi(\frac{t}{\delta})d(\delta)(t + n^{-1})^{-1} + (1 - \chi)(\frac{t}{\delta})t^{-1}d(t).$$

Note that $s_n^{\delta} \in C^{\infty}([0, +\infty[)$, $|\partial_t^{\alpha} s_n^{\delta}(t)| \leq C(\alpha, n, \delta)$, $\alpha \in \mathbb{N}$. To the functions s^{δ}, s_n^{δ}, we associate the vector fields on $\mathbb{R}^d$:

$$\vec{s}^{\delta}(k) := s^{\delta}(|k|)k, \ \vec{s}_n^{\delta}(k) := s_n^{\delta}(|k|)k, \ k \in \mathbb{R}^d.$$

6.2 The semigroup on the one-particle space

We now construct a C_0-semigroup of isometries associated to the vector field $\vec{s}^{\delta}$.

To the vector fields $\vec{s}^{\delta}$ and $\vec{s}_n^{\delta}$ we associate the operators:

$$\begin{aligned} a^{\delta} &= -\tfrac{1}{2}(\vec{s}^{\delta} \cdot D_k + D_k \cdot \vec{s}^{\delta}), \\ a_n^{\delta} &= -\tfrac{1}{2}(\vec{s}_n^{\delta} \cdot D_k + D_k \cdot \vec{s}_n^{\delta}), \end{aligned}$$

acting on $\mathfrak{s} = C_0^{\infty}(\mathbb{R}^d \backslash \{0\})$. The operators a_n^{δ} are essentially selfadjoint on $\mathfrak{s}$, and we will still denote by a_n^{δ} their closures. It is easy to verify that $\mathcal{D}(a_n^{\delta}) = \{h \in \mathfrak{h} | k \cdot \nabla_k h \in \mathfrak{h}\}$.

The operator a^{δ} is symmetric on $\mathfrak{s}$ but has no selfadjoint extension. To describe its closure it is convenient to introduce polar coordinates as in Subsection 2.2. The unitary map T defined in (2.2) sends $C_0^{\infty}(\mathbb{R}^d \backslash \{0\})$ into $C_0^{\infty}(\mathbb{R}^+ \backslash \{0\}) \otimes C^{\infty}(S^{d-1})$. We have:

$$\begin{aligned} Ta^{\delta}T^{-1} &= \mathrm{i}(m^{\delta}(r)\partial_r + \tfrac{1}{2}(m^{\delta})'(r)) =: \tilde{a}^{\delta}, \\ Ta_n^{\delta}T^{-1} &= \mathrm{i}(m_n^{\delta}(r)\partial_r + \tfrac{1}{2}(m_n^{\delta})'(r)) =: \tilde{a}_n^{\delta}, \end{aligned}$$

on $C_0^{\infty}(\mathbb{R}^+ \backslash \{0\}) \otimes C^{\infty}(S^{d-1})$ where:

$$\begin{aligned} m^{\delta}(r) &:= \ rs^{\delta}(r) = \chi(\tfrac{r}{\delta})d(\delta) + (1 - \chi)(\tfrac{r}{\delta})d(r), \\ m_n^{\delta}(r) &:= \ rs_n^{\delta}(r) = \chi(\tfrac{r}{\delta})d(\delta)r(r + n^{-1})^{-1} + (1 - \chi)(\tfrac{r}{\delta})d(r). \end{aligned} \tag{6.2}$$

Let us note the following easy properties of m^δ:

$$1 \leq m_n^\delta(r) \leq m^\delta(r) \leq C(\delta), \ |\partial_r^\alpha m^\delta(r)| \leq C(\alpha, \delta), \ \alpha \in \mathbb{N}. \tag{6.3}$$

We extend the function m^δ to $\mathbb{R}$ by setting $d(-r) := d(r)$ for $r > 0$ and consider the vector field $m^\delta(r)\frac{\partial}{\partial r}$ as a vector field on $\mathbb{R}$. Let $\mathbb{R} \ni r \mapsto \phi_t(r)$ the associated flow.

For $u \in \tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, \mathrm{d}r) \otimes L^2(S^{d-1})$, $t \geq 0$ we set:

$$\tilde{w}_t^\delta u(r, \theta) := 1\!\!1_{\mathbb{R}^+}(\phi_{-t}(r))|\phi'_{-t}(r)|^{\frac{1}{2}} u(\phi_{-t}(r), \theta). \tag{6.4}$$

Note that since $m^\delta(r) \geq 0$, $\phi_t(r) \geq 0$ if $r, t \geq 0$ and hence:

$$\mathbb{R}^+ \ni t \mapsto \tilde{w}_t^\delta$$

is a C_0-semigroup of isometries of $\tilde{\mathfrak{h}}$. Its generator $\tilde{a}^\delta$ is:

$$\tilde{a}^\delta = \mathrm{i}(m^\delta(r)\frac{\partial}{\partial r} + \frac{1}{2}m^\delta(r)'), \ \mathcal{D}(\tilde{a}^\delta) = H_0^1(\mathbb{R}^+) \otimes L^2(S^{d-1}), \tag{6.5}$$

where $H_0^1(\mathbb{R}^+)$ is the closure of $C_0^\infty(]0, +\infty[)$ in $H^1(\mathbb{R})$. The adjoint semigroup is:

$$\tilde{w}_t^{\delta *} u(r, \theta) = 1\!\!1_{\mathbb{R}^+}(r)|\phi'_t(r)|^{\frac{1}{2}} u(\phi_t(r), \theta), \ t \geq 0, \tag{6.6}$$

with generator

$$\tilde{a}^{\delta *} = -\mathrm{i}(m^\delta(r)\frac{\partial}{\partial r} + \frac{1}{2}m^\delta(r)'), \ \mathcal{D}(\tilde{a}^{\delta *}) = H^1(\mathbb{R}^+) \otimes L^2(S^{d-1}).$$

We now define the corresponding objects on $\mathfrak{h}$ by setting:

$$w_t^\delta := T^{-1}\tilde{w}_t^\delta T, \ w_t^{\delta *} = T^{-1}\tilde{w}_t^{*\delta} T.$$

The closure of a^δ on s is the infinitesimal generator of $\{w_t^\delta\}$ which will be still denoted by a^δ. Hence we have:

$$a^\delta := T^{-1}\tilde{a}^\delta T, \ a^{\delta *} = T^{-1}\tilde{a}^{\delta *} T.$$

6.3 Auxiliary results

We start with an elementary lemma.

Lemma 6.1 *Let* $-\frac{\partial^2}{\partial r^2}$ *be the Laplacian on* $L^2(\mathbb{R}^+, \mathrm{d}r)$ *with Dirichlet condition at* 0. *Then*

$$\tilde{a}^{\delta *}\tilde{a}^\delta \leq -C(\delta)\frac{\partial^2}{\partial r^2}.$$

Proof. By an easy computation we have:

$$\tilde{a}^{\delta *}\tilde{a}^\delta = -\partial_r(m^\delta)^2\partial_r - \frac{1}{2}m^\delta m^{\delta\,\prime\prime} - \frac{1}{4}(m^{\delta\,\prime})^2.$$

Now $m^\delta \leq C(\delta)$ by (6.3) and $m^{\delta\,\prime}$ has compact support (depending on δ) since $d(r) \equiv 1$ in $r \geq 1$. Applying then Poincaré's inequality we obtain the lemma. □

We now prove some consequences of the hypotheses on the interaction which will be useful later.

Lemma 6.2 *Assume hypotheses* **(I1)** *and* **(I2)**. *Then:*

i) $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(a^{\delta}))$, $a^{\delta} v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(\omega^{-\frac{1}{2}})) \cap \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathcal{D}(\omega^{-\frac{1}{2}}))$

ii) $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(a_n^{\delta}))$, $a_n^{\delta} v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(\omega^{-\frac{1}{2}}))$, $\forall\, 0 < \delta \le \frac{1}{2}, n \in \mathbb{N}$;

iii) $(\phi(\mathrm{i} a_n^{\delta} v) \to \phi(\mathrm{i} a^{\delta} v))$, *as quadratic forms on* $\mathcal{D}(|H|^{\frac{1}{2}})$ *when* $n \to \infty$.

iv) $\|(H+b)^{-\frac{1}{2}} \phi(a^{\delta} v)(H+b)^{-\frac{1}{2}}\| \le C$, *uniformly in* $0 < \delta \le \frac{1}{2}$.

Assume in addition hypothesis **(I3)**. *Then:*
v) $a^{\delta} v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathcal{D}(a^{\delta}))$.

Proof. We first investigate some bounds and convergence properties of the functions m^{δ} and m_n^{δ}. We have:

$$\begin{aligned}(m^{\delta}(r))' &= \delta^{-1}\chi'(\tfrac{r}{\delta})d(\delta) - \delta^{-1}\chi'(\tfrac{r}{\delta})d(r) + (1-\chi)(\tfrac{r}{\delta})d'(r),\\ (m_n^{\delta}(r))' &= d(\delta)\delta^{-1}\chi'(\tfrac{r}{\delta})\tfrac{r}{r+n^{-1}} + ((1-\chi)(\tfrac{r}{\delta})d(r))' + \chi(\tfrac{r}{\delta})d(\delta)r^{-1}\tfrac{nr}{(nr+1)^2}.\end{aligned} \tag{6.7}$$

We first observe that since $r \le \delta$ on $\operatorname{supp}\chi$:

$$m^{\delta}(r) \le d(r),\ |(1-\chi)(\frac{r}{\delta})d'(r)| \le \frac{C}{r}d(r),\ |\delta^{-1}\chi'(\frac{r}{\delta})d(\delta)| \le \delta^{-1}|\chi'(\frac{r}{\delta})d(r)| \le \frac{C}{r}d(r), \tag{6.8}$$

uniformly in $0 < \delta \le \frac{1}{2}$. This yields:

$$m_n^{\delta}(r) \le m^{\delta}(r) \le d(r),\ |(m_n^{\delta}(r))'| \le \frac{C}{r}d(r). \tag{6.9}$$

Next:

$$m_n^{\delta}(r) \to m^{\delta}(r),\ (m_n^{\delta}(r))' \to (m^{\delta}(r))' \text{ a.e. when } n \to \infty, \tag{6.10}$$

using (6.2) and (6.7).

Let us now prove *i)*. We set $\tilde{v} = (\mathbb{1}_{\mathcal{K}} \otimes T)v$, $w = \tilde{v}(K+1)^{-\frac{1}{2}}$. It suffices then to prove that $w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes H_0^1(\mathbb{R}^+) \otimes L^2(S^{d-1}))$ and that $\tilde{a}^{\delta} w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{D}(r^{-\frac{1}{2}}))$. That $\tilde{a}^{\delta}(K+1)^{-\frac{1}{2}}\tilde{v}$ belongs to $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{D}(r^{-\frac{1}{2}}))$ can be proved similarly by considering the operator $(K+1)^{-\frac{1}{2}}\tilde{v}$.

Since $(1 + r^{-\frac{1}{2}})d(r)$ is bounded below, hypothesis **(I2)** implies that $w, \partial_r w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, i.e. $w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes H^1(\mathbb{R}^+) \otimes L^2(S^{d-1}))$. By Sobolev's embedding theorem, this implies that for $\psi_1, \psi_2 \in \mathcal{K}$, $(\psi_2, w\psi_1)_{\mathcal{K}} \in C^0(\mathbb{R}^+) \otimes L^2(S^{d-1})$, and hence for $r \ge 0$ the expression $(\psi_2, w\psi_2)_{\mathcal{K}}(r)$ is well defined as an element of $L^2(S^{d-1})$. It suffices to show that $(\psi_2, w\psi_2)_{\mathcal{K}}(0) = 0$ for all $\psi_1, \psi_2 \in \mathcal{K}$ to prove that $w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes H_0^1(\mathbb{R}^+) \otimes L^2(S^{d-1}))$. If there exists ψ_1, ψ_2 such that $(\psi_2, w\psi_2)_{\mathcal{K}}(0) \ne 0$, then $\|(\psi_2, w\psi_1)_{\mathcal{K}}(r)\|_{L^2(S^{d-1})} \ge c > 0$ for $0 \le r \ll 1$. But this contradicts hypothesis **(I2)** which implies that $(1 + r^{-\frac{1}{2}})r^{-1}d(r)(\psi_2, w\psi_1)_{\mathcal{K}}(r) \in L^2(\mathbb{R}^+) \otimes L^2(S^{d-1})$.

