Transfer operators for $\Gamma_0(n)$ and the Hecke operators for the period functions of $\mathrm{PSL}(2,\mathbb{Z})$ [1]

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Abstract

In this article we report on a surprising relation between the transfer operators for the congruence subgroups $\Gamma_0(n)$ and the Hecke operators on the space of period functions for the modular group $\mathrm{PSL}(2,\mathbb{Z})$. For this we study special eigenfunctions of the transfer operators with eigenvalues ∓ 1, which are also solutions of the Lewis equations for the groups $\Gamma_0(n)$ and which are determined by eigenfunctions of the transfer operator for the modular group $\mathrm{PSL}(2,\mathbb{Z})$. In the language of the Atkin-Lehner theory of old and new forms one should hence call them old eigenfunctions or old solutions of Lewis equation. It turns out that the sum of the components of these old solutions for the group $\Gamma_0(n)$ determine for any n a solution of the Lewis equation for the modular group and hence also an eigenfunction of the transfer operator for this group.

Our construction gives in this way linear operators in the space of period functions for the group $\mathrm{PSL}(2,\mathbb{Z})$. Indeed these operators are just the Hecke operators for the period functions of the modular group derived previously by Zagier and Mühlenbruch using the Eichler-Manin-Shimura correspondence between period polynomials and modular forms for the modular group.

1 Introduction

This paper has three main ingredients. The first is the transfer operator from statistical mechanics which plays an important role in the ergodic theory of dynamical systems and especially in the theory of dynamical zeta functions (see [**Ma3**], [**Ru**]). Here we are interested in the transfer operators for the geodesic flow on the surfaces $\Gamma/\mathbb{H}$ for Γ any of the congruence subgroups $\Gamma_0(n)$. These operators have been introduced in [**CM**], [**CM1**] in the study of Selbergs zeta function for these groups.

The second ingredient are certain functions holomorphic in the cut plane, introduced by J. B. Lewis in [**Le**] in his study of the Maass wave forms for $\mathrm{PSL}(2,\mathbb{Z})$. They were later named period functions by Zagier (see [**Za1**]) because of their close relation to the classical period polynomials in the Eichler, Manin , Shimura theory of periods for cusp forms. Period functions for the modular group are solutions of the so called Lewis equation

$$\phi(z) = \phi(z+1) + \lambda z^{-2s}\phi\left(1+\frac{1}{z}\right) \tag{1}$$

with $\lambda = \pm 1$, which fulfill certain growth conditions at infinity depending on the weight s. When this weight satisfies $\Re(s) = \frac{1}{2}$, these solutions are in 1-1 correspondence with the

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Maass cusp forms (see [**LZ2**]). There is a simple relation between the transfer operator for $\mathrm{PSL}(2,\mathbb{Z})$ and the period functions: they are just the eigenfunctions of this operator with eigenvalue ± 1 (see [**CM1**]). When s is a negative integer $s = -n$ the space of polynomial solutions of the Lewis equation is in 1-1 correspondence with the space of period polynomials of cusp forms for $\mathrm{PSL}(2,\mathbb{Z})$ (see [**Za1**]). The Eichler-Shimura-Manin theory of periods however tells us that this space of period polynomials modulo a certain one dimensional space is isomorphic to the direct sum of two copies of the space of cusp forms of weight $2n+2$ in the half plane.

The space of cusp forms is extensively studied in number theory and in particular we have the Hecke algebra acting on it.

A Theorem by Choie and Zagier (see [**CZ**] §3 Theorem 2) gives a criterion to find an explicit realization of the corresponding Hecke operators when acting on the space of period polynomials or more generally period functions. Generalizing the description of Hecke operators for Maass wave forms by Manin in [**Ma**], Choie and Zagier found (see [**CZ**], Theorem 3) an explicit form for these Hecke operators in the space of period polynomials. Their matrices, however, from which the Hecke operators are constructed via the well known slash action of the group $\mathrm{Mat}_n(2,\mathbb{Z})$ on smooth functions, have negative entries and hence their action is defined only for entire weights.

The third important ingredient in our paper is a new realization of the Hecke operators on period functions due to T. Mühlenbruch (to appear in his thesis, see [**Mu**]). He uses the matrices in the set

$$S_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a > c \geq 0,\ d > b \geq 0,\ ad - bc = n \right\}$$

such that the Hecke operators T_n have the form

$$T_n := \sum_{A \in S_n} A \tag{2}$$

acting on the period functions via the slash operator, and where the sum is taken in the free abelian group generated by S_n. All the matrices from the set S_n have positive entries and so one can define the Hecke operators on period functions for any weight $s \in \mathbb{C}$.

The relation between the transfer operator and the period functions was considered originally only for the modular group $\mathrm{PSL}(2,\mathbb{Z})$. In this case the Lewis equation is a scalar equation for scalar functions and its solutions can be related explicitly to the period functions of modular and Maass wave forms. In order to extend this theory to more general Fuchsian groups Chang and Mayer began in a series of papers (see [**CM1**] and its references) to investigate the transfer operator approach to congruence subgroups like $\Gamma_0(n)$, $\Gamma^0(n)$ or $\Gamma(n)$. This lead them to transfer operators acting in Banach spaces of vector valued holomorphic functions. The eigenfunctions of these operators then fulfill general Lewis equations in vector spaces whose dimension is just the index in $\mathrm{PSL}(2,\mathbb{Z})$ of the corresponding subgroup.

In the present paper we discuss special solutions of these Lewis equations for the groups $\Gamma_0(n)$ which are in fact determined by the solutions of the Lewis equation for the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Hence our construction is somehow reminiscent of the theory of Atkin and Lehner of old and new forms (see [**AL**]). The exact connection will be discussed in a forthcoming paper.

To state our main results and to sketch the content of each section we have to fix the notations used throughout the text. For each integer n let $\mathrm{Mat}_n(2,\mathbb{Z})$ (resp. $\mathrm{Mat}_*(2,\mathbb{Z})$) be the set of 2×2-matrices with integer entries and determinant n (resp. nonzero determinant) and $\mathcal{R}_n := \mathbb{Z}[\mathrm{Mat}_n(2,\mathbb{Z})]$ (resp. $\mathcal{R} := \mathbb{Z}[\mathrm{Mat}_*(2,\mathbb{Z})]$) the set of finite linear combinations (with coefficients in $\mathbb{Z}$) of the elements of $\mathrm{Mat}_n(2,\mathbb{Z})$ (resp. $\mathrm{Mat}_*(2,\mathbb{Z})$). Note that $\mathcal{R} = \cup_{n\in\mathbb{Z}}\mathcal{R}_n$ and $\mathcal{R}_n \cdot \mathcal{R}_m \subset \mathcal{R}_{nm}$. By definition we have

$$\mathrm{GL}(2,\mathbb{Z}) = \mathrm{Mat}_1(2,\mathbb{Z}) \cup \mathrm{Mat}_{-1}(2,\mathbb{Z}).$$

The following four elements of $\mathrm{GL}(2,\mathbb{Z})$ will play a prominent role in this paper:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It turns out that instead of the groups $\Gamma_0(n)$ it is more convenient to use their extensions $\overline{\Gamma}_0(n)$ in $\mathrm{GL}(2,\mathbb{Z})$

$$\overline{\Gamma}_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid c \equiv 0 \mod n \right\} = \Gamma_0(n) \cup \Gamma_0(n) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In § 4 we recall the definition of the transfer operators for $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ as used by Chang and Mayer in [**CM**], [**CM1**], respectively by Manin and Marcolli in [**MM**], discuss briefly their relation and derive the Lewis equation for the eigenfunctions of the operator of Manin and Marcolli. This operator is defined on a space of holomorphic functions with values in the representation space of $\mathrm{GL}(2,\mathbb{Z})$ induced from the trivial representation of $\overline{\Gamma}_0(n)$. In order to describe and solve the corresponding Lewis equation it turns out that the appropriate indexing of the components of these functions by the set

$$I_n := \overline{\Gamma}_0(n)\backslash\mathrm{GL}(2,\mathbb{Z})$$

helps a lot. The group $\mathrm{GL}(2,\mathbb{Z})$ acts on this coset space on the right in a canonical way. The detailed structure of I_n will be studied in § 5 (in particular see Propositions 5.4 and 5.8). The different components of the Lewis equation can then be written for $i \in I_n$ as follows

$$\phi_i(z) - \phi_{(iT^{-1})}(z+1) - \lambda z^{-2s}\phi_{(iT^{-1}M)}\left(1+\frac{1}{z}\right) = 0. \tag{3}$$

These equations have to be solved simultaneously with functions ϕ_i holomorphic in the cut plane $\mathbb{C}\setminus(-\infty,0]$ for all $i \in I_n$.

Let $\mathcal{I}^\lambda := (I - T - \lambda TM)\mathcal{R}$ be the right ideal generated by $(I - T - \lambda TM)$ in $\mathcal{R}$. Consider then the following system of equations in the right $\mathcal{R}$-module $\mathcal{I}^\lambda\backslash\mathcal{R}$

$$\psi_i - \psi_{(iT^{-1})}T - \lambda\psi_{(iT^{-1}M)}TM = 0 \mod \mathcal{I}^\lambda, \quad \forall i \in I_n, \tag{4}$$

which obviously is closely related to (3). Here the ψ_i's are unknown elements in $\mathcal{R}$. Note the two different matrix actions in these equations: on the one hand matrices acting from the right on the index i of ψ_i and on the other hand matrices acting from the right on elements ψ_i via the ring multiplication of $\mathcal{R}$. Moreover, in (3) we have the familiar slash operation formally defined for $s \in \mathbb{C}$ and $R \in \mathcal{R}$ by

$$\phi \mid_s R(z) = |\det R|^s (cz+d)^{-2s}\phi\left(\frac{az+b}{cz+d}\right). \tag{5}$$

Now suppose $\psi_i, i \in I_n$ solves (4) and ϕ is a solution of the Lewis equation (1) for $\mathrm{PSL}(2,\mathbb{Z})$. For s an integer the left hand side of (4) can act on ϕ via the usual slash-operator and one obtains a solution $(\phi_i)_{i\in I_n}$ of (3) by setting

$$\phi_i := \phi \mid_s \psi_i$$

since $\phi \mid_s \mathcal{I}^\lambda = 0$.

It is well known that $\frac{1}{z}$ is up to a constant factor the only solution of the scalar Lewis equation (1) for $\lambda = 1$ and $s = 1$ (see [**Ma4**]). It follows from a result by Y. Manin and M. Marcolli (see Proposition 4.2 and Remark 7.1) that for these parameter values $(\phi_i)_{i\in\mathcal{I}_n}$ with $\phi_i(z) = \frac{1}{z}$ for all $i \in I_n$ is, up to a trivial scalar factor, also the unique solution of (3). Hence, if $(\psi_i)_{i\in I_n}$ solves (4), then there exists a constant κ such that

$$\frac{1}{z} \mid_1 \psi_i = \kappa \frac{1}{z} \quad \forall i \in I_n$$

must hold. Suppose furthermore that $\psi_i = \sum_{A\in P_i} A$, where P_i is some finite subset of $\mathrm{Mat}_n(2,\mathbb{Z})$. Then the above equality reads $\sum_{A\in P_i} \frac{1}{(az+b)(cz+d)} = \kappa\frac{1}{z}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The right hand side of this expression obviously has a pole and a zero only at 0 and ∞. Hence other poles and zeroes of the left hand side must cancel. This means, however, that the matrices $A \in P_i$ have to be chosen in a very specific way. Explicit calculations for the groups $\Gamma_0(n)$ for small n lead us to an operator $K : A \mapsto K(A)$ which attaches to every matrix $A \in P_i$ another matrix KA whose action just cancels the poles and zeros generated by the action of A. In all cases considered only a finite number of matrices A were necessary to get the correct pole and zero structure. We later found that an operator similar to K was indeed used already by Choie and Zagier [**CZ**] and also by Mühlenbruch (see [**Mu**], Lemma 9) in their completely different derivation of the Hecke operators.

The explicit form of the map K is given as

$$\begin{aligned} K : S_n \setminus Y_n &\rightarrow S_n \setminus X_n \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto T^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c + \lceil \frac{d}{b} \rceil a & -d + \lceil \frac{d}{b} \rceil b \\ a & b \end{pmatrix}, \end{aligned}$$

where

$$X_n := \left\{ \begin{pmatrix} c & a \\ 0 & \frac{n}{c} \end{pmatrix},\ c \mid n,\ 0 \le a < \frac{n}{c} \right\}, \quad Y_n := \left\{ \begin{pmatrix} c & 0 \\ a & \frac{n}{c} \end{pmatrix},\ c \mid n,\ 0 \le a < c \right\}$$

and where for a real r we have denoted by $\lceil r \rceil$ the integer satisfying $\lceil r \rceil - 1 < r \le \lceil r \rceil$. With the usual notation of Gauss brackets we obtain $\lceil r \rceil = -[-r]$. The inverse of K is given by

$$K^{-1} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto MT^{\lceil \frac{a}{c} \rceil} QM \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a + \lceil \frac{a}{c} \rceil c & -b + \lceil \frac{a}{c} \rceil d \end{pmatrix}$$

(see Proposition 6.1 for all this). Borrowing terminology from algebraic geometry one may call K a rational automorphism of S_n.

To each index $i \in I_n$ we will attach a matrix (see Definition 5.9)

$$A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix},$$

where $c \geq 1$, $c \mid n$ and $0 \leq b < \frac{n}{c}$ satisfy $\gcd(c, \frac{n}{c}, b) = 1$. The numbers c and b are then uniquely determined by the index i.

