Exponentially small splitting for whiskered tori in Hamiltonian systems: Continuation of transverse homoclinic orbits

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Abstract

We consider an example of singular or weakly hyperbolic Hamiltonian, with 3 degrees of freedom, as a model for the behaviour of a nearly-integrable Hamiltonian near a simple resonance. The model consists of an integrable Hamiltonian possessing a 2-dimensional hyperbolic invariant torus with fast frequencies $\omega/\sqrt{\varepsilon}$ and coincident whiskers, plus a perturbation of order $\mu = \varepsilon^p$. We choose ω as the golden vector. Our aim is to obtain asymptotic estimates for the splitting, proving the existence of transverse intersections between the perturbed whiskers for ε small enough, by applying the Poincaré–Melnikov method together with a accurate control of the size of the error term.

The good arithmetic properties of the golden vector allow us to prove that the splitting function has 4 simple zeros (corresponding to nondegenerate critical points of the splitting potential), giving rise to 4 transverse homoclinic orbits. More precisely, we show that a shift of these orbits occurs when ε goes across some critical values, but we establish the continuation (without bifurcations) of the 4 transverse homoclinic orbits for all values of $\varepsilon \to 0$.

Keywords. Poincaré–Melnikov method, arithmetic properties of frequencies, transverse homoclinic orbits.

1 Introduction and main results

The application of the *Poincaré–Melnikov method* to the effective detection of *transverse homoclinic orbits*, associated to an n-dimensional whiskered torus of a Hamiltonian system with $n+1$ degrees of freedom, needs to be carefully validated in the case of *exponentially small splitting* of separatrices, since this problem becomes *singular* with respect to the parameter ε of perturbation.

Such a validation requires to ensure that the Poincaré–Melnikov approximation dominates the error term and, consequently, provides the right estimates of the size of the splitting. This was first done in the case of hyperbolic periodic orbits of Hamiltonian systems with 2 degrees of freedom ($n = 1$), and saddle fixed points of area preserving maps, by estimating the error term with the help of *complex parameterizations* (see for instance [DS97, DR98], although such a technique was first introduced in a somewhat different case by Lazutkin [Laz84, Gel99], for the standard map).

When the dimension n of the whiskered torus is 2 or more, the validation of the method is even more involved because of the fundamental rôle played by the small divisors associated to the frequencies of the torus in the size of the Poincaré–Melnikov approximation, as noticed in [Loc90]. In fact, the most important feature is that the dominant Fourier coefficient of the quasiperiodic splitting function changes when so does the perturbation parameter ε, in a way that depends on the arithmetic properties of the frequencies of the whiskered torus. This phenomenon was detected in [Sim94], and rigorously established later [DGJS97] for the quasiperiodically forced pendulum.

A related important fact [Eli94, DG00a] is that, for n-dimensional whiskered tori of a Hamiltonian with $n+1$ degrees of freedom, the splitting vector distance and the Melnikov vector function are the gradient of scalar functions, called respectively splitting potential and Melnikov potential. This implies that transverse homoclinic orbits correspond to nondegenerate critical points of the splitting potential.

In this context, the existence of transverse homoclinic orbits (in the case $n = 2$) is proved in [Sau01, LMS99], but not for all values of $\varepsilon \to 0$, since at some sequence of values of ε the dominant harmonics of the splitting function change, and homoclinic bifurcations could take place. Some examples of such bifurcations have been described in [SV01].

We consider in this paper some concrete perturbations in the case of 3 degrees of freedom ($n = 2$). We choose (like other papers quoted above) the *golden vector* of frequencies, and a perturbation with an *infinite* number of harmonics. In this situation, small divisors appear in the Melnikov function but, thanks to the simple arithmetic properties of the golden vector, it is possible to carry out an accurate analysis of the Melnikov function and its dominant harmonics. Applying the accurate bounds for the size of the error term obtained (from the use of flow-box coordinates) in our paper [DGS02], we show that the dominant harmonics of the splitting function correspond to the dominant harmonics in the Melnikov approximation, and from this we provide *asymptotic estimates* for the splitting. Such estimates allow us to show the existence of exactly 4 transverse homoclinic orbits, and their *continuation* for all values of the perturbation parameter $\varepsilon \to 0$ (with no bifurcations).

Next we give a more precise description of the setting, the results from [DGS02] to be applied, and the new results obtained in the present paper.

1.1 Setup: A concrete example of singular Hamiltonian with 3 degrees of freedom

We consider a Hamiltonian system, with 3 degrees of freedom, depending on two perturbation parameters ε and μ. In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, with the symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\varphi \wedge \mathrm{d}I$, our Hamiltonian is defined by

$$H(x,y,\varphi,I) = H_0(x,y,I) + \mu H_1(x,\varphi), \tag{1}$$

$$H_0(x,y,I) = \langle \omega_\varepsilon, I\rangle + \frac{1}{2}\langle \Lambda I, I\rangle + \frac{y^2}{2} + \cos x - 1, \tag{2}$$

$$H_1(x,\varphi) = h(x) f(\varphi). \tag{3}$$

The concrete vector ω_ε of *fast frequencies* considered in (2) is given by the *golden number*:

$$\Omega = \frac{\sqrt{5}-1}{2}, \qquad \omega = (1, \Omega), \qquad \omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}} \tag{4}$$

(with $\varepsilon > 0$; we will also assume $\mu > 0$ with no loss of generality). We have denoted Λ any symmetric (2×2)-matrix, such that H_0 satisfies the condition of *isoenergetic nondegeneracy*:

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{5}$$

We also consider, in (3), the following concrete analytic periodic functions:

$$h(x) = \cos x - \nu, \text{ where } \nu = 0 \text{ or } \nu = 1, \tag{6}$$

$$f(\varphi) = \sum_{k \in \mathcal{Z}} e^{-\rho|k|} \sin\langle k, \varphi\rangle, \tag{7}$$

where, to avoid repetitions in the Fourier expansion of $f(\varphi)$, we denote

$$\mathcal{Z} = \left\{ k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \text{ or } (k_2 = 0, k_1 \geq 0) \right\}; \tag{8}$$

note that the constant $\rho > 0$ in (7) gives the complex width of analiticity of $f(\varphi)$. In fact, our results will be valid for a somewhat more general function $f(\varphi)$; see Section 3.

The Hamiltonian defined in (1–7) is a particular case, with 3 degrees of freedom, of the model considered in our paper [DGS02]. Our choice in (4) of the frequency vector given by the golden number Ω is motivated by its simple arithmetic properties. On the other hand, as in [DGS02] the two parameters ε and μ will not be independent. On the contrary, they will be linked by a relation of the type $\mu = \varepsilon^p$ with a suitable $p > 0$ (the smaller p the better), i.e. we consider a *singular* problem for $\varepsilon \to 0$ (also called *weakly hyperbolic*, or *a-prori stable*). See for instance [DGS02, DG01] for a motivation of this singular setting.

1.2 Background: The Poincaré–Melnikov method with a bound for the error term

As in [DGS02], our approach is to work with (1–7) considering ε fixed and μ as the perturbation parameter. Our aim is to show that a smallness condition of the type $\mu \le \varepsilon^p$ is enough for the validity of the Poincaré–Melnikov method, despite the fact that the Melnikov function is exponentially small. The key point will be to carry out the bounds on complex domains, and use the quasiperiodicity properties of the splitting.

We recall here that the Hamiltonian H_0 (that corresponds to $\mu = 0$) has a 2-parameter family of 2-dimensional whiskered tori given by the equations $I = \text{const}$, $x = y = 0$. The stable and unstable whiskers of each torus coincide, forming in this way a unique homoclinic whisker. We shall focus our attention on a concrete *whiskered torus*, located at $I = 0$, having the golden vector of frequencies ω_ε. The homoclinic whisker of this torus (denoted $\mathcal{W}_0$) can be parameterized as

$$\mathcal{W}_0 : \qquad Z_0(s,\theta) = (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R},\ \theta \in \mathbb{T}^n, \tag{9}$$

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}, \tag{10}$$

and the inner flow is given by $\dot{s} = 1$, $\dot{\theta} = \omega_\varepsilon$.

We can check that hypotheses (H1–H4) of the paper [DGS02] are fulfilled by our example:

(H1) is the isoenergetic condition (5),

(H2) is the *Diophantine condition*:

$$|\langle k, \omega \rangle| \ge \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\}\,; \tag{11}$$

this holds with $\tau = 1$ and some $\gamma > 0$ (see the remark at the end of Section 2).

(H3) is satisfied with $l = 1$ (the degree of the trigonometric polynomial $h(x)$).

(H4) holds with $\alpha = 2$; this is a bound of the type $\|f\|_{\rho-\delta} \le c/\delta^\alpha$, related with the complex width of analyticity of $f(\varphi)$ (see Section 3).

Under conditions (5) and (11), the *hyperbolic KAM theorem* implies that, for μ small enough, the whiskered torus persists, as well as its local whiskers. We point out that the difference between the two values of ν in (6) is that in the case $\nu = 0$ the whiskered torus persists with some shift and deformation, whereas in the case $\nu = 1$ it remains fixed under the perturbation, though the whiskers do suffer some deformation. A precise statement of the hyperbolic KAM theorem with the corresponding estimates (which are somewhat better in the case of a fixed torus) is given in [DGS02, Th. 1]. The parameters describing the flow on the whiskered torus and in a neighbourhood of it are slightly perturbed. Indeed, the Lyapunov exponent of the torus, which initially is 1, becomes a close amount b. Besides, in the isoenergetic KAM theory the (fast) frequency vector ω_ε introduced in (4), describing the flow on the torus, becomes perturbed to a vector $\tilde{\omega}_\varepsilon$ that is close and proportional to the initial one:

$$\tilde{\omega}_\varepsilon = b' \omega_\varepsilon = \frac{b' \omega}{\sqrt{\varepsilon}} \tag{12}$$

(in fact, $b' = 1$ in the case $\nu = 1$). A bound for $|b - 1|$ and $|b' - 1|$ is given below in Theorem 0.

When the local whiskers are extended to global ones, one can expect in general the existence of *splitting* between the stable and unstable whiskers (denoted $\mathcal{W}^+$ and $\mathcal{W}^-$), since they will no longer coincide. To study this splitting, symplectic *flow-box coordinates* (S, E, φ, I) are introduced in [DGS02], in some neighbourhood containing a piece of both whiskers (and excluding the torus, where such coordinates are not valid). In the flow-box coordinates, the

Hamiltonian equations become very simple ($\dot{S} = b$, $\dot{E} = 0$, $\dot{\varphi} = \tilde{\omega}_\varepsilon$, $\dot{I} = 0$). Besides, those coordinates can be constructed in such a way that the stable whisker is given by a coordinate plane,

$$\mathcal{W}^+ : \qquad W^+(s,\theta) = (s,0,\theta,0), \tag{13}$$

where the parameters (s,θ) are inherited from (9–10). Then, the unstable whisker can be parameterized, in the same neighbourhood, as

$$\mathcal{W}^- : \qquad W^-(s,\theta) = (s,\mathcal{E}(s,\theta),\theta,\mathcal{M}(s,\theta)), \tag{14}$$

and the inner flow on both whiskers is given by $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$. To study the splitting, it is enough to consider the vector function $\mathcal{M}$, called the *splitting function* (the function $\mathcal{E}$ is directly related to $\mathcal{M}$ by the energy conservation).

The use of flow-box coordinates implies the quasiperiodicity of the splitting function $\mathcal{M}$, an important property related with its exponential smallness. More precisely, the function $\mathcal{M}$ is $\hat{\omega}_\varepsilon$*-quasiperiodic*:

$$\mathcal{M}(s,\theta) = \mathcal{M}\left(0,\theta - \hat{\omega}_\varepsilon s\right), \tag{15}$$

where the vector $\hat{\omega}_\varepsilon$ is also close and proportional to the initial frequency vector:

$$\hat{\omega}_\varepsilon := \frac{\tilde{\omega}_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}. \tag{16}$$

Another important property of $\mathcal{M}$ is that it is the gradient of a scalar function $\mathcal{L}$, called the *splitting potential*:

$$\mathcal{M}(s,\theta) = \partial_\theta \mathcal{L}(s,\theta)$$

(and hence $\mathcal{M}$ has zero average). Then, the transverse homoclinic orbits can be studied on $s = 0$ (or any other section $s = \text{const}$), as nondegenerate critical points of $\mathcal{L}(0,\theta)$.