Since $m^{\delta}(r) \le d(r)$, $|(m^{\delta}(r))'| \le C(\delta)$, it follows then from hypothesis **(I2)** that $\tilde{a}^{\delta} w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{D}(\omega^{-\frac{1}{2}}))$.

Let us now prove *ii)*. Going to polar coordinates this is equivalent to

$$w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{D}(\tilde{a}_n^{\delta})),\ \tilde{a}_n^{\delta} w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{D}(r^{-\frac{1}{2}})). \tag{6.11}$$

Note that $\tilde{a}_n^\delta$ is the closure of $\mathrm{i}(m_n^\delta(r)\partial_r + \frac{1}{2}(m_n^\delta(r)')$ on $C_0^\infty(\mathbb{R}^+\backslash\{0\}) \otimes C^\infty(S^{d-1})$. Using the fact that $m_n^\delta(r)$ vanishes at 0, it is easy to show that $\mathcal{D}(\tilde{a}_n^\delta) = \{u \in \tilde{\mathfrak{h}} | \tilde{a}_n^\delta u \in \tilde{\mathfrak{h}}\}$. Now

$$\tilde{a}_n^\delta w = \mathrm{i} m_n^\delta(r)\partial_r w + \frac{\mathrm{i}}{2} m_n^\delta(r)' w.$$

Since $(1+r^{-\frac{1}{2}})d(r)\partial_r w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$ we obtain using (6.9) that $(1+r^{-\frac{1}{2}})m_n^\delta(r)\partial_r w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$. Similarly since $(1+r^{-\frac{1}{2}})r^{-1}d(r)w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$ we obtain that $(1+r^{-\frac{1}{2}})m_n^\delta(r)'w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$, which proves (6.11) and completes the proof of *ii)*.

Let us now prove *iii)*. We recall the bound from Proposition 4.1 *i)*:

$$\|(H+b)^{-\frac{1}{2}}\phi(h)(H+b)^{-\frac{1}{2}}\| \leq C\|\mathbb{1}_\mathcal{K} \otimes \omega^{-\frac{1}{2}} h (K+1)^{-\frac{1}{2}}\| \tag{6.12}$$

for $h \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h})$. Using (6.9), (6.10) and the dominated convergence theorem, we obtain that:

$$(1+r^{-\frac{1}{2}})m_n^\delta(r)\partial_r w \to (1+r^{-\frac{1}{2}})m^\delta(r)\partial_r w,$$

$$(1+r^{-\frac{1}{2}})(m_n^\delta(r))'w \to (1+r^{-\frac{1}{2}})(m^\delta(r))'w$$

in $L^2(\mathbb{R}^d; \mathcal{B}(\mathcal{K}))$ when $n \to \infty$. Using (6.12) this proves *iii)*.

Let us now prove *iv)*. We have by (6.8):

$$\|(1+r^{-\frac{1}{2}})m^\delta(r)\partial_r w\| \leq \|(1+r^{-\frac{1}{2}})d(r)\partial_r w\| \leq C,$$

uniformly in $0 < \delta \leq \frac{1}{2}$. Similarly by (6.9):

$$\|(1+r^{-\frac{1}{2}})m^\delta(r)'w\| \leq \|(1+r^{-\frac{1}{2}})r^{-1}d(r)w\| \leq C,$$

uniformly in $0 < \delta \leq \frac{1}{2}$. This yields:

$$\|\mathbb{1}_\mathcal{K} \otimes \omega^{-\frac{1}{2}} a^\delta v (K+1)^{-\frac{1}{2}}\| \leq C,$$

uniformly in $0 < \delta \leq \frac{1}{2}$, which using (6.12) completes the proof of *iv)*.

To prove *v)*, we have to show that $\tilde{a}^\delta w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes H_0^1(\mathbb{R}^+) \otimes L^2(S^{d-1}))$. We have:

$$\partial_r \tilde{a}^\delta w = \frac{3\mathrm{i}}{2}(m^\delta(r))'\partial_r w + \frac{\mathrm{i}}{2}(m^\delta(r))''w + \mathrm{i}m^\delta(r)\partial_r^2 w.$$

Using (6.3) and hypothesis **(I3)**, we obtain that $\partial_r \tilde{a}^\delta w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$, i.e. $\tilde{a}^\delta w \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes H^1(\mathbb{R}^+) \otimes L^2(S^{d-1}))$. As in the proof of *i)*, it follows that for $\psi_1, \psi_2 \in \mathcal{K}$, $(\psi_2, \tilde{a}^\delta w \psi_1)_\mathcal{K}(r)$ is well defined as an element of $L^2(S^{d-1})$, and it remains to prove that $(\psi_2, \tilde{a}^\delta w\psi_1)_\mathcal{K}(0) = 0$. As in the proof of *i)*, if $(\psi_2, \tilde{a}^\delta w\psi_1)_\mathcal{K}(0) \neq 0$ we have $\|(\psi_2, \tilde{a}^\delta w\psi_1)_\mathcal{K}(r)\| \geq c > 0$ for $0 \leq r \ll 1$. Using the fact that $(m^\delta)'$ vanishes near 0 in conjunction with (6.5) and (6.3), we see that this implies that $\|(\psi_2, \partial_r w\psi_1)_\mathcal{K}(r)\| \geq c > 0$ for $0 \leq r \ll 1$. But this contradicts the fact that $(1+r^{-\frac{1}{2}})d(r)\partial_r w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\tilde{\mathfrak{h}})$. □

6.4 The semigroup on Fock space

We now extend the C_0-semigroup w_t^δ to $\mathcal{H}$ by second quantization, as in Subsection 4.3. We set:

$$W_t^\delta := 1\!\!1_{\mathcal{K}} \otimes \Gamma(w_t^\delta),\ \ W_t^{\delta *} = 1\!\!1_{\mathcal{K}} \otimes \Gamma(w_t^{\delta *}).$$

Clearly W_t^δ is a C_0-semigroup of isometries on $\mathcal{H}$. We denote by A^δ its generator. Similarly we set for $n \in \mathbb{N}$:

$$A_n^\delta := 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(a_n^\delta), \tag{6.13}$$

which is the generator of the unitary group $1\!\!1_{\mathcal{K}} \otimes \Gamma(\mathrm{e}^{ita_n^\delta})$

6.5 Estimates of first commutators

We set

$$M^\delta := 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(m^\delta(|k|)),\ M_n^\delta := 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(m_n^\delta),$$

and:

$$R^\delta := -\phi(\mathrm{i}a^\delta v),\ R_n^\delta := -\phi(\mathrm{i}a_n^\delta v).$$

As in Subsection 2.1 we consider a Pauli-Fierz Hamiltonian $H = K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(|k|) + \phi(v)$ acting on $\mathcal{H}$.

Proposition 6.3 *Assume* **(I1)** *and* **(I2)***. Then $H \in C^1(A_n^\delta)$ and:*

$$[H, \mathrm{i}A_n^\delta]^0 = M_n^\delta + R_n^\delta.$$

Proof. We apply Corollary 4.12, checking conditions (4.23) and (4.24). Let $\phi_{n,t} : \mathbb{R}^d \to \mathbb{R}^d$ be the flow associated to the vector field $\vec{s}_n^\delta$. Note that $\vec{s}_n^\delta$ satisfies:

$$|\vec{s}_n^\delta(k)| \leq C(n,\delta)|k|,\ |\partial_k^\alpha \vec{s}_n^\delta(k)| \leq C(\alpha, n, \delta),\ |\alpha| \geq 1. \tag{6.14}$$

We have:

$$\omega_t = \mathrm{e}^{-\mathrm{i}ta_n^\delta}\omega\mathrm{e}^{ita_n^\delta} = \omega(\phi_{n,t}(k)).$$

Using (6.14) we obtain:

$$|\phi_{n,t}(k) - k| \leq C\int_0^t |\phi_{n,s}(k)|\mathrm{d}s, \tag{6.15}$$

which implies that:

$$|\phi_{n,t}(k)| \leq |k| + C\int_0^t |\phi_{n,s}(k)|\mathrm{d}s. \tag{6.16}$$

By Gronwall's lemma we deduce from (6.16) that

$$|\phi_{n,t}(k)| \leq C|k|,\ 0 \leq |t| \leq 1,$$

which by (6.15) gives:

$$|\phi_{n,t}(k) - k| \leq C|t||k|,\ 0 \leq |t| \leq 1,$$

and hence:

$$|\omega(\phi_{n,t}(k)) - \omega(k)| \leq C|t|\omega(k),$$

which is (4.23). It remains to check (4.24). But this follows from hypothesis **(I2)** and Lemma 6.2 *ii)*. Finally noting that $[\omega, \mathrm{i}a_n^\delta]^0 = m_n^\delta$, we obtain the proposition. □

We now check hypothesis **(M1)** and identify the operator H' and the spaces $\mathcal{D}$ and $\mathcal{G}$.

Lemma 6.4 *Assume hypotheses* **(I1)** *and* **(I2)**. *Then:*

i) $H \in C^1(M^\delta)$, $\mathcal{D}(H) \cap \mathcal{D}(M^\delta)$ *is a core for* M^δ *and* R^δ *is bounded and symmetric on* $\mathcal{D}(H)$.

ii) Let ${H^\delta}'$ *be the closure of* $M^\delta + R^\delta$ *on* $\mathcal{D}(M^\delta) \cap \mathcal{D}(H)$. *Then* $H, {H^\delta}'$ *satisfy hypothesis* **(M1)**.

iii) Let $B := K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(b)$, *for* $b = (k^2+1)^{\frac{1}{2}}$. *We have:* $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}({H^\delta}') = \mathcal{D}(M^\delta) \cap \mathcal{D}(H) = \mathcal{D}(B)$, $\mathcal{G} = \mathcal{D}(B^{\frac{1}{2}})$, *where* $\mathcal{G}$ *is defined in Subsection 5.3.*

Proof. Let us first prove *i)*. To prove that $H \in C^1(M^\delta)$ we apply Corollary 4.12 with $a = m^\delta$. Condition (4.23) is clearly satisfied since $[\omega, m^\delta] = 0$. Condition (4.24) follows from **(I1)** and the fact that m^δ is bounded and $[\omega, m^\delta] = 0$. The fact that R^δ is bounded and symmetric on $\mathcal{D}(H)$ follows from Lemma 6.2 *i)* and Proposition 4.1 *i)*.

We note next that by (6.3) $\mathcal{D}(M^\delta) = \mathcal{K} \otimes \mathcal{D}(N)$. Using that $\mathcal{D}(H) = \mathcal{D}(H_0)$, we obtain $\mathcal{D}(H) \cap \mathcal{D}(M^\delta) = \mathcal{D}(B)$. Clearly $\mathcal{D}(B)$ is a core for M^δ, which completes the proof of *i)*.

Using *i)* and Lemma 5.13, we obtain *ii)*. Let us now prove *iii)*. We have already seen that $\mathcal{D} = \mathcal{D}(B)$ in the proof of *i)*. To prove the second statement of *iii)*, we use the fact that by Lemma 6.2 *iv)*, R^δ is $H-$ form bounded. It follows that the norm $\|u\|_{\mathcal{G}}$ on $\mathcal{D}$ is equivalent to the norm $\sqrt{(u, (M^\delta + H_0 + 1)u)}$, which is equivalent to the norm $\|B^{\frac{1}{2}} u\|$. Since $\mathcal{D} = \mathcal{D}(B)$ is a form core for B, we obtain that $\mathcal{G} = \mathcal{D}(B^{\frac{1}{2}})$ as claimed. □

Proposition 6.5 *Assume hypotheses* **(I1)** *and* **(I2)**. *Then if* $u \in \mathcal{D}(H)$ *is an eigenvector of* H, *we have* $u \in \mathcal{D}(N^{\frac{1}{2}})$ *and* $(u, (M^\delta + R^\delta)u) = 0$.

Proof. It suffices to verify that H and ${H^\delta}'$ satisfy the hypotheses of Proposition 5.16 for the sequence of selfadjoint operators $\{A_n^\delta\}$ defined in (6.13).