Starting now with a matrix $A_i \in X_n$ we apply K repeatedly until we get an element of Y_n where the iteration stops. Since K is injective, two such chains of elements in S_n are either equal or disjoint. For $i \in I_n$ we denote by k_i the number such that $K^j A_i$ is well-defined for $j \leq k_i$ and $K^{k_i} A_i \in Y_n$ (see Definition 6.2). Obviously each element in $X_n \cap Y_n$ forms a one-element chain so that $k_i = 0$ for $A_i \in X_n \cap Y_n$. Our main result then is

Theorem 1.1. *The matrices*

$$\psi_i = \sum_{j=0}^{k_i} K^j(A_i), \quad i \in I_n$$

determine a solution of equations (4). Acting by these matrices through the slash operator on a solution ϕ of the Lewis equation (1) for the group $\mathrm{PSL}(2,\mathbb{Z})$ *with weight s gives a solution of equation (3) for the group $\overline{\Gamma}_0(n)$ with the same weight s.*

In the second part of the theorem we used the fact that the slash operator with weight s an arbitrary complex number is indeed well defined for the elements of $\mathcal{R}$ defining the ψ_i. Details will be discussed in § 3. In particular, Lemma 3.5 gives a condition which ensures the equation

$$\phi \mid_s (I - T - \lambda TM)R = \big(\phi \mid_s (I - T - \lambda TM)\big) \mid_s R = 0$$

for $R \in \mathcal{R}$ to make sense and thus enables us to construct solutions of (3) from solutions of (4) in the way explained above.

Theorem 1.1 shows that any solution ϕ of the scalar Lewis equation (1) for $PSL(2,\mathbb{Z})$ determines a solution $(\phi \mid_s \psi_i)_{i \in I_n}$ of the system (3) of Lewis equations corresponding to $\overline{\Gamma}_0(n)$. Since the sum of the components of any solution of (3) is again a solution of the scalar Lewis equation, this fact together with Theorem 1.1 allows us to define a linear operator $\tilde{T}_n$ mapping the space of solutions of the scalar Lewis equation for $\mathrm{PSL}(2,\mathbb{Z})$ to itself. For these operators we find

Proposition 1.2. *The operators $\tilde{T}_n$ and the Hecke operators T_n defined in (2) are related through*

$$T_n = \sum_{d^2 \mid n} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tilde{T}_{\frac{n}{d^2}}.$$

In particular they coincide if and only if n is a product of distinct primes.

Thereby we identified the matrices T_n and $\tilde{T}_{\frac{n}{d^2}}$ with the operators they define via the slash action. The operators $\tilde{T}_n$ have been constructed through special solutions of the Lewis equation for the congruence subgroups $\overline{\Gamma}_0(n)$. There arises immediately the question if this is also the case for the Hecke operators T_n. Indeed, it turns out that also the operators $\tilde{T}_{\frac{n}{d^2}}$ appearing in the above Proposition 1.2 can be related to special solutions of the Lewis equation for the group $\overline{\Gamma}_0(n)$: one shows quite generally that any solution of the Lewis equation for a group $\overline{\Gamma}_0(m)$ determines a solution of the corresponding equation for the group $\overline{\Gamma}_0(ml)$ for arbitrary $l \in \mathbb{N}$. Its components are just copies of the former's components (see Proposition 9.8.) Taking then as the solution for the group $\overline{\Gamma}_0(m)$ the solution of Theorem 1.1 we get in this way a solution for the group $\overline{\Gamma}_0(ml)$. The sum of its components gives just μ-times the operator $\tilde{T}_m$ where μ is the index of $\overline{\Gamma}_0(ml)$ in $\overline{\Gamma}_0(m)$. This shows that also the operators $\tilde{T}_{\frac{n}{d^2}}$ can be constructed from special solutions of the Lewis equation for the group $\overline{\Gamma}_0(n)$ and hence from special eigenfunctions of the transfer operator for this group.

Our results depend in a crucial way on a modified one-sided continued fraction expansion for rational numbers and closely related partitions of $\mathcal{R}$ described in § 2.

The technical results about the slash-operation are provided in § 3 and the transfer operators for $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ are introduced in § 4. The indexing coset space $\overline{\Gamma}_0(n)\backslash \mathrm{GL}(2,\mathbb{Z})$ is studied in detail in § 5. In § 6 we derive and discuss the operator K and in § 7 we describe various versions of the Lewis equations for the groups $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ and relate them to each other. This allows us to construct special solutions of these equations. Finally, in § 8 we show how our results lead to a completely new approach to the Hecke operators on the space of period functions for $\mathrm{PSL}(2,\mathbb{Z})$ which basically only uses the transfer operators for the congruence subgroups $\Gamma_0(n)$ respectively $\overline{\Gamma}_0(n)$. Work on the extension of this approach to the Hecke operators also for other groups like the congruence subgroups is going on at the moment.

2 A modified continued fraction expansion

This section is basically inspired by the work of Mühlenbruch in [**Mu**] adapted appropriately to our needs. Mühlenbruch introduces in [**Mu**] a modified continued fraction expansion for positive rational numbers and attaches to each $x \in \mathbb{Q}^+$ a suitable chain of elements of $\mathcal{R}$ called a partition of x. To explain his construction we begin by collecting some facts which are standard in the theory of continued fractions (see [**HW**]). Consider the finite continued fraction expansion of x

$$x = [a_0, a_1, \ldots, a_N] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_N}}}$$

and put $\frac{p_n}{q_n} := [a_0, a_1, \ldots, a_n]$ for $0 \le n \le N$. Then $\gcd(p_n, q_n) = 1$, $q_n \ge 0$, and the recursion formulas

$$p_n = a_n p_{n-1} + p_{n-2} \tag{6}$$

$$q_n = a_n q_{n-1} + q_{n-2} \tag{7}$$

hold. In particular, we have

$$q_0 \le q_1 < \ldots < q_N. \tag{8}$$

Moreover, the following equations

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots \le x \le \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1} \tag{9}$$

and

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \; p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n \tag{10}$$

hold. We are going to fill the above sequence (9) with more rational numbers. We do that for the left hand side of the sequence, the case we later use. Assume that n is even. The sequence of numbers

$$[a_0, a_1, \dots, a_{n-1}, t] = \frac{tp_{n-1} + p_{n-2}}{tq_{n-1} + q_{n-2}}, \quad \text{for } t = 0, \dots, a_n$$

is then strictly increasing from $\frac{p_{n-2}}{q_{n-2}}$ to $\frac{p_n}{q_n}$. We insert these numbers into the left hand side of (9) and obtain the longer sequence :

(11)
$$\dots < \frac{p_{n-2}}{q_{n-2}} < \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} < \dots < \frac{(a_n - 1)p_{n-1} + p_{n-2}}{(a_n - 1)q_{n-1} + q_{n-2}} < \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n} < \dots$$

Here we have used the convention $\frac{1}{a + \frac{1}{0}} = 0$. If we denote the rational numbers x_j in this sequence by $x_j = \frac{p'_j}{q'_j}$ then $\gcd(p'_j, q'_j) = 1$ and two consecutive numbers $\frac{p'_j}{q'_j} < \frac{p'_{j+1}}{q'_{j+1}}$ satisfy $q'_j < q'_{j+1}$ and

$$p'_{j+1} q'_j - p'_j q'_{j+1} = 1, \tag{12}$$

where the last equality is a consequence of (10). Recall (see [**HW**]) that for $x \in \mathbb{Q}^+$ there is a unique sequence $a_0, \dots, a_n \in \mathbb{N}$ such that $a_n > 1$ and $x = [a_0, \dots, a_n]$: If $x = [b_0, \dots, b_{m-1}, 1]$ for $b_0, \dots, b_{m-1} \in \mathbb{N}$, then obviously $x = [b_0, \dots, b_{m-1} + 1]$, and hence $m = n + 1$ and $a_0 = b_0, \dots, a_{n-1} = b_{n-1}, a_n = b_n + 1$. This will be used in the following definitions.

Definition 2.1. Given $x \in \mathbb{Q}^+$, the modified continued fraction expansion of x is the sequence $x_j,\ j = 0, 1, \dots$ recursively defined by:

- $x_0 := x = [a_0, \dots, a_N]$ with $a_N > 1$.
- If $x_{j-1} = [b_0, \dots, b_m]$ with $b_m > 1$, then

$$x_j := \begin{cases} [b_0, b_1, \dots, b_{m-1}] & \text{if } 2 \not| m \\ [b_0, b_1, \dots, b_m - 1] & \text{if } 2 \mid m. \end{cases}$$

If $x_{j-1} = 0$, then $x_j = -\infty$ and the sequence stops. □

Note that the length of the modified continued fraction expansion of $x = [a_0, \dots, a_N]$ with $a_N > 1$ is not greater than $\sum_{i=1,\text{even}}^{N} a_i$.

Proposition 2.2. *Let $x \in \mathbb{Q}^+$ and $x_0, x_1, \dots, x_{k-1}, x_k$ with $x_0 = x$ and $x_k = -\infty$ be its modified continued fraction expansion. If $x_j = \frac{p_j}{q_j}$ with $\gcd(p_j, q_j) = 1$ and $q_j \geq 0$, then we have $p_{j-1}q_j - p_j q_{j-1} = 1$ for $j = 1, \dots, k$ and $q_0 > q_1 > \dots > q_{k-1} > q_k = 0$.*

Proof. Suppose that $x_{j-1} = \frac{p_{j-1}}{q_{j-1}} = [b_0, \dots, b_m]$ with $b_m > 1$. If m is odd we have $x_j = \frac{p_j}{q_j} = [b_0, \dots, b_{m-1}]$ and the relation $p_{j-1}q_j - p_j q_{j-1} = 1$ follows from (10) applied to the continued fraction $[b_0, \dots, b_m]$, whereas $q_{j-1} > q_j$ is a consequence of the inequalities in (8) for $[b_0, \dots, b_m]$.

In the case where m is even the same calculation leading to (12) can be used to derive $p_{j-1}q_j - p_j q_{j-1} = 1$ from the recursion relations for the continued fraction $[b_0, \dots, b_m]$. Here $q_{j-1} > q_j$ follows also from the recursion relations for the continued fraction $[b_0, \dots, b_m]$. □

Definition 2.3. A sequence $x_0, x_1, \ldots, x_{k-1}, x_k$ of rational numbers is called an *admissible sequence of length* $k+1$ if the following property holds: if $x_j = \frac{p_j}{q_j}$, where $\gcd(p_j, q_j) = 1$ and $q_j \geq 0$, then

$$\det \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} = 1 \quad \forall j = 1, 2, \ldots, k. \tag{13}$$

Let x be a positive rational number. A *partition* P of x is an admissible sequence $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x_0 = x$ and $x_k = -\infty$. The number $k+1$ is called the *length* of the partition. We use the convention $-\infty = \frac{-1}{0}, 0 = \frac{0}{1}$. A partition P of x is called a *minimal partition* if

$$q_0 > q_1 > \ldots > q_{k-1} > q_k = 0 \tag{14}$$

□

Remark 2.4. From (13) it follows that $p_{j-1}q_j > p_j q_{j-1}$ which implies $x_{j-1} > x_j$ for all $j = 1, 2, \ldots, k$. Moreover, (13) shows that the equation $p_{j-1}q_j \equiv 1 \mod q_{j-1}$ has a unique solution q_j with $0 \leq q_j < q_{j-1}$. Therefore each $x \in \mathbb{Q}^+$ has a unique minimal partition, which we denote by P_x. According to Proposition 2.2 the modified continued fraction expansion of $x \in \mathbb{Q}^+$ satisfies (13) and (14). Therefore it agrees with the minimal partition P_x. We will show in Proposition 2.6 that there is indeed no partition whose length is less than the length of the minimal partition which justifies the name minimal partition. □

Throughout this paper we will use the notations introduced in Definition 2.3.

Remark 2.5. Let $x = x_0, x_1, \ldots, x_{k-1}, x_k$ be a partition of $x \in \mathbb{Q}^+$ and $x_j = \frac{p_j}{q_j}$ with $\gcd(p_j, q_j) = 1$ and $q_j \geq 0$.

(i) The equation $p_{k-1}q_k - p_k q_{k-1} = 1$ implies that $-p_k = 1 = q_{k-1}$. If the partition is minimal Remark 2.4 and the construction of the modified continued fraction expansion of x shows that in addition we have $p_{k-1} = 0$.

(ii) If $q_{j-1} = q_j$ for some $j \in \{1, \ldots, k-1\}$, then (13) shows that $q_{j-1} = q_j = 1$, i.e., $x_j = p_j = p_{j-1} - 1 = x_{j-1} - 1$.

□

For a partition P of x of length $k+1$ given by $x_0, x_1, \cdots, x_{k-1}, x_k$ and any index $1 \leq l \leq k-1$, a simple calculation shows that for $x_j = \frac{p_j}{q_j}$ with $\gcd(p_j, q_j) = 1$ for $j = 0, 1, \ldots, k$ the sequence

$$x_0, \ldots, x_{l-1}, \frac{p_{l-1} + p_l}{q_{l-1} + q_l}, x_l, \ldots, x_{k-1}, x_k \tag{15}$$

defines a new longer partition $P(l)$ of x. We call it a Farey extension of partition P. One can also introduce the inverse of this construction: if a partition P contains a triple of the type $\frac{p_{l-1}}{q_{l-1}}, \frac{p_{l-1}+p_l}{q_{l-1}+q_l}, \frac{p_l}{q_l}$, then one can delete $\frac{p_{l-1}+p_l}{q_{l-1}+q_l}$ and obtains in this way a shorter partition $\check{P}(l)$ of x called a Farey reduction of P.

Proposition 2.6. *Every partition P of a rational number $x \in \mathbb{Q}^+$ can be obtained from the minimal partition P_x of x by a finite number of Farey extensions $P(l)$. The minimal partition P_x can be derived from any partition P by a finite number of Farey reductions $\check{P}(l)$.*

Proof. Given a partition $x_0, x_1, \ldots, x_{k-1}, x_k$ of x with $x_j = \frac{p_j}{q_j}$ and $\gcd(p_j, q_j) = 1$ it is enough to prove that if the sequence $(q_j)_{j=0,\ldots,k}$ is not decreasing, then there exists a number $l \in \{1, \ldots, k-1\}$ such that

$$\frac{p_l}{q_l} = \frac{p_{l+1} + p_{l-1}}{q_{l+1} + q_{l-1}}.$$

Since $q_k = 0$ there exists for $(q_j)_{j=0,\ldots,k}$ not strictly decreasing an index $l \in \{1, \ldots, k-1\}$ such that

$$q_l > q_{l+1} \quad \text{but} \quad q_l \geq q_{l-1}.$$

If $q_l > q_{l-1}$, then the triple x_{l-1}, x_l, x_{l+1} must be of the form $\frac{p_{l-1}}{q_{l-1}}, \frac{e+mp_{l-1}}{f+mq_{l-1}}, \frac{e+(m-1)p_{l-1}}{f+(m-1)q_{l-1}}$, where $m \in \mathbb{N}$ and $\frac{e}{f}$ is the unique rational number such that $p_{l-1}f - q_{l-1}e = 1$ and $0 \leq e < q_{l-1}$.