Applying the Poincaré–Melnikov method, it is possible to give a first order approximation in μ for the splitting. The *Melnikov potential* and the *Melnikov function* are defined in [DGS02] as follows:

$$\begin{aligned} L(s,\theta) &= -\int_{-\infty}^{\infty} \left[h(x_0(s+bt)) - h(0)\right] \cdot f\left(\theta + \tilde{\omega}_\varepsilon t\right) \mathrm{d}t + \text{const}, \qquad (17)\\ M(s,\theta) &= \partial_\theta L(s,\theta). \end{aligned}$$

These functions are also $\hat{\omega}_\varepsilon$-quasiperiodic, since they are defined in terms of the perturbed Lyapunov exponent b and the perturbed frequencies $\tilde{\omega}_\varepsilon$ introduced in (12). As a consequence, the *error term* defined as

$$\mathcal{R}\left(s,\theta\right) = \mathcal{M}\left(s,\theta\right) - \mu M\left(s + s_0,\theta\right) \tag{18}$$

is also $\hat{\omega}_\varepsilon$-quasiperiodic, a crucial property that we use in the present paper. The amount s_0, not very relevant, has been introduced in order to compensate a translation of the parameterizations (13–14) with respect to the initial parameterization (9).

The key point in order to obtain exponentially small estimates for the functions involved is to carry out the bounds on *complex domains* of the parameters (s,θ). We define the domain

$$\mathcal{P}_{\kappa,\nu,\rho} = \{(s,\theta) : |\mathrm{Re}\, s| \le \kappa,\ |\mathrm{Im}\, s| \le \nu,\ \mathrm{Re}\,\theta \in \mathbb{T}^n,\ |\mathrm{Im}\,\theta| \le \rho\}$$

and, for a function $g(s,\theta)$, we denote $|g|_{\kappa,\nu,\rho}$ its supremum norm on this domain.

Initially, the whiskers can be defined for $|\mathrm{Im}\, s| < \pi/2$, $|\mathrm{Im}\,\theta| < \rho$. These restrictions are due to the singularity of (10) at $s = \pm i\pi/2$, and to the expression (7) involving ρ as the width of analyticity. This domain is reduced along the successive steps leading to define the splitting function and potential. One of the main achievements of [DGS02] is to construct the flow-box coordinates in such a way that the loss of complex domain is controlled by a free small parameter δ, with $\delta \ll \pi/2$ and $\delta \ll \rho$. Then, choosing $\delta = \varepsilon^a$ for some $a > 0$ and using that the involved functions are analytic, quasiperiodic and with zero average, it is possible to obtain exponentially small estimates.

With all these ingredients, we give below essentially the statement of [DGS02, Th. 10] (plus a bound for $|b-1|$ and $|b'-1|$ coming from [DGS02, Th. 1]), restricted to the particular case concerned in the present paper: $\tau = 1$, $l = 1$, $\alpha = 2$. The exponents $q_1,\ldots,q_4$ correspond to the exponents appearing in [DGS02], easily computed from the general expressions given there. We stress that some improvement of the exponents can be given for the case of a fixed torus. Because of this, we have different estimates if we consider $\nu = 0$ or $\nu = 1$ in (6).

To express the bounds of functions we use the notation $|f| \preceq |g|$, which means that we can bound $|f| \leq c\,|g|$, with some constant c not depending on any of the parameters that will be relevant to us: ε, μ, δ. In this way, we do not describe the (usually complicated) dependence on amounts like $n, \tau, \rho, \ldots$ and include this dependence in the 'constants'.

Theorem 0 *For a given $\delta > 0$, assuming the conditions*

$$\varepsilon \preceq 1, \qquad \mu \preceq \delta^{q_1}, \qquad \mu \preceq \delta^{q_2+1}\sqrt{\varepsilon}, \tag{19}$$

the splitting function $\mathcal{M}(s,\theta) = \partial_\theta \mathcal{L}(s,\theta)$ is analytic on $\mathcal{P}_{\kappa,\pi/2-\delta,\rho-\delta}$ $(\kappa > 0)$, and $\hat{\omega}_\varepsilon$-quasiperiodic with $\hat{\omega}_\varepsilon$ as in (16). For the amounts b and b' in (12) and (16), one has the bounds

$$|b-1|, |b'-1| \preceq \frac{\mu}{\delta^{q_2}}.$$

The error term $\mathcal{R}(s,\theta)$ defined in (18) is also $\hat{\omega}_\varepsilon$-quasiperiodic, and satisfies the following bound:

$$|\mathcal{R}|_{\kappa,\pi/2-\delta,\rho-\delta} \preceq \frac{\mu^2}{\delta^{q_3}} + \frac{\mu^2}{\delta^{q_4}\sqrt{\varepsilon}}. \tag{20}$$

The exponents $q_1, \ldots, q_4$, in the case $\nu = 0$, are given by

$$q_1 = 8, \qquad q_2 = 4, \qquad q_3 = 14, \qquad q_4 = 12$$

and, in the case $\nu = 1$, the exponents are given by

$$q_1 = 6, \qquad q_2 = 2, \qquad q_3 = 10, \qquad q_4 = 8.$$

We remark that, in the quoted result [DGS02, Th. 10], the splitting function is defined in a complex strip whose widths in $|\mathrm{Im}\, s|$ and $|\mathrm{Im}\,\theta|$ are written as $\nu_3 = \pi/2 - 3\delta$ and $\rho_6 = \rho - 6\delta$. This comes from the number of successive reductions carried out until the splitting function can be defined. Of course, the value of δ can be rescaled. On the other hand, the width in $|\mathrm{Re}\, s|$ is some κ_3 that we have renamed as κ (related to the reach of the flow-box coordinates).

1.3 Description of the results

The main goal of this paper is to show that, for a singular Hamiltonian with the two parameters linked by a relation of the type $\mu = \varepsilon^p$, the splitting can be approximated by the Melnikov function. This was shown in [DG00a], but assuming that μ is exponentially small in ε. In the present singular case $\mu = \varepsilon^p$, we need to show that, in the Poincaré–Melnikov approximation (18) for whole splitting function $\mathcal{M}(s,\theta)$, the term $\mu M(s+s_0,\theta)$ (exponentially small in ε) dominates, in some sense, the error term $\mathcal{R}(s,\theta)$. A natural approach to this is to provide asymptotic estimates (or at least lower bounds) of the dominant harmonics of the Melnikov potential L. Such estimates have to be big enough in order to be valid also for the dominant harmonics of splitting potential $\mathcal{L}$ (recall that $\mathcal{M} = \partial_\theta \mathcal{L}$), showing that they dominate the part coming from $\mathcal{R}$. It is then possible to prove the existence of simple zeros of $\mathcal{M}$ (with asymptotic estimates of the associated eigenvalues of $\partial_\theta \mathcal{M}$), and hence transverse homoclinic orbits.

We carry out this scheme for the example with 3 degrees of freedom defined in (1–7). We have considered a very concrete frequency vector in (4), the golden one, for which a careful analysis of the associated small divisors can be done, making it possible to give the above mentioned asymptotic estimates. Then, applying previous results, summarized in Section 1.2, we prove the existence of exactly 4 transverse homoclinic orbits.

As a difference with respect to previous works [Sau01, LMS99, DG00a], we show that there exist exactly 4 transverse homoclinic orbits *for any* $\varepsilon > 0$ small enough. So we have a continuation of the homoclinic orbits as ε goes to 0, with no bifurcations.

In fact, we prove this result for a slight generalization of the example (1–7). Indeed, we replace the function $f(\varphi)$ in (7) by a function having a more general Fourier expansion (33), introduced in Section 3. In order to prove the continuation of the transverse homoclinic orbits, we assume a quite general condition (explicited in Section 6) on the phases of the Fourier expansion of the function $f(\varphi)$. This assumption on the phases excludes the case of a reversible perturbation (given by an even function $f(\varphi)$), often considered in related papers [Gal94, GGM99, RW98]. In such a case, bifurcations of some of the homoclinic orbits can occur when ε goes across some critical values. This kind of bifurcations has been described in [SV01], where the Arnold example (slightly modified) is considered.

Let us give now a short summary of the results presented. As said above, one of the main features of the example considered is that the frequency vector ω, given by the golden number, has nice arithmetic properties (in fact, other quadratic irrationals could also be considered). To start, in Section 2 we review these *arithmetic properties*, putting a special emphasis on a classification of the associated resonances.

In Section 3, we define the generalization of the function $f(\varphi)$, in terms of arbitrary phases, and compute the Fourier coefficients of its Melnikov potential L. Next, in Section 4 we carry out an accurate analysis of the coefficients and find the *dominant harmonics* of the Fourier expansion in θ of L. Since we look for nondegenerate critical points on $\mathbb{T}^2$ of the potential, we need at least the 2 most dominant harmonics. We show that, when ε goes across some critical values ε_n (defined below), some changes in the dominance occur. In fact, for ε close to ε_n, we have to consider the 3 most dominant harmonics because the second and third ones are of the same magnitude. We estimate the size of these dominant harmonics, and show that the sum of all other harmonics is much smaller. In Section 5, these estimates are translated from the Melnikov potential L to the splitting potential $\mathcal{L}$, using the bound on the error term given above in Theorem 0.

Finally, in Section 6 we study the nondegenerate critical points of $\mathcal{L}$ (which correspond to simple zeros of $\mathcal{M}$) and obtain the *main result* of this paper, which implies the existence of exactly 4 *transverse homoclinic orbits*. This result is valid in both the cases of 2 or 3 dominant harmonics, and ensures the *continuation* (without bifurcations) of the 4 homoclinic orbits for all values of $\varepsilon \to 0$.

We state below as Theorem 1 the main result of this paper. The quasiperiodicity (15) of the splitting function $\mathcal{M}(s,\theta)$ allows us to restrict ourselves to the section $s=0$, and the (simple) zeros of $\mathcal{M}(0,\theta)$ give rise to (transverse) homoclinic orbits. These (simple) zeros are given by (nondegenerate) critical points of the splitting potential $\mathcal{L}(0,\theta)$. The theorem gives an asymptotic estimate for the maximum of the function $\mathcal{M}(0,\theta)$, and ensures that it has 4 simple zeros $\theta_{(j)}$, $j=1,2,3,4$, giving also an asymptotic estimate for the minimum eigenvalue (in modulus) of the *splitting matrix* $\partial_\theta \mathcal{M}\left(0,\theta_{(j)}\right)$, for each zero. This eigenvalue provides a measure of the transversality of the homoclinic orbits.

To give a precise statement, we define

$$\varepsilon_0 = \frac{\pi^2(2-\Omega)^2\Omega^4}{4\rho^2}, \qquad \varepsilon_n = \Omega^{4n}\varepsilon_0, \tag{21}$$

the intervals

$$\mathcal{I} = \left[\Omega^2\varepsilon_1, \Omega^{-2}\varepsilon_1\right], \qquad \mathcal{I}' = \left[\Omega^2\varepsilon_1, \varepsilon_1\right], \qquad \mathcal{I}'' = \left[\varepsilon_1, \Omega^{-2}\varepsilon_1\right],$$

and the functions

$$h_1(\varepsilon) = \frac{1}{2}\left[\left(\frac{\varepsilon}{\varepsilon_1}\right)^{1/4} + \left(\frac{\varepsilon_1}{\varepsilon}\right)^{1/4}\right] = \cosh\left(\frac{\ln\varepsilon - \ln\varepsilon_1}{4}\right), \qquad \varepsilon\in\mathcal{I}, \tag{22}$$

$$h_2(\varepsilon) = \begin{cases} \dfrac{1}{2}\left[\left(\dfrac{\varepsilon}{\varepsilon_2}\right)^{1/4} + \left(\dfrac{\varepsilon_2}{\varepsilon}\right)^{1/4}\right] = \cosh\left(\dfrac{\ln\varepsilon - \ln\varepsilon_2}{4}\right), & \varepsilon\in\mathcal{I}', \\[2ex] \dfrac{1}{2}\left[\left(\dfrac{\varepsilon}{\varepsilon_0}\right)^{1/4} + \left(\dfrac{\varepsilon_0}{\varepsilon}\right)^{1/4}\right] = \cosh\left(\dfrac{\ln\varepsilon - \ln\varepsilon_0}{4}\right), & \varepsilon\in\mathcal{I}'', \end{cases} \tag{23}$$

We extend these functions from the interval $\mathcal{I}$ to any $\varepsilon>0$, in such a way that $h_i\left(\Omega^4\varepsilon\right) = h_i(\varepsilon)$. Then, the functions $h_i(\varepsilon)$ obtained are continuous and $4\ln\Omega$-periodic in $\ln\varepsilon$, and satisfy the inequalities $1 \le h_1(\varepsilon) \le 1.029086 \le h_2(\varepsilon) \le 1.118034$ (for an illustration, see Figure 1 in Section 4). We also define the constant

$$C_0 = \sqrt{\frac{2\pi\rho}{2-\Omega}}. \tag{24}$$

The next theorem states the main properties of the splitting function, in the section $s=0$. We need an exponent $p>p^*$, where p^* depends on the value of ν chosen in (6), i.e. whether the whiskered torus remains fixed or not.

We use the notation $f \sim g$ if we can bound $c_1 |g| \leq |f| \leq c_2 |g|$ with positive constants c_1, c_2 not depending on the parameters ε, μ, δ.