For $u \in \mathcal{D}$, we have $(u, {H^\delta}' u) = (u, M^\delta u) + (u, R^\delta u)$. By Lemma 6.2 *iv)*, we know that $(u, R^\delta u)$ is a bounded quadratic form on $\mathcal{D}(H)$. By the discussion before Proposition 5.16, we see that the closure of $(u, M^\delta u)$ on the Hilbert space $\mathcal{D}(H)$ with domain $\mathcal{D}$ is the quadratic form $(u, M^\delta u)$ with domain $\mathcal{D}(H) \cap \mathcal{D}((M^\delta)^{\frac{1}{2}})$. Hence the closure $(u, \dot{H} u)$ of $(u, {H^\delta}' u)$ is the quadratic form $(u, \dot{H} u) = (u, M^\delta u) + (u, R^\delta u)$, with domain $\mathcal{D}(H) \cap \mathcal{D}((M^\delta)^{\frac{1}{2}})$.

Next from Proposition 6.3 we know that $H \in C^1(A_n^\delta)$ and $[H, \mathrm{i}A_n^\delta]^0 = M_n^\delta + R_n^\delta$ is a bounded quadratic form on $\mathcal{D}(H)$. We observe that m_n^δ is increasing w.r.t. n and $m^\delta(k) = \sup_n m_n^\delta(k)$. Using monotone convergence this implies that

$$\lim_{n \to \infty} (u, M_n^\delta u) = (u, M^\delta u), \tag{6.17}$$

as quadratic forms on the Hilbert space $\mathcal{D}(H)$, where on the r.h.s. we consider $(u, M^\delta u)$ with domain $\mathcal{D}(H) \cap \mathcal{D}((M^\delta)^{\frac{1}{2}})$. Finally by Lemma 6.2 *iii)*, we obtain that

$$\lim_{n \to \infty} R_n^\delta = R^\delta$$

as bounded quadratic forms on $\mathcal{D}(H)$. Using (6.17) and the description of $(u, \dot{H} u)$ given above, we see that the hypotheses of Proposition 5.16 are satisfied. □

Proposition 6.6 *Assume* **(I1)** *and* **(I2)**. *Then:*

i) $\{W_t\}$ *and* $\{W_t^*\}$ $b-$*preserve* $\mathcal{G}$;

ii) $H \in C^1(A^\delta; \mathcal{G}, \mathcal{G}^*)$ *and* $[H, \mathrm{i}A^\delta]^0 = {H^\delta}'$ *on* $\mathcal{D}$.

Consequently the hypotheses **(M3)** *and* **(M4)** *are satisfied.*

Proof. Recall that $\mathcal{D}(H)\cap\mathcal{D}(H^{\delta\,\prime})$ equals $\mathcal{D}(B)$. To prove *i)* and *ii)* we will apply Proposition 4.10. To check the assumptions, it is convenient to use polar coordinates by conjugation with the unitary map $T:\mathfrak{h}\to\tilde{\mathfrak{h}}$ introduced in (2.2) and to work with the C_0-semigroup $\tilde{w}_t^\delta$. Let us denote again by b and ω the operators of multiplication by $(r^2+1)^{\frac{1}{2}}$ and r on $\tilde{\mathfrak{h}}$.

Using (6.4) and (6.6) we have:

$$\tilde{w}_t^* b\tilde{w}_t = b\circ\phi_t(r),\ \ \tilde{w}_t b\tilde{w}_t^* = 1\!\!1_{\mathrm{I\!R}^+}\circ\phi_{-t}(r)b\circ\phi_{-t}(r)$$

(recall that the flow ϕ_t was extended to a flow on $\mathrm{I\!R}$). Since $|m^\delta(r)|\leq C$ we have:

$$|\phi_t(r)-r|\leq C|t|,\ 0\leq|t|\leq 1, \tag{6.18}$$

and

$$|b\circ\phi_t(r)-b(r)|\leq\|\nabla b\|_\infty|\phi_t(r)-r|.$$

This yields:

$$b\circ\phi_t(r)\leq C(1+|t|)b(r),\ 0\leq|t|\leq 1,$$

since $b(r)\geq 1$. We see that condition (4.19) in Proposition 4.10 satisfied and thus $\{W_t\}$ and $\{W_t^*\}$ $b-$preserve $\mathcal{G}$. This completes the proof of *i)*.

Let us now prove *ii)*. Clearly $\omega\leq Cb$. We have $\omega w_t - w_t\omega = (\omega-\omega\circ\phi_{-t})w_t$. By (6.18) we obtain:

$$|\phi_t(r)-r|\leq C|t|b(r),\ 0\leq t\leq 1,$$

and hence

$$|(\omega-\omega\circ\phi_{-t})|\leq C|t|b.$$

Since $\{w_t\}$ b-preserves $\mathcal{D}(b^{\frac{1}{2}})$, this implies condition(4.21). Finally by **(I2)** and Lemma 6.2 we know that $v\in\mathcal{B}(\mathcal{D}((K)^{\frac{1}{2}}),\mathcal{K}\otimes\mathcal{D}(a^\delta))$ so condition (4.22) holds. This shows that $H\in C^1(A^\delta;\mathcal{G},\mathcal{G}^*)$ and that $[H,\mathrm{i}A^\delta]^0 = M^\delta+R^\delta$, as elements of $\mathcal{B}(\mathcal{G},\mathcal{G}^*)$. Since M^δ and R^δ are also bounded operators on $\mathcal{D}$, this completes the proof of the proposition. □

6.6 Estimates of second commutators

Proposition 6.7 *Assume hypotheses* **(I1)**, **(I2)** *and* **(I3)**. *Let us still denote by $H^{\delta\,\prime}$ the operator $[H,\mathrm{i}A^\delta]^0\in\mathcal{B}(\mathcal{G},\mathcal{G}^*)$. Then $H^{\delta\,\prime}\in C^1(A^\delta;\mathcal{G},\mathcal{G}^*)$ and*

$$[H^{\delta\,\prime},\mathrm{i}A^\delta]^0 = 1\!\!1_{\mathcal{K}}\otimes\mathrm{d}\Gamma(m^\delta\partial_r m^\delta)-\phi((a^\delta)^2v).$$

Consequently the hypothesis **(M5)** *is satisfied.*

Proof. Since $H^{\delta\,\prime}=1\!\!1\otimes\mathrm{d}\Gamma(m^\delta)-\phi(\mathrm{i}a^\delta v)$, we will apply Proposition 4.10 to $H^{\delta\,\prime}$ with ω replaced by m^δ and v replaced by $-\mathrm{i}a^\delta v$ and b as before. Again we introduce polar coordinates and work with the C_0-semigroup $\tilde{w}_t^\delta$. We still denote by $m^\delta=m^\delta(r)$ the operator $Tm^\delta T^{-1}$. Clearly $m^\delta\leq Cb$ and $m^\delta\tilde{w}_t^\delta-\tilde{w}_t^\delta m^\delta=(m^\delta-m^\delta\circ\phi_{-t})w_t$. Using (6.18), we get $|m^\delta-m^\delta\circ\phi_{-t}|\leq C\|\partial_r m^\delta\|_\infty tb$ for $0\leq t\leq 1$. Since $\{\tilde{w}_t^\delta\}$ b-preserves $\mathcal{D}(b^{\frac{1}{2}})$ this implies condition (4.21).

Using then **(I3)** and Lemma 6.2 *v)*, we see that condition (4.22) is also satisfied. Applying Proposition 4.10 we obtain the proposition. □

7 The Mourre estimate for Pauli-Fierz Hamiltonians

This section is devoted to the proof of the Mourre estimate for Pauli-Fierz Hamiltonians, i.e. to the verification of condition **(M2)** in Subsection 5.3, for a Pauli-Fierz Hamiltonian H and the operator $H^{\delta\,\prime} = M^{\delta} + R^{\delta}$ introduced in Subsection 6.4. In all this section, we will assume conditions **(H0)**, **(I1)** and **(I2)**. We recall that the dispersion relation ω is equal to $|k|$.

Moreover to simplify notation, if w is a coupling function in $\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ and a is an operator on $\mathfrak{h}$, we will denote by aw the coupling function$(\mathbb{1}_{\mathcal{K}}\otimes a)w$.

We first describe some abstract results allowing to deduce from a local Mourre estimate with a compact error term a uniformly local Mourre estimate without error. This part is analogous to a standard step in the proof of the Mourre estimate for N-particle Schrödinger operators. We then proceed to the proof of the Mourre estimate using position space and momentum space decompositions and an induction argument.

7.1 Local positivity of quadratic forms

In this subsection we collect various abstract results about quadratic forms related to the Mourre estimate.

The basic objects are a selfadjoint operator H, a closed densely defined positive quadratic form M on the Hilbert space $\mathcal{D}(H)$ and a bounded quadratic form R on $\mathcal{D}(H)$. In addition we assume the following virial relation:

$$(u, (M+R)u) = 0, \text{ if } u \in \mathcal{D}(H) \text{ and } Hu = \lambda u,\ \lambda \in \mathbb{R}. \tag{7.1}$$

We denote again by M the selfadjoint operator associated to the quadratic form M. We will later apply the results of this subsection to $M = M^{\delta}$, $R = R^{\delta}$ (note that by Proposition 6.5 (7.1) is then satisfied).

Let us now fix some notation and introduce a definition. We fix a cutoff function $f \in C_0^{\infty}(\mathbb{R})$, $0 \le f \le 1$, $f(\lambda) \equiv 1$ if $|\lambda| \le \frac{1}{2}$, $f(\lambda) \equiv 0$ if $|\lambda| \ge 1$. For $E \in \mathbb{R}, \kappa > 0$, we set

$$f_{E,\kappa}(\lambda) := f(\frac{\lambda - E}{\kappa}).$$

Definition 7.1 *We say that the* Mourre estimate *holds at E if $\forall\, \epsilon_0 > 0$, $\exists\, C, \kappa > 0$, K compact such that:*

$$M + f_{E,\kappa}(H) R f_{E,\kappa}(H) \ge (1-\epsilon_0)\mathbb{1} - C(1 - f_{E,\kappa}(H))^2 - K.$$

We say that the strict Mourre estimate *holds at E if $\forall\, \epsilon_0 > 0$, $\exists\, C, \kappa > 0$ such that:*

$$M + f_{E,\kappa}(H) R f_{E,\kappa}(H) \ge (1-\epsilon_0)\mathbb{1} - C(1 - f_{E,\kappa}(H))^2.$$

The inequalities in Def. 7.1 should be understood as inequalities on $\mathcal{D}(M)$. In fact the term $f_{E,\kappa}(H) R f_{E,\kappa}(H)$ is bounded, since R is bounded on $\mathcal{D}(H)$.

Note that in Def. 7.1 we use a non standard formulation of the Mourre estimate (the optimal constant in the r.h.s. being equal to 1). This formulation turns out to be necessary for our later induction proof. Its connection with the formulation of the Mourre estimate in **(M2)** is given in the following easy lemma.

Lemma 7.2 *Assume in addition that the quadratic form R is bounded on $\mathcal{D}(|H|^{\frac{1}{2}})$. Then if the strict Mourre estimate holds at E there exists an open interval $J \ni E$ and $a, b > 0$ such that:*

$$M + R \geq \left(a\mathbb{1}_J(H) - b\mathbb{1}_J^{\perp}(H) \right) \langle H \rangle.$$

Proof. Set $f = f_{E,\kappa}(H)$, $f^{\perp} = (1 - f_{E,\kappa})(H)$ for E, κ as in Def 7.1 and let J an bounded open interval such that $\mathbb{1}_J \leq f$. Then we have for $\epsilon > 0$

$$fRf^{\perp} + f^{\perp}Rf \geq -\epsilon fR\langle H \rangle^{-1}Rf - \epsilon^{-1}f^{\perp}\langle H \rangle f^{\perp} \geq -C\epsilon f^2 - \epsilon^{-1}(f^{\perp})^2\langle H \rangle.$$

Choosing $\epsilon \ll 1$ and using the strict Mourre estimate, this yields:

$$M + R \geq \frac{1}{2}f^2 - C(f^{\perp})^2\langle H \rangle \geq \left(a\mathbb{1}_J(H) - b\mathbb{1}_J^{\perp}(H) \right) \langle H \rangle,$$

for some $a > 0$, since $\mathbb{1}_J(H) \leq f$ and $\mathbb{1}_J^{\perp}(H) \geq f^{\perp}$. $\Box$

Lemma 7.3 *The following estimates hold:*

i) $(1 - f_{E',\kappa'}(H)) \geq (1 - f_{E,\kappa}(H))$, *for* $|E - E'| \leq \kappa/4$, $0 < \kappa' \leq \kappa/4$,

ii) $\forall\, \epsilon > 0, \exists\, C$ *such that* $f_{E',\kappa'}(H)Rf_{E',\kappa'}(H) \geq f_{E,\kappa}(H)Rf_{E,\kappa}(H) - \epsilon - C(1 - f_{E',\kappa'}(H))^2$,

uniformly for $|E - E'| \leq \kappa/4$, $0 < \kappa' \leq \kappa/4$.