If $q_l = q_{l-1}$, then Remark 2.5 shows that $q_{l-1} = q_l = 1$ and x_{l-1}, x_l, x_{l+1} is of the form $\frac{p_{l-1}}{1}, \frac{p_{l-1}-1}{1}, \frac{p_{l+1}}{q_{l+1}}$. But then (13) shows that $p_{l+1} = (p_{l-1} - 1)q_{l+1} - 1$ so that

$$\frac{p_{l-1} + p_{l+1}}{q_{l-1} + q_{l+1}} = p_{l-1} - 1,$$

which implies the claim also in this case. □

Lemma 2.7. *Let $P_x = (x_0, x_1, \ldots, x_k)$ be the minimal partition of $x \in \mathbb{Q}^+$. If $x_j = \frac{p'_j}{q'_j}$ with $\gcd(p'_j, q'_j) = 1$ and $q'_j \geq 0$, then we have*

(i) $x < \frac{p'_{j-1} - p'_j}{q'_{j-1} - q'_j}$ *for $j = 1, 2, \ldots, k$.*

(ii) $\lceil \frac{xq'_{j+1} - p_{j+1}}{xq'_j - p'_j} \rceil = p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1}$ *for $j = 1, 2, \ldots, k-1$.*

Proof. (i) Let $x_0 = [a_0, \ldots, a_N]$ be the continued fraction expansion of x with $a_N > 1$. If p_n and q_n are the corresponding denumerators and denominators defined by (6) and (7), then Remark 2.4 shows that the sequence $\ldots > \frac{p'_{j-1}}{q'_{j-1}} > \frac{p'_j}{q'_j} > \frac{p'_{j+1}}{q'_{j+1}} > \ldots$ is the same as (11) which can also be rewritten as

(16)
$$\ldots < \frac{p_{n-2}}{q_{n-2}} = \frac{p_n - a_n p_{n-1}}{q_n - a_n q_{n-1}} < \frac{p_n - (a_n - 1)p_{n-1}}{q_n - (a_n - 1)q_{n-1}} < \ldots < \frac{p_n - p_{n-1}}{q_n - q_{n-1}} < \frac{p_n}{q_n} < \ldots,$$

where n is even. For two consecutive elements

$$\frac{p'_j}{q'_j} = \frac{(k-1)p_{n-1} + p_{n-2}}{(k-1)q_{n-1} + q_{n-2}}, \quad \text{and} \quad \frac{p'_{j-1}}{q'_{j-1}} = \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}$$

in (16) we have

$$\frac{p'_{j-1} - p'_j}{q'_{j-1} - q'_j} = \frac{(kp_{n-1} + p_{n-2}) - ((k-1)p_{n-1} + p_{n-2})}{(kq_{n-1} + q_{n-2}) - ((k-1)q_{n-1} + q_{n-2})} = \frac{p_{n-1}}{q_{n-1}}$$

and since n is even (9) shows that this is larger than x.

(ii) There are two possible forms for three consecutive elements in the sequence (16). The first is

$$(17)\quad \frac{p'_{j-1}}{q'_{j-1}} = \frac{p_n-(k-1)p_{n-1}}{q_n-(k-1)q_{n-1}}, \quad \frac{p'_j}{q'_j} = \frac{p_n-kp_{n-1}}{q_n-kq_{n-1}}, \quad \frac{p'_{j+1}}{q'_{j+1}} = \frac{p_n-(k+1)p_{n-1}}{q_n-(k+1)q_{n-1}},$$

where $k = 1, 2, \ldots, a_n - 1$. Then, using (10) and n even, we obtain

$$\begin{aligned} & p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1} = \\ = \ & (p_n-(k-1)p_{n-1})(q_n-(k+1)q_{n-1}) - (p_n-(k+1)p_{n-1})(q_n-(k-1)q_{n-1}) \\ = \ & 2. \end{aligned}$$

On the other hand, using $k \geq 1$ and $p_{n-1} - xq_{n-1}, xq_n - p_n > 0$ (again recall that n is even), we calculate

$$\begin{aligned} \lceil \tfrac{xq'_{j+1}-p_{j+1}}{xq'_j-p'_j} \rceil &= \lceil \tfrac{x(q_n-(k+1)q_{n-1})-(p_n-(k+1)p_{n-1})}{x(q_n-kq_{n-1})-(p_n-kp_{n-1})} \rceil \\ &= 1 + \lceil \tfrac{p_{n-1}-xq_{n-1}}{k(p_{n-1}-xq_{n-1})+xq_n-p_n} \rceil \\ &= 2. \end{aligned}$$

Thus (ii) if proved for triples of the form (17).

The second type of triples appearing in (16) is

$$(18)\qquad \frac{p'_{j-1}}{q'_{j-1}} = \frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \quad \frac{p'_j}{q'_j} = \frac{p_n}{q_n}, \quad \frac{p'_{j+1}}{q'_{j+1}} = \frac{p_n-p_{n+1}}{q_n-q_{n+1}}$$

with even n. This time we have

$$\begin{aligned} p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1} &= (p_n+p_{n+1})(q_n-q_{n-1}) - (p_n-p_{n-1})(q_n+q_{n+1}) \\ &= a_{n+1}+2 \end{aligned}$$

and

$$\lceil \tfrac{xq'_{j+1}-p_{j+1}}{xq'_j-p'_j} \rceil = \lceil \tfrac{x(q_n-q_{n-1})-(p_n-p_{n-1})}{xq_n-p_n} \rceil = 1 + \lceil \tfrac{p_{n-1}-xq_{n-1}}{xq_n-p_n} \rceil.$$

But an easy calculation again using (10) shows that $\lceil \frac{p_{n-1}-xq_{n-1}}{xq_n-p_n} \rceil = a_{n+1} + 1$ if and only if $\frac{p_n+p_{n+1}}{q_n+q_{n+1}} \leq x < \frac{p_{n+1}}{q_{n+1}}$, which, according to (11), is indeed the case.

□

Definition 2.8. Consider an admissible sequence $P = (x_0, \ldots, x_k)$ of $x_0 = x \in \mathbb{Q}^+$ with $x_j = \frac{p_j}{q_j}$ such that $\gcd(p_j, q_j) = 1$ and $q_j \geq 0$. To this partition we attach the following element $m(P)$ of $\mathbb{Z}[\mathcal{R}_1] = \mathbb{Z}[\mathrm{SL}(2,\mathbb{Z})]$

$$(19)\quad m(P) = \begin{pmatrix} q_0 & -p_0 \\ q_1 & -p_1 \end{pmatrix} + \ldots + \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix} + \begin{pmatrix} q_l & -p_l \\ q_{l+1} & -p_{l+1} \end{pmatrix} + \ldots + \begin{pmatrix} q_{k-1} & -p_{k-1} \\ q_k & -p_k \end{pmatrix}.$$

□

Given two admissible sequences $P_1 = (x_0, x_1, \ldots, x_k)$ and $P_2 = (y_0, y_1, \ldots, y_l)$ with $x_k = y_0$ we can define the *join*

$$P_1 \vee P_2 = (x_0, x_1, \ldots, x_k, y_1, \ldots, y_l) \tag{20}$$

of P_1 and P_2, which is again admissible. Note that in this case we have

$$m(P_1 \vee P_2) = m(P_1) + m(P_2). \tag{21}$$

$\mathrm{GL}(2,\mathbb{Z})$ acts on rational numbers from the left in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax+b}{cx+d}.$$

For the next lemma we will need the corresponding right action:

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} x = \frac{dx-b}{-cx+a}$$

Lemma 2.9. *Let* $P = (x_0, x_1, \ldots, x_k)$ *be an admissible sequence and* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ *with*

$$\frac{a}{c} \geq x_i, \ i = 0, 1, 2, \ldots, k \tag{22}$$

(which for $c = 0$ *simply means* $a > 0$*). Then*

$$P \cdot A := \begin{cases} (x_0 A, x_1 A, \ldots, x_{k-1} A, x_k A) & \text{for } \det A = 1 \\ (x_k A, x_{k-1} A, \ldots x_1 A, x_0 A) & \text{for } \det A = -1 \end{cases}$$

defines an admissible sequence with the property

$$m(P)A = \begin{cases} m(P \cdot A) & \text{for } \det A = 1 \\ Mm(P \cdot A) & \text{for } \det A = -1, \end{cases}$$

where $(m(P), A) \mapsto m(P)A$ *is the multiplication in* $\mathcal{R}$*.*

Proof. Condition (22) implies that for $x_j = \frac{p_j}{q_j}$ with $\gcd(p_j, q_j) = 1$ and $q_j \geq 0$, the number $x_j A = \frac{dp_j - bq_j}{aq_j - cp_j} \in \mathbb{Q}$ and $\gcd(dp_j - bq_j, aq_j - cp_j)$ =1, since $(r,s)\begin{pmatrix} p \\ q \end{pmatrix} = 1$ implies $\big((r,s)A\big)\big(A^{-1}\begin{pmatrix} p \\ q \end{pmatrix}\big) = 1$. Moreover, for $\det A = 1$ the matrix

$$\begin{pmatrix} aq_{j-1} - cp_{j-1} & -dp_{j-1} + bq_{j-1} \\ aq_j - cp_j & -dp_j + bq_j \end{pmatrix} = \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{23}$$

has determinant 1, which implies that $P \cdot A$ is indeed admissible. The equality $m(P)A = m(P \cdot A)$ is immediate from (23). The case $\det A = -1$ can be treated similarly. $\square$

To simplify the notation we define for $r \in \mathbb{Z}$ and $m \in \mathbb{N}$ the number $(r)_m \in \{0, \ldots, m-1\}$ as

$$(r)_m \equiv r \mod m.$$

Moreover, having fixed $n \in \mathbb{N}$ once and for all, we attach to each $i \in \{1, \ldots, n-1\}$ relative prime to n the number $\hat{\imath} \in \{1, \ldots, n-1\}$ with $i\hat{\imath} \equiv 1 \mod n$.

Lemma 2.10. *Suppose that* $c, c', i \in \mathbb{N}_0$ *satisfy* $c, c' \geq 1, \quad c, c' \mid n$ *and* $\gcd(c, c') = 1 = \gcd(i, n)$.

(i) *The matrix*

$$X := \begin{pmatrix} (c\hat{\imath} - c')_{\frac{n}{c'}} + c' & c' \\ \frac{n}{c'} & 0 \end{pmatrix} \cdot \begin{pmatrix} c & (c'i)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}^{-1}$$

is contained in $\mathrm{GL}(2, \mathbb{Z})$.

(ii) *Set*

$$x := \left\{ \frac{c'i - c}{\frac{n}{c}} \right\}, \quad y := \left\{ \frac{c'i}{\frac{n}{c}} \right\}, \quad z := \left\{ \frac{c\hat{\imath} - c'}{\frac{n}{c'}} \right\}.$$

and

$$s := \frac{((c'i - c)_{\frac{n}{c}} + c - (c'i)_{\frac{n}{c}})}{n/c}.$$

Then $P_z \cdot X$ *is an admissible sequence beginning with* y *and ending in* $x - s$. *Moreover, the join* $(P_z \cdot X) \vee (P_x \cdot T^s)$ *is well-defined and a partition of* y.

(iii)

$$\begin{aligned} m(P_x) \begin{pmatrix} c & (ic' - c)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} T + M\, m(P_z) \begin{pmatrix} c' & (\hat{\imath}c - c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} TM = \\ = \quad m\big((P_z \cdot X) \vee (P_x \cdot T^s)\big) \begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}. \end{aligned}$$

Proof. (i) Note first that

$$\begin{aligned} X &= \frac{1}{n} \begin{pmatrix} (c\hat{\imath} - c')_{\frac{n}{c'}} + c' & c' \\ \frac{n}{c'} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{n}{c} & -(c'i)_{\frac{n}{c}} \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} \frac{((c\hat{\imath} - c')_{\frac{n}{c'}} + c')\frac{n}{c}}{n} & \frac{-((c\hat{\imath} - c')_{\frac{n}{c'}} + c')(c'i)_{\frac{n}{c}} + cc'}{n} \\ \frac{\frac{n}{c}\frac{n}{c'}}{n} & \frac{-\frac{n}{c'}(c'i)_{\frac{n}{c}}}{n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{((c\hat{\imath} - c')_{\frac{n}{c'}} + c')}{c} & \frac{-((c\hat{\imath} - c')_{\frac{n}{c'}} + c')(c'i)_{\frac{n}{c}} + cc'}{n} \\ \frac{n}{cc'} & -\frac{(c'i)_{\frac{n}{c}}}{c'} \end{pmatrix}. \end{aligned}$$

Since c and c' are relative prime and divide n we have $cc' \mid n$ so that $c \mid \frac{n}{c'}$ and $c' \mid \frac{n}{c}$. But then

$$c \mid (c' + (\hat{\imath}c - c')_{\frac{n}{c'}} \quad \text{iff} \quad c \mid (c' + \hat{\imath}c - c')$$

and the latter is evident. Similarly $c' \mid (ic')_{\frac{n}{c}}$ reduces to $c' \mid ic'$ which is clear and

$$n \mid \big(-(c' + (\hat{\imath}c - c')_{\frac{n}{c'}})(ic')_{\frac{n}{c}} + cc' \big)$$

reduces to $n \mid (-\hat{\imath}cic' + cc')$ which again is evident. Thus the entries of X are all integral. Since $\det X = -1$ is immediate from the definition of X, this proves the claim.