Theorem 1 *In the example (1–7), assume ε small enough and $\mu = \varepsilon^p$ with $p > p^*$, where we define $p^* = 2$ if $\nu = 1$, and $p^* = 3$ if $\nu = 0$.*

(a) $\max\limits_{\theta \in \mathbb{T}^2} |\mathcal{M}(0,\theta)| \sim \dfrac{\mu}{\sqrt{\varepsilon}} \exp\left\{ -\dfrac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}} \right\}.$

(b) *The splitting function $\mathcal{M}(0,\theta)$ has exactly 4 zeros $\theta_{(j)}$, all simple, and (the modulus of) the minimum eigenvalue of $\partial_\theta \mathcal{M}(0,\cdot)$ at each zero satisfies*

$$m_{(j)} \sim \mu \varepsilon^{1/4} \exp\left\{ -\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \right\}, \qquad j = 1,2,3,4.$$

In fact, the result established in Section 6 (Theorem 8) is a generalization of this one, in the sense that the function $f(\varphi)$ in (7) is replaced by a function with a more general Fourier expansion (see Section 3) satisfying a suitable condition on the phases of its Fourier expansion. Such a conditon will be clearly fulfilled in the concrete case (7).

2 The golden vector

In order to carry out the analysis of the small divisors, the simplest case is that of the *golden vector*:

$$\omega = (1, \Omega), \qquad \Omega = \frac{\sqrt{5}-1}{2} = 0.618034 \tag{25}$$

(the number considered in [DG00a] was Ω^{-1}, but this makes no important difference). Note that $\Omega^2 = 1 - \Omega$ with $0 < \Omega < 1$. This frequency vector allows us to take advantage of the nice properties of the Fibonacci numbers. However, it is possible to generalize our results to other quadratic irrationals instead of Ω.

It is well-known that the golden vector ω satisfies the Diophantine condition (11), with $\tau = 1$ and some $\gamma > 0$ (in fact, this can be deduced from Proposition 3). For every $k \in \mathbb{Z}^2 \setminus \{0\}$, we define its associated *"numerator"* as

$$\gamma_k = \gamma_k(\omega) := |\langle k, \omega \rangle| \cdot |k| \tag{26}$$

(we write $|k| = |k|_1 = |k_1| + |k_2|$), and note that always $\gamma_k \geq \gamma$. The aim of this section is to provide a simple classification of the small divisors of ω according to the size of their numerators γ_k.

Consider the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = -\left(T^\top\right)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of T are Ω^{-1} and $-\Omega$, with eigenvectors $(1, \Omega) = \omega$ and $\left(1, -\Omega^{-1}\right)$ respectively. Since $T = T^\top$, the matrix U has the same eigenvectors, and their respective eigenvalues are $-\Omega$ and Ω^{-1}.

We point out that T is a *unimodular* matrix (i.e. T is a square matrix with integer entries and $|\det T| = 1$). Besides, we stress that the matrix T has a unique eigenvalue with modulus greater than 1, whose associated eigenvector is our frequency vector. A matrix having such properties can also be constructed if Ω in (25) is replaced by another quadratic irrational. A generalization of the matrix T to higher dimensions is considered in [Koc99].

We say that an integer vector $k \in \mathbb{Z}^2 \setminus \{0\}$ is *admissible* (with respect to ω) if $|\langle k, \omega \rangle| < 1/2$, and denote $\mathcal{A}$ the set of admissible integer vectors. The analysis of the small divisors can be restricted to the set $\mathcal{A}$, since for any $k \notin \mathcal{A}$ one has $|\langle k, \omega \rangle| > 1/2$ and hence $\gamma_k > |k|/2$.

The following equality is going to play a fundamental rôle:

$$\langle Uk, \omega \rangle = \left\langle k, U^\top \omega \right\rangle = -\Omega \langle k, \omega \rangle. \tag{27}$$

It implies that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that k is *primitive* if $k \in \mathcal{A}$ but $U^{-1}k \notin \mathcal{A}$. We deduce from (27) that the primitive vectors are exactly the ones satisfying

$$\frac{\Omega}{2} < |\langle k, \omega \rangle| < \frac{1}{2}. \tag{28}$$

It is clear that the admissible vectors are those of the form $\hat{k}(j) = (-m(j), j)$, where $j \neq 0$ is an integer and $m(j)$ is the closest integer to $j\Omega$. We shall always assume that $j > 0$ with no restriction. When $\hat{k}(j)$ is primitive, we also say that j is primitive. We denote

$$d(j) := \left\langle \hat{k}(j), \omega \right\rangle = j\Omega - m(j).$$

For any given primitive j, we can define the following sequence of (admissible) integer vectors:

$$s(j,n) := U^{n-1}\hat{k}(j), \qquad n \geq 1. \tag{29}$$

The following simple result says that the sequences generated by primitives cover the whole set of admissible vectors.

Lemma 2 *For any $k \in \mathcal{A}$, there exist a primitive j and an integer $n \geq 1$, both unique, such that $k = s(j,n)$.*

Proof. If $k \in \mathcal{A}$, one finds a unique primitive vector in the sequence $U^{-n}k$, $n \geq 0$. Indeed, using (27) one has the equality $|\langle U^{-n}k, \omega \rangle| = \Omega^{-n} |\langle k, \omega \rangle|$, and hence only one of the vectors $U^{-n}k$ satisfies (28). □

It is not hard to see from (29), applying induction with respect to n, that

$$s(j,n) = (-f(j,n-1), f(j,n)), \tag{30}$$

where $f(j,n)$ is a "generalized" Fibonacci sequence, $f(j,n) = f(j,n-1) + f(j,n-2)$, starting from $f(j,0) = m(j)$ and $f(j,1) = j$. Note that for $j = 1$ we have the (classical) Fibonacci sequence, since $m(1) = 1$.

The motivation for defining the sequences $s(j,\cdot)$ is that they provide a classification of the small divisors, because the numerators $\gamma_{s(j,n)}$ become nearly constant when $n \to \infty$. Indeed, since $s(j,n) \in \mathcal{A}$, for n large we have $f(j,n) \simeq f(j,n-1)\Omega^{-1}$ (recall that $f(j,n-1)$ is the closest integer to $f(j,n)\Omega$), and hence $|s(j,n)| \simeq |s(j,n-1)|\,\Omega^{-1}$. On the other hand, $|\langle s(j,n), \omega \rangle| = \Omega\,|\langle s(j,n-1), \omega \rangle|$ and we obtain $\gamma_{s(j,n)} \simeq \gamma_{s(j,n-1)}$.

In fact the numerators $\gamma_{s(j,n)}$ oscillate around a *"limit numerator"*, which we denote γ_j^*. In the next result we rigorously find this limit for each j, and show the oscillations of $|s(j,n)|$ and $\gamma_{s(j,n)}$ around their limit behaviours.

Proposition 3 *For any primitive vector $\hat{k}(j) = (-m(j), j)$ with $j > 0$, its associated limit numerator is given by*

$$\gamma_j^* = \lim_{n \to \infty} \gamma_{s(j,n)} = \frac{\left|j^2 - jm(j) - m(j)^2\right|}{2 - \Omega},$$

and one has:

(a) $|s(j,n)| = f(j,n+1) = \Omega^{-(n+1)} \left(m(j) + \dfrac{d(j)}{2-\Omega} \right) - (-\Omega)^{n+1} \dfrac{d(j)}{2-\Omega} \quad \forall n \geq 1.$

(b) $\gamma_{s(j,n)} = \gamma_j^* + \left(-\Omega^2\right)^n \dfrac{|d(j)|\,d(j)}{2-\Omega} \quad \forall n \geq 1.$

(c) $\dfrac{j}{2(2-\Omega)} < \gamma_j^* < \dfrac{3j}{2(2-\Omega)}.$

Proof. We start by writing $\hat{k}(j)$ as the following linear combination of the eigenvectors of U:

$$\hat{k}(j) = \begin{pmatrix} -m(j) \\ j \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \Omega \end{pmatrix} - c_2 \begin{pmatrix} 1 \\ -\Omega^{-1} \end{pmatrix},$$
$$c_1 = \frac{d(j)}{2-\Omega}, \quad c_2 = m(j) + c_1$$

(note that $|c_1| < 1/2$ and $c_2 > 1$). The following equalities are easily deduced from definition (29):

$$
\begin{aligned}
s(j,n) &= (-\Omega)^{n-1} c_1 \begin{pmatrix} 1 \\ \Omega \end{pmatrix} - \Omega^{-(n-1)} c_2 \begin{pmatrix} 1 \\ -\Omega^{-1} \end{pmatrix}, \\
|s(j,n)| &= f(j,n+1) = \Omega^{-(n+1)} c_2 - (-\Omega)^{n+1} c_1, \\
|\langle s(j,n), \omega \rangle| &= \Omega^{n-1} (2-\Omega) \, |c_1| = \Omega^{n-1} \, |d(j)| \,,
\end{aligned}
$$

and from the expression for $|s(j,n)|$ we get statement (a). We also obtain

$$
\gamma_{s(j,n)} = |d(j)| \left(\Omega^{-2} c_2 + \left(-\Omega^2 \right)^n c_1 \right),
$$

whose limit for $n \to \infty$ is

$$
\gamma_j^* = |d(j)| \, \Omega^{-2} c_2 = \Omega^{-2} \left| d(j) \left(m(j) + \frac{d(j)}{2-\Omega} \right) \right| = \frac{\left| j^2 - j m(j) - m(j)^2 \right|}{2-\Omega},
$$

and we also deduce part (b).

Finally, part (c) is easily proved in the following rough way:

$$
\begin{aligned}
\left| j^2 - j m(j) - m(j)^2 \right| &= |d(j)((1+2\Omega) j - d(j))| > \frac{\Omega}{2} \left((1+2\Omega) j - \frac{1}{2} \right) \\
&= \left(1 - \frac{\Omega}{2} \right) j - \frac{\Omega}{4} \geq \left(1 - \frac{3\Omega}{4} \right) j > \frac{j}{2},
\end{aligned}
$$

where we have used that $\Omega/2 < |d(j)| < 1/2$, according to (28). One obtains the upper inequality in a similar way. □

To illustrate this result, we write down the (integer) limits

$$
\tilde{\gamma}_j^* = (2-\Omega) \gamma_j^* = \left| j^2 - j m(j) - m(j)^2 \right| \tag{31}
$$

for the sequences generated by the first few primitives $\hat{k}(j)$ with $j > 0$, as well as a lower bound for the remaining ones. Note that the lower and upper bounds given in Proposition 3 imply a wide separation among the limit numerators in the sequences, except for some of them whose limits coincide.

$\hat{k}(j)$	$\tilde{\gamma}_j^*$
$(-1,1)$	1
$(-2,4)$	4
$(-4,7)$	5
$(-6,9)$	9
$(-7,12)$	11
$(-9,14)$	11

$\hat{k}(j)$	$\tilde{\gamma}_j^*$
$(-11,17)$	19
$(-12,20)$	16
$(-14,22)$	20
$(-15,25)$	25
$(-17,27)$	19
$j \geq 28$	> 14

In fact, it is easy to see that there exist infinite primitives. Indeed, taking two primitives $\hat{k}(j_1)$ and $\hat{k}(j_2)$ such that $d(j_1)$ and $d(j_2)$ have the same sign, and denoting $k = \hat{k}(j_1) + \hat{k}(j_2)$, one sees that $\Omega < |\langle k, \omega \rangle| < 1$, and one of Uk or $U^2 k$ is a new primitive vector, whose second component is greater than $j_1 + j_2$.

Remark. It is an obvious consequence of Proposition 3 that the exponent in the Diophantine condition (11) is $\tau = 1$. Besides, it is not hard to find the constant γ as the minimum of the numerators γ_k, $k \neq 0$. A simple inspection (including vectors $k \notin \mathcal{A}$) gives the rigorous value $\gamma = \gamma_{(0,1)} = \Omega$. Nevertheless, it is more significant to give for γ the following asymptotic value (see also remark 3 after Theorem 8):

$$
\gamma \simeq \liminf_{|k| \to \infty} \gamma_k = \min \left\{ \gamma_j^* : j \text{ is primitive} \right\} = \gamma_1^* = \frac{1}{2-\Omega} = 0.723607. \tag{32}
$$

3 The Melnikov potential of a generalized example

The Hamiltonian we consider in this and the subsequent sections, with 3 degrees of freedom, is given as in (1–6) but we consider, instead of (7), a more general perturbation:

$$f(\varphi) = \sum_{k\in\mathcal{Z}} f_k \cos(\langle k,\varphi\rangle - \sigma_k), \quad \text{with } f_k = e^{-\rho|k|} \text{ and } \sigma_k \in \mathbb{T}, \tag{33}$$

where $\mathcal{Z} \subset \mathbb{Z}^2$ is defined as in (8). Recall the difference between the two values of ν in (6): for $\nu = 1$ the whiskered torus remains fixed under the perturbation, and for $\nu = 0$ it does not.