Proof. If $|E - E'| \leq \kappa/4$, $0 < \kappa' \leq \kappa/4$, then $f_{E',\kappa'}(H)f_{E,\kappa}(H) = f_{E',\kappa'}(H)$, which proves *i)*. Next

$$\begin{aligned}
& f_{E',\kappa'}(H)Rf_{E',\kappa'}(H) \\
= \; & f_{E,\kappa}(H)Rf_{E,\kappa}(H) - (1 - f_{E',\kappa'}(H))f_{E,\kappa}(H)Rf_{E',\kappa'}(H) \\
& -f_{E',\kappa'}(H)Rf_{E,\kappa}(H)(1 - f_{E',\kappa'}(H)) - (1 - f_{E',\kappa'}(H))f_{E,\kappa}(H)Rf_{E,\kappa}(H)(1 - f_{E',\kappa'}(H)) \\
\geq \; & f_{E,\kappa}(H)Rf_{E,\kappa}(H) - \epsilon\|f_{E,\kappa}(H)Rf_{E,\kappa}(H)\|^2 \\
& -(\epsilon^{-1} + \|f_{E,\kappa}(H)Rf_{E,\kappa}(H)\|)(1 - f_{E',\kappa'}(H))^2. \; \Box
\end{aligned}$$

We deduce from Lemma 7.3 that

Proposition 7.4 *Assume that the Mourre estimate (resp. the strict Mourre estimate) holds at E. Then for all $\epsilon_0 > 0$ there exists $\kappa_0, C > 0$, K compact such that for all $0 < \kappa \leq \kappa_0$:*

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq (1 - \epsilon_0)\mathbb{1} - C(1 - f_{E,\kappa}(H))^2 - K,$$

(resp.)

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq (1 - \epsilon_0)\mathbb{1} - C(1 - f_{E,\kappa}(H))^2.$$

Proposition 7.5 *Assume that the Mourre estimate holds at E. Then:*

i) For κ' small enough $\mathrm{Tr}\mathbb{1}^{\mathrm{pp}}_{[E-\kappa',E+\kappa']}(H) < \infty$,

ii) If $E \notin \sigma_{\mathrm{pp}}(H)$, the strict Mourre estimate holds at E.

Proof. Let us first prove *i)*. Take $\kappa' \leq \kappa/2$ and assume the contrary. Let u_n be eigenvectors of H with $u_n = \mathbb{1}_{[E-\kappa', E+\kappa']}(H)u_n$, $u_n \to 0$ weakly. Since $f_{E,\kappa}(H)u_n = u_n$, we obtain by (7.1):

$$\begin{aligned} 0 = &\ (u_n, (M+R)u_n) \\ = &\ (u_n, (M + f_{E,\kappa}(H)Rf_{E,\kappa}(H))u_n) \\ \geq &\ (1-\epsilon_0)\|u_n\|^2 - (u_n, Ku_n). \end{aligned}$$

Since K is compact, $Ku_n \to 0$ strongly, which gives a contradiction.

Let us now prove *ii)*. We write:

$$\begin{aligned} K = &\ \tfrac{1}{2}(Kf_{E,\kappa}(H) + K(1 - f_{E,\kappa}(H)) + \mathrm{hc}) \\ \geq &\ \tfrac{1}{2}(Kf_{E,\kappa}(H) + \mathrm{hc}) - \epsilon\|K\|^2 - \epsilon^{-1}(1 - f_{E,\kappa}(H))^2, \end{aligned}$$

which proves the statement since $f_{E,\kappa}(H)$ tends strongly to 0 when $\kappa \to 0$. □

The following proposition is an abstract version of [MS, Lemma 4.4].

Proposition 7.6 *Assume that the Mourre estimate holds at E. Then $\forall\, \epsilon_0 > 0$, $\exists\, C, \kappa$ such that*

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq -\epsilon_0\mathbb{1} - C(1 - f_{E,\kappa}(H))^2.$$

Proof. If $E \notin \sigma_{\mathrm{pp}}(H)$ the result is clear by Proposition 7.5 *ii)*.

Assume now that $E \in \sigma_{\mathrm{pp}}(H)$ and let $P = \mathbb{1}_{\{E\}}(H)$. By (7.1), $P : \mathcal{H} \to \mathcal{D}(M^{\frac{1}{2}})$ and hence $(1-P) : \mathcal{D}(M^{\frac{1}{2}}) \to \mathcal{D}(M^{\frac{1}{2}})$. Moreover P is compact by Proposition 7.5.

Set

$$B(E,\kappa) := M + f_{E,\kappa}(H)Rf_{E,\kappa}(H).$$

As quadratic forms on $\mathcal{D}(M^{\frac{1}{2}})$, we have:

$$\begin{aligned} B(E,\kappa) = &\ PB(E,\kappa)P + 2\mathrm{Re}PB(E,\kappa)(1-P) \\ &\ +(1-P)B(E,\kappa)(1-P). \end{aligned} \tag{7.2}$$

We will estimate separately the three terms in the r.h.s of (7.2). By (7.1) we have

$$PB(E,\kappa)P = 0. \tag{7.3}$$

By the Mourre estimate, since $(1-P)(1-f_{E,\kappa}(H))=(1-f_{E,\kappa}(H))$:

$$\begin{array}{rl} & (1-P)B(E,\kappa)(1-P)\\ \geq & -\epsilon_0-(1-P)K(1-P)-C(1-f_{E,\kappa}(H))^2\\ = & -\epsilon_0-\Big((1-P)f_{E,\kappa}(H)K(1-P)\\ & +(1-P)(1-f_{E,\kappa}(H))K(1-P)\Big)-C(1-f_{E,\kappa}(H))^2\\ = & -\epsilon_0-\frac{1}{2}\Big((1-P)f_{E,\kappa}(H)K(1-P)\\ & +(1-f_{E,\kappa}(H))K(1-P)+\mathrm{hc}\Big)-C(1-f_{E,\kappa}(H))^2\\ \geq & -\epsilon_0-\frac{1}{2}((1-P)f_{E,\kappa}(H)K(1-P)+\mathrm{hc})-\epsilon\|K(1-P)\|^2\\ & -\epsilon^{-1}(1-f_{E,\kappa}(H))^2-C(1-f_{E,\kappa}(H))^2\\ \geq & -2\epsilon_0-\frac{1}{2}((1-P)f_{E,\kappa}(H)K(1-P)+\mathrm{hc})-C'(1-f_{E,\kappa}(H))^2. \end{array}$$

Next we use that $(1-P)f_{E,\kappa}(H)\to 0$ strongly when $\kappa\to 0$, which yields

$$(1-P)B(E,\kappa)(1-P)\geq -3\epsilon_0-C'(1-f_{E,\kappa}(H))^2, \tag{7.4}$$

for κ small enough.

We consider now the term $PB(E,\kappa)(1-P)$. We have:

$$\begin{array}{rl} & PB(E,\kappa)(1-P)\\ = & PM(1-P)+Pf_{E,\kappa}(H)Rf_{E,\kappa}(H)(1-P)\\ =: & I_1+I_2. \end{array}$$

We write I_2 as:

$$I_2=P(PRf_{E,\kappa_0}(H))f_{E,\kappa}(H)(1-P),\ \text{ for } \kappa\leq 2\kappa_0,$$

and use that $PRf_{E,\kappa_0}(H)$ is bounded, P is compact and $f_{E,\kappa}(H)(1-P)\to 0$ strongly when $\kappa\to 0$ to obtain:

$$\mathrm{Re}I_2\geq -\epsilon_0,\ \text{ for } \kappa \text{ small enough.} \tag{7.5}$$

Next we write I_1 as:

$$\begin{array}{rl} PM(1-P)= & P\mathbb{1}_{[n_0,\infty[}(M)M-P\mathbb{1}_{[n_0,\infty[}(M)MP+P\mathbb{1}_{[0,n_0[}(M)M(1-P)\\ = & P(PM^{\frac{1}{2}}\mathbb{1}_{[n_0,\infty[}(M))M^{\frac{1}{2}}-P(PM^{\frac{1}{2}}\mathbb{1}_{[n_0,\infty[}(M))M^{\frac{1}{2}}P\\ & +(P\mathbb{1}_{[0,n_0[}(M)M)f_{E,\kappa}(H)(1-P)+(P\mathbb{1}_{[0,n_0[}(M)M)(1-f_{E,\kappa}(H))\\ =: & R_1+R_2+R_3+R_4. \end{array}$$

Let us first bound R_1+R_2: if we set $a=PPM^{\frac{1}{2}}\mathbb{1}_{[n_0,\infty[}(M)$, $b^*=M^{\frac{1}{2}}$ or $M^{\frac{1}{2}}P$ and use the bound $ab^*+ba^*\geq -\rho aa^*-\rho^{-1}bb^*$, we obtain:

$$\mathrm{Re}(R_1+R_2)\geq -\rho P(PM^{\frac{1}{2}}\mathbb{1}_{[n_0,\infty[}(M)M^{\frac{1}{2}}P)P-\frac{1}{2}\rho^{-1}M-\frac{1}{2}\rho^{-1}PMP.$$

We claim that

$$\text{s-}\lim_{n_0\to\infty} PM^{\frac{1}{2}}1\!\!1_{[n_0,\infty[}(M)M^{\frac{1}{2}}P = 0. \tag{7.6}$$

In fact P sends $\mathcal{H}$ into $\mathcal{D}(M^{\frac{1}{2}})$ hence $M^{\frac{1}{2}}P$ is bounded on $\mathcal{H}$ and $1\!\!1_{[n_0,\infty[}(M)$ tends strongly to 0. Since $PM^{\frac{1}{2}}$ is bounded we obtain (7.6).

Using (7.6) and the fact that P is compact, we see that for any $\epsilon_0, \sigma > 0$, we can choose $\rho \gg 1$ and then fix $n_0 \gg 1$ so that:

$$\mathrm{Re}(R_1 + R_2) \geq -\epsilon_0 - \sigma M. \tag{7.7}$$

Let us now bound R_3: $P1\!\!1_{[0,n_0[}(M)M$ is compact, $f_{E,\kappa}(H)(1-P) \to 0$ strongly when $\kappa \to 0$, so for $0 < \kappa < \kappa_0$, we have:

$$\mathrm{Re}R_3 \geq -\epsilon_0. \tag{7.8}$$

Finally let us bound R_4: we have:

$$\mathrm{ReR}_4 \geq -\frac{1}{2}\rho P1\!\!1_{[0,n_0[}(M)M^2P - \frac{1}{2}\rho^{-1}(1-f_{E,\kappa}(H))^2. \tag{7.9}$$

Collecting (7.3)—(7.9), we obtain finally that for any $\epsilon_0, \sigma > 0$ there exist C, κ_0 such that for $0 < \kappa \leq \kappa_0$:

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq -8\epsilon_0 - \sigma M - C(1-f_{E,\kappa}(H))^2.$$

This yields:

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq -\frac{8\epsilon_0}{1+\sigma} + \frac{\sigma}{1+\sigma}f_{E,\kappa}(H)Rf_{E,\kappa}(H) - \frac{C}{1+\sigma}(1-f_{E,\kappa}(H))^2.$$

Now we use that $\|f_{E,\kappa}(H)Rf_{E,\kappa}(H)\| \leq C_0$, uniformly for $0 < \kappa \leq 1$, which yields

$$\begin{array}{rl} & M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \\ \geq & -\frac{8\epsilon_0 + C_0\sigma}{1+\sigma} - \frac{C}{1+\sigma}(1-f_{E,\kappa}(H))^2 \\ \geq & -10\epsilon_0 - C(1-f_{E,\kappa}(H))^2, \end{array}$$

for σ small enough. This completes the proof of the proposition. □

Lemma 7.7 *Assume that the Mourre estimate holds at E. Then for all $\epsilon_0 > 0$, there exists $\delta, \kappa, C > 0$, such that for all E' with $|E - E'| \leq \delta$:*

$$M + f_{E',\kappa}(H)Rf_{E',\kappa}(H) \geq -\epsilon_0 1\!\!1 - C(1-f_{E',\kappa}(H))^2.$$

Proof. this follows immediately from Proposition 7.6 and Lemma 7.3. □

From Lemma 7.7 and a covering argument, we deduce the following uniform version of Proposition 7.6.