(ii) Suppose that P_z is given by $(z = z_0, \ldots, z_m)$. According to Remark 2.4 we have $z = z_0 > z_1 > \ldots > z_m$ Note that for X condition (22) is satisfied for P_z. In fact, the number $\frac{a}{c}$ in (22) is

$$\frac{\left(\frac{((c\hat{i}-c')_{\frac{n}{c'}}+c')}{c}\right)}{\left(\frac{n}{cc'}\right)} = z + \frac{{c'}^2}{n} > z \geq z_j \quad \forall j = 0, \ldots, m.$$

Now Lemma 2.9 shows that $P_z \cdot X$ exists and since $\det X = -1$ the first element of $P_z \cdot X$ is given by

$$z_m X = -\infty X = \frac{-\frac{(c'i)_{\frac{n}{c}}}{c'}}{-\frac{n}{cc'}} = \frac{(c'i)_{\frac{n}{c}}}{\frac{n}{c}} = y,$$

whereas the last element of $P_z \cdot X$ is given by

$$\begin{aligned}
z_0 X &= \frac{(c\hat{i}-c')_{\frac{n}{c'}}}{\frac{n}{c'}} \begin{pmatrix} \frac{((c\hat{i}-c')_{\frac{n}{c'}}+c')}{c} & \frac{-((c\hat{i}-c')_{\frac{n}{c'}}+c')(c'i)_{\frac{n}{c}}+cc'}{n} \\ \frac{n}{cc'} & -\frac{(c'i)_{\frac{n}{c}}}{c'} \end{pmatrix} \\
&= \frac{(c'i)_{\frac{n}{c}} - c}{\frac{n}{c}} \\
&= \frac{(c'i-c)_{\frac{n}{n}}}{\frac{n}{c}} - \frac{((c'i-c)_{\frac{n}{c}} + c - (c'i)_{\frac{n}{c}})}{\frac{n}{c}} \\
&= x - s.
\end{aligned}$$

Note that for T^s the number $\frac{a}{c}$ in (22) is $\frac{1}{0} = \infty$. Since $\det T^s = 1$ Lemma 2.9 shows that $P_x \cdot T^s$ exists, starts with $xT^s = x - s$ and ends at $-\infty T^s = -\infty$. Thus the join $(P_z \cdot X) \vee (P_x \cdot T^s)$ exists and is a partition of y.

(iii) An elementary calculation shows

$$\begin{pmatrix} c & (c'i-c)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} T = T^s \begin{pmatrix} c & (c'i)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}$$

and using the formula for X derived in (i) we also find

$$\begin{pmatrix} c' & (c\hat{i}-c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} TM = \begin{pmatrix} (c\hat{i}-c')_{\frac{n}{c'}} + c' & c' \\ \frac{n}{c'} & 0 \end{pmatrix} = X \begin{pmatrix} c & (c'i)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}.$$

Now using (ii), (21), and Lemma 2.9 we calculate

$$\begin{aligned}
&m(P_x) \begin{pmatrix} c & (ic'-c)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} T + M\, m(P_z) \begin{pmatrix} c' & (\hat{i}c-c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} TM \\
&= [m(P_x)T^s + M\, m(P_z)X] \begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} \\
&= [m(P_x T^s) + m(P_z X)] \begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} \\
&= m\big((P_z \cdot X) \vee (P_x \cdot T^s)\big) \begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}.
\end{aligned}$$

□

3 The slash operator for complex weight s

Let $\mathcal{F}$ be the set of functions ϕ holomorphic in the domain $\mathbb{C}\backslash(-\infty, r]$ for some $r = r_\phi$ which we call a branching point of ϕ. Note that this does not rule out that ϕ extends to the point r as a holomorphic function. In $\mathcal{F}$ we have the usual addition and multiplication of functions. If $\phi_1, \phi_2 \in \mathcal{F}$, then one can find $r_{\phi_1\phi_2}, r_{\phi_1+\phi_2}$ such that $r_{\phi_1\phi_2}, r_{\phi_1+\phi_2} \leq \max\{r_{\phi_1}, r_{\phi_2}\}$. We fix the branch of $\log z$ in $\mathbb{C}\backslash(-\infty, 0]$ which coincides with the ordinary logarithm on $(0,\infty)$ and set $z^s := e^{s\log z}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $s \in \mathbb{C}$. For each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}$ with

$$\mathcal{G} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_*(2,\mathbb{Z}) \mid (c > 0 \text{ or } (c = 0 \ \& \ a, d > 0)) \right\}$$

one has $(cz+d)^s \in \mathcal{F}$. If $c = 0$ and $\phi \in \mathcal{F}$, then also $\phi(\frac{az+b}{d}) \in \mathcal{F}$. Consider the subset $\mathcal{DS}$ of $\mathcal{F} \times \mathcal{G}$ consisting of those pairs $\left(\phi, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ such that there exists a branching point r_ϕ for ϕ with

$$a - cr_\phi > 0 \tag{24}$$

Proposition 3.1. *Fix $s \in \mathbb{C}$. Then the formula*

$$(\phi \mid_s R)(z) = |\det R|^s (cz+d)^{-2s} \phi\left(\frac{az+b}{cz+d}\right) \tag{25}$$

defines a map

$$\begin{array}{rcl} \mathcal{DS} & \to & \mathcal{F} \\ (\phi, R) & \mapsto & \phi \mid_s R. \end{array}$$

If r_ϕ is a branching point for ϕ satisfying (24) for $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}$, then

$$\max\left\{ \frac{dr_\phi - b}{a - cr_\phi}, -\frac{d}{c} \right\}$$

is a branching point for $\phi \mid_s R$, where we interprete $-\frac{d}{c}$ as $-\infty$ if $c = 0$.

Proof. Suppose that $z > -\frac{d}{c}$. Then

$$\frac{az+b}{cz+d} > r_\phi \quad \Leftrightarrow \quad (a - cr_\phi)z > dr_\phi - b$$

and if (24) is satisfied the last inequality is equivalent to

$$z > \frac{dr_\phi - b}{a - cr_\phi}.$$

Now the claim is immediate. ☐

Remark 3.2. The *slash-operation* from Proposition 3.1 can be extended by linearity to the subset $\mathcal{DS}_\mathbb{Z}$ of $\mathcal{F} \times \mathbb{Z}[\mathcal{G}]$ consisting of those pairs $\left(\phi, \sum_{j=1}^m n_j R_m\right)$ for which all $(\phi, R_j) \in \mathcal{DS}$. In fact, suppose that (24) is satisfied for (ϕ, R_j) with branching points $r_{\phi,j}$ for ϕ, then (24) is satisfied for all (ϕ, R_j) with branching points $\min_j r_{\phi,j}$. ☐

Proposition 3.3. *Suppose that* $R_1, R_2, R_1R_2 \in \mathcal{G}$ *and* $(\phi, R_1), (\phi\ |_s\ R_1, R_2), (\phi, R_1R_2) \in \mathcal{DS}$. *Then for each* $s \in \mathbb{C}$ *we have*

$$(\phi\ |_s\ R_1)\ |_s\ R_2 = \phi\ |_s\ (R_1R_2).$$

Proof. We argue by analytic continuation. Note first that for $R_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ we have $R_1R_2 = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$ and since $R_1, R_2, R_1R_2 \in \mathcal{G}$ the functions

$$\big((c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)\big)^{-2s}$$

and

$$(c_2z + d_2)^{-2s}\left(c_1\left(\tfrac{a_2z+b_2}{c_2z+d_2}\right) + d_1\right)^{-2s} = (c_2z + d_2)^{-2s}\left(\tfrac{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}{c_2z+d_2}\right)^{-2s}$$

are holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and agree on $(0, \infty)$, hence agree everywhere. But then

$$\begin{aligned}
&\big((\phi\ |_s\ R_1)\ |_s\ R_2\big)(z) = \\
= \;& |\det R_2|^s (c_2z + d_2)^{-2s} (\phi\ |_s\ R_1)\left(\tfrac{a_2z+b_2}{c_2z+d_2}\right) \\
= \;& |\det R_2|^s (c_2z + d_2)^{-2s} |\det R_1|^s \left(c_1\left(\tfrac{a_2z+b_2}{c_2z+d_2}\right) + d_1\right)^{-2s} \phi\left(\frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_1}\right) \\
= \;& |\det R_1R_2|^s (c_2z + d_2)^{-2s} \left(c_1\left(\tfrac{a_2z+b_2}{c_2z+d_2}\right) + d_1\right)^{-2s} \phi\left(\tfrac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}\right) \\
= \;& \big(\phi\ |_s\ (R_1R_2)\big)(z).
\end{aligned}$$

□

Remark 3.4. Set

$$\mathcal{G}^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} \mid a > 0; b, d \geq 0)) \right\}$$

and

$$\mathcal{F}_0 := \{\phi \in \mathcal{F} \mid 0 \text{ is a branching point of } \phi\}.$$

Then $\mathcal{G}^+$ is a multiplicative subsemigroup of $\mathrm{Mat}_*(2, \mathbb{Z})$ and we have $\mathcal{F}_0 \times \mathcal{G}^+ \subseteq \mathcal{DS}$. Moreover the slash-operation $|_s$ induces a semigroup action $\mathcal{F}_0 \times \mathcal{G}^+ \to \mathcal{F}_0$. In fact, given $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}^+$ and $\phi \in \mathcal{F}_0$, Proposition 3.1 shows that $0 \geq \max\{-\frac{b}{a}, -\frac{d}{c}\}$ is a branching point for $\phi\ |_s\ R$. Then Proposition 3.3 implies the identity $(\phi\ |_s\ R_1)\ |_s\ \mathcal{R}_2 = \phi\ |_s\ (R_1R_2)$ for all $R_1, R_2 \in \mathcal{G}^+$. Of course we can extend the action to $\mathbb{Z}[\mathcal{G}] \subset \mathcal{R}$ by linearity. Note, finally, that I, T, TM and MTM are contained in $\mathcal{G}^+$, but M is not. □

Let

$$\mathcal{T} := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} \mid a > 0\}$$

Lemma 3.5. *For all* $\phi \in \mathcal{F}_0$ *and* $P \in \mathbb{Z}[\mathcal{T}]$ *the following equality is well-defined:*

$$\phi\ |_s\ (I - T - \lambda TM)P = \big(\phi\ |_s\ (I - T - \lambda TM)\big)\ |_s\ P$$

Proof. According to Remark 3.4 we have $\phi \mid_s (I - T - \lambda TM) \in \mathcal{F}_0$. Therefore, in view of Proposition 3.3 it suffices to check that

$$TA = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}, TMA = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix}, A \in \mathcal{G}$$

and all three satisfy (24) with $r_\phi = 0$. But this just means

$$\frac{a+c}{c}, \frac{a+c}{a}, \frac{a}{c} > 0,$$

which follows immediately from the hypothesis. Here, of course, we interprete $\frac{a}{c}$ as ∞ for $c = 0$. ☐

Remark 3.6. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $A \in \mathcal{T}$, then

$$TMA = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix}, \quad MTMA = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix} \in \mathcal{T},$$

and if $A \in \mathrm{Mat}_*(2, \mathbb{Z}^+ \cup \{0\})$, then

$$MATM = \begin{pmatrix} c+d & c \\ a+b & a \end{pmatrix}, \quad ATM = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix} \in \mathcal{T}.$$

We will need these facts in the proof of our main theorem. ☐

4 Transfer operators for $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$

Let W be a μ-dimensional complex vector space and $A, B \in \mathrm{Aut}_{\mathbb{C}}(W)$. We assume the isomorphisms $A^n \in \mathrm{Aut}_{\mathbb{C}}(W)$ to be uniformly bounded in $n \in \mathbb{N}$ w.r.t. one and hence any norm on $\mathrm{Aut}_{\mathbb{C}}(W)$. Consider the Banach space $\mathcal{B}(D)$ of holomorphic functions in the disc $D = \{z \in \mathbb{C} : \ |z - 1| < \frac{3}{2}\}$ which are continuous on $\overline{D}$ with the sup norm. Then the operator $\mathcal{L}_s : \mathcal{B}(D) \otimes W \to \mathcal{B}(D) \otimes W$ with

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} (z+n)^{-2s} A^{n-1} B f\left(\frac{1}{z+n}\right) \tag{26}$$

is a nuclear operator for $\Re(2s) > 1$ in this Banach space and $\mathcal{L}_s$ extends to a meromorphic family of nuclear operators in the whole s-plane with possible poles of order one at the points $s = \frac{1-k}{2}$ with $k \in \mathbb{N}_0$. The proof follows the same line of arguments as in [**CM**]. In fact, using the k-th Taylor polynomial of f at 0 we have:

$$\mathcal{L}_s f(z) = \mathcal{L}_{s+\frac{k+1}{2}} \tilde{f}(z) + \sum_{i=0}^{k} \zeta_{A,B}(i + 2s, z + 1) \frac{f^i(0)}{i!}, \tag{27}$$

where

$$\tilde{f}(z) := z^{-k-1}\big(f(z) - \sum_{i=0}^{k} \frac{f^i(0)}{i!} z^i\big)$$

and

$$\zeta_{A,B}(a, b) = \sum_{n=0}^{\infty} \frac{A^{n-1} B}{(b+n)^a}$$

is a kind of Hurwitz zeta function. The first term on the right hand side in expression (27) is holomorphic in $\Re(s) > \frac{1-(k+1)}{2}$ and the second term has poles of order one at $\frac{1-i}{2}$, $i = 0, 1, \ldots, k$ (the proof of this last statement is as for the usual Hurwitz zeta function, [**La**], Chapter XIV). This proves our assertion.

