THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF FORCED BURGERS EQUATION ON THE CIRCLE

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Abstract. — We describe the asymptotic behaviour of entropy solutions of unviscid Burgers equation on the circle with time-periodic forcing. These solutions converge to periodic states, but the period of these limit states may be greater than the period of the forcing. We obtain as a corollary a new result on the flow of the associated Hamiltonian system.

Résumé. — On décrit les comportement asymptotique des solutions entropiques d'équations de Burgers sans viscosité avec un potentiel dépendant périodiquement du temps. Ces solutions convergent vers des états périodiques, dont la période est un multiple de celle du potentiel. On obtient un nouveau résultat sur la dynamique du flot du système Hamiltonien associé.

1. Introduction

1.1. The standard circle $\mathbb{R}/\mathbb{Z}$ is noted $\mathbb{T}$. The cotangent bundle $T^*\mathbb{T}$ is identified with $\mathbb{T}\times\mathbb{R}$. Given a function $f(t,x)$ of two variables, we will note f_t the function $x \longmapsto f(t,x)$. The partial derivative with respect to the variable t will be denoted $\partial_t f$. In all this paper, we will consider a time-periodic Hamiltonian $H(t,x,p) : \mathbb{R}\times T^*\mathbb{T} = \mathbb{R}\times\mathbb{T}\times\mathbb{R} \longrightarrow \mathbb{R}$, the associated time-periodic vector-field of $\mathbb{T}\times\mathbb{R}$ is noted X. We have

$$X(t,x,p) = \big(\partial_p H(t,x,p), -\partial_x H(t,x,p)\big).$$

1.2. The following standard hypotheses will be assumed :
i. The Hamiltonian H is C^2 and 1-periodic in t.
ii. The Hamiltonian H is convex in p, and $\partial_{pp}H > 0$.
iii. The Hamiltonian has superlinear growth in p, *i.e.* $\lim_{|p|\longrightarrow\infty} H(t,x,p)/p = \infty$ for each (t,x).
iv. The Hamiltonian flow is complete. More precisely for all (t_0,x_0,p_0), there exists a C^1 curve $\gamma(t) = (x(t),p(t)) : \mathbb{R} \longrightarrow \mathbb{T}\times\mathbb{R}$ such that $(x(t_0),p(t_0)) = (x_0,p_0)$ and $\dot\gamma(t) = X(t,\gamma(t))$ for all $t\in\mathbb{R}$. The mapping $\gamma(t_0) \longmapsto \gamma(t)$ is a diffeomorphism of $\mathbb{T}\times\mathbb{R}$, denoted $\phi_{t_0,t}$. We will pay a special attention to the diffeomorphism $\phi = \phi_{0,1}$. Note that the completeness Hypothesis is satisfied if there exists a constant C such that $|H_t| \leqslant C(1+H)$.

1.3. A typical example of Hamiltonian satisfying our hypotheses is

$$H(t,x,p) = \frac{1}{2}p^2 + V(t,x)$$

with a C^2 potential V periodic in t.

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1.4. We consider the equation

$$\partial_t y+\partial_x\big(H(t,x,y)\big)=0 \tag{B}$$

of the unknown function $y(t,x):\mathbb{R}\times\mathbb{T}\longrightarrow\mathbb{R}$. This equation will be called the Burgers equation in the sequel. Note that in case $H=p^2/2+V(t,x)$, we have the standard forced unviscid Burgers equation

$$\partial_t y+y\partial_x y=-\partial_x V(t,x).$$

1.5. The Burgers equation is quasi-linear, and its characteristics (see [A], chapter 2) are the trajectories of X. In other words, if $y(t,x):[a,b]\times\mathbb{T}\longrightarrow\mathbb{R}$ is a C^1 solution of the Burgers equation, then for each $a\leqslant t_0\leqslant t_1\leqslant b$, the graph of the function $x\longmapsto y(t_1,x)$ is the image by the diffeomorphism ϕ_{t_0,t_1} of the graph of the function $x\longmapsto y(t_0,x)$.

1.6. Still assuming that y is a C^1 solution of the Burgers equation, we obtain that

$$c(t)=\int_{\mathbb{T}} y(t,x)dx$$

is a constant, that we denote c. The function y can be written $y(t,x)=c+\partial_x u(t,x)$, where $u(t,x):I\times\mathbb{T}\longrightarrow\mathbb{R}$ satisfies the Hamilton-Jacobi equation

$$\partial_t u+H(t,x,c+\partial_x u)=0. \tag{HJc}$$

1.7. It is known that there exists in general no classical solution of Burgers equation defined on $\mathbb{R}^+\times\mathbb{T}$ satisfying a given initial condition $y(0,x)=y_0(x)$. However, the Cauchy problem is well-posed in the sense of entropy solutions. More precisely, for all $s\in\mathbb{R}$ and all function $u_s\in L^\infty(\mathbb{T})$ there exists a unique entropy solution $u(t,x)\in C([s,\infty),L^1(\mathbb{T}))$ such that $u(s,.)=u_s$, see 3.2. The operator which, to each function u_0, associates the function $u_1=u(1,.)$, where $u(t,x)$ is the entropy solution with initial condition $u(0,.)=u_0$, can be extended to a continuous operator $E:L^1(\mathbb{T})\longrightarrow L^1(\mathbb{T})$, see 3.3.