In fact, a quite general condition on the phases σ_k will have to be fulfilled in order to guarantee the continuation of the homoclinic orbits (see this condition in Section 6). For instance, taking $\sigma_k = \pi/2$ for every k, one has the example (7), and the condition on the phases will be fulfilled for this concrete case.

On the other hand, we have $\alpha = 2$ in hypothesis (H4) of [DGS02, Sect. 1.1], since α is the exponent obtained in the following estimate:

$$\|f\|_{\rho-\delta} := \sum_k |f_k| \, e^{(\rho-\delta)|k|} = \sum_k e^{-\delta|k|} \sim \frac{1}{\delta^2},$$

where we have considered a norm which takes into account the Fourier expansion in the angular variables (see more details in [DGS02, Sect. 1.5]). This value of α has been used to give the exponents in Theorem 0.

We can obtain the Fourier expansion of the Melnikov potential using (17), together with (16). We have:

$$\begin{aligned} L(s,\theta) &= -\sum_{k\in\mathcal{Z}\setminus\{0\}} f_k \int_{-\infty}^{\infty} (\cos x_0(s+bt) - 1)\cos\left(\langle k, \theta + \tilde\omega_\varepsilon t\rangle - \sigma_k\right) \mathrm{d}t \\ &= \sum_{k\in\mathcal{Z}\setminus\{0\}} L_k \cos\left(\langle k, \theta - \hat\omega_\varepsilon s\rangle - \sigma_k\right), \end{aligned}$$

$$L_k = 2f_k \int_{-\infty}^{\infty} \frac{\cos\langle k,\tilde\omega_\varepsilon t\rangle}{\cosh^2 bt} \mathrm{d}t = \frac{2\pi \langle k,\hat\omega_\varepsilon\rangle f_k}{b\sinh\left(\frac{\pi}{2}\langle k,\hat\omega_\varepsilon\rangle\right)} = \frac{2\pi \left|\langle k,\hat\omega_\varepsilon\rangle\right| e^{-\rho|k|}}{b\sinh\left|\frac{\pi}{2}\langle k,\hat\omega_\varepsilon\rangle\right|} \tag{34}$$

(we take $L_0 = 0$ to have zero average). The integral has been computed by residues, and we have also used the formula $\cos x_0(bt) - 1 = -2/\cosh^2 bt$. Notice that the value of ν does not influence the Melnikov potential, and that the phases σ_k in the Fourier expansion of $L(0,\theta)$ are the same as in the function $f(\varphi)$ given in (33).

4 Dominant harmonics of the Melnikov potential

To show that the Melnikov potential has nondegenerate critical points, we have to consider at least the 2 most dominant harmonics in its Fourier expansion. As we see below, which the dominant harmonics are depends on ε. Nevertheless, for some values of ε we will have to consider the 3 most dominant harmonics because the second and the third ones can be of the same magnitude.

Studying the size of the coefficients L_k requires to use the arithmetic properties of our frequency vector. Taking into account the definition of γ_k in (26), we have

$$\left|\langle k,\hat\omega_\varepsilon\rangle\right| = \left|\left\langle k, \frac{b'\omega}{b\sqrt{\varepsilon}}\right\rangle\right| = \frac{b'\gamma_k}{b\,|k|\,\sqrt{\varepsilon}}.$$

Recall from Theorem 0 that b and b' are $1+\mathcal{O}\left(\mu\delta^{-q_2}\right)$. Using the equality $\sinh x = e^x(1-e^{-2x})/2$, we obtain from (34) the following expression for the coefficients:

$$L_k = \alpha_k e^{-\beta_k}, \tag{35}$$

where

$$\alpha_k = \frac{4\pi b'\gamma_k}{b^2\,|k|\,\sqrt{\varepsilon}\left[1-\exp\left\{-\frac{\pi b'\gamma_k}{b|k|\sqrt{\varepsilon}}\right\}\right]}, \qquad \beta_k = \rho\,|k| + \frac{\pi b'\gamma_k}{2b\,|k|\,\sqrt{\varepsilon}}. \tag{36}$$

The largest coefficients L_k will be given by the smallest exponents β_k.

A more suitable expression for those exponents is:

$$\begin{aligned} \beta_k &= \frac{C_\mu\sqrt{\tilde\gamma_k}}{2\varepsilon^{1/4}}\left(\frac{|k|\,\varepsilon^{1/4}}{D_\mu\sqrt{\tilde\gamma_k}}+\frac{D_\mu\sqrt{\tilde\gamma_k}}{|k|\,\varepsilon^{1/4}}\right), \\ C_\mu &= \sqrt{\frac{2\pi b'\rho}{b(2-\Omega)}}, \quad D_\mu = \sqrt{\frac{\pi b'}{2b(2-\Omega)\rho}} = \frac{C_\mu}{2\rho}, \end{aligned} \tag{37}$$

where we denote $\tilde\gamma_k = (2-\Omega)\gamma_k$, analogously to (31). Note that $C_\mu = C_0 + \mathcal{O}(\mu\delta^{-q_2})$, $D_\mu = D_0 + \mathcal{O}(\mu\delta^{-q_2})$, where C_0 is given in (24) and $D_0 = C_0/2\rho$. We deduce from (37) the lower bound

$$\beta_k \geq \frac{C_\mu\sqrt{\tilde\gamma_k}}{\varepsilon^{1/4}}, \tag{38}$$

which suggests that the size of the exponent β_k is strongly related to the sequence $s(j,\cdot)$, defined in (29), to which k belongs, due to the fact that the numerators tend to a constant for each sequence. Indeed, recall from Proposition 3 that, for k belonging to the (classical) Fibonacci sequence $s(1,\cdot)$, the limit of the $\tilde\gamma_k$ is $\tilde\gamma_1^* = 1$, whereas the limits for all other sequences $s(j,\cdot)$, $j>1$, are $\tilde\gamma_j^* \geq \tilde\gamma_4^* = 4$ (recall that the $\tilde\gamma_j^*$ have been defined in (31)). This says that the smallest exponents can be found among the Fibonacci sequence. We show below in Lemma 4 that the 3 dominant harmonics are given by 3 consecutive Fibonacci harmonics, and we provide an estimate for their size through the associated exponents β_k. We also provide an estimate for the difference between the Melnikov potential and its approximation by the 2 or 3 dominant harmonics, in terms of the following dominant harmonic.

We recall the decreasing sequence ε_n previously defined in (21), and define a new sequence ε'_n as follows:

$$\varepsilon_n = ((2-\Omega)\Omega^{n+1}D_0)^4 = \Omega^{4n}\varepsilon_0, \qquad \varepsilon'_n = \sqrt{\varepsilon_n\varepsilon_{n-1}} = \Omega^{4n-2}\varepsilon_0.$$

We now introduce the functions

$$g_n(\varepsilon) = \frac{1}{2}\left[\left(\frac{\varepsilon}{\varepsilon_n}\right)^{1/4}+\left(\frac{\varepsilon_n}{\varepsilon}\right)^{1/4}\right] = g_0(\Omega^{-4n}\varepsilon).$$

It is clear that each g_n has its minimum at $\varepsilon=\varepsilon_n$. Notice that, as a function of $\ln\varepsilon$, the graph of g_n is simply the graph of g_0 translated a distance $4n\ln\Omega$. This is illustrated in Figure 1, using logarithmic scale for ε for the sake of clarity.

We also define the following functions:

$$\begin{array}{llll} h_1(\varepsilon) = g_n(\varepsilon), & & & \text{for } \varepsilon\in\left[\varepsilon'_{n+1},\varepsilon'_n\right], \\ h_2(\varepsilon) = g_{n+1}(\varepsilon), & h_3(\varepsilon) = g_{n-1}(\varepsilon), & h_4(\varepsilon) = g_{n+2}(\varepsilon), & \text{for } \varepsilon\in\left[\varepsilon'_{n+1},\varepsilon_n\right], \\ h_2(\varepsilon) = g_{n-1}(\varepsilon), & h_3(\varepsilon) = g_{n+1}(\varepsilon), & h_4(\varepsilon) = g_{n-2}(\varepsilon), & \text{for } \varepsilon\in\left[\varepsilon_n,\varepsilon'_n\right]. \end{array} \tag{39}$$

By connecting the successive intervals $\left[\varepsilon'_{n+1},\varepsilon'_n\right]$, we get that these functions are continuous on the whole interval $(0,\infty)$, and satisfy the equality $h_i\left(\Omega^4\varepsilon\right) = h_i(\varepsilon)$ for any $\varepsilon>0$. In other words, the functions h_i are $4\ln\Omega$-periodic in $\ln\varepsilon$. The functions $h_i(\varepsilon)$ are also illustrated in Figure 1. It is not hard to check that the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ are the ones defined in (22–23).

We can easily check that

$$1 \leq h_1(\varepsilon) \leq 1.029086 \leq h_2(\varepsilon) \leq 1.118034 \leq h_3(\varepsilon) \leq 1.272020 \leq h_4(\varepsilon) \leq 1.5, \tag{40}$$

and equalities can take place only for $\varepsilon=\varepsilon_n,\varepsilon'_n$. More precisely, for $\varepsilon=\varepsilon_n$ we have $h_1<h_2=h_3<h_4$, and for $\varepsilon=\varepsilon'_n$ we have $h_1=h_2<h_3=h_4$.

For any given $\varepsilon<\varepsilon_1$, we define $N_i=N_i(\varepsilon)$, $i=1,2,3,4$, as the 4 integers $n\geq 1$ minimizing $g_n(\varepsilon)$. This means that

$$g_{N_1}(\varepsilon) \leq g_{N_2}(\varepsilon) \leq g_{N_3}(\varepsilon) \leq g_{N_4}(\varepsilon) \leq g_n(\varepsilon) \quad \forall n\neq N_1,N_2,N_3,N_4.$$

For ε belonging to a concrete interval $\left(\varepsilon'_{n+1},\varepsilon'_n\right)$, the first mimimum is given by $N_1=n$. The second and third mimima are $N_2=n\pm 1$, $N_3=n\mp 1$ depending on the subinterval to which ε belongs: $\left(\varepsilon'_{n+1},\varepsilon_n\right)$ or $\left(\varepsilon_n,\varepsilon'_n\right)$. The fourth minimum is $N_4 = n \pm 2$. The main fact that we shall use is that the values of the 4 minima are given by the functions h_i defined in (39). Indeed, one easily checks that

$$g_{N_i}(\varepsilon) = h_i(\varepsilon), \quad i = 1, 2, 3, 4.$$

Notice that there is some ambiguity in the definition of $N_i(\varepsilon)$ at the endpoints of the intervals, but the important fact is that they are critical values at which some of the $N_i(\varepsilon)$ giving the minima change when ε goes across them. On the other hand, for a fixed ε it is easy to check that $g_n(\varepsilon)$ grows geometrically for $n \neq N_i$:

$$g_n(\varepsilon) \sim \Omega^n \varepsilon^{-1/4} \quad \text{for } n < N_i, \tag{41}$$

$$g_n(\varepsilon) \sim \Omega^{-n} \varepsilon^{1/4} \quad \text{for } n > N_i \tag{42}$$

(the notation '$\sim$' was introduced just before Theorem 1).

For the sake of shortness, we also denote

$$S_i = S_i(\varepsilon) := s(1, N_i(\varepsilon)), \quad i = 1, 2, 3, 4,$$

the terms of the Fibonacci sequence indexed by the minimizing integers. As a consequence of Proposition 3 and the definition of N_i, one easily deduces the following estimate, to be used later:

$$|S_i| \sim \Omega^{-N_i} \sim \varepsilon^{-1/4}, \quad i = 1, 2, 3, 4. \tag{43}$$

The next lemma implies that the 3 most dominant harmonics of the Melnikov potential are the ones corresponding to S_1, S_2, S_3, giving an estimate for their coefficients L_{S_i}. Besides, the lemma provides an estimate for the sum of all the L_k (recall that they are all positive) except the l dominant ones ($0 \leq l \leq 3$), in terms of the first neglected harmonic. This can be considered a bound of the difference between the Melnikov potential and the main part of it, given by the dominant harmonics. In fact, since we are interested in some derivative of the Melnikov potential, we consider the sum of (positive) amounts of the type $|k|^m L_k$. Notice that the constant C_0 in the exponentials is the one defined in (24).

We recall that the notations '$\preceq$' and '$\sim$' were introduced just before Theorems 0 and 1 respectively.