By a direct calculation we have

$$\mathcal{L}_s f(z) - (A\mathcal{L}_s) f(z+1) = (z+1)^{-2s} B f\left(\frac{1}{z+1}\right)$$

Therefore any eigenvector f of $\mathcal{L}_s$ with eigenvalue λ satisfies the following three term functional equation:

$$\lambda(f(z) - Af(z+1)) = (z+1)^{-2s} B f\left(\frac{1}{z+1}\right).$$

It is convenient to make the change of variable $z \mapsto z-1$ and introduce the new function $\Phi(z) = f(z-1)$. For $\lambda \neq 0$ the above equation then takes the form:

$$\Phi(z) - A\Phi(z+1) = \lambda^{-1} z^{-2s} B \Phi\left(1 + \frac{1}{z}\right) \tag{28}$$

Since f is defined in the disk D, Φ is defined in the shifted disk $\{z : |z-2| \le \frac{3}{2}\}$. As in [**CM1**] one shows that any eigenfunction f of the operator $\mathcal{L}_s$ can be extended holomorphically to the entire complex plane $\mathbb{C}$ cut along the line $(-\infty, -1]$. Hence the corresponding function $\Phi(z)$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. In what follows we are interested in solutions of (28) in the domain $\mathbb{C} \setminus (-\infty, 0]$ for the eigenvalues $\lambda = \pm 1$. In the scalar case $\mu = 1$ with $A, B = I$ equation (28) was introduced by J. Lewis in [**Le**]. The derivation of his equation via the transfer operator appeared independently in [**Ma2**]. There one can also find the conditions under which a holomorphic solution of equation (28) determines an eigenfunction of the transfer operator with eigenvalue λ. An interesting property of the solutions of equation (28) is described by the following proposition:

Proposition 4.1. *If $\lambda = \pm 1$ and $(BA^{-1})^2 = I$, then any solution of equation (28) in $\mathbb{C} \setminus (-\infty, 0]$ satisfies*

$$\Phi(z) = \lambda z^{-2s} B A^{-1} \Phi\left(\frac{1}{z}\right). \tag{29}$$

Proof. The domain $\mathbb{C} \setminus (-\infty, 0]$ is invariant under $z \mapsto \frac{1}{z}$. We insert $\frac{1}{z}$ in (28), multiply it by $\lambda z^{-2s} B A^{-1}$ and then subtract the result from (28). Using the hypotheses we get the equality in (29). □

Of special interest for the following is the case $s = 1$: For this let us suppose that A and B are two invertible real matrices with non-negative entries which satisfy

$$A\mathbb{I} = \mathbb{I} = B\mathbb{I}, \tag{30}$$

where $\mathbb{I}$ is a μ-dimensional vector with all components equal to 1. This is for instance the case for A and B permutation matrices. Then the vector $\Phi' = \Phi'(z)$ with all entries equal to $\frac{1}{z}$ is obviously a solution of (28) with $\lambda = 1$ and $s = 1$.

Generalizing the analogous result for the scalar case $\mu = 1$ in [**Ma4**] one has

Proposition 4.2. *Φ' is up to a constant factor the unique solution of (28) for $\lambda = 1$ and $s = 1$ in the Banach space $\mathcal{B}(D) \otimes W$. There does not exist any other solution of equation (28) in this space for the parameter values $s = 1$ and λ with $\mid \lambda \mid = 1$.*

Proof. The proof is a straightforward adaption from [**Ma3**], Appendix C, and [**MM**]. □

Induced representations: Let G be a group and H be a subgroup of finite index $\mu = [G : H]$ of G. For each representation $\chi : H \to \mathrm{End}(V)$ we consider the induced representation $\chi_G : G \to \mathrm{End}(V_G)$, where

$$V_G := \{f \colon G \to V \mid f(hg) = \chi(h)f(g) \quad \forall g \in G, h \in H\}$$

and the action of G is given by

$$\big(\chi_G(g)f\big)(x) = f(xg) \quad \forall x, g \in G.$$

If $V = \mathbb{C}$ and the initial representation is trivial, the induced representation χ_G is the right regular representation $\rho \colon G \to \mathrm{GL}(\mathbb{C}^{H\backslash G})$. In fact, in this case V_G is the space of complex valued left H-invariant functions on G or, what is the same, complex valued functions on $H\backslash G$, and the action is by right translation in the argument. This also shows that we can view ρ as a homomorphism $G \to \mathrm{GL}(\mathbb{Z}^{H\backslash G})$. Moreover, for each $g \in G$ the operators $\rho(g)^n \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{H\backslash G})$ are uniformly bounded in $n \in \mathbb{N}$.

Remark 4.3. One can identify V_G with V^μ using a set $\{g_1, g_2, \dots, g_\mu\}$ of representatives for $H\backslash G$, i.e.

$$H\backslash G = \{Hg_1, Hg_2, \dots, Hg_\mu\}.$$

Then

$$\begin{array}{rcl} V_G & \to & V^\mu \\ f & \mapsto & \big(f(g_1), \dots, f(g_\mu)\big) \end{array}$$

is a linear isomorphism which transports χ_G to the linear G-action on V^μ given by

$$g \cdot (v_1, \dots, v_\mu) = \big(\chi(g_1 g g_{k_1}^{-1}) v_{k_1}, \dots, \chi(g_\mu g g_{k_\mu}^{-1}) v_{k_\mu}\big),$$

where $k_j \in \{1, \dots, \mu\}$ is the unique index such that $Hg_j g = Hg_{k_j}$. To see this one simply calculates

$$\chi_G(g) f(g_j) = f(g_j g) = f(g_j g g_{k_j}^{-1} g_{k_j}) = \chi(g_j g g_{k_j}^{-1}) f(g_{k_j}).$$

In the case of the right regular representation the identification $V_G \cong \mathbb{C}^\mu$ yields a matrix realization

$$\rho(g) = (\delta(g_i g g_j^{-1}))_{i,j=1,\dots,\mu},$$

where $\delta(g) = 1$ if $g \in H$ and $\delta(g) = 0$ otherwise. Note for the following that the matrix $\rho(g)$ is a permutation matrix for all $g \in G$ □

In this article we are primarily intested in the subgroups $\Gamma_0(n) \subset \mathrm{PSL}(,2,\mathbb{Z})$, respectively their extensions $\overline{\Gamma}_0(n) \subset \mathrm{GL}(2,\mathbb{Z})$. The representation χ is in both cases the trivial representation of $\Gamma_0(n)$, respectively $\overline{\Gamma}_0(n)$. The transfer operators for the groups $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ have been introduced by Chang and Mayer (see [**CM**], [**CM1**]), respectively Manin and Marcolli (see [**MM**]). Taking in expression (26) for A the matrix $\rho(QT^{\pm 1}Q)$ and for B the matrix $\rho(QT^{\pm 1})$ we get the transfer operators $\mathcal{L}_{s,\pm}$ for $\Gamma_0(n)$ whereas for

$A = \rho(T^{-1})$ and $B = \rho(T^{-1}M)$ we have the transfer operator $\mathcal{L}_s$ for the group $\overline{\Gamma}_0(n)$. An easy calculation shows that the operators $\mathcal{L}_{s,+}\mathcal{L}_{s,-}$ and $\mathcal{L}_s^2$ can be conjugated by the matrix $\rho(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$. On the other hand it was shown in [**CM**] that the Selberg zeta function $Z_{\Gamma_0(n)}(s)$ for the group $\Gamma_0(n)$ can be expressed in terms of the Fredholm determinant of the operator $\mathcal{L}_{s,+}\mathcal{L}_{s,-}$ as $Z_{\Gamma_0(n)}(s) = \det(1 - \mathcal{L}_{s,+}\mathcal{L}_{s,-})$ and hence also as $Z_{\Gamma_0(n)}(s) = \det(1 - \mathcal{L}_s^2) = det(1 + \mathcal{L}_s)\det(1 - \mathcal{L}_s)$. This shows that using the operator $\mathcal{L}_s$ the Selberg zeta function for the group $\Gamma_0(n)$ factorizes as in the case of the modular group and hence this transfer operator facilitates also the discussion of the period functions for $\Gamma_0(n)$. In the following we will therefore use this operator. The Lewis equation for $\Gamma_0(n)$ derived from the eigenfunction equation for $\mathcal{L}_s$ then has the form

$$\Phi(z) - \rho(T^{-1})\Phi(z+1) - \lambda^{-1}z^{-2s}\rho(T^{-1}M)\Phi\left(1 + \frac{1}{z}\right) = 0 \tag{31}$$

For the transfer operators considered above one finds $(BA^{-1})^2 = I$ since $BA^{-1} = \rho(QTQT^{\pm 1}Q)$, respectively $BA^{-1} = \rho(T^{-1}MT)$, and hence the two term equation (29) holds. Note that the matrices in both examples are permutation matrices and so also the scalar equations in (29) involve only two terms.

5 The indexing coset space

In this section we study the fine structure of $\overline{\Gamma}_0(n)\backslash \mathrm{GL}(2,\mathbb{Z})$ as a right $\mathrm{GL}(2,\mathbb{Z})$-space. To do this we embed $\overline{\Gamma}_0(n)\backslash \mathrm{GL}(2,\mathbb{Z})$ into a natural $\mathrm{GL}(2,\mathbb{Z})$-space with an action by a kind of linear fractional transformations. We start with $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ on which $\mathrm{GL}(2,\mathbb{Z})$ acts via

$$(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+cy, bx+dy).$$

We define an equivalence relation $\sim_n$ on $\mathbb{Z} \times \mathbb{Z}$ via

$$(x,y) \sim_n (x',y') \quad :\Leftrightarrow \quad (\exists k \in \mathbb{Z})\ \gcd(k,n) = 1,\ \begin{array}{ll} kx \equiv x' & \mod n \\ ky \equiv y' & \mod n. \end{array}$$

Then the linearity of the action shows that it preserves $\sim_n$ so that the space $[\mathbb{Z} : \mathbb{Z}]_n := (\mathbb{Z} \times \mathbb{Z})/\sim_n$ of equivalence classes inherits a right $\mathrm{GL}(2,\mathbb{Z})$-action. If, for fixed n, the equivalence class of (x,y) is denoted by $[x:y]$, then this action is given by

$$[x:y]\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax+cy : bx+dy]$$

which is of course very reminiscent of linear fractional transformations. Note, however, that even for $n = p$ prime the space $[\mathbb{Z} \times \mathbb{Z}]_p$ is not the projective space $\mathbb{P}^1(\mathbb{Z}_p)$ since we have not excluded the pairs of numbers both divisible by p.

Remark 5.1. The stabilizer of the point $[0:1] \in [\mathbb{Z} \times \mathbb{Z}]_n$ is $\overline{\Gamma}_0(n)$ since $(0,1) \sim_n (c,d)$ if and only if $c \equiv 0 \mod n$ and $\gcd(d,n) = 1$. Thus the orbit map

$$\begin{array}{rcl} \mathrm{GL}(2,\mathbb{Z}) & \to & [\mathbb{Z} \times \mathbb{Z}]_n \\ g & \mapsto & [0:1]g \end{array}$$

factors to the equivariant injection

$$\begin{array}{rcl} \overline{\pi}\colon \overline{\Gamma}_0(n)\backslash \mathrm{GL}(2,\mathbb{Z}) & \to & [\mathbb{Z}\times\mathbb{Z}]_n \\ \overline{\Gamma}_0(n)\begin{pmatrix} a & b \\ c & d\end{pmatrix} & \mapsto & [c:d] \end{array}$$

□

Now we set $I_n := \mathrm{Im}(\overline{\pi}) \subseteq [\mathbb{Z}\times\mathbb{Z}]_n$ and note that I_n is $\mathrm{GL}(2,\mathbb{Z})$-invariant.

Proposition 5.2. $I_n = \{[x:y] \mid \gcd(x,y,n) = 1\}$.

Proof. "$\supseteq$": If $\gcd(x,y,n) = 1$ set $m := \gcd(x,y)$ and $x' := \frac{x}{m}, y' := \frac{y}{m}$. Then $\gcd(m,n) = \gcd(x',y') = 1$ and one can find $a,b\in\mathbb{Z}$ such that $ay' - bx' = 1$. Therefore $g := \begin{pmatrix} a & -b \\ x' & y'\end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ and

$$[0:1]g = [x':y'] = [mx':my'] = [x:y].$$

"$\subseteq$": If $[x:y] = [0:1]\begin{pmatrix} a & b \\ c & d\end{pmatrix} = [c:d]$, then there exist $k,r,s\in\mathbb{Z}$ such that $\gcd(k,n) = 1$ and

$$\begin{array}{rcl} kc - x & = & rn \\ kd - y & = & sn. \end{array}$$

If now $a = \gcd(x,y,n)$, then $a \mid \gcd(c,d) = 1$ since $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ has determinant $ad - bc = 1$. □

Lemma 5.3. *Given $m,n\in\mathbb{Z}$ and $u,v\in\mathbb{Z}$ such that $c = um + vn = \gcd(m,n)$ one can find $t\in\mathbb{Z}$ such that*

$$\gcd(u + \frac{n}{c}t, n) = 1.$$

Proof. Let $n = \prod_{j=1}^{s} p_j^{\alpha_j}$ be the decomposition into prime factors and suppose that they are arranged in such a way that $c = \prod_{j=1}^{s} p_j^{\beta_j}$ with $\alpha_j = \beta_j$ for $j \le s_1$ and $\alpha_j > \beta_j$ for $j > s_1$. Then $u\frac{m}{c} + v\frac{n}{c} = 1$ implies that u cannot contain a prime factor p_j with $j > s_1$ so that $\gcd(u,n) = \prod_{j=1}^{s_1} p_j^{\gamma_j}$ with $0 \le \gamma_j \le \alpha_j$. We may assume w.l.o.g. that $\gamma_j > 0$ for $j \le s_2$ and $\gamma_j = 0$ for $s_2 < j \le s_1$, i.e.

$$\gcd(u,n) = \prod_{j=1}^{s_2} p_j^{\gamma_j}.$$

Now we pick $t = \prod_{j=s_2+1}^{s_1} p_j$ and comparing which p_j divide respectively u, t, and $\frac{n}{c}$, we see that no p_j divides $u + \frac{n}{c}t$. □

Proposition 5.4. *Each element of I_n can be written as $[c:d]$, $c \ge 1$, $c \mid n$. Here c is determined uniquely, whereas d is determined only up to an integer multiple of $\frac{n}{c}$. It is possible to choose $d = kd'$ with $d' \ge 1, d' \mid n$, $1 \le k < n$ and $\gcd(c,d') = 1 = \gcd(k,n)$.*

Proof. For $[x : y] \in I_n$ set $c = \gcd(x, n)$ and choose $u, v \in \mathbb{Z}$ such that $ux + vn = c$. Using Lemma 5.3 we can find $t \in \mathbb{Z}$ such that $\gcd(u + \frac{n}{c}t, n) = 1$. Set $\nu := u + \frac{n}{c}t$. Then we have

$$\nu x = c + n\left(t\frac{x}{c} - v\right) \equiv c \mod n$$

so that with $d := \nu y$ we obtain $[x : y] = [c : d]$. Now we set $d' := \gcd(n, d)$ and use Lemma 5.3 in order to find a ν' with $\gcd(\nu', n) = 1$ such that $\nu' d = d' \mod n$. Choosing $k \in \{1, \ldots, n-1\}$ such that $k\nu' \equiv 1 \mod n$ we have $d \equiv kd' \mod n$ and find $[x : y] = [c : kd']$. This proves the existence part of the proposition since $\gcd(c, d') = \gcd(x, y, n) = 1$.