1.8. We want to describe the asymptotic behaviour of entropy solutions. Let us first recall that for each c there exists a 1-periodic solution of average c. More precisely, there exists a continuous and increasing function $c\longmapsto y^c$ from $\mathbb{R}$ to $L^1(\mathbb{T})$ which, to each c, associates a fixed point of E of average c, see [KO] or [JKM]. Note however that there may exist more than one fixed point of a given average c. It is natural to ask whether all solutions are attracted by these fixed points. The answer is negative, there are examples where there exist periodic points of E which are not fixed point, see [FM], that is periodic entropy solutions of minimal period greater than one. These subharmonic solutions in turn attract all other solutions, as we now state.

1.9. THEOREM *Let $y(t,x):[t_0,\infty)\times\mathbb{T}\longrightarrow\mathbb{R}$ be an entropy solution of Burgers equation. There exists an integer T and an entropy solution $\omega(t,x):\mathbb{R}\times\mathbb{T}\longrightarrow\mathbb{R}$ which is T-periodic in t and such that*

$$y_{t+nT}\longrightarrow\omega_t$$

in $L^1(\mathbb{T})$ as n goes to infinity.

If H is a function of p only, then $\omega(t,x)$ is the constant $\int y(t,x)dx$. The result in this special case has been obtained by Lax [L]. If H does not depend on t, then the asymptotic solution ω does not depend on t either, the result in this case follows from works of Roquejoffre [Ro] and Fathi [Fa3]. The theorem will be proved in section 3 as a consequence of a similar result for viscosity solutions of (HJc) obtained in [Be], see also [BR].

1.10. One can compare the situation with the viscous case. If one considers the parabolic equation

$$\partial_t y+\partial_x\big(H(t,x,y)\big)=\mu\partial_{xx}y \tag{B_μ}$$

with $\mu>0$, the behaviour is much simpler. One can prove in the line of [JKM], see also [B], that for each c, there exists a unique solution y^c of average c which is 1-periodic in time. This solution attracts all the solutions of average c. More precisely, if $y : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ is a solution of (B_μ), and if $\int y_t dx = c$, then $y_{t+n} \longrightarrow y^c_t$ uniformly as $n \in \mathbb{N} \longrightarrow \infty$.

1.11. The result in the unviscid case can be used to study the dynamics of the diffeomorphism $\phi = \phi_{0,1}$. Note that this diffeomorphism is a finite composition of area preserving right twist maps, and that any finite composition of right twist maps can be obtained that way. This correspondence between the twist property and the convexity of the Hamiltonian has been described by Moser, see [Mo].

1.12. In order to give a more geometrical meaning to Theorem 1.9, we consider the set $\mathcal{E}$ of functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ which can be locally written as the sum of a continuous and of a decreasing function. A function $f \in \mathcal{E}$ has a right limit $f^-(x)$ and a left limit $f^+(x)$ at each point x. These limits satisfy $f^-(x) \leqslant f^+(x)$, with a strict inequality on an at most countable set. Let $G^-(f)$ and $G^+(f)$ be the graphs, in $\mathbb{T} \times \mathbb{R}$ of the functions f^- and f^+. We define the graph $G(f) = G^-(f) \cap G^+(f)$ of f. Note that $\overline{G(f)} = G^-(f) \cup G^+(f)$. It is also useful to consider the set

$$H(f) = \bigcup_x \{x\} \times [f^-(x), f^+(x)] \subset \mathbb{T} \times \mathbb{R},$$

which is a Jordan curve containing $\overline{G(f)}$. The Hausdorff distance $d_H(f, g)$ between the compact sets $H(f)$ and $H(g)$ defines a distance d_H on $\mathcal{E}$ (one should take the quotient of $\mathcal{E}$ by the relation of almost everywhere equality).

1.13. The link between entropy solutions and the dynamic of ϕ can now be detailed, see 3.5. For each $y \in L^1$, we have $E(y) \in \mathcal{E}$. If in addition $y \in \mathcal{E}$, then

$$\overline{G(E(y))} \subset \phi(G(y)).$$

This property has striking consequences. For example, if y^c is a fixed point of E, then $y^c \in \mathcal{E}$ and $\overline{G(y^c)}$ is negatively invariant. As a consequence, the α-limit of $\phi_{|G(y^c)}$ is a non-empty compact set which is fully invariant by ϕ. It is an Aubry-Mather set. The rotation number $\rho(c) \in \mathbb{R}$ of the orbits of this set depends only on c, and the function $c \longmapsto \rho(c)$ is non decreasing and continuous, see 3.6.