Proposition 7.8 *Assume that the Mourre estimate holds at all $E \in I$, I compact interval. Then $\forall\, \epsilon_0 > 0$, $\exists\, C, \kappa$ such that $\forall\, E \in I$:*

$$M + f_{E,\kappa}(H)Rf_{E,\kappa}(H) \geq -\epsilon_0 1\!\!1 - C(1-f_{E,\kappa}(H))^2.$$

7.2 Position space decomposition

Following [DG1], we now describe a geometric decomposition of the quadratic form

$$B^\delta(E,\kappa) = M^\delta - f_{E,\kappa}(H)\phi(\mathrm{i}a^\delta v)f_{E,\kappa}(H)$$

which will be useful to prove a Mourre estimate. This decomposition amounts to treat separately the bosons close to the origin and those close to infinity.

Let $j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_\infty \in C^\infty(\mathbb{R}^d)$, $0 \leq j_0$, $0 \leq j_\infty$, $j_0^2 + j_\infty^2 = 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). We denote again by j_0, j_∞ the operators $j_0(x), j_\infty(x)$, where $x = \mathrm{i}\nabla_k$.

We set $j := (j_0, j_\infty)$ and for $R \geq 1$, $j^R = (j_0^R, j_\infty^R)$, where $j_0^R(x) = j_0(\frac{x}{R})$, $j_\infty^R(x) = j_\infty(\frac{x}{R})$.

To j^R we associate the isometric operator defined in Subsection 3.2:

$$\check{\Gamma}(j^R) : \mathcal{H} \to \mathcal{H}^{\mathrm{ext}} = \mathcal{H} \otimes \Gamma(\mathfrak{h}).$$

We define also the following Hamiltonians acting on $\mathcal{H}^{\mathrm{ext}}$:

$$\begin{aligned}
H^{\mathrm{ext}} &:= H \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}} \otimes \mathrm{d}\Gamma(\omega),\\
H_0^{\mathrm{ext}} &:= H_0 \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}} \otimes \mathrm{d}\Gamma(\omega),\\
N^{\mathrm{ext}} &:= N \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}} \otimes N,\\
M^{\delta\,\mathrm{ext}} &:= M^\delta \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}} \otimes M^\delta,\\
M_0^\delta &:= M^\delta \otimes 1\!\!1_{\Gamma(\mathfrak{h})},\\
M_\infty^\delta &:= 1\!\!1_{\mathcal{H}} \otimes M^\delta.
\end{aligned}$$

We note that $\check{\Gamma}(j^R)N = N^{\mathrm{ext}}\check{\Gamma}(j^R)$, so that $\check{\Gamma}(j^R)$ sends $\mathcal{D}(N^\alpha)$ into $\mathcal{D}((N^{\mathrm{ext}})^\alpha$, $\alpha = \frac{1}{2}, 1$.

We set

$$B^{\delta\,\mathrm{ext}}(E,\kappa) := M^{\delta\,\mathrm{ext}} - f_{E,\kappa}(H^{\mathrm{ext}})\phi(\mathrm{i}a^\delta v) \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f_{E,\kappa}(H^{\mathrm{ext}}),$$

as a quadratic form on $\mathcal{D}((M^{\delta\,\mathrm{ext}})^{\frac{1}{2}})$.

Proposition 7.9 *Assume* **(H0)** *and* **(I1)**. *Let $w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ such that*

$$w \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathcal{D}(\omega^{-\frac{1}{2}})) \cap \mathcal{B}(\mathcal{K}, \mathcal{D}(K^{\frac{1}{2}})^* \otimes \mathcal{D}(\omega^{-\frac{1}{2}})), \tag{7.10}$$

and let $f \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}
f(H) = &\ \check{\Gamma}(j^R)^* f(H^{\mathrm{ext}})\check{\Gamma}(j^R) + N^{\frac{1}{2}} o_f(R^0) N^{\frac{1}{2}},\\
M^\delta + f(H)\phi(w)f(H) = &\ \check{\Gamma}(j^R)^*\Big(M^{\delta\,\mathrm{ext}} + f(H^{\mathrm{ext}})\phi(w)\otimes 1\!\!1_{\Gamma(\mathfrak{h})} f(H^{\mathrm{ext}})\Big)\check{\Gamma}(j^R)\\
&+ N^{\frac{1}{2}} o_{f,\delta}(R^0) N^{\frac{1}{2}}.
\end{aligned}$$

Remark 7.10 If we apply Proposition 7.9 to $w = -\mathrm{i}a^\delta v$, using Lemma 6.2 *i)*, we obtain that

$$B^\delta(E,\kappa) = \check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E,\kappa)\check{\Gamma}(j^R) + N^{\frac{1}{2}} o_{\delta,E,\kappa}(R^0) N^{\frac{1}{2}}. \tag{7.11}$$

Proof. For $z \in \mathbb{C} \backslash \mathbb{R}$ we have:

$$\begin{array}{rl} & (z - H^{\mathrm{ext}})^{-1} \check{\Gamma}(j^R) - \check{\Gamma}(j^R)(z - H)^{-1} \\ = & (z - H^{\mathrm{ext}})^{-1} \Big(H^{\mathrm{ext}} \check{\Gamma}(j^R) - \check{\Gamma}(j^R) H \Big) (z - H)^{-1}. \end{array}$$

By [DG1, Lemma 2.16] the following identity holds on $\mathcal{D}(H) \cap \mathcal{D}(N)$:

$$H_0^{\mathrm{ext}} \check{\Gamma}(j^R) - \check{\Gamma}(j^R) H_0 = \mathrm{d}\check{\Gamma}(j^R, \check{\mathrm{ad}}_{|k|} j^R),$$

where $\check{\mathrm{ad}}_{|k|} j^R$ is the operator $(\mathrm{ad}_{|k|} j_0^R, \mathrm{ad}_{|k|} j_\infty^R)$. We recall also the following bound from [DG1, Lemma 2.16]:

$$(7.12) \qquad \|(N^{\mathrm{ext}} + 1)^{-\frac{1}{2}} \mathrm{d}\check{\Gamma}(j^R, k) u\| \leq \|\mathrm{d}\Gamma(k_0^* k_0 + k_\infty^* k_\infty)^{\frac{1}{2}} u\|.$$

By the same argument as in [G2, Lemma 5.2], we have:

$$(7.13) \qquad \|[|k|, j_\epsilon^R]\| = O(R^{-1}), \ \epsilon = 0, \infty.$$

Next we know by [G2, Lemma 3.9] that $(z - H)^{-1}$ preserves $\mathcal{D}(N)$, and

$$(7.14) \qquad \|(N + 1)(z - H)^{-1}(N + 1)^{-1}\| \leq C|\mathrm{Im} z|^{-2}, \ z \in U \Subset \mathbb{C}.$$

By interpolation we have also:

$$(7.15) \qquad \|(N + 1)^{\frac{1}{2}}(z - H)^{-1}(N + 1)^{-\frac{1}{2}}\| \leq C|\mathrm{Im} z|^{-2}, \ z \in U \Subset \mathbb{C}.$$

Applying (7.12), (7.13) and (7.15), we obtain:

$$(7.16) \qquad \begin{array}{rl} & \|(N^{\mathrm{ext}} + 1)^{-\frac{1}{2}}(z - H^{\mathrm{ext}})^{-1} \Big(H_0^{\mathrm{ext}} \check{\Gamma}(j^R) - \check{\Gamma}(j^R) H_0 \Big) (z - H)^{-1} (N + 1)^{-\frac{1}{2}}\| \\ = & O(R^{-1}) |\mathrm{Im} z|^{-2}, \end{array}$$

for $z \in U \Subset \mathbb{C}$. Next again by [DG1, Lemma 2.16]:

$$(7.17) \qquad \begin{array}{rl} & \phi(w) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} \check{\Gamma}(j^R) - \check{\Gamma}(j^R) \phi(w) \\ = & \frac{1}{\sqrt{2}} \Big((a^*((1 - j_0^R)w) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} - \mathbb{1}_{\Gamma(\mathfrak{h})} \hat{\otimes} a^*(j_\infty^R w)) \check{\Gamma}(j^R) - \check{\Gamma}(j^R) a((1 - j_0^R) w) \Big), \end{array}$$

where the twisted tensor product $\hat{\otimes}$ is defined as follows:

let $T : \mathcal{K} \otimes \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \mathcal{K} \otimes \Gamma(\mathfrak{h})$ be the unitary operator defined by

$$T \psi \otimes u_1 \otimes u_2 = u_1 \otimes \psi \otimes u_2.$$

Then if B is an operator on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$, we set

$$\mathbb{1}_{\Gamma(\mathfrak{h})} \hat{\otimes} B := T^{-1} (\mathbb{1}_{\Gamma(\mathfrak{h})} \otimes B) T.$$

We now apply (7.17) to $w = v$. Note that $\text{s-}\lim_{R \to \infty} j_\infty^R = \text{s-}\lim_{R \to \infty} (1 - j_0^R) = 0$ in $\mathcal{B}(\mathfrak{h})$ and hence since $v \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}), \mathcal{K} \otimes \mathfrak{h})$ we have $\text{s-}\lim_{R \to \infty} j_\infty^R v (K + 1)^{-\frac{1}{2}} = \text{s-}\lim_{R \to \infty} (1 - j_0^R) v (K + 1)^{-\frac{1}{2}} = 0$ in $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. Since by hypothesis **(H0)** $(K + 1)^{-\frac{1}{2}}$ is compact on $\mathcal{K}$, we obtain that

$$(7.18) \qquad \|(1 - j_0^R) v (K + 1)^{-1}\| + \|j_\infty^R v (K + 1)^{-1}\| = o(R^0).$$

Using also (7.15), and the fact that $(K+1)(z-H)^{-1}$ is bounded, we obtain:

$$\begin{aligned}&\|(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}(z-H^{\mathrm{ext}})^{-1}\Big(\phi(v)\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)\phi(v)\Big)(z-H)^{-1}(N+1)^{-\frac{1}{2}}\|\\ &= o(R^0)|\mathrm{Im}z|^{-2},\end{aligned}\tag{7.19}$$

for $z\in U\Subset \mathbb{C}$. Combining (7.16) and (7.19) we obtain:

$$\begin{aligned}&\|(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}\Big((z-H^{\mathrm{ext}})^{-1}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)(z-H)^{-1}\Big)(N+1)^{-\frac{1}{2}}\|\\ &= o(R^0)|\mathrm{Im}z|^{-2},\end{aligned}\tag{7.20}$$

for $z\in U\Subset \mathbb{C}$. We recall the functional calculus formula:

$$\chi(A)=\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}(z)(z-A)^{-1}\mathrm{d}z\wedge \mathrm{d}\,\overline{z},\tag{7.21}$$

where A is a selfadjoint operator and $\tilde{\chi}\in C_0^\infty(\mathbb{C})$ is an almost-analytic extension of χ satisfying

$$\begin{aligned}&\tilde{\chi}_{|\mathbb{R}}=\chi,\\ &|\partial_{\overline{z}}\tilde{\chi}(z)|\leq C_n|\mathrm{Im}z|^n,\quad n\in\mathbb{N}.\end{aligned}$$

Using (7.21) we get:

$$f(H)=\check{\Gamma}(j^R)^*f(H^{\mathrm{ext}})\check{\Gamma}(j^R)+N^{\frac{1}{2}}o_f(R^0)N^{\frac{1}{2}},\tag{7.22}$$

which proves the first identity of the proposition.

Let us now prove the second identity. We note first that applying again (7.17) and (7.15), arguing as in the proof of (7.19), we obtain:

$$(H^{\mathrm{ext}}+\mathrm{i})^{-1}\Big(\phi(w)\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)\phi(w)\Big)(H+\mathrm{i})^{-1}=(N^{\mathrm{ext}})^{\frac{1}{2}}o(R^0)N^{\frac{1}{2}}.\tag{7.23}$$

Next we consider the term M^δ. By [DG1, Lemma 2.16], we have:

$$M^{\delta\,\mathrm{ext}}\check{\Gamma}(j^R)=\check{\Gamma}(j^R)M^\delta+\mathrm{d}\check{\Gamma}(j^R,\check{\mathrm{ad}}_{m^\delta}j^R),$$

as bounded operators from $\mathcal{D}(N)$ into $\mathcal{H}^{\mathrm{ext}}$.