To prove uniqueness suppose that $[c : d] = [c' : d']$. Then we have $c = lc' + rn, d = ld' + sn$ for some $r, s, l \in \mathbb{Z}$ with $\gcd(l, n) = 1$. If now $c, c' \geq 1,\ c, c' \mid n$ the first equality implies that $c = c'$ and $l = -r\frac{n}{c} + 1$. Inserting this l into the second equality we obtain the uniqueness of d up to $\frac{n}{c}\mathbb{Z}$. □

Unfortunately the parametrization of the elements of I_n by $[c : d]$ with $c \geq 1$ and $c \mid n$ is not unique as shown in Proposition 5.4. To achive an unique parametrization we proceed as follows:

Definition 5.5. For fixed $n \in \mathbb{N}$ and $c \in \{1, \ldots, n\}$ with $c \mid n$ choose $b \in \{0, \ldots, \frac{n}{c} - 1\}$. We call the pair (c, b) *n-admissible* if there exists $k \in \{0, \ldots, c-1\}$ with $\gcd(c, b + k\frac{n}{c}) = 1$. For such a pair we set

$$d(c, b) := min\{c + b + k\frac{n}{c} : k \in \{0, \ldots, c-1\}, \gcd(c, b + k\frac{n}{c}) = 1\} \tag{32}$$

□

Remark 5.6. (a) If $\gcd(c, b) = 1$, then $d(c, b) = c + b$.

(b) If (c, b) is n-admissible, then $\gcd(c, b, \frac{n}{c}) = 1$.

(c) the pair (c, b) is n-admissible iff $\exists k \in \mathbb{Z}$ with $\gcd(c, b + k\frac{n}{c}) = 1$.

□

We need the following lemma[5]:

Lemma 5.7. *Given the numbers a,b,c* $\in \mathbb{Z}$ *then* $\gcd(a, b, c) = 1$ *iff there exists a* $k \in \mathbb{Z}$ *such that* $\gcd(a, b + kc) = 1$.

Proof. If $\gcd(a, b + kc) = 1$ for some $k \in \mathbb{Z}$ then there exist $x, y \in \mathbb{Z}$ such that $ax + (b + kc)y = 1$ and hence $\gcd(a, b, c) = 1$.
Conversely, if $\gcd(a, b, c) = 1$ define

$$t_{ab} := \gcd(a, b), \quad t_{bc} := \gcd(b, c), \quad t_{ac} := \gcd(a, c),$$

respectively

$$t_a := \frac{a}{t_{ab}t_{ac}}, \quad t_b := \frac{b}{t_{ab}t_{bc}}, \quad t_c := \frac{c}{t_{ac}t_{bc}}.$$

Then $\gcd(t_x, t_y) = \gcd(t_{xy}, t_{xz}) = \gcd(t_x, t_{yz}) = 1$ for all $x \neq y \neq z \in \{a, b, c\}$. Obviously

$$a = t_a t_{ab} t_{ac}, \ b = t_b t_{ab} t_{bc}, \ c = t_c t_{ac} t_{bc}.$$

---

[5] We thank Ch. Elsholtz for showing us how to prove this lemma

To determine $k \in \mathbb{Z}$ with $\gcd(a, b + kc) = 1$ we proceed as follows: In the case $\gcd(t_{ab}, t_a) = 1$ one finds for $a = t_a t_{ab} t_{ac}$ and $b + kc = t_b t_{ab} t_{bc} + k t_c t_{ac} t_{bc}$ with $k = t_a$

$$\begin{aligned}
1 &= \gcd(t_{ac}, b) = \gcd(t_{ac}, b + kc),\\
1 &= \gcd(t_{ab}, t_a t_c t_{ac} t_{bc}) = \gcd(t_{ab}, b + kc),\\
1 &= \gcd(t_a, b) = \gcd(t_a, b + kc).
\end{aligned}$$

In the case $d = \gcd(t_a, t_{ab}) > 1$ with $t_a = dt'_a$ and $t_{ab} = dt'_{ab}$ write $t'_a = sk$ with $s \mid d$ and $\gcd(k, d) = 1$. Then $\gcd(t_{ab}, k) = 1$. Otherwise $\gcd(k, t'_{ab}) > 1$ and hence $\gcd(t_{ab}, t_a) = kd$ in contradiction to the definition of d . Therefore

$$\begin{aligned}
1 &= \gcd(t_{ac}, b) = \gcd(t_{ac}, b + kc),\\
1 &= \gcd(t_{ab}, kc) = \gcd(t_{ab}, b + kc),\\
1 &= \gcd(d, k t_c t_{ac}) = \gcd(d, d t_b t'_{ab} + k t_c t_{ac}) =\\
&= \gcd(ds, d t_b t'_{ab} + k t_c t_{ac}) = \gcd(dsk, d t_b t_{ab} + k t_c t_{ac}) =\\
&= \gcd(t_a, d t_b t'_{ab} t_{cb} + k t_c t_{ac} t_{cb}) = \gcd(t_a, b + kc).
\end{aligned}$$

□

This now allows an unique parametrization of the elements in I_n.

Proposition 5.8. *There is a bijection from the set*

$$P_n = \{(c, b) : c \geq 1,\ c \mid n, \quad b \in \{0, \ldots, \frac{n}{c} - 1\},\ (c, b)\ n - admissible\}$$

to the set I_n. The map is given by

$$(c, b) \mapsto [c : d(c, b)]$$

with $d(c, b)$ from Definition 5.5.

Proof. We show first that the above map is surjective. For any $[x : y] \in I_n$ by Proposition 5.4 there exist an unique $c \geq 1,\ c \mid n$ and d' with $[x : y] = [c : d']$. Define $b \in \{0, \ldots, \frac{n}{c} - 1\}$ through $d' \equiv (b + c) \mod \frac{n}{c}$. We claim (c, b) is n-admissible. Indeed, from Proposition 5.2 we see $\gcd(c, d', n) = 1$. Assume $\lambda = \gcd(c, b, \frac{n}{c}) > 1$. But $\lambda \mid \gcd(d', n)$ and hence $\lambda \mid \gcd(c, d', n)$. Hence by Lemma 5.7 there exists $k \in \mathbb{Z}$ with $\gcd(c, b + k\frac{n}{c}) = 1$.
Next we claim $d' \equiv d(c, b) \mod \frac{n}{c}$. Indeed $d' \equiv (b+c) \mod \frac{n}{c}$ and $d(c, b) \equiv (b+c) \mod \frac{n}{c}$ implies $d' = d(c, b) + l\frac{n}{c}$. Choose $r, s, t \in \mathbb{Z}$ with $rd' - sc - tn = l$. An easy calculation then gives $d(c, b) = d' - l\frac{n}{c} = (1 - \frac{n}{c}r)d' + (s + t\frac{n}{c})n$ and trivially $c = (1 - \frac{n}{c}r)c + rn$. We claim $\gcd((1 - \frac{n}{c}r), n) = 1$. Obviously $\gcd((1 - \frac{n}{c}r), \frac{n}{c}) = 1$. Assume then $\gcd((1 - \frac{n}{c}r), n) = m > 1$. Then $\gcd((1 - \frac{n}{c}r), c) = m$. Since $d(c, b) = d' - l\frac{n}{c} = (1 - \frac{n}{c}r)d' + (s + t\frac{n}{c})n$ the number m divides also $d(c, b)$ and hence $\gcd(d(c, b), c)) > 1$ in contradiction to the definition of $d(c, b)$. Hence $[c : d'] = [c, d(c, b)]$.

To show injectivity of the map $(c, b) \mapsto [c : d(c, b)]$ lets assume (c', b') maps to $[c' : d(c', b')]$ and $[c : d(c, b)] = [c' : d(c', b')]$. Since $c, c' \geq 1$ and $c, c' \mid n$, Proposition 5.4 shows $c = c'$. But $b \equiv (d(c, b) - c) \mod \frac{n}{c}$ and $b' \equiv (d(c', b') - c) \mod \frac{n}{c}$. By Proposition 5.4 we know that $d(c, b) \equiv d(c', b') \mod \frac{n}{c}$. Therefore also $b \equiv b' \mod \frac{n}{c}$ and hence $b = b'$ since both $b, b' \in \{0, \ldots, \frac{n}{c} - 1\}$. □

The set P_n can be ordered lexicographically by saying $(c,b) < (c',b')$ iff $c < c'$ or $c = c'$ and $b < b'$.

Definition 5.9. Proposition 5.8 allows us to identify each element $[r:s] \in I_n$ with a pair $(c,b) \in P_n$. Then we set

$$A_{[r:s]} := \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}, \quad B_{[r:s]} := \begin{pmatrix} \frac{n}{c} & 0 \\ b & c \end{pmatrix} \tag{33}$$

and define the rational number

$$x([r:s]) := \frac{b}{\frac{n}{c}}.$$

Obviously there is a one to one correspondence between the sets I_n, P_n and the sets of matrices $A \in \mathrm{Mat}_n(2,\mathbb{Z})$ which are upper triangular, respectively those which are lower triangular, and whose entries have greatest common divisor 1. $\square$

6 The operator K

Recall from the introduction the sets of matrices

$$\begin{aligned} S_n &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a > c \geq 0,\ d > b \geq 0,\ ad - bc = n \right\} \\ X_n &= \left\{ \begin{pmatrix} c & a \\ 0 & \frac{n}{c} \end{pmatrix} :\ c \mid n,\ 0 \leq a < \frac{n}{c} \right\} \\ Y_n &= \left\{ \begin{pmatrix} c & 0 \\ a & \frac{n}{c} \end{pmatrix} :\ c \mid n,\ 0 \leq a < c \right\}. \end{aligned}$$

Proposition 6.1. *The formula*

$$K \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c + \lceil \frac{d}{b} \rceil a & -d + \lceil \frac{d}{b} \rceil b \\ a & b \end{pmatrix} \tag{34}$$

defines a bijection $K : S_n \setminus Y_n \to S_n \setminus X_n$ *with inverse given by the formula*

$$K^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = M T^{\lceil \frac{a'}{c'} \rceil} Q M \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} c' & d' \\ -a' + \lceil \frac{a'}{c'} \rceil c' & -b' + \lceil \frac{a'}{c'} \rceil d' \end{pmatrix}. \tag{35}$$

Proof. We denote the right hand side of (34) by $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. The condition $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_n \setminus Y_n$ implies

$$a > c \geq 0, \quad d > b > 0, \quad ad - bc = n.$$

From this it is clear that $c' > 0$ so that $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ is not contained in X_n. To show that it is in S_n we note

$$a' = \lceil \tfrac{d}{b} \rceil a - c \geq a - c \geq a = c' > 0,$$

$$0 \leq b' = \lceil \tfrac{d}{b} \rceil b - d = (\lceil \tfrac{d}{b} \rceil - \tfrac{d}{b}) b < b = d',$$

and

$$a'd' - b'c' = (-c + \lceil \tfrac{d}{b} \rceil a) b - (-d + \lceil \tfrac{d}{b} \rceil b) a = ad - bc = n.$$

Thus K is well defined. That

$$MT^{\lceil \frac{a'}{c'} \rceil} Q M T^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

follows from

$$MT^{\lceil r \rceil} Q M T^{\lceil s \rceil} Q = \begin{pmatrix} 1 & 0 \\ r-s & 1 \end{pmatrix}$$

and $\lceil \frac{a'}{c'} \rceil = \lceil \frac{d}{b} \rceil$ which in turn is a consequence of $\frac{a'}{c'} = -\frac{c}{a} + \lceil \frac{d}{b} \rceil$ and $-1 < -\frac{c}{a} \leq 0$. Similarly we see that

$$MT^{\lceil \frac{d''}{b''} \rceil} Q M T^{\lceil \frac{a'}{c'} \rceil} Q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ denotes the right hand side of (35). All that remains to be seen is that $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in S_n \setminus Y_n$ if $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in S_n \setminus X_n$, but that can be checked similarly as the well-definedness of K. $\square$

An operator slightly different from the above operator K was used also by Choie and Zagier in [**CZ**] and by Mühlenbruch in [**Mu**] in their derivation of the Hecke operators within the Eichler, Manin and Shimura theory of period polynomials. In the following we will use this operator to attach to any index $[c:d] \in I_n$ a sequence of elements in $\mathcal{R}_n$ which on the other hand are closely related to the minimal partition of the rational number $x([c:d])$.

Definition 6.2. For $i \in I_n$ we denote by k_i the natural number with the property that $K^j(A_i)$ (cf. Definition 5.9) is well-defined for $j \leq k_i$ and $K^{k_i}(A_i) \in Y_n$. We call

$$A_i, K(A_i), \ldots, K^{k_i}(A_i)$$

the *chain* associated with $i \in I_n$. $\square$

If $A_i \in X_n \cap Y_n$, then clearly A_i forms a chain in itself so that $k_i = 0$ in this case.

