Resonances Width in Crossed Electric and Magnetic Fields

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Abstract

We study the spectral properties of a charged particle confined to a two-dimensional plane and submitted to homogeneous magnetic and electric fields and an impurity potential V. We use the method of complex translations to prove that the life-times of resonances induced by the presence of electric field are at least Gaussian long as the electric field tends to zero.

1 Introduction

The purpose of this paper is to study the dynamics of an electron in two dimensions in the presence of crossed magnetic and electric fields and a potential type perturbation. We assume that the magnetic field acts in the direction perpendicular to the electron plane with a constant intensity B and that the electric field of constant intensity F points in the $x-$direction. The perturbation $V(x,y)$ is supposed to satisfy certain localisation conditions. The corresponding quantum Hamiltonian reads as follows

$$H(F) = H(0) - Fx = H_L + V - Fx,$$

where H_L is the Landau Hamiltonian of an electron in a homogeneous magnetic field of intensity B. Its spectrum is given by the infinitely degenerate eigenvalues (Landau levels) $(2n+1)B$, $n \in \mathbb{N}$.

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When $F = 0$, the impurity potential V creates generically an infinite number of eigenvalues of $H(0)$ in between the Landau levels. These eigenvalues, which correspond to the so-called impurity states, then accumulate at Landau levels. This holds for any sign definite, bounded V, which tends to zero at infinity, see [Ra], [MR]. Classically, such impurity states represent the electron motion on localised trajectories. The main question that we address is what happens with these localised states when a constant electric field is switched on. In particular one would like to know, whether the eigenvalues of $H(0)$ may survive in the presence of a nonzero electric field and if not, what is the characteristic time in which they dissolve.

Answer to this question is well known for the hydrogen atom in a homogeneous electric field, in which case the corresponding Schrödinger operator has no eigenvalues, [Ti]. The localised states turn into so-called Stark resonances, whose life-times are exponentially long as $F \to 0$. This was first computed by Oppenheimer in [Op] and later rigorously proved in [HaSi]. The Oppenheimer formula was then partially generalised also for many body and non Coulombic potentials, see [Sig] and references therein.

On the other hand, results concerning systems with simultaneous constant magnetic and electric fields are scarce. Such a model is considered in [GM] where the impurity V is supposed to act as a $\delta-$potential. Using the special properties of a two-dimensional $\delta-$interaction, the authors of [GM] compute the spectral density of $H(F)$ in the neighbourhood of the discrete spectrum of $H(0)$ and prove that all impurity states are unstable. Their life-times are then shown to be of order $\exp[\frac{B}{F^2}]$ as $F \to 0$ and it is conjectured that such a behaviour holds in general. It is our motivation to extend this result for continuous impurity potentials when the method of [GM] is no longer applicable. In particular, we will prove under some assumptions on V that the life-times of magnetic Stark resonances are for F small enough at least Gaussian long, i.e. we find a lower bound compatible with the asymptotics obtained in [GM].

Let us now describe the content of our paper more in detail. Basic mathematical tool we use is the method of complex translations for Stark Hamiltonians, which was introduced in [AH] as a modification of the original theory of complex scaling [AC], [BC]. Following [AH] we consider the transformation $U(\theta)$, which acts as a translation in $x-$direction; $(U(\theta)\psi)(x) = \psi(x+\theta)$. For non real θ the translated operator $H(F,\theta) = U(\theta)H(F)U^{-1}(\theta)$ is non-selfadjoint and therefore can have some complex eigenvalues. The complex eigenvalues of $H(F,\theta)$ with $\Im\theta > 0$ are called the spectral resonances of $H(F)$, see e.g. [HS], and the corresponding resonance widths are given by their imaginary parts. Moreover, the result of [FK] tells us that if ϕ is an eigenfunction of $H(0)$, then $(\phi, e^{-itH(F)}\,\phi)$ decays exponentially at the rate given by the imaginary parts of the eigenvalues of $H(F,\theta)$.