Applying (7.12), we obtain:

$$\|(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}\mathrm{d}\check{\Gamma}(j^R,\check{\mathrm{ad}}_{m^\delta}j^R)(N+1)^{-\frac{1}{2}}\|\leq \|[m^\delta,j_0^R]\|+\|[m^\delta,j_\infty^R]\|=O_\delta(R^{-1}).$$

This yields:

$$M^\delta=\check{\Gamma}(j^R)^*M^{\delta\,\mathrm{ext}}\check{\Gamma}(j^R)+N^{\frac{1}{2}}O_\delta(R^{-1})N^{\frac{1}{2}}.\tag{7.24}$$

We can now complete the proof of the proposition. We first claim that

$$\|N^{\frac{1}{2}}\phi(w)f(H)(N+1)^{-\frac{1}{2}}\|<\infty.\tag{7.25}$$

In fact this follows by writing

$$\begin{aligned}&N^{\frac{1}{2}}\phi(w)f(H)(N+1)^{-\frac{1}{2}}\\ =\;&N^{\frac{1}{2}}\phi(w)(N+1)^{-\frac{1}{2}}(H+\mathrm{i})^{-1}\times(H+\mathrm{i})(N+1)^{\frac{1}{2}}f(H)(N+1)^{-\frac{1}{2}}.\end{aligned}$$

The first factor is bounded using (7.10) and Proposition 4.1 and the second also by [G2, Lemma 3.9].

We now write

$$\begin{aligned} & f(H)\phi(w)f(H) \\ = \; & \check{\Gamma}(j^R)^* f(H^{\text{ext}})\check{\Gamma}(j^R)\phi(w)f(H) + N^{\frac{1}{2}}o(R^0)N^{\frac{1}{2}}, \end{aligned}$$

using (7.22) and (7.25). Next:

$$\check{\Gamma}(j^R)^* f(H^{\text{ext}})\check{\Gamma}(j^R)\phi(w)f(H) = \; \check{\Gamma}(j^R)^* f(H^{\text{ext}})\phi(w) \otimes 1\!\!1_{\Gamma(\mathfrak{h})}\check{\Gamma}(j^R)f(H) + N^{\frac{1}{2}}o(R^0)N^{\frac{1}{2}},$$

by (7.23). Finally

$$\begin{aligned} & \check{\Gamma}(j^R)^* f(H^{\text{ext}})\phi(w) \otimes 1\!\!1_{\Gamma(\mathfrak{h})}\check{\Gamma}(j^R)f(H) \\ = \; & \check{\Gamma}(j^R)^* f(H^{\text{ext}})\phi(w) \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f(H^{\text{ext}})\check{\Gamma}(j^R) + N^{\frac{1}{2}}o(R^0)N^{\frac{1}{2}}, \end{aligned}$$

using (7.22) and the analog of (7.25) for the Hamiltonians $H^{\text{ext}}, N^{\text{ext}}$. This yields:

$$\begin{aligned} & f(H)\phi(w)f(H) \\ = \; & \check{\Gamma}(j^R)^* f(H^{\text{ext}})\phi(w) \otimes 1\!\!1_{\Gamma(\mathfrak{h})} f(H^{\text{ext}})\check{\Gamma}(j^R) + N^{\frac{1}{2}}o(R^0)N^{\frac{1}{2}}, \end{aligned} \tag{7.26}$$

which combined with (7.24) completes the proof of the proposition. □

7.3 Momentum space decomposition

To prove the Mourre estimate, we will need an additional decomposition in momentum space. This decomposition will take place on the extended Hilbert space $\mathcal{H}^{\text{ext}} = \mathcal{H} \otimes \Gamma(\mathfrak{h})$ and concern only the component $\Gamma(\mathfrak{h})$ describing the bosons close to infinity. Note that this decomposition is slightly different from the one used in Subsection 7.2 because the associated map $\check{\Gamma}(F^\delta)$ is unitary and not only isometric. To construct this decomposition, we set:

$$F^\delta := (1\!\!1_{[0,\delta[}(|k|), 1\!\!1_{[\delta,\infty[}(|k|)),$$

$$\mathfrak{h}_\delta^< := L^2(\{|k| < \delta\}, \mathrm{d}k),\ \mathfrak{h}_\delta^> := L^2(\{|k| \geq \delta\}, \mathrm{d}k).$$

The maps

$$\begin{aligned} & F^\delta : \mathfrak{h} \to \mathfrak{h}_\delta^< \oplus \mathfrak{h}_\delta^>, \\ & \check{\Gamma}(F^\delta) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}_\delta^<) \otimes \Gamma(\mathfrak{h}_\delta^>) \end{aligned}$$

are unitary. The map

$$1\!\!1_{\mathcal{H}} \otimes \check{\Gamma}(F^\delta) : \mathcal{H}^{\text{ext}} = \mathcal{H} \otimes \Gamma(\mathfrak{h}) \to \mathcal{H} \otimes \Gamma(\mathfrak{h}_\delta^<) \otimes \Gamma(\mathfrak{h}_\delta^>) =: \hat{\mathcal{H}}^{\text{ext}}$$

is unitary.

On the space $\hat{\mathcal{H}}^{\text{ext}}$, we define the following operators:

$$\hat{H}^{\text{ext}} := H \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^<)} \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^>)} + 1\!\!1_{\mathcal{H}} \otimes \mathrm{d}\Gamma(|k|) \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^>)} + 1\!\!1_{\mathcal{H}} \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathrm{d}\Gamma(|k|),$$

$$\hat{M}^{\delta\,\text{ext}} := M^\delta \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^<)} \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^>)} + 1\!\!1_{\mathcal{H}} \otimes M^\delta \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^>)} + 1\!\!1_{\mathcal{H}} \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\delta^<)} \otimes M^\delta.$$

Applying again the identities in [DG1, Lemma 2.16], we obtain:

$$B^{\delta\,\mathrm{ext}}(E,\kappa) = (\mathbb{1}_{\mathcal{H}}\otimes\check{\Gamma}(F^{\delta}))^{*}\hat{B}^{\delta\,\mathrm{ext}}(E,\kappa)\mathbb{1}_{\mathcal{H}}\otimes\check{\Gamma}(F^{\delta}), \tag{7.27}$$

for

$$\hat{B}^{\delta\,\mathrm{ext}}(E,\kappa) = \hat{M}^{\delta\,\mathrm{ext}} - f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\phi(\mathrm{i}a^{\delta}v)\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{<})}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{>})}f_{E,\kappa}(\hat{H}^{\mathrm{ext}}).$$

Note that there are no error terms in (7.27) because $[m^{\delta},F^{\delta}]=[|k|,F^{\delta}]=0$. A similar identity that we will need is:

$$M_{\infty}^{\delta} = \mathbb{1}_{\mathcal{H}}\otimes\check{\Gamma}(F^{\delta})^{*}\left(\mathbb{1}_{\mathcal{H}}\otimes M^{\delta}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{>})}+\mathbb{1}_{\mathcal{H}}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{<})}\otimes M^{\delta}\right)\mathbb{1}_{\mathcal{H}}\otimes\check{\Gamma}(F^{\delta}). \tag{7.28}$$

In the sequel we will use the following easy observation

Lemma 7.11 *i) We have:*

$$\mathbb{1}_{[0,1[}(M^{\delta}) = \mathbb{1}_{\{0\}}(M^{\delta}) = \mathbb{1}_{\{0\}}(N).$$

ii) As an identity on $\Gamma(\mathfrak{h}_{\delta}^{<})\otimes\Gamma(\mathfrak{h}_{\delta}^{>})$*, we have:*

$$\mathbb{1}_{[1,\infty[}(M^{\delta}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}^{>})}+\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes M^{\delta}) = \mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes\mathbb{1}_{[1,\infty[}(M^{\delta})+\mathbb{1}_{[1,\infty[}(M^{\delta})\otimes\mathbb{1}_{\{0\}}(N).$$

Proof. *i)* follows from the fact that $m^{\delta}\geq 1$. To prove *ii)*, we write:

$$\begin{array}{rl}
& \mathbb{1}_{[1,\infty[}(M^{\delta}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}^{>})}+\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes M^{\delta}) \\
= & \mathbb{1}_{[1,\infty[}(M^{\delta}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}^{>})}+\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes M^{\delta})\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes\mathbb{1}_{[1,\infty[}(M^{\delta}) \\
& +\mathbb{1}_{[1,\infty[}(M^{\delta}\otimes\mathbb{1}_{\Gamma(\mathfrak{h}^{>})}+\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes M^{\delta})\mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes\mathbb{1}_{[0,1[}(M^{\delta}) \\
= & \mathbb{1}_{\Gamma(\mathfrak{h}^{<})}\otimes\mathbb{1}_{[1,\infty[}(M^{\delta})+\mathbb{1}_{[1,\infty[}(M^{\delta})\otimes\mathbb{1}_{\{0\}}(N),
\end{array}$$

using *i)*. □

7.4 Proof of the Mourre estimate

This subsection is devoted to the proof of the Mourre estimate stated in Theorem 7.12 below. Note that this is the only place where hypothesis **(H0)** is used.

Theorem 7.12 *Assume hypotheses* **(H0)**, **(I1)** *and* **(I2)**. *For all* $E_0<\infty$ *there exists* $0<\delta\leq\frac{1}{2}$ *such that:*

i) For all $E\leq E_0,\epsilon_0>0$ *there exist* $C,\kappa>0$ *and a compact operator* K_0 *such that:*

$$M^{\delta}-f_{E,\kappa}(H)\phi(\mathrm{i}a^{\delta}v)f_{E,\kappa}(H)\geq(1-\epsilon_0)\mathbb{1}-C(1-f_{E,\kappa}(H))^{2}-K_0. \tag{7.29}$$

ii) For all $E\leq E_0,\epsilon_0>0$*,* $E\notin\sigma_{\mathrm{pp}}(H)$*, there exist* $C,\kappa>0$ *such that:*

$$M^{\delta}-f_{E,\kappa}(H)\phi(\mathrm{i}a^{\delta}v)f_{E,\kappa}(H)\geq 1-\epsilon_0-C(1-f_{E,\kappa}(H))^{2}. \tag{7.30}$$

We will deduce Theorem 7.12 from the following proposition.

Proposition 7.13 *For all* $E_0 < \infty$ *there exists* $0 < \delta \leq \frac{1}{2}$ *such that the following assertion holds: Assume that the Mourre estimate (7.29) holds at all energies* $E \leq E_1$ *for some* $E_1 \leq E_0 + 1$*. Then (7.29) holds at all energies* $E \leq E_1 + \delta$*.*

Proof of Theorem 7.12. Note first that *ii)* follows from *i)* by Proposition 7.5, so it suffices to prove *i)*. Using Proposition 7.13 and an induction argument, it suffices to show that the Mourre estimate holds at all energies $E \leq \inf \sigma(H) - 2$. If $E \leq \inf \sigma(H) - 2$, $\kappa \leq 1$, then $f_{E,\kappa}(H) = 0$ and

$$B^{\delta}(E, \kappa) = M^{\delta} \geq 0 \geq (1 - \epsilon_0)\mathbb{1} - C(1 - f_{E,\kappa}(H))^2,$$

for all $\epsilon_0 > 0, C \geq 1$. □

Before starting the proof of Proposition 7.13, we state an auxiliary lemma.