Lemma 6.3. *Let $[c:d] \in I_n$ and $P_{x([c:d])} = (x_0, x_1, \ldots, x_{k-1}, x_k)$ the minimal partition of $x_0 = x([c:d]) = \frac{b}{\frac{n}{c}}$ (cf. Definition 5.9 and Remark 2.4). Suppose that $x_j = \frac{p_j}{q_j}$,* $\gcd(p_j, q_j) = 1$, *and $q_j \geq 0$. Then we have $k_{[c:d]} = k-1$ and*

$$K^j(A_{[c:d]}) = \begin{pmatrix} q_{k-1-j} & -p_{k-1-j} \\ q_{k-j} & -p_{k-j} \end{pmatrix} A_{[c:d]} \quad \forall j = 0, \ldots, k-1.$$

Proof. Recall the definition of $b \in \{0, \ldots, \frac{n}{c} - 1\}$ attached to $[c:d]$ in Definition 5.9. We assume $c \geq 1, \quad c \mid n$ and hence

$$A_{[c:d]} = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}.$$

We claim

$$\begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_{[c:d]} = \begin{pmatrix} cq_{j-1} & bq_{j-1} - \frac{n}{c} p_{j-1} \\ cq_j & bq_j - \frac{n}{c} p_j \end{pmatrix} \in S_n \quad \forall j = 1, \ldots, k. \tag{36}$$

In fact, using Lemma 2.7(i) and minimality of the partition we find

$$\left(bq_j - \frac{n}{c}p_j\right) - \left(bq_{j-1} - \frac{n}{c}p_{j-1}\right) = (q_j - q_{j-1})\left(\frac{b}{\frac{n}{c}} - \frac{p_{j-1}-p_j}{q_{j-1}-q_j}\right) > 0,$$

whereas

$$\frac{b}{\frac{n}{c}} = x_0 \geq x_{j-1} = \frac{p_{j-1}}{q_{j-1}}$$

implies $bq_{j-1} - \frac{n}{c}p_{j-1} \geq 0$ and even

$$bq_j - \frac{n}{c}p_j > 0 \quad \forall j = 1, \ldots, k. \tag{37}$$

Since the determinant condition is trivially satisfied we have proved (36). But there is more detailed information available: Since $\frac{p_0}{q_0} = \frac{b}{\frac{n}{c}}$ and $cq_1 < cq_0$ we have

$$\begin{pmatrix} q_0 & -p_0 \\ q_1 & -p_1 \end{pmatrix}\begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} = \begin{pmatrix} cq_0 & bq_0 - \frac{n}{c}p_0 \\ cq_1 & bq_1 - \frac{n}{c}p_1 \end{pmatrix} = \begin{pmatrix} cq_0 & 0 \\ cq_1 & bq_1 - \frac{n}{c}p_1 \end{pmatrix} \in Y_n.$$

Moreover, Remark 2.5 shows that $\begin{pmatrix} q_{k-1} & -p_{k-1} \\ q_k & -p_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\begin{pmatrix} q_{k-1} & -p_{k-1} \\ 0 & p_k \end{pmatrix}\begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} \in X_n.$$

On the other hand, by (37) none of the $\begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_{[c:d]}$ in (36) with $j = 1, \ldots, k-1$ can be in $X_n \cup Y_n$ since $q_j \neq 0$ for these j. Now it suffices to prove the identities

$$K : \begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{[c:d]} \mapsto \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_{[c:d]}. \tag{38}$$

To prove (38) note first that for an arbitrary matrix $A \in S_n \setminus Y_n$ we have

$$K(A)A^{-1} = T^{\lceil \frac{d'}{b'} \rceil} Q,$$

where $A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. For $A := \begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{[c:d]}$ by Lemma 2.7(ii) we have

$$\lceil \tfrac{d'}{b'} \rceil = \lceil \tfrac{bq_{j+1} - \frac{n}{c}p_{j+1}}{bq_j - \frac{n}{c}p_j} \rceil = \lceil \tfrac{xq_{j+1} - p_{j+1}}{xq_j - p_j} \rceil = p_{j-1}q_{j+1} - p_{j+1}q_{j-1}$$

and calculate

$$\begin{aligned} &\begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_{[c:d]}(\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{[c:d]})^{-1} = \\ &= \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix}\begin{pmatrix} -p_{j+1} & p_j \\ -q_{j+1} & q_j \end{pmatrix} \\ &= \begin{pmatrix} p_{j-1}q_{j+1} - p_{j+1}q_{j-1} & -1 \\ 1 & 0 \end{pmatrix} \\ &= T^{(p_{j-1}q_{j+1} - p_{j+1}q_{j-1})} Q \\ &= T^{\lceil \frac{d'}{b'} \rceil} Q \end{aligned}$$

proving

$$K(A) = \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_{[c:d]}.$$

□

Remark 6.4. A construction rather similar to the one in Lemma 6.3 has been used also by L. Merel in [**Me**], where he discussed the connection between the ordinary Hecke operators for the group $\Gamma_0(n)$ and continued fractions. □

7 The Lewis equations for the group $\Gamma_0(n)$

Consider the right $\mathcal{R}$-module

$$\mathcal{R}^{I_n} := \{\psi \colon I_n \to \mathcal{R}\} = \mathcal{R} \otimes \mathbb{Z}^{I_n},$$

whose elements we also denote by $\psi = (\psi_i)_{i \in I_n}$. The module $\mathcal{R}^{I_n}$ is equipped with a natural left $\mathrm{GL}(2,\mathbb{Z})$-action given by $(g \cdot \psi)_i = \psi_{ig}$. Recall the right regular representation ρ of $\mathrm{GL}(2,\mathbb{Z})$ on $\overline{\Gamma}_0(n)\backslash\mathrm{GL}(2,\mathbb{Z})$ which we identify with I_n. Then we have

$$g \cdot (R \otimes w) = R \otimes \rho(g)w \quad \forall g \in \mathrm{GL}(2,\mathbb{Z}), w \in \mathbb{Z}^{I_n}, R \in \mathcal{R}$$

and, by abuse of notation we write $\rho(g)\psi$ for $g \cdot \psi$. It is important to note that

$$(\rho(g)\psi)R = \rho(g)(\psi R) \quad \forall g \in \mathrm{GL}(2,\mathbb{Z}), \psi \in \mathcal{R}^{I_n}, R \in \mathcal{R}.$$

Therefore all terms in the equation

$$\psi - \rho(T^{-1})\psi T - \lambda\rho(T^{-1}M)\psi TM \equiv 0 \mod (\mathcal{I}^{\lambda})^{I_n} \tag{39}$$

for the unknown $\psi \in \mathcal{R}^{I_n}$ are unambiguous. Comparing (39) with equation (31) shows that it is reasonable to call (39) the *Lewis equation* in $(\mathcal{I}^{\lambda}\backslash\mathcal{R})^{I_n}$ corresponding to the transfer operator for the group $\Gamma_0(n)$.

Remark 7.1. To rewrite (39) as a system of scalar equations we have to determine the actions of T^{-1} and $T^{-1}M$ on the indexing coset space I_n. They are given by

$$[c:d]T^{-1} = [c:d-c] \quad \text{and} \quad [c:d]T^{-1}M = [d-c:c].$$

Thus the corresponding system of scalar equations is

$$\psi_{[c:d]} - \psi_{[c:d-c]}T - \lambda\psi_{[d-c:c]}TM \equiv 0 \mod \mathcal{I}^{\lambda}, \quad \forall [c:d] \in I_n. \tag{40}$$

Replacing $[c:d]$ by $[c:d]\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = [d-c:d]$, multiplying the resulting equation from the right by λM, and then subtracting it from the original equation we get

$$\psi_{[c:d]} \equiv \lambda\psi_{[d-c:d]}M \mod \mathcal{I}^{\lambda} \tag{41}$$

We call $\psi_{[c:d]}, \psi_{[d-c:d]}$ a *symmetric pair.* It is easy to see that $[c:d] = [d-c:d]$ if and only if $c = 1$ and $n \mid d(d-2)$. In particular, $[1:0]$ and $[1:2]$ have this property. We call an element $\psi_{[c:d]}$ with this property a *self-symmetric element.* □

There is a close relation between the equations (40) and (31). To explain this relation we have to write (31) as a system of scalar equations.

Remark 7.2. Let $\Phi \in \mathcal{B}(D) \otimes \mathbb{C}^{I_n}$ be a solution of (31). We write it as $\Phi = (\phi_i)_{i \in I_n}$ with $\phi_i \in \mathcal{B}(D)$. Similarly as in Remark 7.1 we see that the ϕ_i satisfy the following system of scalar equations

$$\phi_{[c:d]} - \phi_{[c:d-c]}T - \lambda \phi_{[d-c:c]}TM = 0 \quad \forall [c:d] \in I_n. \tag{42}$$

□

Remark 7.3. Let ϕ be a solution of the scalar Lewis equation (1). Then according to [**LZ2**] we have $\phi \in \mathcal{F}_0$ so that in view of Remark 3.4 equation (1) can be rewritten as

$$\phi \mid_s (I - T - \lambda TM) = 0 \tag{43}$$

Next we write equation (40) as

$$\psi_{[c:d]} - \psi_{[c:d-c]}T - \lambda \psi_{[d-c:c]}TM = (I - T - \lambda TM)P_{[c:d]}, \tag{44}$$

with $P_{[c:d]}$ some element in $\mathcal{R}$. Now we want both sides of (44) to act on ϕ via the slash action. If this were possible, by (43) the right hand side would annihilate ϕ, so that $\Phi := (\phi \mid_s \psi_i)_{i \in I_n}$ in view of Remark 7.2 were a solution of (31). Lemma 3.5 shows that this is indeed possible as long as all the matrices occurring in $P_{[c:d]}$ satisfy the hypotheses of this lemma. □

Remark 7.4. For $\lambda = \pm 1$ we have

$$\lambda I - M = (I - T - \lambda TM)(\lambda I - M) \equiv 0 \mod \mathcal{I}^\lambda \tag{45}$$

and so

$$(I - T - MTM) = (I - T - \lambda TM) + (\lambda I - M)TM = (I - T - \lambda TM)(I + \lambda TM - MTM)$$

which implies that $\mathcal{I} := (I - T - MTM)\mathcal{R} \subset \mathcal{I}^\lambda$. Replacing $[c:d]$ by $[c:d+c] = [c:d]\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in (40) we arrive at the system

$$\psi_{[c:d+c]} - \psi_{[c:d]}T - \lambda \psi_{[d:c]}TM \equiv 0 \mod \mathcal{I}^\lambda \quad \forall [c:d] \in I_n \tag{46}$$

of scalar equations. A further modification is suggested by (45):

$$\psi_{[c:d+c]} - \psi_{[c:d]}T - M\psi_{[d:c]}TM \equiv 0 \mod \mathcal{I} \quad \forall [c:d] \in I_n. \tag{47}$$

This is the equation we will solve and from where we will construct a solution of (46). More precisely, if $\psi_{[c:d+c]} - \psi_{[c:d]}T - M\psi_{[d:c]}TM = (I - T - MTM)R,\ R \in \mathcal{R}$, then

$$\begin{aligned} & \psi_{[c:d+c]} - \psi_{[c:d]}T - \lambda \psi_{[d:c]}TM = \\ = \;& (I - T - \lambda TM)((1 + \lambda TM - MTM)R + (M - \lambda I)\psi_{[d:c]}TM) \\ \equiv \;& 0 \mod \mathcal{I}^\lambda. \end{aligned}$$

In view of Remark 3.6 if $R \in \mathbb{Z}[\mathcal{T}]$ and $\psi_{[d:c]} \in \mathbb{Z}[\mathrm{Mat}_*(2, \mathbb{Z}^+ \cup \{0\})]$ then

$$(1 + \lambda TM - MTM)R + (M - \lambda I)\psi_{[d:c]}TM \in \mathbb{Z}[\mathcal{T}]$$

This means that if for $[c:d] \in I_n$ we have that $\psi_{[c:d]} \in \mathbb{Z}[\mathrm{Mat}_*(2, \mathbb{Z}^+ \cup \{0\})]$ are solutions of (47) fulfilling the condition of Lemma 3.5, then these $\psi_{[c:d]},\ [c:d] \in I_n$ solve also (46) and satisfy the condition of Lemma 3.5. □

Lemma 7.5. *For any two partitions P_1, P_2 of $x \in \mathbb{Q}^+$ we have*

$$m(P_1) - m(P_2) \equiv 0 \mod \mathcal{I}.$$

If $s \in \mathbb{C}$ and ϕ is a solution of the scalar Lewis equation (1), then we have the following (well-defined) equality

$$0 = \phi \mid_s \big(m(P_1) - m(P_2)\big).$$

Proof. By Lemma 2.6 it is enough to prove the lemma for a partition P and its modification $P(l)$ in (15). In the notation of (15) the element $m\big(P(l)\big)$ is given by

$$\ldots + \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l + q_{l-1} & -p_l - p_{l-1} \end{pmatrix} + \begin{pmatrix} q_l + q_{l-1} & -p_l - p_{l-1} \\ q_l & -p_l \end{pmatrix} + \begin{pmatrix} q_l & -p_l \\ q_{l+1} & -p_{l+1} \end{pmatrix} \ldots$$

so that

$$\begin{aligned} & m\big(P(l)\big) - m(P) = \\ = \quad & \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l + q_{l-1} & -p_l - p_{l-1} \end{pmatrix} + \begin{pmatrix} q_l + q_{l-1} & -p_l - p_{l-1} \\ q_l & -p_l \end{pmatrix} - \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix} \\ = \quad & (MTM + T - I) \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix}. \end{aligned}$$

If $q_l = 0$, then we have $-q_{l-1}p_l = 1$ so that $q_{l-1} > 0$ and $-p_l > 0$, which in turn shows that $\begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix}$ satisfies the conditions of Lemma 3.5. If $q_l > 0$ and $q_{l-1} = 0$, then $q_l p_{l-1} = 1$ so that $p_{l-1} > 0$ and $x_l = \infty$. This contradiction shows that $q_{l-1} > 0$ and hence also in this case $\begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix}$ satisfies the conditions of Lemma 3.5. As a result we obtain that $\phi \mid_s \big(m\big(P(l)\big) - m(P)\big)$ is well-defined and in fact equal to 0 since ϕ is a solution of (1) □

Proposition 7.6. *For $[c:d] \in I_n$ set*

$$\widetilde{\psi}_{[c:d]} := \sum_{j=0}^{k_{[c:d]}} K^j(A_{[c:d]}).$$

(i) *$\widetilde{\psi} \in \mathcal{R}^{I_n}$ is a solution of (47).*

(ii) *If $s \in \mathbb{C}$ and ϕ is a solution of the scalar Lewis equation (1), then*

$$0 = \phi \mid_s \big(\widetilde{\psi}_{[c:d]} T + M \widetilde{\psi}_{[d:c]} TM - \widetilde{\psi}_{[c:c+d]}\big).$$