1.14. Having defined the rotation number $\rho(c)$ allows us to complement Theorem 1.9. The asymptotic behaviour of solutions depend strongly on their space average c and on the associated rotation number $\rho(c)$. If $\rho(c)$ is irrational, then there exists a single fixed point of E of average c, see [E]. We will prove that it attracts all the trajectories of average c, that is one can take $T = 1$ in Theorem 1.9. If $\rho(c)$ is rational, $\rho(c) = p/q$ in lowest terms, then one can take $T = q$ in Theorem 1.9. It is thus natural to define the integer $T(c)$ by $T(c) = 1$ if $\rho(c)$ is irrational and $T(c) = q$ if $\rho(c) = p/q$ in lowest terms, and we have the following refinement of Theorem 1.9 :

Theorem *For each $y \in L^1(\mathbb{T})$, and $c = \int y$, there exists a fixed point $l \in \mathcal{E}$ of $E^{T(c)}$ such that $\|E^{nT(c)}(y) - l\|_{L^1} \longrightarrow 0$, which implies that $d_H(E^{nT(c)}(y), l) \longrightarrow 0$.*

1.15. We obtain a new result on the dynamics of ϕ, which may be seen as a converse to the celebrated result of Birkhoff (see [Ma2], [HF], [Si]) stating that a rotational (not homotopic to a constant) Jordan curve in $\mathbb{T} \times \mathbb{R}$ which is invariant by ϕ has to be the graph of a Lipshitz function $y : \mathbb{T} \longrightarrow \mathbb{R}$.

Corollary *Let $G \subset \mathbb{T} \times \mathbb{R}$ be the graph of a continuous function $y : \mathbb{T} \longrightarrow \mathbb{R}$. Assume that there exists an increasing sequence n_k of positive integers such that $\phi^{n_k}(G)$ is the graph of a continuous function. Then there exists a positive integer T such that $\phi^T(G) = G$, and the function y is Lipschitz. In addition, if $\phi(G) \neq G$, then the rotation number of $\phi^T_{|G}$ is an integer, hence G contains a T-periodic point of ϕ.*

Let us mention that it is certainly possible to give a more direct proof of this Corollary. One could use a topological approach, as suggested to me by by P. Le Calvez or a variational approach, as suggested by J. Xia. The proof presented here in 3.8 and 3.9 is however extremely short.

1.16. In the rest of the paper, we will detail the outline given above. We will obtain all the important properties of entropy solutions of (B) as consequences of properties of the viscosity solutions of (HJc). Hence we first describe these viscosity solutions in section 2, and draw our conclusions in section 3.

2. Calculus of variations and Hamilton-Jacobi equation

In the present section, we describe the main properties of viscosity solutions of the equation (HJc). These properties follow from the study of extremals via the Hopf-Lax-Oleinik formula, a global reference is the work of Fathi, [Fa1] and [Fa2]. We also state a result analogous to Theorem 1.9 for these solutions.

2.1. It is useful to introduce the Lagrangian $L : \mathbb{R} \times \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ associated to H. It is defined by

$$L(t,x,v) = \sup_p pv - H(t,x,p),$$

and has the following properties, which follow easily from the analogous properties 1.2 of H:
i. The Lagrangian L is C^2 and 1-periodic in t.
ii. The Lagrangian L is convex in v, and $\partial_{vv} L > 0$.
iii. The Lagrangian has superlinear growth in v, *i.e.* $\lim_{|v| \longrightarrow \infty} L(t,x,v)/v = \infty$ for each (t,x).

The Lagrangian associated to the modified Hamiltonian $H(t,x,p+c)$ is $L(t,x,v) - cv$, it satisfies the three properties above.

2.2. For each c and each $t_0 \leqslant t$, we have the Hopf-Lax-Oleinik operator $V^c_{t_0,t} : C(\mathbb{T},\mathbb{R}) \longrightarrow C(\mathbb{T},\mathbb{R})$ defined by

$$V^c_{t_0,t}(u)(x) = \min \left(u(x(t_0)) + \int_{t_0}^{t} L(s,x(s),\dot{x}(s)) - c\dot{x}(s)\, ds \right)$$

where the minimum is taken on the set of absolutely continuous curves $x : [t_0,t] \longrightarrow \mathbb{T}$ such that $x(t) = x$. Any curve realising the above minimum is C^2 and is the projection of a trajectory of X. More precisely, the curve $(x(s), \partial_v L(s,x(s),\dot{x}(s)))$ is a trajectory of X. These operators clearly satisfy the Markov property

$$V^c_{t_1,t_2} \circ V^c_{t_0,t_1} = V^c_{t_0,t_2}.$$

Note that this operator has been used in the study of viscosity solutions for quite a long time, see for instance [Fl].

2.3. A function is called semi-concave if it is locally the sum of a smooth function and of a concave function. A function $u(x) : \mathbb{T} \longrightarrow \mathbb{R}$ is called K-semi-concave if $\partial_{xx} u \leqslant K$ in the sense of distributions.