In Section 5 we show that the eigenvalues of $H(F,\theta)$ are located in the Gaussian small vicinity of real axis as $F \to 0$, see Theorem 5.2. In order to prove this we employ a geometric resolvent equation in the form developed in [BG] for the study of Stark Wannier Ladders. The idea of our proof is based on the fact that the eigenfunctions of $H(0)$ have a Gaussian-like decay at infinity and therefore "feel" the electric field only locally. That leads us to a construction of the reference Hamiltonian $H_2(F)$, which describes the system where the electric field is localised in the vicinity of impurity potential V by a suitable cut-off function. For a precise definition of $H_2(F)$ see Section 3. When $F \to 0$ we let the cut-off function tend to 1 at the rate proportional to $F^{-1+\varepsilon}$ ($\varepsilon > 0$), which assures the convergence of spectra of $H_2(F)$ to that of $H(0)$. It follows from the general theory of complex deformations that the discrete spectrum of $H_2(F)$ is not affected by the transformation $U(\theta)$. Moreover, for $H_2(F)$ also the essential spectrum does not change under $U(\theta)$. Therefore $\sigma(H_2(F,\theta))$ remains real even when θ becomes complex. The geometric resolvent equation, (4.5), then allows us to deduce that for F small enough the resolvent $R(z;\theta) = (z - H(F,\theta))^{-1}$ is bounded except in a small neighbourhood of the eigenvalues of $H_2(F,\theta)$. More precisely, we show that the norm of $R(z;\theta)$ remains bounded as long as the distance between z and $\sigma(H_2(F,\theta))$ is at least of order

$$e^{-\frac{B\,C}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0, \tag{1.1}$$

where C is a strictly positive constant and ε can be taken arbitrarily small. Moreover, we prove that on the energy intervals well separated from Landau levels the spectral projector of $H(F,\theta)$ converges uniformly to that of $H_2(F,\theta)$ as $F \to 0$. These results give us the existence of eigenvalues of $H(F,\theta)$ and an upper bound on their imaginary parts. Let us note, that our result does not exclude the existence of point spectrum of $H(F)$. In other words, we do not answer the question whether all impurity states become unstable once the electric field with finite intensity is switched on. Although the quantum tunnelling phenomenon leads us to believe that it is indeed the case, a rigorous proof is missing and the question remains open.

2 The Model

We work in the system of units, where $m = 1/2$, $e = 1$, $\hbar = 1$. The crossed fields Hamiltonian is then given by

$$H_1(F) = H_L - Fx = (-i\partial_x + By)^2 - \partial_y^2 - Fx, \quad \text{on} \quad L^2(\mathbb{R}^2). \tag{2.1}$$

Here we use the Landau gauge with $\mathbf{A}(x, y) = (-By, 0)$. A straightforward application of [RS, Thm. X.37] shows that $H_1(F)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, see also [RS, Prob. X.38]. Moreover, one can easily check that

$$\sigma(H_1(F)) = \sigma_{ac}(H_1(F)) = \mathbb{R} \tag{2.2}$$

As mentioned in the Introduction we employ the translational analytic method developed in [AH]. We introduce the translated operator $H_1(F, \theta)$ as follows:

$$H_1(F, \theta) = U(\theta)H_1(F)U^{-1}(\theta) \tag{2.3}$$

where

$$\left(U(\theta)f\right)(x, y) := \left(e^{ip_x\theta} f\right)(x, y) = f(x + \theta, y) \tag{2.4}$$

An elementary calculation shows that

$$H_1(F, \theta) = H_1(F) - F\theta \tag{2.5}$$

Operator $H_1(F, \theta)$ is clearly analytic in θ. Following [AH] we define the class of $H_1(F)-$translation analytic potentials.

Definition 2.1. *Suppose that $V(z, y)$ is analytic in the strip $|\Im z| < \beta$, $\beta > 0$ independent of y. We then say that V is $H_1(F)-$translation analytic if $V(x+z, y)(H_1(F)+i)^{-1}$ is a compact analytic operator valued function of z in the given strip.*

We can thus formulate the conditions to be imposed on V:

(a) $V(x, y)$ is $H_1(F)-$translation analytic in the strip $|\Im z| < \beta$.

(b) There exists $\beta_0 \le \beta$ such that for $|\Im z| \le \beta_0$ the function $V(x + z, y)$ satisfies

$$|V(x+z, y)| \le \begin{cases} V_0 & \text{if } x \in [-a_0 - \Re z, a_0 - \Re z], y \in [-a_1, a_1] \\ V_0\, e^{-\nu\,(x+\Re z)^2},\ \nu > 0 & \text{if } x \notin [-a_0 - \Re z, a_0 - \Re z] \end{cases}$$

and

$$|V(x + z, y)| = 0, \quad y \notin [-a_1, a_1]$$

for given positive constants a_0, a_1, independent of F.

In order to characterise the potential class for which the above conditions are fulfilled let us assume for the moment, that the integral kernel of $(H_1(F) + i)^{-1}$ has at most a local logarithmic singularity at the origin. This is a very plausible hypothesis, see Lemma 4.3 in [FK], it then follows that any $L^2(\mathbb{R}^2)$ function that can be analytically continued in a given strip $|\Im z| < \beta$ satisfies the condition (a). If in addition the analytic continuation satisfy (b), both assumptions are satisfied.