Lemma 4 *Assuming $\varepsilon \preceq 1$ and $\mu \preceq \delta^{q_2} \varepsilon^{1/4}$, one has:*

(a) $L_{S_i} \sim \dfrac{1}{\varepsilon^{1/4}} \exp\left\{-\dfrac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}\right\}, \quad i = 1, 2, 3, 4.$

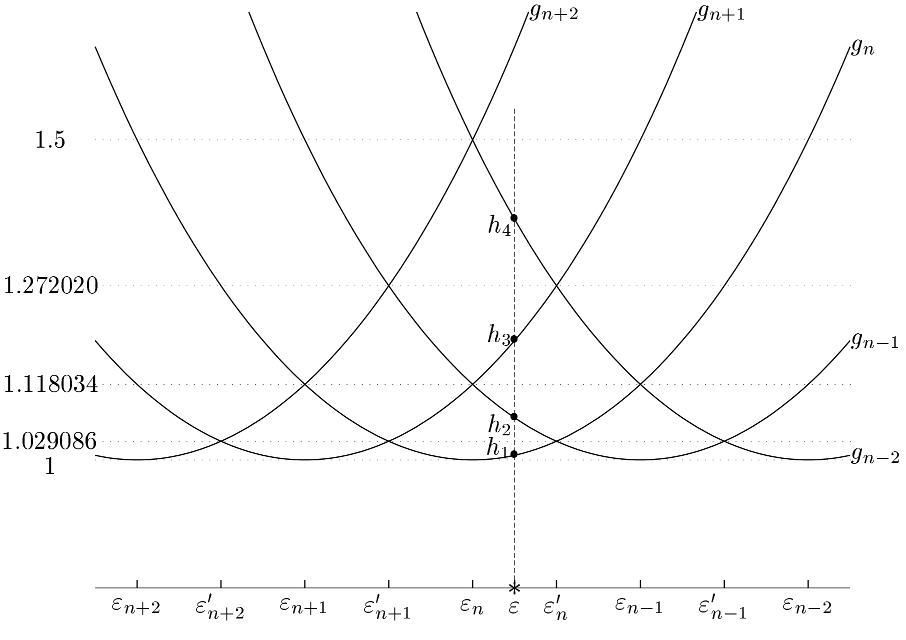


Figure 1: The functions $h_i(\varepsilon)$ (using logarithmic scale for ε)

(b) $\displaystyle\sum_{k\neq S_1,\dots,S_l} |k|^m L_k \sim \frac{1}{\varepsilon^{m/4}} L_{S_{l+1}}, \quad 0\le l\le 3, \quad m\ge 0.$

Proof. We need to show that the largest coefficients L_k are the ones given by $k = S_i$. According to (35–36), this corresponds essentially to find the smallest exponents β_k. To begin, we study the coefficients indexed by the Fibonacci sequence: $k = s(1,n)$. We deduce from Proposition 3, using $m(1) = 1$ and $d(1) = \Omega - 1$, that

$$|s(1,n)| = \frac{\Omega^{-(n+1)}}{2-\Omega} + \mathcal{O}\left(\Omega^{n+1}\right), \qquad \tilde{\gamma}_{s(1,n)} = 1 + \mathcal{O}\left(\Omega^{2n}\right). \tag{44}$$

Then, from (37) and the definition of ε_n we obtain the approximation

$$\beta_{s(1,n)} \simeq \frac{C_0}{2\varepsilon^{1/4}}\left(\frac{\Omega^{-(n+1)}\varepsilon^{1/4}}{(2-\Omega)D_0} + \frac{(2-\Omega)D_0}{\Omega^{-(n+1)}\varepsilon^{1/4}}\right) = \frac{C_0 g_n(\varepsilon)}{\varepsilon^{1/4}}, \tag{45}$$

and hence the functions g_n contain the main information on the size of the exponents $\beta_{s(1,n)}$.

Since the 4 smallest values of $g_n(\varepsilon)$, for ε small enough, are the ones obtained for $n = N_i(\varepsilon)$, and these smallest values coincide with the functions $h_i(\varepsilon)$, the smallest exponents among the $\beta_{s(1,n)}$ are

$$\beta_{S_i} \simeq \frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}, \qquad i = 1,2,3,4.$$

In fact, when we look at the true exponents β_{S_i}, we actually have a perturbation of this situation. Substituting (44) into (37) and using that $\Omega^{2N_i} \sim \sqrt{\varepsilon}$, we obtain:

$$\beta_{S_i} = \frac{C_0 h_i(\varepsilon) + \mathcal{O}\left(\sqrt{\varepsilon}, \mu\delta^{-q_2}\right)}{\varepsilon^{1/4}},$$

and the perturbative part can be neglected under the smallness conditions for ε and μ, since we have $\sqrt{\varepsilon} \preceq \varepsilon^{1/4}$ and $\mu\delta^{-q_2} \preceq \varepsilon^{1/4}$.

We have shown that the exponents β_{S_i} are the smallest among the exponents β_k for k in the Fibonacci sequence $s(1,\cdot)$, but it is also important to stress (and we use this fact below in this proof) that the exponents β_{S_i}, $i = 1,2,3,4$, are also the smaller than any non-Fibonacci exponent β_k. Indeed, if k belongs to another sequence $s(j,\cdot)$, we use the lower bound (38) and the fact that the limit of the $\sqrt{\tilde{\gamma}_k}$ is given by

$$\sqrt{\tilde{\gamma}_j^*} \ge 2 \gg h_4(\varepsilon), \qquad \forall j > 1. \tag{46}$$

On the other hand, for a non-admissible k, i.e. not belonging to any sequence $s(j,\cdot)$ (see Section 2), we have $\gamma_k > |k|/2$ and we always find in (36) that $\beta_k \succeq 1/\sqrt{\varepsilon}$ for this case.

Once we have found the smallest exponents β_k, we show that the dominance in the coefficients L_k is not affected by the multiplicative term α_k in (35–36). Indeed, we deduce from (36) that $\alpha_k \preceq \beta_k$ in general, and hence $|\ln \alpha_k| \ll \beta_k$. In fact, the only possible exception could occur if the denominator $[1 - \exp\{\cdots\}]$ in (36) is too small. But this only happens if $|k| \succeq \gamma_k/\sqrt{\varepsilon}$, and one then obtains $\alpha_k \preceq 1$. For the dominant coefficients L_{S_i}, we see that $[1 - \exp\{\cdots\}] \sim 1$ and easily find the estimate $\alpha_{S_i} \sim \varepsilon^{-1/4}$. This estimate, together with the one for β_{S_i}, implies part (a).

Let us prove the bound of part (b) for $l = 1$. To bound the sum of the coefficients L_k for $k \neq S_1$, we first consider the ones corresponding to Fibonacci harmonics: $k = s(1,n)$, with $n \neq N_1$, and divide them into the ones previous to S_1 and the ones after S_1. We call S_2' the last Fibonacci previous to S_1, and S_2'' the first Fibonacci after S_1; notice that S_2 is one of S_2', S_2''. For any $s(1,n)$ previous to S_1 (i.e. $n < N_1$), we use (45) and (41):

$$\beta_{s(1,n)} \sim \frac{\Omega^n}{\sqrt{\varepsilon}},$$

and we get that the corresponding coefficients $L_{s(1,n)}$ grow faster than geometrically, and the same can be said for the coefficients $|s(1,n)|^m L_{s(1,n)}$ of the derivative. Then, their sum can be (essentially) estimated by the last coefficient:

$$\sum_{n<N_1} |s(1,n)|^m L_{s(1,n)} \sim |S_2'|^m L_{S_2'} \sim \frac{1}{\varepsilon^{m/4}} L_{S_2'}. \tag{47}$$

For any $s(1,n)$ after S_1 (i.e. $n > N_1$), we proceed analogously, but using (42), and deduce that the corresponding coefficients $L_{s(1,n)}$ decrease faster than in a geometric series. As before, we obtain

$$\sum_{n>N_1} |s(1,n)|^m L_{s(1,n)} \sim |S_2''|^m L_{S_2''} \sim \frac{1}{\varepsilon^{m/4}} L_{S_2''}. \tag{48}$$

To bound the sum of the non-Fibonacci coefficients we may proceed in a similar way, dividing them into the cases $|k| \preceq \varepsilon^{-1/4}$ and $|k| \succeq \varepsilon^{-1/4}$. In the first case, we can use that the number of such terms is $\mathcal{O}(1/\sqrt{\varepsilon})$, together with the bound (38) and the inequality (46), obtaining an upper bound smaller than (47–48). In the second case, the same is true because those terms can be bounded by a geometric series.

Finally, we point out that the proof of (b) in the other cases $l \neq 1$ is analogous. The only difference is that one has to exclude from the sum some dominant Fibonacci harmonics, which are consecutive, instead of excluding only the most dominant harmonic. □

5 Dominant harmonics of the splitting potential

The estimates given in Lemma 4 for the dominant harmonics of the Melnikov potential $L(s,\theta)$ can be used as a first approximation to study the simple zeros of the Melnikov function $M = \partial_\theta L$, on a concrete section $s = \text{const}$. Now, we want to show that, assuming $\mu = \varepsilon^p$ for a suitable $p > 0$, the dominant harmonics remain essentially unchanged when one considers the whole splitting function $\mathcal{M}(s,\theta)$, including the error term (18).

Recalling that $\mathcal{M} = \partial_\theta \mathcal{L}$, and taking into account that the splitting potential $\mathcal{L}$ is $\hat\omega_\varepsilon$-quasiperiodic, we can write

$$\mathcal{L}(s,\theta) = \sum_{k\in\mathbb{Z}^2} \mathcal{L}_k^* e^{i\langle k,\theta-\hat\omega_\varepsilon s\rangle} = \sum_{k\in\mathcal{Z}} \mathcal{L}_k \cos\left(\langle k,\theta-\hat\omega_\varepsilon s\rangle - \tau_k\right), \tag{49}$$

where $\mathcal{L}_k$, τ_k are real, $\mathcal{L}_k \geq 0$ (recall that $\mathcal{Z}$ is defined in (8)). For every $k \in \mathcal{Z}$, the coefficients of the exponential form and the trigonometric form are related by $\mathcal{L}_k^* = \frac{1}{2}\mathcal{L}_k e^{-i\tau_k}$, $\mathcal{L}_{-k}^* = \overline{\mathcal{L}_k^*} = \frac{1}{2}\mathcal{L}_k e^{i\tau_k}$. The analogous coefficients for the splitting function $\mathcal{M}(s,\theta)$ are $\mathcal{M}_k^* = ik\mathcal{L}_k^*$ (in the exponential form). In the next lemma, we use the bound on the error term $\mathcal{R}(s,\theta)$, given in Theorem 0, to compare the coefficients of the splitting potential with those of the Melnikov potential. We also compare, for the dominant coefficients, the phases τ_k, σ_k of both potentials (this is affected by the translation s_0 that appears in (18)).

Lemma 5 *Assuming*

$$\varepsilon \preceq 1, \qquad \mu = \varepsilon^p \quad \textit{with } p > p^* := \frac{q_3 - 2}{4} \tag{50}$$

(we defined q_3 in Theorem 0), one has:

(a) $\mathcal{L}_{S_i} \sim \dfrac{\mu}{\varepsilon^{1/4}} \exp\left\{-\dfrac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}\right\}$,
$\left|\tau_{S_i} - \sigma_{S_i} - s_0\langle S_i, \hat\omega_\varepsilon\rangle\right| \preceq \dfrac{\mu}{\varepsilon^{p^*}}$, $\quad i = 1,2,3,4$.

(b) $\displaystyle\sum_{k\neq S_1,\ldots,S_l} |k|^m \mathcal{L}_k \sim \frac{1}{\varepsilon^{m/4}} \mathcal{L}_{S_{l+1}}, \quad 0 \leq l \leq 3, \quad m \geq 0.$

Proof. Assuming the smallness conditions (19) on μ, with δ to be chosen, we know from Theorem 0 that the splitting function $\mathcal{M}(s,\theta)$ can be defined on $\mathcal{P}_{\kappa,\pi/2-\delta,\rho-\delta}$, and we have the upper bound (20) for the error term $|\mathcal{R}|_{\kappa,\pi/2-\delta,\rho-\delta}$. Since $\mathcal{R}$ is $\hat\omega_\varepsilon$-quasiperiodic, applying to it a standard result (see, for instance, [DGS02, Lemma 11]) we get the following bound for its Fourier coefficients:

$$|\mathcal{R}_k^*| \preceq \left(\frac{\mu^2}{\delta^{q_3}} + \frac{\mu^2}{\delta^{q_4}\sqrt{\varepsilon}}\right) e^{-\tilde\beta_k}, \qquad \tilde\beta_k = (\rho-\delta)|k| + \frac{(\pi/2-\delta)b'\gamma_k}{b|k|\sqrt{\varepsilon}},$$

for any $k \neq 0$. Since $\mathcal{R}_k^* = ik\left(\mathcal{L}_k^* - \mu L_k^* e^{-is_0\langle k,\hat\omega_\varepsilon\rangle}\right)$, taking modulus and argument we get

$$|\mathcal{L}_k - \mu L_k| \preceq \frac{|\mathcal{R}_k^*|}{|k|}, \qquad |\tau_k - \sigma_k - s_0\langle k,\hat\omega_\varepsilon\rangle| \preceq \frac{|\mathcal{R}_k^*|}{|k|\,\mu L_k}.$$