Proposition *The following properties are equivalent for a function* $u \in C([t_0,t_1] \times \mathbb{T},\mathbb{R})$ *:*
i. The function u is a viscosity solution of (HJc) in the classical sense, (see [Ba]*).*
ii. The function u is locally Lipschitz on $]t_0,t_1[\times\mathbb{T}$ and it satisfies (HJc) almost everywhere. In addition, there exists a non increasing function $K :]t_0,t_1] \longrightarrow]0,\infty[$ such that the functions u_t is $K(t)$-semi-concave for $t_0 < t \leqslant t_1$.
iii. For each t and t' such that $t_0 \leqslant t \leqslant t' \leqslant t_1$, we have $u_{t'} = V^c_{t,t'}(u_t)$.

Corollary *For each initial condition u_{t_0}, there exists one and only one viscosity solution $u : [t_0,\infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ of (HJc), it is given by*

$$u(t,x) = V^c_{t_0,t} u_{t_0}(x).$$

There exists a non-increasing function $K(t) :]t_0, \infty[\longrightarrow]0, \infty[$ *such that* u_t *is* $K(t)$*-semi-concave and such that* u *is* $K(t)$*-Lipschitz on* $[t, \infty) \times \mathbb{T}$ *for each* $t > t_0$*. In addition, if* u_{t_0} *is Lipschitz, then so is* u *on* $[t_0, \infty) \times \mathbb{T}$.

PROOF. It is standard that *i.* $\Longleftrightarrow$ *iii.*, see a good exposition in [FMa]. Let us recall a possible sketch of proof. One can first prove using variations around the maximum principle that there is at most one function satisfying *i.* with a given initial condition u_{t_0} (see [Ba], 2.4.). On the other hand, it is obvious that there exists one and only one function satisfying *iii.*, namely $(t, x) \longmapsto V^c_{t_0,t} u_{t_0}(x)$. One can prove (see [Fa2]) that this function also satisfies *i.* It is then the only one to do so, by uniqueness. It is also classical that *iii.* $\Longrightarrow$ *ii.*, see [Fa2]. We shall prove more carefully that *ii.* $\Longrightarrow$ *i.*, which seems less classical. Let us fix $(S, Q) \in]t_0, \infty) \times \mathbb{T}$, it is enough to prove (see [Ba], 5.3) that all C^1 function ϕ such that $u - \phi$ has a local minimum at (S, Q), satisfies the equation at (S, Q). If such a function ϕ exists, then $\partial_x u(S, Q)$ exists and is equal to $\partial_x \phi(t, x)$. It follows from the Lemma below that u is differentiable at (S, Q), and satisfies the equation at this point, which implies that ϕ also satisfies the equation at (S, Q). The additional conclusions of the Corollary follow in a classical way from the analysis of calibrated curves, as defined in 2.4, see [Fa2]. □

LEMMA *Let* $u(t, x)$ *be a function satisfying condition ii. of the proposition. If* $(S, Q) \in]t_0, \infty) \times \mathbb{T}$ *is a point where* $\partial_x u$ *exits, then the function* u *is differentiable and satisfies* (HJc) *at* (S, Q).

PROOF. Let us fix a time $t_2 \in]t_0, S[$. In view of the fact that all the functions $u_t, t \geqslant t_2$ are $K(t_2)$-semi-concave, it is not hard to prove that $\partial_x u(s_n, q_n) \longrightarrow \partial_x u(T, Q)$ when (s_n, q_n) is a sequence of points of differentiability of u converging to (S, Q). If we assume in addition that (HJc) holds at (s_n, q_n), we obtain that $\partial_t u(s_n, q_n)$ has a limit $H(t, x, c + \partial_x u(t, x))$. Let us denote L the linear form $(s, q) \longmapsto q\partial_x u(S, Q) + sH(S, Q, \partial_x u(S, Q))$. We have proved that there exists a modulus of continuity δ and a set $K \subset \mathbb{R} \times \mathbb{T}$ of full measure in a neighbourhood of (S, Q) such that, for each $(S + s, Q + q) \in K$, the function u is differentiable at $(S + s, Q + q)$ and $\|du(S + s, Q + q) - L\| \leqslant \delta\big(\|(s, q)\|\big)$. It follows that we have the estimate $|u(S + s, Q + q) - u(S, Q) - L(s, q)| \leqslant \|(s, q)\|\delta(\|(s, q)\|)$ for all (s, q) small enough, hence u is differentiable at (S, Q), and $du(S, Q) = L$. □

2.4. Let $u(t, x) : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ be a viscosity solution of (HJc), and let $t_0 \leqslant t < t'$. An absolutely continuous curve $x(s) : [t, t'] \longrightarrow \mathbb{T}$ is said calibrated by u if

$$u(t', x(t')) = u(t, x(t)) + \int_t^{t'} L(s, x(s), \dot{x}(s)) - c\dot{x}(s)\, ds.$$