Lemma 7.14 *Let* $E_0 \in \mathbb{R}$*. There exists* $C > 0$ *such that for all* $N_0 \in \mathbb{N}^*, \epsilon, R > 0$ *there exists* $K(N_0, R)$ *compact such that for all* $E \leq E_0, 0 < \kappa \leq 1$*:*

i) $-\check{\Gamma}(j^R)^* \mathbb{1}_{\{0\}}(M^{\delta}_{\infty})\check{\Gamma}(j^R) \geq -\epsilon\mathbb{1} - \epsilon^{-1}(1 - f_{E,\kappa}(H))^2 - CN_0^{-1}N - K(N_0, R)$
ii) $\check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E, \kappa)\mathbb{1}_{\{0\}}(M^{\delta}_{\infty})\check{\Gamma}(j^R) \geq -\epsilon\mathbb{1} - \epsilon^{-1}(1 - f_{E,\kappa}(H))^2 - CN_0^{-1}N - K(N_0, R).$

Proof. Let first $j \in C_0^{\infty}(\mathbb{R}^d)$, $0 \leq j \leq 1$ and $j^R(x) = j(\frac{x}{R})$. We claim that for each $N_0 \in \mathbb{N}^*$, $R \geq 1$, $E_0 \in \mathbb{R}$ there exists a compact operator $K(N_0, R, E_0)$ such that

$$\Gamma(j^R)^2 \leq K(N_0, R, E_0) + \epsilon + \epsilon^{-1}(1 - f_{E,\kappa})(H)^2 + N_0^{-1}N, \tag{7.31}$$

uniformly for $\epsilon > 0$ $E \leq E_0$, $0 < \kappa \leq 1$. In fact we write:

$$\begin{aligned}
\Gamma(j^R)^2 = {} & \Gamma(j^R)^2\mathbb{1}_{[0,N_0]}(N) + \Gamma(j^R)^2\mathbb{1}_{]N_0,\infty[}(N) \\
\leq {} & \Gamma(j^R)^2\mathbb{1}_{[0,N_0]}(N) + N_0^{-1}N \\
= {} & \Gamma(j^R)^2\mathbb{1}_{[0,N_0]}(N)\mathbb{1}_{]-\infty,E_0+1]}(H) + \Gamma(j^R)^2\mathbb{1}_{[0,N_0]}(N)\mathbb{1}_{]E_0+1,\infty[}(H) + N_0^{-1}N \qquad (7.32)\\
\leq {} & K(N_0, R, E_0) + \epsilon + \epsilon^{-1}\mathbb{1}_{]E_0+1,\infty[}(H) + N_0^{-1}N \\
\leq {} & K(N_0, R, E_0) + \epsilon + \epsilon^{-1}(1 - f_{E,\kappa}(H))^2 + N_0^{-1}N,
\end{aligned}$$

for $K(N_0, R, E_0) = |\Gamma(j^R)^2\mathbb{1}_{[0,N_0]}(N)\mathbb{1}_{]-\infty,E_0+1]}(H)|$. The operator $K(N_0, R, E_0)$ is compact using the fact that $(K + \mathrm{i})^{-1}$ is compact on $\mathcal{K}$, j^R has compact support and $\omega(k) = |k| \to +\infty$ when $k \to \infty$.

Let us now prove the lemma. Note first that since $m^{\delta} \geq 1$, we have $\mathbb{1}_{\{0\}}(M^{\delta}_{\infty}) = \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\{0\}}(N)$. It follows also from [DG1, Subsection 2.13] that:

$$\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(M_{\infty})\check{\Gamma}(j^R) = \Gamma(j_0^R)^2,\ \mathbb{1}_{\{0\}}(M_{\infty})\check{\Gamma}(j^R) = \mathbb{1}_{\{0\}}(M_{\infty})\check{\Gamma}(j^R)\Gamma(j_1^R),$$

if $j_1 \in C_0^{\infty}(\mathbb{R}^d)$ is such that $j_1 j_0 = j_0$.

Part *i)* of the lemma follows then from (7.31) for $j = j_0$. To prove *ii)* we recall that:

$$B^{\delta\,\mathrm{ext}}(E, \kappa) := M^{\delta\,\mathrm{ext}} - f_{E,\kappa}(H^{\mathrm{ext}})\phi(\mathrm{i}a^{\delta}v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} f_{E,\kappa}(H^{\mathrm{ext}}). \tag{7.33}$$

Since $M^{\delta,\,\mathrm{ext}} \geq 0$, it suffices to bound from below the second term in (7.33). Since by Lemma 6.2 *iv)* $\|(H+\mathrm{i})^{-\frac{1}{2}}\phi(\mathrm{i}a^{\delta}v)(H+\mathrm{i})^{-\frac{1}{2}}\| \leq C$ uniformly in $0 \leq \delta \leq \frac{1}{2}$, we have:

$$\|f_{E,\kappa}(H^{\mathrm{ext}})\phi(\mathrm{i}a^{\delta}v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} f_{E,\kappa}(H^{\mathrm{ext}})\| \leq C_0,$$

uniformly in $0 < \delta \leq \frac{1}{2}$, $E \leq E_0$, $0 < \kappa \leq 1$. This yields:

$$\check{\Gamma}(j^R)^* \mathbb{1}_{\{0\}}(M^{\delta}_{\infty}) f_{E,\kappa}(H^{\mathrm{ext}})\phi(\mathrm{i}a^{\delta}v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} f_{E,\kappa}(H^{\mathrm{ext}})\check{\Gamma}(j^R) \geq -C_0 \Gamma(j_1^R)^2.$$

Applying then (7.31) for $j = j_1$, we obtain *ii)*. □

Proof of Proposition 7.13. Let us first explain how to fix the parameter δ. By Lemma 6.2 *iv)*, we have:

$$\sup_{0<\delta\leq\frac{1}{2}} \|(H+\mathrm{i})^{-\frac{1}{2}}\phi(\mathrm{i}a^{\delta}v)(H+\mathrm{i})^{-\frac{1}{2}}\| \leq C < \infty.$$

For $E_0 \in \mathbb{R}$, since $\lim_{t\to 0} d(t) = +\infty$, we can fix $0 < \delta < \frac{1}{2}$ small enough so that:

$$d(\delta) \geq \sup_{0<\delta\leq\frac{1}{2}} \|\mathbb{1}_{]-\infty,E_0+2]}(H)\phi(\mathrm{i}a^{\delta}v)\mathbb{1}_{]-\infty,E_0+2]}(H)\| + 1. \tag{7.34}$$

We start by considering the extended objects H^{ext} and $B^{\delta\,\mathrm{ext}}(E,\kappa)$. By Lemma 7.11 and (7.27), (7.28) we have:

$$\mathbb{1}_{[1,+\infty[}(M^{\delta}_{\infty})B^{\delta\,\mathrm{ext}}(E,\kappa) = (\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta}))^*(I_1 + I_2)(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta})),$$

for

$$\begin{aligned} I_1 = {} & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{<})} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta})\hat{B}^{\delta\,\mathrm{ext}}(E,\kappa), \\ I_2 = {} & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N)\hat{B}^{\delta\,\mathrm{ext}}(E,\kappa). \end{aligned}$$

We start by bounding I_2 from below. We can write

$$I_2 = I_3 + I_4,$$

for

$$\begin{aligned} I_3 = {} & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N)\hat{M}^{\delta\,\mathrm{ext}} \\ \geq {} & (1-\epsilon)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N)\Big(\mathbb{1}_{\mathcal{H}} \otimes M^{\delta} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{>})} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{<})} \otimes M^{\delta}\Big) \\ & + \epsilon d(\delta)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N). \end{aligned} \tag{7.35}$$

In fact $m^{\delta}_{|\mathfrak{h}_{\delta}^{<}} \geq d(\delta)$ which implies that

$$\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N)\hat{M}^{\delta\,\mathrm{ext}} \geq d(\delta)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N).$$

Next:

$$\begin{aligned} I_4 = {} & -\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N) \\ & \times f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\phi(\mathrm{i}a^{\delta}v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{<})} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_{\delta}^{>})} f_{E,\kappa}(\hat{H}^{\mathrm{ext}}) \\ \geq {} & -C_0\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^{\delta}) \otimes \mathbb{1}_{\{0\}}(N), \end{aligned}$$

uniformly for $E \leq E_0 + 1$, $0 < \kappa \leq 1$, $0 < \delta \leq \frac{1}{2}$, for

$$C_0 = \sup_{0<\delta\leq\frac{1}{2}} \|\mathbb{1}_{]-\infty,E_0+2]}(H)\phi(\mathrm{i}a^\delta v)\mathbb{1}_{]-\infty,E_0+2]}(H)\| < \infty. \tag{7.36}$$

This yields

$$\begin{aligned} I_2 \geq\ & (1-\epsilon)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^\delta) \otimes \mathbb{1}_{\{0\}}(N)\Big(\mathbb{1}_{\mathcal{H}} \otimes M^\delta \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes M^\delta\Big) \\ & + (\epsilon d(\delta) - C_0)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{[1,\infty[}(M^\delta) \otimes \mathbb{1}_{\{0\}}(N)), \end{aligned} \tag{7.37}$$

for all $\epsilon > 0$. Let us now bound I_1. We have:

$$\begin{aligned} I_1 =\ & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta) \\ & \times\Big(\hat{M}^{\delta\,\mathrm{ext}} - f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\phi(\mathrm{i}a^\delta v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\Big). \end{aligned}$$

Since by Lemma 7.11 *i)* $\mathbb{1}_{[1,\infty[}(M^\delta) = \mathbb{1}_{[1,\infty[}(N)$, and $|k| \geq \delta$ on $\mathfrak{h}_\delta^>$, we obtain that the operator

$$\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta)\Big(M^\delta \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} - f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\phi(\mathrm{i}a^\delta v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\Big),$$

acting on $\mathcal{H} \otimes \Gamma(\mathfrak{h}_\delta^<) \otimes \Gamma(\mathfrak{h}_\delta^>)$ is unitarily equivalent to

$$\int_{[\delta,\infty[}^{\oplus} M^\delta + f_{E-\lambda,\kappa}(H)\phi(\mathrm{i}a^\delta v) f_{E-\lambda,\kappa}(H)\mathrm{d}\lambda.$$

By the induction assumption the Mourre estimate holds at all energies $E' \leq E_1$. By Proposition 7.8, for any $\epsilon_0 > 0$ there exists $C, \kappa > 0$ such that for all $E' \leq E_1$

$$M^\delta - f_{E',\kappa}(H)\phi(\mathrm{i}a^\delta v) f_{E',\kappa}(H) \geq -\epsilon_0/4\mathbb{1} - C(1 - f_{E',\kappa})(H)^2. \tag{7.38}$$

We fix κ such that (7.38) holds and we obtain that

$$\begin{aligned} & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta)\Big(M^\delta \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} \\ - & f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\phi(\mathrm{i}a^\delta v) \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} f_{E,\kappa}(\hat{H}^{\mathrm{ext}})\Big) \\ \geq\ & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta)\Big(-\epsilon_0/4\mathbb{1} - C(1 - f_{E,\kappa})(\hat{H}^{\mathrm{ext}})^2\Big), \end{aligned}$$

uniformly for $E \leq E_1 + \delta$. Using also that $\mathbb{1}_{[1,\infty[}(M^\delta)M^\delta \geq d(\delta)\mathbb{1}_{[1,\infty[}(M^\delta)$, we obtain finally:

$$\begin{aligned} I_1 \geq\ & \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta) \\ & \times\Big((1-\epsilon)(\mathbb{1}_{\mathcal{H}} \otimes M^\delta \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^>)} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes M^\delta)\Big) \\ & + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta)(\epsilon d(\delta) - \epsilon_0/4) \\ & - C\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\delta^<)} \otimes \mathbb{1}_{[1,\infty[}(M^\delta)(1 - f_{E,\kappa})(\hat{H}^{\mathrm{ext}})^2, \end{aligned} \tag{7.39}$$

for all $\epsilon > 0$, uniformly for $E \leq E_1 + \delta$.

Using now (7.37) and the functorial properties of $\check{\Gamma}(F^\delta)$, we obtain:

$$\begin{array}{rl} & 1\!\!1_{[1,\infty[}(M_\infty^\delta) B^{\delta\,\mathrm{ext}}(E,\kappa) \\ \geq & (1-\epsilon) M_\infty^\delta 1\!\!1_{[1,\infty[}(M_\infty^\delta) + \min\{\epsilon d(\delta)-\epsilon_0/4, \epsilon d(\delta) - C_0)\} 1\!\!1_{[1,\infty[}(M_\infty^\delta) \\ & -C(1-f_{E,\kappa})(H^{\mathrm{ext}})^2. \end{array}$$

Going back to the original Hilbert space $\mathcal{H}$, this yields:

$$(7.40)\qquad \begin{array}{rl} & \check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E,\kappa) 1\!\!1_{[1,\infty[}(M_\infty^\delta)\check{\Gamma}(j^R) \\ \geq & (1-\epsilon)\check{\Gamma}(j^R)^* M_\infty^\delta 1\!\!1_{[1,\infty[}(M_\infty^\delta)\check{\Gamma}(j^R) \\ & +\min\{\epsilon d(\delta)-\epsilon_0/4, \epsilon d(\delta) - C_0\}\check{\Gamma}(j^R)^* 1\!\!1_{[1,\infty[}(M_\infty^\delta)\check{\Gamma}(j^R) \\ & -C\check{\Gamma}(j^R)^*(1-f_{E,\kappa})(H^{\mathrm{ext}})^2\check{\Gamma}(j^R), \end{array}$$

for all $\epsilon > 0$. We have now completed the main part of the proof, and it remains only to collect the error terms.