Proceeding as in (37), but now with $\pi/2-\delta$ and $\rho-\delta$ instead of $\pi/2$ and ρ, we have:

$$\tilde\beta_k = \frac{C_{\mu,\delta}\sqrt{\tilde\gamma_k}}{2\varepsilon^{1/4}}\left(\frac{|k|\,\varepsilon^{1/4}}{D_{\mu,\delta}\sqrt{\tilde\gamma_k}} + \frac{D_{\mu,\delta}\sqrt{\tilde\gamma_k}}{|k|\,\varepsilon^{1/4}}\right),$$

now with $C_{\mu,\delta} = C_0 + \mathcal{O}(\mu\delta^{-q_2},\delta)$, $D_{\mu,\delta} = D_0 + \mathcal{O}(\mu\delta^{-q_2},\delta)$. Proceeding as in the proof of Lemma 4, we obtain

$$\tilde\beta_{S_i} = \frac{C_0 h_i(\varepsilon) + \mathcal{O}(\sqrt{\varepsilon},\mu\delta^{-q_2},\delta)}{\varepsilon^{1/4}},$$

and the perturbative part can be neglected if $\mu \preceq \delta^{q_2}\varepsilon^{1/4}$ and $\delta \preceq \varepsilon^{1/4}$. So we choose

$$\delta = \varepsilon^{1/4},$$

and the smallness conditions (19) can be rewritten as $\mu \preceq \varepsilon^{q_1/4}$. Note that the condition on μ containing the exponent q_2 in (19) can be ignored, since $q_1 \geq q_2 + 3$. Then, using also (43), we get

$$|\mathcal{L}_{S_i} - \mu L_{S_i}| \preceq \frac{\mu^2}{\varepsilon^{(q_3-1)/4}} \exp\left\{-\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}\right\}. \tag{51}$$

As before, the term containing the exponent q_4 can be ignored, since $q_3 \geq q_4 + 2$. The upper bound (51) is dominated by the term $|\mu L_{S_i}|$, estimated in Lemma 4, if one takes $\mu = \varepsilon^p$ with $p > (q_3-2)/4$. Since $q_3 - 2 \geq q_1$, it is enough to consider p^* as defined in (50), and this implies the first statement of (a). The second statement of (a) is proved in a similar way.

The remaining coefficients $|\mathcal{L}_k - \mu L_k|$, for $k \neq S_i$, can be bounded from above analogously to the coefficients L_k in the proof of Lemma 4, and this gives statement (b). □

Remarks.

1. Notice that the exponent p^* is the one announced in Section 1.3 (see Theorem 1): for $\nu = 0$ we have $q_3 = 14$ and hence $p^* = 3$, and for $\nu = 1$ we have $q_3 = 10$ and $p^* = 2$.
2. In fact, we should recall that the results of this paper rely on the application of the hyperbolic KAM theorem [DGS02, Th. 1] with $n = 2$ as the number of frequencies, and with $\tau = 1$ as the exponent in the Diophantine condition (11). To be strictly rigorous, the quoted theorem cannot be directly applied because the condition $\tau > n-1$ should be required. Nevertheless, it is clear that ω also satisfies (11) with any $\tau > 1$, and one easily sees that this does not affect the final restriction $p > p^*$ in (50).

6 Critical points of the splitting potential: transverse homoclinic orbits and their continuation

We are going to use in this section the estimates given in Lemma 5, to show that the splitting potential $\mathcal{L}(0,\theta)$ has nondegenerate critical points (fixing $s = 0$). First, we will study the critical points for the approximations given by the 2 or 3 most dominant harmonics:

$$\mathcal{L}^{(2)}(\theta) = \sum_{i=1,2} \mathcal{L}_{S_i}\cos\left(\langle S_i,\theta\rangle - \tau_{S_i}\right), \qquad \mathcal{L}^{(3)}(\theta) = \sum_{i=1,2,3} \mathcal{L}_{S_i}\cos\left(\langle S_i,\theta\rangle - \tau_{S_i}\right). \tag{52}$$

Afterwards, we discuss the validity of these critical points in the whole function $\mathcal{L}(0,\theta)$, by means of a quantitative version of the implicit function theorem, given in the Appendix. As the functions $h_i(\varepsilon)$ defined in (39) suggest, it is enough to consider the 2 dominant harmonics for most values of ε, but for ε close to a critical value ε_n we have to consider the 3 dominant harmonics.

To start, we study the function $\mathcal{L}^{(2)}$ for $\varepsilon \neq \varepsilon_n$, and the function $\mathcal{L}^{(3)}$ for $\varepsilon \neq \varepsilon'_n$. To fix ideas, we look at concrete intervals: we assume $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$ in the first case, and $\varepsilon \in \left(\varepsilon'_{n+1}, \varepsilon'_n\right)$ in the second case. Recalling Figure 1, note that

$$\begin{array}{llll} S_1 = s(1,n), & S_2 = s(1,n+1), & S_3 = s(1,n-1), & \text{for } \varepsilon \in \left(\varepsilon'_{n+1}, \varepsilon_n\right), \\ S_1 = s(1,n), & S_2 = s(1,n-1), & S_3 = s(1,n+1), & \text{for } \varepsilon \in (\varepsilon_n, \varepsilon'_n), \\ S_1 = s(1,n-1), & S_2 = s(1,n), & & \text{for } \varepsilon \in (\varepsilon'_n, \varepsilon_{n-1}). \end{array}$$

In order to have a simpler expression for the functions $\mathcal{L}^{(i)}(\theta)$, we carry out in both cases the linear change $(\theta_1, \theta_2) \mapsto (\psi_1, \psi_2)$ defined by

$$\psi_1 = \langle s(1,n-1), \theta \rangle - \tau_{s(1,n-1)}, \qquad \psi_2 = \langle s(1,n), \theta \rangle - \tau_{s(1,n)}, \tag{53}$$

which can be written as

$$\psi = \mathcal{A}_n \theta - b_n, \qquad \text{where } \mathcal{A}_n = \begin{pmatrix} s(1,n-1)^\top \\ s(1,n)^\top \end{pmatrix}, \; b_n = \begin{pmatrix} \tau_{s(1,n-1)} \\ \tau_{s(1,n)} \end{pmatrix}.$$

This change is one-to-one on $\mathbb{T}^2$, because $\det \mathcal{A}_n = (-1)^{n-1}$ as one may check by induction. With this change, the functions $\mathcal{L}^{(2)}(\theta)$, $\mathcal{L}^{(3)}(\theta)$ move respectively to the following ones:

$$\begin{aligned} \mathcal{K}^{(2)}(\psi) &= A \cos \psi_1 + B \cos \psi_2, \\ \mathcal{K}^{(3)}(\psi) &= B\eta(1-Q) \cos \psi_1 + B \cos \psi_2 + B\eta Q \cos(\psi_1 + \psi_2 - \triangle\tau), \end{aligned} \tag{54}$$

where we denote

$$\begin{aligned} & A = \mathcal{L}_{s(1,n-1)}, \quad B = \mathcal{L}_{s(1,n)}, \\ & \eta = \frac{\mathcal{L}_{s(1,n-1)} + \mathcal{L}_{s(1,n+1)}}{\mathcal{L}_{s(1,n)}}, \qquad Q = \frac{\mathcal{L}_{s(1,n+1)}}{\mathcal{L}_{s(1,n-1)} + \mathcal{L}_{s(1,n+1)}}, \\ & \triangle\tau = \tau_{s(1,n+1)} - \tau_{s(1,n)} - \tau_{s(1,n-1)} \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}. \end{aligned} \tag{55}$$

Note that A, B, η and Q are positive, because so are the coefficients $\mathcal{L}_k$ in (49). Looking at $\mathcal{K}^{(2)}$, we have $B = \mathcal{L}_{S_1}$, $A = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon_n, \varepsilon'_n)$, and $A = \mathcal{L}_{S_1}$, $B = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon'_n, \varepsilon_{n-1})$, i.e. the first and second dominant harmonics swap when ε goes across the value ε'_n.

Instead, when looking at $\mathcal{K}^{(3)}$ we have $B = \mathcal{L}_{S_1}$ for any $\varepsilon \in \left(\varepsilon'_{n+1}, \varepsilon'_n\right)$. So the first dominant harmonic of $\mathcal{K}^{(3)}$ is always $\cos \psi_2$, whereas the second and third ones swap when ε goes across ε_n. Note that η measures the size of the second and third harmonics with respect to the first one, and Q is an indicator of the relative weight of the second and third harmonics ($0 < Q < 1$). We study $\mathcal{K}^{(3)}$ in terms of η and Q, considering η as a perturbation parameter (note that $\eta \sim \mathcal{L}_{S_2}/\mathcal{L}_{S_1}$ is small except for ε close to the endpoints ε'_n, ε'_{n+1}). However, we have to point out that η and Q are not independent parameters, because they are both linked to ε.

In the next lemma, we show the existence of 4 critical points for η small enough and any Q, provided the difference of phases $\triangle\tau \in \mathbb{T}$ is not very close to 0 or π (mod 2π). To measure this closeness, we denote

$$\tau^* = \min(|\triangle\tau|, |\triangle\tau - \pi|).$$

Lemma 6

(a) *The function* $\mathcal{K}^{(2)}$ *has exactly 4 critical points, all nondegenerate:* $\psi^{(2)}_{(1)} = (0,0)$, $\psi^{(2)}_{(2)} = (0,\pi)$, $\psi^{(2)}_{(3)} = (\pi,0)$, $\psi^{(2)}_{(4)} = (\pi,\pi)$. *At the critical points,* $\left|\det \mathrm{D}^2\mathcal{K}^{(2)}\left(\psi^{(2)}_{(j)}\right)\right| = AB$.

(b) *Assume* $\tau^* > 0$ *and define* $E^{(\pm)}$, $\alpha^{(\pm)}$ *by*

$$\begin{aligned} E^{(\pm)} &= \sqrt{1 - 2Q(1-Q)(1 \mp \cos\triangle\tau)}, \\ \cos \alpha^{(\pm)} &= \frac{(1-Q) \pm Q \cos\triangle\tau}{E^{(\pm)}}, \quad \sin \alpha^{(\pm)} = \frac{\pm Q \sin\triangle\tau}{E^{(\pm)}}. \end{aligned}$$

Then, for any $Q \in [0,1]$ *and* $0 < \eta \preceq \tau^*$ *the function* $\mathcal{K}^{(3)}$ *has exactly 4 critical points, all nondegenerate:* $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, $j = 1,2,3,4$, *where* $\psi^{(3)}_{(1),0} = \left(\alpha^{(+)}, 0\right)$, $\psi^{(3)}_{(2),0} = \left(\alpha^{(-)}, \pi\right)$, $\psi^{(3)}_{(3),0} = \left(\alpha^{(+)} + \pi, 0\right)$, $\psi^{(3)}_{(4),0} =$

$\left(\alpha^{(-)}+\pi,\pi\right)$. *At the critical points,*

$$\left|\det \mathrm{D}^2\mathcal{K}^{(3)}\left(\psi^{(3)}_{(1,3)}\right)\right| = B^2\left(E^{(+)}\eta + \mathcal{O}\left(\eta^2\right)\right),$$
$$\left|\det \mathrm{D}^2\mathcal{K}^{(3)}\left(\psi^{(3)}_{(2,4)}\right)\right| = B^2\left(E^{(-)}\eta + \mathcal{O}\left(\eta^2\right)\right).$$

Proof. We do not prove part (a), because it is very simple. Instead, the proof of (b) requires some more work. The critical points of $\mathcal{K}^{(3)}$ are the solutions of the following system of equations:

$$\sin\psi_2 = \eta(1-Q)\sin\psi_1, \qquad (1-Q)\sin\psi_1 + Q\sin(\psi_1+\psi_2-\triangle\tau) = 0. \tag{56}$$

It is clear that, for η small enough, the solutions of the first equation of (56) are two curves in $\mathbb{T}^2$, defined by

$$\psi_2 = f_\eta(\psi_1) = \mathcal{O}(\eta), \qquad \psi_2 = \pi - f_\eta(\psi_1).$$

Replacing $\psi_2 = f_\eta(\psi_1)$ into the second equation of (56), one obtains the equation $F^{(+)}_\eta(\psi_1) = 0$, with

$$\begin{aligned} F^{(+)}_\eta(\psi_1) &= (1-Q)\sin\psi_1 + Q\sin(\psi_1-\triangle\tau) + \mathcal{O}(\eta) \\ &= E^{(+)}\sin\left(\psi_1-\alpha^{(+)}\right) + \mathcal{O}(\eta). \end{aligned}$$

For $\eta = 0$, the solutions are clearly $\alpha^{(+)}$ and $\alpha^{(+)}+\pi$, except for the case that $E^{(+)} = 0$ (which happens if $\triangle\tau = \pi$, $Q = 1/2$). Note that $E^{(+)} \geq \sqrt{(1+\cos\triangle\tau)/2} \succeq \tau^*$ and, consequently, these solutions persist for $\eta \preceq \tau^*$ (in fact, this is a simple consequence of Proposition 9). The perturbed solutions obtained give rise to the critical points $\psi^{(3)}_{(1)}$, $\psi^{(3)}_{(3)}$.