If $x(s)$ is a calibrated curve, then it is C^2 and the curve $\big(x(s), \partial_v L(s, x(s), \dot{x}(s)\big)$ is a trajectory of the Hamiltonian vector field X. By extension, we say that a curve $\gamma(s) = (x(s), p(s)) : [t, t'] \longrightarrow \mathbb{T} \times \mathbb{R}$ is calibrated by u if $x(s)$ is calibrated by u and if $p(s) = \partial_v L(s, x(s), \dot{x}(s))$. It is then a trajectory of X. A curve is said to be calibrated by u on an interval I it is calibrated by u on $[t, t']$ for all $[t, t'] \subset I$. It was proved by Fathi that if $\gamma(s) = (x(s), p(s)) : [t, t'] \longrightarrow \mathbb{T} \times \mathbb{R}$ is calibrated, then the function u is differentiable at $(s, x(s))$ for $s \in]t, t'[$ and satisfies

$$\partial_x u(t, x(s)) = p(s).$$

It is not hard to prove in the same way that if u_0 is Lipschitz, then there exists a constant K such that each calibrated curve $\gamma(s) = (x(s), p(s)) : [t_0, t] \longrightarrow \mathbb{T} \times \mathbb{R}$ with $t > t_0$ satisfies $|p(s)| \leqslant K$ for $s \in [t_0, t]$. We obtain from this remark that u is Lipschitz if u_0 is.

2.5. We will note $V^c : C(\mathbb{T}, \mathbb{R}) \longrightarrow C(\mathbb{T}, \mathbb{R})$ the operator $V^c_{0,1}$.

PROPOSITION *The operator* V^c *is a contraction,*

$$\|V^c(u) - V^c(v)\|_\infty \leqslant \|u - v\|_\infty.$$

In addition, For each $C > 0$, there is a constant $K > 0$ such that, if $|c| \leqslant C$, the elements of the image of V^c are K-Lipschitz and K-semi-concave.

It will also be useful to consider the operator $\tilde{V} : \mathbb{R} \times C(\mathbb{T}) \longrightarrow \mathbb{R} \times C(\mathbb{T})$ defined by

$$\tilde{V}(c, u) = (c, V^c(u)).$$

This operator is continuous and compact.

2.6. There exists a C^1 convex and super linear function $\alpha(c) : \mathbb{R} \longrightarrow \mathbb{R}$ such that, for all solution $u(t, x) : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ of (HJc), the function $u(t, x) + t\alpha(c)$ is bounded on $[t_0, \infty) \times \mathbb{T}$. This function is the alpha function of Mather, also called the effective Hamiltonian. Let us illustrate a bit more the meaning of this function:

LEMMA *If $u(t, x) : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ is a viscosity solution of (HJc), the function $v(\theta, x) : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}$ defined by*

$$v(\theta, x) = \liminf_{t \ mod \ 1 = \theta} \big(u(t, x) - t\alpha(c)\big)$$

is a viscosity solution of the Hamilton-Jacobi equation

$$\partial_\theta v + H(\theta, x, c + \partial_x v) = \alpha(c),$$

where we have noted θ for t mod 1.

COROLLARY *The Hamilton-Jacobi equation*

$$\partial_\theta u + H(\theta, x, c + \partial_x u) = a$$

has a 1-periodic viscosity solution if and only if $a = \alpha(c)$.

A proof of the Lemma is written in [BR], the corollary is by now classical.

2.7. The number $\rho(c) = \alpha'(c)$ has an important dynamical meaning:
PROPOSITION *For each viscosity solution $u(t, x) : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ of (HJc), there exist curves $\gamma : [t_0, \infty) \longrightarrow \mathbb{T}$ which are calibrated by u. These curves all have the same rotation number*

$$\lim \frac{1}{t} \int_{t_0}^{t} \dot{x} = \rho(c) = \alpha'(c).$$

PROOF. Let $u_c : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ be a viscosity solution of (HJc) and let $x(t) : [t_0, \infty) \longrightarrow \mathbb{T}$, be calibrated by u_c. We have

$$u_c(t, x(t)) - u_c(t_0, x(t_0)) = \int_{t_0}^{t} L(s, x(s), \dot{x}(s))ds - c \int_{t_0}^{t} \dot{x}(s)ds.$$

In view of the definition of $\alpha(c)$ in 2.6, we obtain

$$\lim_{t \longrightarrow \infty} \Big(\frac{c}{t} \int_{t_0}^{t} \dot{x}(s)ds - \frac{1}{t} \int_{t_0}^{t} L(s, x(s), \dot{x}(s))ds\Big) = \alpha(c).$$

Let us now consider a viscosity solution $u_e : [t_0, \infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ of (HJe). With the same curve $x(s)$, we have

$$u_e(t, x(t)) - u_e(t_0, x(t_0)) \leqslant \int_{t_0}^{t} L(s, x(s), \dot{x}(s))ds - e \int_{t_0}^{t} \dot{x}(s)ds$$

hence

$$\alpha(e) \geqslant \limsup \Big(\frac{e}{t} \int_{t_0}^{t} \dot{x}(s)ds - \frac{1}{t} \int_{t_0}^{t} L(s, x(s), \dot{x}(s))ds\Big),$$

which implies

$$\alpha(e) - \alpha(c) \geqslant \limsup \Big(\frac{e - c}{t} \int_{t_0}^{t} \dot{x}(s)ds\Big).$$