Let us fix $\epsilon = 1$ in (7.40). Using the fact that $d(\delta) \geq 1$, $d(\delta) - C_0 \geq 1$, together with Proposition 7.9 we get:

$$(7.41)\qquad \begin{array}{rl} & \check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E,\kappa) 1\!\!1_{[1,\infty[}(M_\infty^\delta)\check{\Gamma}(j^R) + (1-\epsilon_0/4)\check{\Gamma}(j^R)^* 1\!\!1_{\{0\}}(M_\infty^\delta)\check{\Gamma}(j^R) \\ \geq & (1-\epsilon_0/4)1\!\!1 - C(1-f_{E,\kappa})(H)^2 + +N^{\frac12} o_{\delta,E,\kappa}(R^0) N^{\frac12}. \end{array}$$

By Lemma 7.14 there exists $C_1 > 0$ such that for all $N_0 \in \mathbb{N}^*, \epsilon, R > 0$ there exists $K(N_0, R)$ compact such that

$$-\check{\Gamma}(j^R)^* 1\!\!1_{\{0\}}(M_\infty^\delta)\check{\Gamma}(j^R) \geq -\epsilon - \epsilon^{-1}(1-f_{E,\kappa}(H))^2 - C_1 N_0^{-1} N - K(N_0,R),$$

$$\check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E,\kappa) 1\!\!1_{\{0\}}(M_\infty^\delta)\check{\Gamma}(j^R) \geq -\epsilon - \epsilon^{-1}(1-f_{E,\kappa}(H))^2 - C_1 N_0^{-1} N - K(N_0,R).$$

Picking $\epsilon = \epsilon_0/8$ in the two estimates above, we deduce from (7.41) that:

$$\check{\Gamma}(j^R)^* B^{\delta\,\mathrm{ext}}(E,\kappa)\check{\Gamma}(j^R) \geq (1-\epsilon_0/2)1\!\!1 - C(1-f_{E,\kappa})(H)^2 - CN_0^{-1}N - K(N_0,R) + N^{\frac12} o_{\delta,E,\kappa}(R^0) N^{\frac12},$$

Applying once more Proposition 7.9, we obtain:

$$B^\delta(E,\kappa) \geq (1-\epsilon_0/2) - C(1-f_{E,\kappa}(H))^2 - C_1 N_0^{-1} N - K(N_0,R) + N^{\frac12} o_{\delta,E,\kappa}(R^0) N^{\frac12}.$$

Since $N \leq M^\delta$, for any $\alpha > 0$ we can finally fix $R, N_0 \gg 1$ such that:

$$B^\delta(E,\kappa) \geq (1-\epsilon_0/2) - C(1-f_{E,\kappa}(H))^2 - K - \alpha M^\delta,$$

where K is compact. This gives:

$$(1+\alpha)B^\delta(E,\kappa) \geq (1-\epsilon_0/2) - C(1-f_{E,\kappa}(H))^2 - K - \alpha\|f_{E,\kappa}(H)\phi(\mathrm{i}a^\delta v)f_{E,\kappa}(H)\|.$$

Since for $E \leq E_0 + 1, 0 < \kappa \leq 1$, $\|f_{E,\kappa}(H)\phi(\mathrm{i}a^\delta v)f_{E,\kappa}(H)\| \leq C_1$, we can choose $\alpha \ll 1$ such that $(1+\alpha)^{-1}(1-\epsilon_0/2 - C_1\alpha) \geq (1-\epsilon_0)$. This completes the proof of the proposition. □

7.5 An improved Mourre estimate

We now formulate an improved version of the Mourre estimate, which will not be used in this paper. Nevertheless, it could be useful to prove sharper propagation estimates on the dynamics $\mathrm{e}^{-\mathrm{i}tH}$.

Theorem 7.15 *For all* $\epsilon_1 > 0$, $E_0 < \infty$, *there exists* $0 < \delta \leq \frac{1}{2}$ *such that for all* $E \leq E_0, \epsilon_0 > 0$ *there exist* $C, \kappa, R > 0$, *and* K *compact such that:*

$$M^\delta - f_{E,\kappa}(H)\phi(\mathrm{i}a^\delta v)f_{E,\kappa}(H) \geq 1 - \epsilon_0 + (1-\epsilon_1)\check{\Gamma}(j^R)^* M^\delta_\infty \check{\Gamma}(j^R) - C(1 - f_{E,\kappa}(H))^2 - K.$$

Proof. We repeat the proof of Proposition 7.13 until we arrive at (7.40) and we fix $\epsilon = \epsilon_1$. We choose now $0 < \delta \leq 1$ such that

$$\epsilon_1 d(\delta) - \epsilon_0/4 \geq 1, \ \epsilon_1 d(\delta) - C_0 \geq 1.$$

We obtain an extra term

$$\begin{aligned} & (1-\epsilon_1)\check{\Gamma}(j^R)^* M^\delta_\infty 1\!\!1_{[1,\infty[}(M^\delta_\infty)\check{\Gamma}(j^R) \\ = \ & (1-\epsilon_1)\check{\Gamma}(j^R)^* M^\delta_\infty \check{\Gamma}(j^R), \end{aligned}$$

by Lemma 7.11 *i)*. Then we argue as in the rest of the proof of Proposition 7.13. □

8 Proof of the main results

In this section we give the proof of the results in Subsection 2.5.

Proof of Proposition 2.2. The fact that H is selfadjoint and bounded below on $\mathcal{D}(H_0)$ follows from Corollary 4.4. The assertion concerning the spectrum follows from Proposition 4.8.

Proof of Theorem 2.3. Theorem 2.3 follows from Proposition 6.5.

Proof of Theorem 2.4. We check the hypotheses of Proposition 7.5 for $M = M^\delta$ and $R = R^\delta$. By Proposition 6.5, we know that the virial relation (7.1) holds. By Theorem 7.12 we know that the Mourre estimate holds for M^δ, R^δ at each energy $E_0 \in \mathbb{R}$. Applying then Proposition 7.5 *i)* we obtain the theorem.

Proof of Theorem 2.5. We first verify the hypotheses **(M1)**— **(M5)** of Theorem 5.15, for $H^{\delta\,\prime}$ defined in Lemma 6.4 and the semigroup $\{W_t\}$ introduced in Subsection 6.4. By Lemma 6.4, Props. 6.6 and 6.7, hypotheses **(M1)**, **(M3)**, **(M4)** and **(M5)** hold. By Theorem 7.12 and Lemma 7.2, for each $E_0 \in \mathbb{R}$ there exists δ such that for each $\lambda \in]-\infty, E_0] \backslash \sigma_{\mathrm{pp}}(H)$, hypothesis **(M2)** holds for a neighborhood J of λ. Therefore the conclusion of Theorem 5.15 holds.

To complete the proof of the theorem, it remains to prove that $(\mathrm{d}\Gamma(b)+1)^{-s}(N+1)^{\frac{1}{2}}$ sends $\mathcal{H}$ into $\mathcal{G}^*_s \equiv \mathcal{G}^*_{s,2}$, where the notation $\mathcal{G}^*_{s,2}$ indicates that the space can also be defined by complex interpolation. Let us set $\mathcal{N} = \mathcal{D}(N^{\frac{1}{2}})$. Clearly $\{W^\delta_t\}$ and $\{W^{\delta *}_t\}$ b-preserve $\mathcal{N}$ since $[N, W^\delta_t] = 0$. It is also easy to see that $\mathcal{N}^* \subset \mathcal{G}^*$, $\mathcal{D}(A^\delta; \mathcal{N}^*) \subset \mathcal{D}(A^\delta; \mathcal{G}^*)$ continuously. Therefore by interpolation $\mathcal{N}^*_{s,2} \subset \mathcal{G}^*_{s,2}$ continuously. Using again that $[W^\delta_t, N] = 0$, we see that $(N+1)^{\frac{1}{2}}(|A^\delta|+1)^{-s}$ is bounded from $\mathcal{H}$ into $\mathcal{G}^*_{s,2}$.

Next by Proposition 3.4 *i)* we know that $A^{\delta *}A^{\delta} \leq \mathrm{d}\Gamma(|a^{\delta}|)^2$. Since by Lemma 6.1 $\tilde{a}^{\delta *}\tilde{a}^{\delta} \leq C(\delta)\tilde{b}^2$, we obtain using Proposition 3.4 *ii)* that $A^{\delta *}A^{\delta} \leq C(\delta)\mathrm{d}\Gamma(b)^2$ and hence $|A^{\delta}|^s \leq C(\delta)|\mathrm{d}\Gamma(b)|^s$ for $0 \leq s \leq 2$. Therefore the operator $(N+1)^{\frac{1}{2}}(\mathrm{d}\Gamma(b)+1)^{-s}$ is bounded from $\mathcal{H}$ into $\mathcal{G}^*_{s,2}$. Then the statements in the theorem follow from corresponding statements in Theorem 5.15.

References

[A] Ammari, Z.: Asymptotic completeness for a renormalized non-relativistic Hamiltonian in quantum field theory: the Nelson model. Math. Phys. Anal. Geom. **3** (2000), 217–285.

[ABG] Amrein, W., Boutet de Monvel, A., Georgescu, V.: *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[Ar] Arai, A.: Ground State of the Massless Nelson Model Without Infrared Cutoff in a Non-Fock Representation, preprint mp-arc 00-478 (2000).

[AH1] Arai, A., Hirokawa, M.: On the existence and uniqueness of ground states of a generalized spin-boson model. J. Funct. Anal. **151** (1997), 455–503.

[AH2] Arai, A., Hirokawa, M.: Ground states of a general class of quantum field Hamiltonians. Rev. Math. Phys. **12** (2000), 1085–1135.

[AHH] Arai, A., Hirokawa, M., Hiroshima, F.: On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff. J. Funct. Anal. **168** (1999), 470–497.

[BFS] Bach, V., Fröhlich, J., Sigal, I.: Quantum electrodynamics of confined non-relativistic particles, Adv. Math. **137** (1998), 299–395.

[BFSS] Bach, V., Fröhlich, J., Sigal, I., Soffer, A.: Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules. Comm. Math. Phys. **207** (1999), 557–587.

[Ca] Cannon, J.: Quantum field theoretic properties of a model of Nelson: Domain and eigenvector stability for perturbed linear operators. J. Funct. Anal. **8** (1971), 101–152.

[DG1] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, Rev. Math. Phys. **11** (1999), 383–450.

[DG2] Dereziński, J., Gérard, C.: Spectral and scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians, Comm. Math. Phys. **213** (2000), 39–125.

[DJ] Dereziński, J. Jaksic, V.: Spectral theory of Pauli-Fierz operators. J. Funct. Anal. **180** (2001), 243–327.

[FGS1] Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field. Adv. Math. **164** (2001), 349–398.

[FGS2] Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic completeness for Rayleigh scattering. Ann. Henri Poincaré **3** (2002), 107–170.

[FGS3] Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic completeness for Compton scattering, preprint mp-arc 01-420 (2001).

[GGM] Georgescu, V., Gérard, C., Møller, J.: Commutators, C_0-semigroups and Resolvent Estimates, preprint.

[G1] Gérard, C. : On the existence of ground states for massless Pauli-Fierz Hamiltonians, Ann. Henri Poincaré **1** (2000), 443–459.

[G2] Gérard, C.: On the scattering theory of massless Nelson models, Rev. Math. Phys. **14** (2002), 1165-1280.

[HS] Hübner, M., Spohn, H.: Spectral properties of the spin-boson Hamiltonian, Ann. Inst. Henri Poincaré **62** (1995), 289–323.

[Ka] Kato, T.: *Perturbation Theory for Linear Operators*, Springer Verlag, 1976.

[LMS] Lörinczi, J., Minlos, R. A., Spohn, H.: The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field. Ann. Henri Poincaré **3** (2002), 269–295.

[MS] Møller, J. S., Skibsted, E.: Spectral theory of time-periodic many-body systems, preprint mp_arc 02-316 (2002).

[Ne] Nelson. E: Interaction of non-relativistic particles with a quantized scalar field, J. Math. Phys. **5** (1964), 1190–1197.

[Sk] Skibsted, E.: Spectral analysis of N-body systems coupled to a bosonic field. Rev. Math. Phys. **10** (1998), 989–1026.

[Sp] Spohn, H.: Asymptotic completeness for Rayleigh scattering. J. Math. Phys. **38** (1997), 2281–2296.