Analogously, one can replace $\psi_2 = \pi - f_\eta(\psi_1)$ into the second equation of (56), obtaining the equation $F^{(-)}_\eta(\psi_1) = 0$, with

$$\begin{aligned} F^{(-)}_\eta(\psi_1) &= (1-Q)\sin\psi_1 - Q\sin(\psi_1-\triangle\tau) + \mathcal{O}(\eta) \\ &= E^{(-)}\sin\left(\psi_1-\alpha^{(-)}\right) + \mathcal{O}(\eta), \end{aligned}$$

and for $\eta = 0$ the solutions are $\alpha^{(-)}$ and $\alpha^{(-)}+\pi$, except for the case that $E^{(-)} = 0$ (which happens if $\triangle\tau = 0$, $Q = 1/2$). In this case, $E^{(-)} \geq \sqrt{(1-\cos\triangle\tau)/2} \succeq \tau^*$ and, proceeding as before, one obtains the critical points $\psi^{(3)}_{(2)}$, $\psi^{(3)}_{(4)}$.

The determinant is easily computed. We have

$$\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi) = B^2\left(\eta\cos\psi_2\cdot((1-Q)\cos\psi_1 + Q\cos(\psi_1+\psi_2-\triangle\tau)) + \mathcal{O}\left(\eta^2\right)\right)$$

for any $\psi \in \mathbb{T}^2$. At the point $\psi^{(3)}_{(1)} = \left(\alpha^{(+)},0\right) + \mathcal{O}(\eta)$, we obtain

$$\det \mathrm{D}^2\mathcal{K}^{(3)}\left(\psi^{(3)}_{(1)}\right) = B^2\left(\eta\left(F^{(+)}_0\right)'\left(\alpha^{(+)}\right) + \mathcal{O}\left(\eta^2\right)\right) = B^2\left(E^{(+)}\eta + \mathcal{O}\left(\eta^2\right)\right),$$

and similarly with $\psi^{(3)}_{(2)}$, $\psi^{(3)}_{(3)}$, $\psi^{(3)}_{(4)}$. □

Remarks.

1. The amount τ^*, which measures the distance from $\triangle\tau$ to the "forbidden" values 0 and π (mod 2π), depends on n, as we see in (55). However, in Theorem 8 we will assume τ^* greater than a concrete positive constant (independent of n) by imposing a simple condition on the phases $\sigma_{s(1,n)}$ of the initial perturbation (33).

2. If $\triangle\tau$ is near to 0 or π, one of the determinants (given at first order by $E^{(\pm)}$) can be very small. Then, studying more carefully the term $\mathcal{O}(\eta)$ neglected from the equations one could show that, near $Q = 1/2$, bifurcations of some of the 4 critical points can take place, giving rise to 6 critical points. This happens, for instance, if the Hamiltonian considered in (33) is reversible, i.e. the function $f(\varphi)$ is even, because one then has $\sigma_k = 0$ for any k. Such bifurcations have been shown in [SV01], where the Hamiltonian considered is (a slight modification of) the classical Arnold's example.

3. It can be interesting to study the character of the critical points found. In both cases $i = 2, 3$, we see that $\mathcal{K}^{(i)}$ has a maximum at $\psi^{(i)}_{(1)}$, a miminum at $\psi^{(i)}_{(4)}$, and saddles at $\psi^{(i)}_{(2)}$, $\psi^{(i)}_{(3)}$.

We can give a description, illustrated in Figure 2, of the *continuation* of the critical points of $\mathcal{K}^{(3)}$ as Q goes from 0 to 1 (recall that this corresponds to transfer the second dominance from the harmonic $\cos\psi_1$ to the harmonic $\cos(\psi_1 + \psi_2 - \triangle\tau)$). Assuming $0 < \triangle\tau < \pi$, the point $\psi^{(3)}_{(1),0}$ drifts on a line from $(0, 0)$ to $(\triangle\tau, 0)$ with the first component increasing. On the other hand, the point $\psi^{(3)}_{(2),0}$ drifts on a line from $(0, \pi)$ to $(\triangle\tau + \pi, \pi)$ with the first component decreasing. Recall that $\psi^{(3)}_{(3,4),0} = \psi^{(3)}_{(1,2),0} + (\pi, 0)$. If one assumes $\pi < \triangle\tau < 2\pi$, the situation is analogous, but the points reverse their motion. When one considers the perturbed points $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, the lines become close curves. This picture does not hold for $\triangle\tau$ near to 0 or π (see remark 2), because in this case there are bifurcations of some of the critical points.

Next, we translate the results of Lemma 6 from the functions $\mathcal{K}^{(i)}(\psi)$ to the functions $\mathcal{L}^{(i)}(\theta)$, through the linear change (53), obtaining the critical points

$$\theta^{(i)}_{(j)} = \mathcal{A}_n^{-1}\left(\psi^{(i)}_{(j)} + b_n\right), \qquad j = 1, 2, 3, 4, \quad i = 2, 3. \tag{57}$$

We also find an estimate for the minimum eigenvalue (in modulus) of $\mathrm{D}^2\mathcal{L}^{(i)}$ at each point $\theta^{(i)}_{(j)}$; we denote $m^{(i)}_{(j)}$ this eigenvalue, closely related with the transversality of the homoclinic orbit associated to the critical point $\theta^{(i)}_{(j)}$.

Lemma 7

(a) *The function $\mathcal{L}^{(2)}$ has exactly 4 critical points $\theta^{(2)}_{(j)}$, given by (57), all nondegenerate and satisfying*

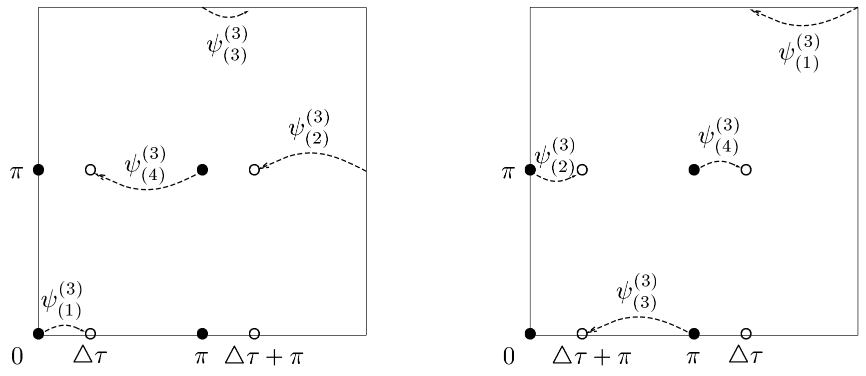
$$m^{(2)}_{(j)} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}, \qquad j = 1, 2, 3, 4.$$

(b) *Assuming $\tau^* > 0$ and $\mathcal{L}_{S_2} \preceq \tau^*\mathcal{L}_{S_1}$, the function $\mathcal{L}^{(3)}$ has exactly 4 critical points $\theta^{(3)}_{(j)}$, given by (57), all nondegenerate and satisfying*

$$\tau^*\sqrt{\varepsilon}\mathcal{L}_{S_2} \preceq m^{(3)}_{(j)} \preceq \sqrt{\varepsilon}\mathcal{L}_{S_2}, \qquad j = 1, 2, 3, 4.$$

Proof. For the minimum eigenvalue (in modulus) of $\mathrm{D}^2\mathcal{L}^{(2)}\left(\theta^{(2)}_{(j)}\right)$, we use the following expression:

$$m^{(2)}_{(j)} = \frac{2\,|D|}{|T| + \sqrt{T^2 - 4D}}, \tag{58}$$



$0 < \triangle\tau < \pi$ $\qquad$ $\pi < \triangle\tau < 2\pi$

Figure 2: Continuation of the critical points from $Q = 0$ to $Q = 1$

where we denote $D = \det \mathrm{D}^2\mathcal{L}^{(2)}\left(\theta^{(2)}_{(j)}\right)$ and $T = \operatorname{tr} \mathrm{D}^2\mathcal{L}^{(2)}\left(\theta^{(2)}_{(j)}\right)$. So we have to find estimates for D and T. It is clear that $\mathrm{D}^2\mathcal{L}^{(2)}\left(\theta^{(2)}_{(j)}\right) = \mathcal{A}_n^\top \mathrm{D}^2\mathcal{K}^{(2)}\left(\psi^{(2)}_{(j)}\right)\mathcal{A}_n$ and, since $|\det \mathcal{A}_n| = 1$, we obtain directly from Lemma 6 that $|D| = AB = \mathcal{L}_{S_1}\mathcal{L}_{S_2}$. On the other hand, writing $\mathrm{D}^2\mathcal{K}^{(2)}\left(\psi^{(2)}_{(j)}\right) = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ we see that

$$\begin{aligned} \mathrm{D}^2\mathcal{L}^{(2)}\left(\psi^{(2)}_{(j)}\right) &= k_{11}s(1,n-1)s(1,n-1)^\top \\ &\quad + k_{12}\left(s(1,n-1)s(1,n)^\top + s(1,n)s(1,n-1)^\top\right) \\ &\quad + k_{22}s(1,n)s(1,n)^\top \end{aligned}$$

and, using (30), we obtain

$$\begin{aligned} T &= k_{11}\left(f(1,n-2)^2 + f(1,n-1)^2\right) + 2k_{12}f(1,n-1)\left(f(1,n-2)+f(1,n)\right) \\ &\quad + k_{22}\left(f(1,n-1)^2 + f(1,n)^2\right). \end{aligned} \tag{59}$$

Using that $|k_{11}| = A$, $|k_{22}| = B$ and $k_{12} = 0$, and also that $\mathcal{L}_{S_1} = \max(A,B)$ and estimate (43), we deduce that

$$|T| \sim \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{S_1}. \tag{60}$$

Since $|D| \ll T^2$, we see from (58) that

$$m^{(2)}_{(j)} \sim \frac{|D|}{|T|} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

To estimate the minimum eigenvalue of $\mathrm{D}^2\mathcal{L}^{(3)}\left(\theta^{(3)}_{(j)}\right)$, we can proceed analogously. Applying Lemma 6, we obtain

$$|D| = B^2\left(E^{(\pm)}\eta + \mathcal{O}\left(\eta^2\right)\right) \sim \mathcal{L}^2_{S_1}E^{(\pm)}\eta \sim E^{(\pm)}\mathcal{L}_{S_1}\mathcal{L}_{S_2} \succeq \tau^*\mathcal{L}_{S_1}\mathcal{L}_{S_2},$$

under the additional condition

$$\eta \sim \frac{\mathcal{L}_{S_2}}{\mathcal{L}_{S_1}} \preceq \tau^*.$$

We have used that $B = \mathcal{L}_{S_1}$ and the fact that $\tau^* \preceq E^{(\pm)}$ (as seen in the proof of Lemma 6). On the other hand, to give an estimate for T we can use equality (59) with $|k_{22}| = B\left(1+\mathcal{O}(\eta)\right) \sim \mathcal{L}_{S_1}$ and $|k_{11}|, |k_{12}| \preceq B\eta \sim \mathcal{L}_{S_2}$, obtaining an estimate like (60) again and, consequently, the expected estimate for $m^{(3)}_{(j)}$. $\square$

Remark. It is also interesting to study, applying the linear change (53) to Figure 2, the continuation of the critical points $\theta^{(3)}_{(j)}$ of $\mathcal{L}^{(3)}$ as Q goes from 0 to 1. Recall that the critical points $\psi^{(3)}_{(j)}$ of $\mathcal{K}^{(3)}$ drift on curves close to the lines $\psi_2 = 0$, $\psi_2 = \pi$. Then, through the linear change (53) we see that the points $\theta^{(3)}_{(j)}$ drift on curves close to 2 lines $\langle s(1,n), \theta\rangle = \text{const}$. Note that their slope is $f(1,n-1)/f(1,n)$, which tends to Ω as $n \to \infty$ (i.e. as $\varepsilon \to 0$).

After having studied the critical points of the approximations $\mathcal{L}^{(2)}$, $\mathcal{L}^{(3)}$, the last step is to show their persistence in the whole splitting potential $\mathcal{L}$. To ensure the continuation of the critical points as $\varepsilon \to 0$, we assume in (33) that the difference of phases

$$\triangle\sigma_n := \sigma_{s(1,n+1)} - \sigma_{s(1,n)} - \sigma_{s(1,n-1)} \tag{61}$$

keeps far away from 0 or π (mod 2π) for any n. The next theorem is formulated in terms of the splitting function, $\mathcal{M}(0,\theta) = \partial_\theta\mathcal{L}(0,\theta)$, which gives a measure of the distance between the whiskers. The result for $\mathcal{M}(0,\theta)$ can be extended to $\mathcal{M}(s,\theta)$ in a natural way, because of the quasiperiodicity (15) of this function.