We conclude

$$\alpha'(c) \leqslant \liminf \frac{1}{t}\int_{t_0}^{t} \dot{x}(s)ds \leqslant \limsup \frac{1}{t}\int_{t_0}^{t} \dot{x}(s)ds \leqslant \alpha'(c).$$

It follows that the curve $x(s)$ has a rotation number $\alpha'(c)$. □

2.8. The asymptotic behaviour of viscosity solution is described by the following theorem, obtained in [Be] (see also [BR] for a better proof, and see [Fa3] and [Ro] for related results). Let $T(c) \in \mathbb{N}$ be defined by:
T(c)=1 if $\rho(c)$ is irrational,
T(c)=q if $\rho(c)$ is the rational p/q in lowest terms.

THEOREM *For each $u \in C(\mathbb{T},\mathbb{R})$, the sequence $\big(V^c\big)^{nT(c)}(u)$ is converging uniformly to a fixed point of $\big(V^c\big)^{T(c)}$.*

3. ENTROPY SOLUTIONS AND CHARACTERISTICS

The relation between classical solutions of Burgers equation and the Hamiltonian dynamics is quite well understood from 1.5. We shall now describe the main properties of entropy solutions, with emphasis on their relation with dynamics. We will also prove Theorem 1.14 and corollary 1.15.

3.1. A function $y : [t_0,\infty) \longrightarrow \mathbb{R}$ is called an entropy solution of Burgers equation (B) if :

i. The functions y and $H(t,x,y(t,x))$ are locally integrable and the equation holds in the sense of distributions :

$$\int_{[t_0,\infty)\times\mathbb{T}} y(t,x)\partial_t\phi(t,x) + H(t,x,y(t,x))\partial_x\phi(t,x)dtdx + \int_{\mathbb{T}} y(t_0,x)\phi(t_0,x)dx = 0$$

for all smooth function $\phi : [t_0,\infty)\times\mathbb{T} \longrightarrow \mathbb{R}$ with compact support (see [S] for details). Note that the space average $\int_{\mathbb{T}} y(t,x)dx$ is then a constant c.

ii. The Oleinik inequalities

$$y_t(x+\delta) - y_t(x) \leqslant K(t)\delta$$

hold for all $t > t_0$, $x \in \mathbb{T}$ and $\delta > 0$, with a positive and decreasing function $K(t)$.

3.2. PROPOSITION *For each $t_0 \in \mathbb{R}$ and each $y_0 \in L^\infty(\mathbb{T})$, there exists a unique entropy solution $y : [t_0,\infty)\times\mathbb{T} \longrightarrow \mathbb{R}$ of Burgers equation satisfying $y(t_0,.) = y_0$. We have*

$$y \in L^\infty([t_0,\infty)\times\mathbb{T}) \cap C\big([t_0,\infty), L^1(\mathbb{T})\big).$$

This solution is given by $y(t,x) = c + \partial_x u(t,x)$, where $c = \int y_0$ and $u(t,x)$ is the viscosity solution of (HJc) of initial condition $u_{t_0}(x) = \int_0^x (y_0 - c)$.

Note that we could associate a function $y \in C(]t_0,\infty[, L^1)$ to any initial condition $y_0 \in L^1$ in exactly the same way. However we do not have $y \in C([t_0,\infty[, L^1)$ if y_0 is not bounded.

PROOF. Let us first deal with uniqueness. The standard method to prove uniqueness is to use the Oleinik inequalities 3.1 ii., via a duality method, see [H], Theorem 2.2.1, or [S], 2.8. We shall use the Hamilton-Jacobi equation. Indeed, let $y(t,x) : [t_0,\infty)\times\mathbb{T} \longrightarrow \mathbb{R}$ be an entropy solution. Note that this function is locally bounded in $]t_0,\infty)\times\mathbb{T}$ in view of the Oleinik inequalities. Define $\tilde{u}(t,x) = \int_0^x (y(t,x)-c)dx - \int_0^1 (y(t,x)-c)dx$, where $c = \int y$. We have, in the sense of distributions, $\partial_{tx}\tilde{u} = \partial_t y = -\partial_x(H(t,x,y))$. Hence the distribution $\partial_t\tilde{u}(t,x) + H(t,x,y(t,x))$ does not depend on x, and is the locally integrable function $f(t) = \int_{\mathbb{T}} H(t,x,y(t,x))dx$. The function $u(t,x) = \tilde{u}(t,x) - \int_{t_0}^t f(s)ds$, satisfies $\partial_t u + H(t,x,y) = 0$ in the sense of distributions, hence it is locally Lipschitz on $]t_0,\infty)\times\mathbb{T}$ and the Hamilton-Jacobi equation $\partial_t u + H(t,x,c+\partial_x u) = 0$

holds almost everywhere. In addition, the function u satisfies condition *ii.* of proposition 2.3 as a consequence of the Oleinik inequalities *ii.* above, so that u has to be *the* viscosity solution of (HJc) of initial condition u_{t_0}. So the only candidate to be an entropy solution of (B) is $y(t,x) = c + \partial_x u(t,x)$.