Proof. Note first that Lemma 6.3, together with the Definitions 5.9 and 2.8, shows

$$\widetilde{\psi}_{[c:d]} = m(P_{x([c:d])}) A_{[c:d]}. \tag{48}$$

Fix $[c:d] \in I_n$. According to Lemma 5.4 we may assume that $d = ic'$ with $1 \le c, c' \mid n$ and $\gcd(c, c') = 1 = \gcd(i, n)$. Thus we are in the situation of Lemma 2.10. But in the notation of that lemma we have

$$\begin{aligned}
&\widetilde{\psi}_{[c:d]}T + M\widetilde{\psi}_{[d:c]}TM = \\
&\quad = \; m(P_x)\begin{pmatrix} c & (ic'-c)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix} T + M\, m(P_z)\begin{pmatrix} c' & (\hat{i}c-c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} TM \\
&\quad = \; m\big((P_z \cdot X) \vee (P_x \cdot T^s)\big)\begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}.
\end{aligned}$$

On the other hand we have

$$\widetilde{\psi}_{[c:c+d]} = m(P_y)\begin{pmatrix} c & (d)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}$$

so that now the first part of Lemma 7.5 shows that

$$\widetilde{\psi}_{[c:d]}T + M\widetilde{\psi}_{[d:c]}TM - \widetilde{\psi}_{[c:c+d]} \equiv 0 \mod \mathcal{I}.$$

This proves (i) and (ii) is now an immediate consequence of the second part of Lemma 7.5. □

Remark 7.7. Replacing $[c:d]$ with $[c:d-c] = [c:d]$ in (47) we find

$$\psi_{[c:d]} - \psi_{[c:d-c]}T - M\psi_{[d-c:c]}TM \equiv 0 \mod \mathcal{I} \quad \forall [c:d] \in I_n \tag{49}$$

and by Remark 7.4 a solution of (49) is also a solution of (44). Thus Proposition 7.6 actually provides a solution of (44). Moreover, together with the last part of Remark 7.4 and Remark 7.3, it provides the equality

$$\phi \mid_s \left(\widetilde{\psi}_{[c:d]}T + \lambda\widetilde{\psi}_{[d:c]}TM - \widetilde{\psi}_{[c:c+d]}\right) = 0$$

so that $\Phi := (\phi \mid_s \psi_i)_{i \in I_n}$ is a solution of (31). Thus we have now proved Theorem 1.1. □

8 Hecke operators

In [**Za1**], [**Za2**] D. Zagier derived a representation of the Hecke operators on the space of period polynomials for the group $\mathrm{PSL}(2,\mathbb{Z})$ by transferring the action of the classical Hecke operators on the space of cusp forms via the Eichler-Shimura-Manin isomorphism to the space of period polynomials. In his thesis T. Mühlenbruch ([**Mu**]) found another representation for these operators in terms of matrices with nonnegative entries which allowed him to extend their action to the space of period functions with arbitrary weight. It turns out that the special solutions of the Lewis equations for the congruence subgroups $\Gamma_0(n)$ we constructed in Proposition 7.6 are closely related to the Hecke operators for $\mathrm{PSL}(2,\mathbb{Z})$ in the form given by Mühlenbruch.

Indeed, since both the maps $T : I_n \to I_n$ and $MT : I_n \to I_n$ are invertible any solution Φ of the Lewis equation (3) for $\Gamma_0(n)$ given by $\phi_i = \phi_i(z), i \in I_n$, determines a solution $\tilde{\phi} = \tilde{\phi}(z)$ of the Lewis equation (1) for the group $\mathrm{PSL}(2,\mathbb{Z})$ with

$$\tilde{\phi}(z) = \sum_{i \in I_n} \phi_i(z). \tag{50}$$

Clearly, it can happen that this function vanishes identically. This just signals that the corresponding solution $\phi_i, i \in I_n$ for the group $\Gamma_0(n)$ is not related to any solution ϕ of the group $\mathrm{PSL}(2,\mathbb{Z})$ and hence, in analogy to the Atkin-Lehner theory, should be called a *new solution* of the Lewis equation for $\Gamma_0(n)$. The special solution, however, determined in Proposition 7.6 leads to a nontrivial solution $\tilde{\phi}$ which furthermore depends linearly on the solution ϕ of equation (1). This shows that the map $\tilde{H}_n : \phi \mapsto \tilde{\phi}$ with $\tilde{\phi}$ as defined in equation (50) determines a linear operator in the space of period functions of the group $\mathrm{PSL}(2,\mathbb{Z})$. To determine the explicit form of the operator $\tilde{H}_n$ we have to characterize the matrices $K^j(A_{[c:d]})$ appearing in the definition of the solutions $\tilde{\psi}_{[c:d]}$ in Proposition 7.6 in more detail.

From the definition of the operator K in Proposition 6.1 it is obvious that all matrix elements of $A \in S_n \setminus Y_n$ have greatest common divisor 1 if and only if the matrix elements of the matrix $K(A)$ have this property. Since the entries of the matrix $A_{[c:d]}$ in Definition 5.9 for $[c:d] \in I_n$ have greatest common divisor 1 all the matrices appearing in the definition of $\tilde{\psi}_{[c:d]}$ in Proposition 7.6 have this property.

Consider next any matrix $A \in S_n \setminus X_n$ whose entries have greatest common divisor 1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $K^{-1}A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $c' < c$ and hence there exists $j \in \mathbb{N}$ with $K^{-j}A \in X_n$. But from Proposition 5.8 it follows that any matrix A in X_n whose entries have only 1 as a common divisor appears as $A_{[c:d]}$ for some $[c:d] \in I_n$. This shows that any matrix A in the set S_n whose entries have no common divisor besides 1 appears exactly once in one of the components $\tilde{\psi}_{[c:d]}$ in Proposition 7.6.

Denote then by $\tilde{T}_n$ the matrix

$$\tilde{T}_n := \sum_{A \in S_n : \gcd(a,b,c,d)=1} A, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then one finds for the operator $\tilde{H}_n$ acting on the space of period functions ϕ for the group $\mathrm{PSL}(2,\mathbb{Z})$

$$\tilde{H}_n \phi = \phi \mid_s \tilde{T}_n$$

Summarizing we have shown:

Theorem 8.1. *For any solution $\phi = \phi(z)$ of the Lewis equation (1) for* $\mathrm{PSL}(2,\mathbb{Z})$ *with arbitrary weight s the function $\tilde{\phi} = \tilde{\phi}(z) = \tilde{H}_n\phi(z) = \phi \mid_s \tilde{T}_n(z)$ is also a solution of equation (1) with weight s.*

Comparing the operators $\tilde{T}_n$ with the Hecke operators T_n of Mühlenbruch and Zagier in (2) we find as a corollary

Corollary 8.2. *The operators $\tilde{T}_n$ and the Hecke operators T_n defined in (2) are related through*

$$T_n = \sum_{d^2 \mid n} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tilde{T}_{\frac{n}{d^2}}.$$

The operators coincide if and only if n is a product of distinct primes.

The operators $\tilde{T}_n$ have been constructed from special solutions of the Lewis equation (3) for the group $\Gamma_0(n)$. It turns out that also the Hecke operators T_n can be derived in this way. To do this consider any $n_1, n_2 \in \mathbb{N}$. For $n_2 \mid n_1$ there is a canonical surjective map

$\sigma_{n_1,n_2} : [\mathbb{Z}\times\mathbb{Z}]_{n_1} \to [\mathbb{Z}\times\mathbb{Z}]_{n_2}$ which is equivariant with respect to the $\mathrm{GL}(2,\mathbb{Z})$-actions, i.e.

$$\sigma_{n_1,n_2}([x:y])A = \sigma_{n_1,n_2}([x:y]A), \quad \forall A \in \mathrm{GL}(2,\mathbb{Z}),\ [x:y] \in [\mathbb{Z}\times\mathbb{Z}]_{n_1} \tag{51}$$

Therefore (cf. Remark 5.1) σ_{n_1,n_2} induces a map $I_{n_1} \to I_{n_2}$ which we still denote by σ_{n_1,n_2} (or simply by σ if n_1 and n_2 are clear from the context).

Proposition 8.3. *If $\psi^1_i,\ i \in I_{n_1}$ solve (40) for $n = n_1$ then $\psi^2_j := \sum_{i\in\sigma^{-1}j} \psi^1_i,\ j \in I_{n_2}$ solve (40) for $n = n_2$. Moreover, if $\psi^2_j,\ j \in I_{n_2}$ solve (40) for $n = n_2$ then $\psi^1_i := \psi^2_{\sigma i},\ i \in I_{n_1}$ solve (40) for $n = n_1$.*

Proof. (51) implies that the fibers $\sigma^{-1}j$ of σ are invariant under the action of $\mathrm{GL}(2,\mathbb{Z})$, and in particular T^{-1} and $T^{-1}M$. ☐

Proposition 8.3 shows that any solution $\Phi = (\phi_i, i \in I_n)$ of equation (40) for the group $\Gamma_0(\frac{n}{d^2})$ determines a solution for this equation for the group $\Gamma_0(n)$ whose components coincide with the components for the former group. Indeed any component shows up μ-times, where μ is the index of $\Gamma_0(\frac{n}{d^2})$ in $\Gamma_0(n)$. Taking for Φ the special solution $\phi \mid_s \tilde{\psi}_i, i \in I_{\frac{n}{d^2}}$ determined in (ii) of Proposition 7.6 we therefore get

Corollary 8.4. *For any solution ϕ of the Lewis equation (1) for the group $\mathrm{PSL}(2,\mathbb{Z})$ with weight s the functions $\tilde{\phi}_{j,d} := \phi \mid_s \tilde{\psi}_{\sigma(j)}, j \in I_n$ define a solution of the Lewis equation (3) for the group $\Gamma_0(n)$ with weight s.*

Hence also the function $\tilde{\phi}_d$ with

$$\tilde{\phi}_d = \frac{1}{\mu}\sum_{j\in I_n} \tilde{\phi}_{j,d} = \phi \mid_s \tilde{T}_{\frac{n}{d^2}}$$

defines a solution of the Lewis equation (1) for the group $\mathrm{PSL}(2,\mathbb{Z})$. Obviously the matrix inducing this solution $\tilde{\phi}_d$ coincides with the matrix $\tilde{T}_{\frac{n}{d^2}}$. This shows that indeed the Hecke operator T_n on the period functions of $\mathrm{PSL}(2,\mathbb{Z})$ for arbitrary weight s can be derived from special solutions of the Lewis equation for the group $\Gamma_0(n)$ with weight s.

The extension of this approach to the Hecke operators on period functions for the congruence subgroups $\Gamma_0(n)$ will be discussed in a forthcoming paper.

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References

[AL] Atkin, A. O. L., Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. 185 (1970), 134–160.

[CM] Chang, C.-H., Mayer, D.: Thermodynamic formalism and Selberg's zeta function for modular groups. Regul. Chaotic Dyn. 5 (2000), no. 3, 281–312.

[**CM1**] Chang, C.-H., Mayer, D.: Eigenfunctions of the transfer operators and the period functions for modular groups. In “Dynamical, spectral, and arithmetic zeta functions” (San Antonio, TX, 1999). Contemp. Math. 290, pp. 1–40, Amer. Math. Soc., Providence, RI, (2001).

[**CZ**] Choie, Y. , Zagier, D.: Rational period functions for PSL$(2,\mathbb{Z})$. IN “ A tribute to Emil Grosswald: number theory and related analysis”. Contemp. Math. 143, pp. 89–108 Amer. Math. Soc., Providence, RI, (1993).

[**HW**] Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers. Oxford Univ. Press, (1954).

[**La**] Lang, S.: Introduction to modular forms. Grund. Math. Wiss. 222. Springer-Verlag, Berlin, (1995).

[**Le**] Lewis, J.B.: Spaces of holomorphic functions equivalent to the even Maass cusp forms. Invent. Math. 127 (1997), no. 2, 271–306.

[**LZ1**] Lewis, J.B.; Zagier, D.: Period functions and the Selberg zeta function for the modular group. IN “The mathematical beauty of physics” (Saclay, 1996), Adv. Ser. Math. Phys. 24, pp. 83–97. World Sci. Publishing, River Edge, NJ, (1997).

[**LZ2**] Lewis, J.B.; Zagier, D.: Period functions for Maass wave forms. I. Ann. of Math. (2) 153 (2001), no. 1, 191–258.

[**MM**] Manin, Yu.I., Marcolli M.: On the distribution of continued fractions and modular symbols, Preprint of MPIM (2001).

[**Ma**] Manin, Yu.I.: Periods of cusp forms, and p-adic Hecke series. Mat. Sb. (N.S.) 92 (134) (1973), 378–401, 503. Eng. Tran. in Math. USSR-Sb. 92 (1973), 371–393.

[**Ma1**] Mayer, D.: On a ζ function related to the continued fraction transformation. Bull. Soc. Math. France, 104 (1976), 195-203.

[**Ma2**] Mayer, D.: The thermodynamic formalism approach to Selberg’s zeta function for PSL$(2,\mathbb{Z})$. Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 1, 55–60.

[**Ma3**] Mayer, D.: The Ruelle-Araki transfer operator in classical statistical mechanics. Lecture Notes in Physics, 123. Springer-Verlag, Berlin-New York, 1980.

[**Ma4**] Mayer, D.: Continued fractions and related transformations. In: Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, pp. 175-222. Eds. T. Bedford et al., Oxford University Press, Oxford 1991.

[**Me**] Merel, L.: Operateurs de Hecke pour $\Gamma_0(N)$ et fractions continues. Ann. Inst. Fourier, Grenoble 41 (1) (1991), 519-537.

[**Mu**] Mühlenbruch, T.: On Hecke operators. Unpublished notes (2002).

[**Ru**] Ruelle, D.: Thermodynamic formalism. Addison-Wesley Publishing Co., Reading, Mass. (1978).

[**VZ**] Venkov, A.B., Zograf, P.G.: Analogues of Artin's factorization formulas in the spectral theory of automorphic functions associated with induced representations of Fuchsian groups. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 6, 1150–1158, 1343. Engl. tran.: Math. USSR-Izv. 21 (1983), no. 3, 435–443.

[**Za1**] Zagier, D.: Periods of modular forms, traces of Hecke operators, and multiple zeta values. In "Research into automorphic forms and L functions" (Kyoto, 1992). Sūrikaisekikenkyūsho Kōkyūroku No. 843, (1993), 162–170.

[**Za2**] Zagier, D.: Hecke operators and periods of modular forms. Israel Math. Conf. Proc. 3 (1990) 321-336.