In this theorem, we establish the existence of 4 simple zeros of $\mathcal{M}(0,\theta)$, denoted $\theta_{(j)}$, $j = 1,2,3,4$, and we provide for these zeros an estimate for the minimum eigenvalue (in modulus) of the *splitting matrix* $\partial_\theta\mathcal{M}\left(0,\theta_{(j)}\right)$, which is clearly symmetric. As pointed out in [DG00b], this minimum eigenvalue provides a lower bound for the transversality of the homoclinic orbit associated to the zero $\theta_{(j)}$.

For the sake of completeness, we have also included a much simpler statement concerning the maximum size (in modulus) of the splitting function $\mathcal{M}(0,\theta)$, giving in this way an estimate for maximum splitting distance. Notice the difference in the exponents of the estimates of parts (a) and (b), illustrated by the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ in Figure 1.

We point out that a concrete example in which the condition that $\triangle\sigma_n$ keeps far away from 0 or π (mod 2π) is the one introduced in (7). Indeed, in this example all phases are $\sigma_k = \pi/2$, and therefore $\triangle\sigma_n = -\pi/2$ for any n. In fact, we gave in Section 1.3 a simplified version of the theorem for this concrete case.

Theorem 8 *In the example defined by (1–6) and (33), assume that the differences (61) satisfy the inequality*

$$\min(|\triangle\sigma_n|, |\triangle\sigma_n - \pi|) \geq \sigma^* > 0,$$

with σ^* independent of n. For any

$$\varepsilon \preceq (\sigma^*)^{1/(p-p^*)}, \qquad \mu = \varepsilon^p \quad \text{with } p > p^*, \tag{62}$$

one has:

(a) $\displaystyle\max_{\theta\in\mathbb{T}^2} |\mathcal{M}(0,\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}.$

(b) *The function $\mathcal{M}(0,\theta)$ has exactly 4 zeros $\theta_{(j)}$, all simple, and (the modulus of) the minimum eigenvalue of $\partial_\theta\mathcal{M}(0,\cdot)$ at each zero satisfies*

$$\sigma^*\mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\} \preceq m_{(j)} \preceq \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}, \qquad j = 1,2,3,4.$$

Proof. We first show that part (a) is a simple consequence of Lemma 5 and the fact that $\mathcal{M} = \partial_\theta\mathcal{L}$. It is enough to consider as in (52) the approximation $\mathcal{L}^{(2)}$ given by the 2 most dominant harmonics. We can easily give an estimate for $\partial_\theta\mathcal{L}^{(2)}$ by writing it in the variables ψ as in (54). Using also Lemma 5, we obtain

$$\left|\partial_\theta\mathcal{L}^{(2)}\right| \sim \frac{1}{\varepsilon^{1/4}}\mathcal{L}_{S_1}, \qquad \left|\partial_\theta\mathcal{L}(0,\cdot) - \partial_\theta\mathcal{L}^{(2)}\right| \preceq \frac{1}{\varepsilon^{1/4}}\mathcal{L}_{S_3}$$

(we are using here the notation $|\cdot|$ for the supremum norm of a function on the real domain $\mathbb{T}^2$), and this gives the expected estimate for $|\mathcal{M}(0,\cdot)|$. We stress that, if ε is not close to some ε'_n, it would have been enough to consider only the first dominant harmonic, and bound the sum of all other harmonics from the second one, but if ε is close to ε'_n we need to consider 2 harmonics.

To prove part (b), we will show that $\mathcal{L}$ has nondegenerate critical points by applying Proposition 9 (the implicit function theorem; see the Appendix) to a suitable approximation, $\mathcal{L}^{(2)}$ or $\mathcal{L}^{(3)}$, depending on the closeness of ε to the values ε_n, ε'_n. In some interval around ε'_n, we consider $\mathcal{L}^{(2)}$ and, near ε_n, we consider $\mathcal{L}^{(3)}$. Our aim is to show that these two intervals intersect and therefore no value of ε (small enough) is excluded.

First, for $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$ we apply Proposition 9 with $G_0(\theta) = \partial_\theta\mathcal{L}^{(2)}(\theta)$, $G(\theta) = \partial_\theta\mathcal{L}(0,\theta) = \mathcal{M}(0,\theta)$. For each zero $\theta^{(2)}_{(j)}$ of G_0, we have given in Lemma 7 an estimate for $m^{(2)}_{(j)}$, the minimum eigenvalue of $\mathrm{D}G_0\left(\theta^{(2)}_{(j)}\right)$. We also have, according to part (b) of Lemma 5, the bounds

$$|G - G_0| \preceq \frac{1}{\varepsilon^{1/4}}\mathcal{L}_{S_3}, \qquad |\mathrm{D}G - \mathrm{D}G_0| \preceq \frac{1}{\varepsilon^{1/2}}\mathcal{L}_{S_3}, \qquad \left|\mathrm{D}^2 G_0\right| \preceq \frac{1}{\varepsilon^{3/4}}\mathcal{L}_{S_1}.$$

Then, the zero $\theta^{(2)}_{(j)}$ of G_0 persists, giving rise to a zero $\theta_{(j)}$ of G, provided the following inequality is satisfied:

$$\mathcal{L}_{S_3} \preceq \frac{\varepsilon^2\mathcal{L}^2_{S_2}}{\mathcal{L}_{S_1}}. \tag{63}$$

Since $p > p^*$ in (62), we can use part (a) of Lemma 5, which provides an estimate for the coefficients $\mathcal{L}_{S_i}$. Taking logarithms, we see that (63) can be written, for some constant $c > 0$, as

$$2h_2(\varepsilon) \leq h_1(\varepsilon) + h_3(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\ln\left(c\varepsilon^2\right). \tag{64}$$

It is not hard to investigate how far this inequality is true. If we restrict ourselves to the subinterval $(\varepsilon_n, \varepsilon'_n)$, we need

$$2g_{n-1}(\varepsilon) \le g_n(\varepsilon) + g_{n+1}(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0} \ln\left(c\varepsilon^2\right).$$

Using the definitions of the functions g_n, we can first solve the equation $2g_{n-1}(\varepsilon) = g_n(\varepsilon) + g_{n+1}(\varepsilon)$ explicitly: its solution is $\varepsilon = \varepsilon_n/(3\Omega - 1)^2$. Note that, in the interval considered, the perturbative term is $\varepsilon^{1/4}\ln\left(c\varepsilon^2\right) \sim n\Omega^n$. So we have an estimate of the part of the subinterval $(\varepsilon_n, \varepsilon'_n)$ where (64) is true. Since the situation on the other subinterval $(\varepsilon'_n, \varepsilon_{n-1})$ is symmetric, we finally see that (64) is true for

$$\frac{\varepsilon_n}{a_n} \le \varepsilon \le \tilde{a}_n \varepsilon_{n-1}, \qquad a_n, \tilde{a}_n = (3\Omega - 1)^2 + \mathcal{O}\left(n\Omega^n\right) \simeq 0.729490. \tag{65}$$

Note that a_n, $\tilde{a}_n$ measure the part of the interval $(\varepsilon_n, \varepsilon_{n-1})$ where the approximation $\mathcal{L}^{(2)}$ is enough ($a_n = \tilde{a}_n = 1$ would mean the whole interval).

For $\varepsilon \in \left(\varepsilon'_{n+1}, \varepsilon'_n\right)$ we apply Proposition 9 again, but now with $G_0(\theta) = \partial_\theta \mathcal{L}^{(3)}(\theta)$. Calling $\triangle\tau_n = \tau_{s(1,n+1)} - \tau_{s(1,n)} - \tau_{s(1,n-1)}$, we get from part (a) of Lemma 5 that $|\triangle\tau_n - \triangle\sigma_n| \preceq \mu\varepsilon^{-p^*}$. Recalling that $\mu = \varepsilon^p$, we deduce from (62) that $\mu\varepsilon^{-p^*} \preceq \sigma^*$, and we get that

$$\min\left(|\triangle\tau_n|, |\triangle\tau_n - \pi|\right) \succeq \sigma^*,$$

which implies a lower estimate for $m^{(3)}_{(j)}$ from Lemma 7. Proceeding as before, we see that the zero $\theta^{(3)}_{(j)}$ persists if

$$\mathcal{L}_{S_4} \preceq \frac{\left(\sigma^*\varepsilon\right)^2 \mathcal{L}^2_{S_2}}{\mathcal{L}_{S_1}}.$$

This condition can be written as

$$2h_2(\varepsilon) \le h_1(\varepsilon) + h_4(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0} \ln\left(c\left(\sigma^*\varepsilon\right)^2\right),$$

which is true for any $\varepsilon \in \left(\varepsilon'_{n+1}, \varepsilon'_n\right)$, as we see from (40). However, in this case we have from Lemma 7 the additional condition $\mathcal{L}_{S_2} \preceq \sigma^* \mathcal{L}_{S_1}$, which can be written as

$$h_1(\varepsilon) \le h_2(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0} \ln\left(c\sigma^*\right).$$

We can find the part of the interval $\left(\varepsilon'_{n+1}, \varepsilon'_n\right)$ where this inequality is true, proceeding as with (64). In this case, the inequality is true in almost the whole interval:

$$\frac{\varepsilon'_{n+1}}{b_n} \le \varepsilon \le \tilde{b}_n \varepsilon'_n, \qquad b_n, \tilde{b}_n = 1 + \mathcal{O}\left(\Omega^n \ln\left(1/\sigma^*\right)\right) \simeq 1. \tag{66}$$

We see that any ε small enough, satisfying (62), belongs to at least one of the intervals (65–66), and hence the simple zeros of $\partial_\theta \mathcal{L}^{(2)}$ or $\partial_\theta \mathcal{L}^{(3)}$ persist in $\partial_\theta \mathcal{L} = \mathcal{M}$, for any ε. We also obtain from Lemma 7 the upper and lower bound for the minimum eigenvalue at each zero. □

Remarks.

1. We recall that the value of the exponent p^*, defined in (50), depends on whether the torus is moved by the perturbation or it remains fixed: $p^* = 3$ if $\nu = 0$, and $p^* = 2$ if $\nu = 1$. This is the exponent appearing in Theorem 1, a simplified version of the one proved above.
2. We have shown that the splitting function can be approximated by its 2 dominant harmonics except for ε close to some ε_n, according to (65). This says that the 4 zeros remain close to constant for ε in a large part of each interval $(\varepsilon_n, \varepsilon_{n-1})$. When ε goes across a critical value ε_n, a quick drift of the zeros takes place, until they again remain close to constant in the next interval.
3. It is interesting to compare the asymptotic estimate given in part (a) with the general upper bound for the splitting function given in [DGS02, Th. 12]. The constant C_0 is closely related to the constant appearing in that upper bound. Indeed, taking in the Diophantine condition $\tau = 1$ and γ as in (32), one has:

$$\left(1 + \frac{1}{\tau}\right)\left(\frac{\pi\tau\rho^\tau\gamma}{2}\right)^{1/(\tau+1)} = \sqrt{\frac{2\pi\rho}{2 - \Omega}} = C_0.$$

This shows the optimality of the constant C_0 in the upper bound of [DGS02, Th. 12]. In fact, in the asymptotic estimates given here this constant has been replaced by the function $C_0 h_1(\varepsilon) \geq C_0$, which provides a refinement of the estimate (recall that $h_1(\varepsilon) = 1$ only at the critical values ε_n).

Appendix: A quantitative implicit function theorem

In order to study the nondegenerate critical points of the splitting potential (i.e. the simple zeros of the splitting function), we use the approximation provided by a few dominant harmonics. Once we have shown that this approximation has nondegenerate critical points, we ensure that they persist when adding all the neglected harmonics. We can do this with the help of a quantitative implicit function theorem, which allows us to take into account the concrete size of the neglected part.

The following quantitative version of the implicit function theorem concerns the persistence of a simple zero of a function under a perturbation. We use $|\cdot|$ to denote the supremum norm of a function (on the real domain $\mathbb{T}^n$). We omit the proof of this result because it is standard.

Proposition 9 *Let $G_0, G : \mathbb{T}^n \longrightarrow \mathbb{R}^n$ functions of class $\mathcal{C}^2$, and $\theta_0 \in \mathbb{T}^n$ such that $G_0(\theta_0) = 0$, satisfying*

$$|\mathrm{D}G_0(\theta_0)h| \geq m\,|h|\,, \quad \forall h \in \mathbb{R}^n.$$

Let $M = \left|\mathrm{D}^2 G_0\right|$, $\eta = |G - G_0|$, $\eta' = |\mathrm{D}G - \mathrm{D}G_0|$, and assume that

$$\eta \leq \frac{m^2}{8M}, \qquad \eta' \leq \frac{m}{4}.$$

Then, the equation $G(\theta) = 0$ has a solution θ^* satisfying $|\theta^* - \theta_0| \leq 2\eta/m$, and this is the unique solution in the neighbourhood $|\theta - \theta_0| \leq m/4M$.

In our case, since the functions involved are gradients of potentials, the matrix $\mathrm{D}G_0(\theta_0)$ is symmetric and we can take m as the minimum eigenvalue (in modulus) of this matrix.

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