It is classical to obtain the existence of entropy solutions as limits of regular solutions of the viscous equation (B_μ). However we shall use once more the Hamilton Jacobi equation, *i.e.* we shall prove that the function $y(t,x) = c + \partial_x u(t,x)$ introduced in the discussion on uniqueness is indeed an entropy solution. Recall that u_{t_0} is Lipschitz hence u is Lipschitz, hence $\partial_t u$ and $y = c + \partial_x u$ are well defined in $L^\infty([t_0,\infty)\times\mathbb{T})$, as well as $H(t,x,y(t,x))$. It is straightforward that *i.* is satisfied, and *ii.* follows from the property *ii.* of Proposition 2.3. Since the the injection $L^\infty \longrightarrow L^1$ is compact, the continuity of $t \longmapsto y_t$ follows from the continuity of $t \longmapsto u_t$ and the fact that the functions y_t are equibounded. □

3.3. We call E the operator $y_0 \longmapsto y_1$, which can be extended to an operator

$$E : L^1(\mathbb{T}) \longrightarrow L^1(\mathbb{T}).$$

Let us recall the major properties of E.
i. The operator E is nondecreasing, and it is a contraction in L^1:

$$\|E(y) - E(z)\|_1 \leqslant \|y - z\|_1$$

ii. The operator E is compact. More precisely, for all $C > 0$, there exists $K > 0$ such that, when $|\int y| \leqslant C$, we have $\|E(y)\|_\infty \leqslant K$, and the Oleinik inequality

$$E(y)(x+\delta) \leqslant E(y)(x) + K\delta$$

for all x and $\delta > 0$.
iii. The operator E preserves the average $c = \int y$.

The contraction property is quoted here for completeness. It can be obtained as a consequence of the Oleinik inequalities using a duality method, see [S]. It could also be proved using the function u, but we shall not write down this proof here. We shall only use this property in the case where all functions y, z, $E(y)$ and $E(z)$ are continuous. In this very easy case, the contraction property is a direct consequence of the area-preservation property of the diffeomorphism ϕ, as will become clear in the sequel.

3.4. Consider an entropy solution $y(t,x) : [t_0,\infty)\times\mathbb{T} \longrightarrow \mathbb{R}$ of Burgers equation, let $c = \int y$ and let $u(t,x) : [t_0,\infty)\times\mathbb{T} \longrightarrow \mathbb{R}$ be a viscosity solution of (HJc) such that $y = c + \partial_x u$. A trajectory $\gamma(s) : [t_0,\infty) \supset [t,t'] \longrightarrow \mathbb{T}\times\mathbb{R}$ of the Hamiltonian vector-field X is called a y-characteristic if $\gamma(s) \in \overline{G(y_s)}$ for each $s \in [t,t']$. The following theorem extends the method of characteristics to entropy solutions :

Theorem (Characteristics)
i. Let us fix $t_0 \leqslant t < t'$. *A curve* $\gamma(s) :]t_0,\infty) \supset [t,t'] \longrightarrow \mathbb{T}\times\mathbb{R}$ *is a* y*-characteristic if and only if it is calibrated by* u.
ii. For every $t > t_0$ *and every* $(x,p) \in \overline{G(y_t)}$*, there exists a unique* y*-characteristic* $\gamma(s) :]t_0,t] \longrightarrow \mathbb{T}\times\mathbb{R}$ *such that* $\gamma(t) = (x,p)$.
iii. Let $\gamma(s) = (x(s),p(s)) :]t_0,t] \longrightarrow \mathbb{T}\times\mathbb{R}$ *be a* y*-characteristic, then for each* $s \in]t_0,t[$*, the function* y_s *is continuous at* $x(s)$ *and* $y(s,x(s)) = p(s)$.
iv. If in addition $y_{t_0} \in \mathcal{E}$*, then for every* $t > t_0$ *and every* $(x,p) \in \overline{G(y_t)}$*, there exists a unique* y*-characteristic* $\gamma(s) = (x(s),p(s)) : [t_0,t] \longrightarrow \mathbb{T}\times\mathbb{R}$ *such that* $\gamma(t) = (x,p)$*. In addition, the function* y_{t_0} *is continuous at* $x(t_0)$*, and* $y(t_0,x(t_0)) = p(t_0)$.

3.5. In terms of the dynamics, using the notations of 1.12, this theorem implies that

$$\overline{G(y_{t'})} \subset \phi_{t,t'}\big(G(y_t)\big)$$

when $t_0 < t < t'$, and that

$$\overline{G(E(y))} \subset \phi\big(G(y)\big)$$

when $y \in \mathcal{E}$.

3.6. Let $y(t,x) : [t_0,\infty) \times \mathbb{T} \longrightarrow \mathbb{R}$ be an entropy solution of Burgers equation, and let $c = \int y$. We say that an absolutely continuous curve $x(s) : [t_0,\infty) \supset [t,t'] \longrightarrow \mathbb{T}$ is a weak y-characteristic if it satisfies the equation $\dot{x}(s) \in [y_s^-(x(s)), y_s^+(x(s))]$ almost everywhere. In view of the Oleinik inequalities, it follows from extended Cauchy-Lipschitz results, see [H], Theorem 1.4.2, that for each $t \geqslant t_0$ and $x_0 \in \mathbb{T}$, there exists one and only one weak characteristic $x(s) : [t,\infty) \longrightarrow \mathbb{T}$ satisfying $x(t) = x_0$. It is clear that y-characteristics are weak y-characteristics, hence Proposition 2.7 implies:

Proposition *All weak y-characteristics $x(s) : [t,\infty) \longrightarrow \mathbb{T}$ have the same rotation number*

$$\lim \frac{1}{t}\int_{t_0}^{t} \dot{x} = \rho(c) = \alpha'(c).$$

3.7. Let us define the operator $U : L^1(\mathbb{T}) \longrightarrow C(\mathbb{T},\mathbb{R})$ by

$$U(y)(x) = \int_0^x y(x)dx - x\int_{\mathbb{T}} y(x)dx.$$

It will also be useful to consider the operator $\tilde{U} : L^1(\mathbb{T}) \longrightarrow \mathbb{R} \times C(\mathbb{T})$ defined by $\tilde{U}(y) = (\int_{\mathbb{T}} y, U(y))$. We then have

$$E = \tilde{U}^{-1} \circ \tilde{V} \circ \tilde{U}.$$

In order to give a meaning to this expression, we have to notice that, if u is in the image of V, then u is Lipschitz so that $\tilde{U}^{-1}(c,u)$ is well defined for all c and given by

$$\tilde{U}^{-1}(c,u)(t,x) = c + \partial_x u(t,x).$$

In addition, the restriction of $\tilde{U}^{-1}$ to the image of $\tilde{V}$ is continuous. Indeed, let us consider a sequence $c_n \longrightarrow c$ of reals and a sequence u_n in the image of V^{c_n}, such that $u_n \longrightarrow u$. The sequence u_n is then equilipshitz, hence $\partial_x u_n \longrightarrow \partial_x u$ in L^1. Theorem 1.14 is now a direct consequence of Theorem 2.8.

3.8. Let us turn our attention to the dynamics of ϕ. It is useful to consider the reversed Hamiltonian $\breve{H}(t,x,p) = H(-t,x,-p)$, which clearly also satisfies the hypotheses 1.2. We denote by $\breve{E}$ the associated entropy operator. Denoting by S the symmetry $(x,p) \longmapsto (x,-p)$, we have for the associated vector-field

$$\breve{X}(t,x,p) = -S(X(-t,x,-p)).$$

Hence the flow satisfies

$$S \circ \breve{\phi} = \phi^{-1} \circ S,$$

so that

$$\overline{G(\breve{E}(-y))} \subset \phi^{-1}\big(G(-y)\big)$$

when $-y \in \mathcal{E}$.

Lemma *If there exists $n > 2$ such that the function $E^n(y)$ is continuous, then we have*

$$\breve{E}^k(-E^n(y)) = -E^{n-k}(y)$$

for $0 < k < n$, hence the functions $E^{n-k}(y)$ are K-Lipschitz.

Proof. By recurrence, it is enough to prove the Lemma for $k = 1$. We have, noting $y_i = E^i(y)$,

$$G(-\breve{E}(-y_n)) \subset \phi^{-1}G(y_n) \subset G(y_{n-1})$$

hence $y_{n-1} = -\breve{E}(-y_n)$. It follows that $-y_{n-1}$ satisfies the inequalities 3.3, ii. so that y_{n-1} is K-Lipschitz since it is also satisfies these inequalities. ☐

3.9. We are now in a position to prove Corollary 1.15 for ϕ. Let us set $c = \int y$. The functions $E^n(y)$ are K-Lipschitz for $n > 0$. In view of Theorem 1.14, the sequence $E^{nT(c)}(y)$ has a limit z which is also K-Lipschitz and satisfies $E^{T(c)}(z) = z$. Let us set for simplicity $y_n = E^{nT(c)}(y)$. It follows from the Lemma that $y_{n-k} = \breve{E}^{kT(c)}(y_n)$ and $z = \breve{E}^{T(c)}(z)$. The contraction property of $\breve{E}$, see 3.3 *i.*, implies that the sequence $\|y_n - z\|_1$ is non decreasing, hence it is identically zero, so that $y = z$. ☐

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