Remark 2.1. *It follows from the proof of our main result, given below, that the localisation of V w.r.t. y could be replaced by a Gaussian decay. However, we use the assumption (b) in order to keep the computations as simple as possible. Note that this assumption is of crucial importance to get the Gaussian upper bound, in $1/F$, on the imaginary part of the eigenvalues of $H(F, ib)$. See in particular Remark A.2 in Appendix A.*

From the well known perturbation argument, [Ka], we see that under assumption (b)

$$H(F,\theta) = U(\theta)H(F)\,U^{-1}(\theta) = H_1(F,\theta) + V\,(x+\theta, y) \tag{2.6}$$

forms an analytic family of type A.

Furthermore, since $V(x+\theta, y)(H_1(F) + i)^{-1}$ is compact by (a), we have [RS, Cor. 2, p. 113]

$$\sigma_{ess}(H(F,\theta) + ibF) = \sigma_{ess}(H_1(F)) = \mathbb{R} \implies \sigma_{ess}(H(F,\theta)) = \mathbb{R} - ibF \tag{2.7}$$

where $\theta = ib$, $b \in \mathbb{R}$. By standard arguments [RS, Prob. XIII.76], all eigenvalues of $H(F, ib)$ lie in the strip $-bF < \Im z \leq 0$ and are independent of b as long as they are not covered by the essential spectrum.
The complex eigenvalues of $H(F,\theta)$ with $\Im\theta > 0$, in $\{z \in \mathbb{C} : -bF < \Im z < 0\}$ are called the spectral *resonances* of $H(F)$, and are intrinsic to $H(F)$, see [HS, Chap. 16]. The corresponding *resonance widths* are given by the imaginary parts of the eigenvalues E_α of $H(F,\theta)$: $\Gamma_\alpha = -2\Im E_\alpha$, and the *lifetimes* by $\tau_\alpha = \Gamma_\alpha^{-1}$.

Next we will show that, for sufficiently weak electric field F, the eigenvalues E_α of $H(F, ib)$ exist and are located in Gaussian small neighbourhood of real axis. In particular, we will prove that

$$|\Im E_\alpha| \leq e^{-\frac{B\tilde{R}_\alpha}{F^{2(1-\varepsilon)}}}$$

where the positive constant $\tilde{R}_\alpha$ depends on the real part of E_α and ε can be made arbitrarily small. The method we employ is based on the decoupling formula developed in [BG], see also [FM].

3 Auxiliary Hamiltonian

The reference Hamiltonian reads

$$H_2(F) = H_L + V - Fxh_F(x)\chi_A(y) \equiv H_L + V + W_F$$

with χ_A being characteristic function of the set $A = [-\bar{y}, \bar{y}]$ ($\bar{y} = y_1 + \frac{1}{F^\tau}$, with y_1 and τ defined in Section 4 below) and

$$h_F(x) = \tfrac{1}{2}\left\{\tanh(\gamma_F(x+\bar{x})) - \tanh(\gamma_F(x-\bar{x}))\right\}$$

where[1] $\gamma_F = \frac{\gamma_0}{F^{1-\varepsilon}} > 0$ and $\bar{x} > 0$ must satisfy

$$F\bar{x} \to 0 \quad \text{as} \quad F \to 0\ . \tag{3.1}$$

This is required because we don't want the local electric field to modify significatively the impurity potential V. We can thus expect that the spectrum of $H_2(F)$ is "close" to that of $H(0)$. We will chose $\bar{x} = \frac{\bar{C}}{F^{1-\varepsilon}} > 0$, for $\varepsilon > 0$.
In Figure 1 we sketch the $x-$section of $V(x,y) - xh_F(x)\chi_A(y)$ for the case of impurity potential given by $V(x,y) = -V_0 e^{-x^2} f(y)$ (f being any locally supported positive bounded function).

Before giving the results on the spectral properties of $H_2(F)$ and its translated correspondent $H_2(F, ib)$ we define the set of $\theta = ib$ for which W_F can be analytically continued in the x variable. Since $\tanh(z)$ has an analytic continuation for $|\Im z| < \frac{\pi}{2}$ we have $\gamma_F|b| < \frac{\pi}{2}$. For our purpose we will consider the family of operator $U(\theta) \equiv U(ib)$ defined in Section 1, with $\theta \in \mathcal{D}_\theta$ where

$$\mathcal{D}_\theta = \{\theta \in \mathbb{C} : \gamma_F|\Im\theta| < \tfrac{\pi}{4}\}$$

Since $\gamma_F = \frac{\gamma_0}{F^{1-\varepsilon}}$ we take

$$b = b_0 F^\alpha, \alpha > 2 \tag{3.2}$$

Proposition 3.1. *Assume V satisfies (a) and (b). Then*

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[1] We will often drop the subscript F.

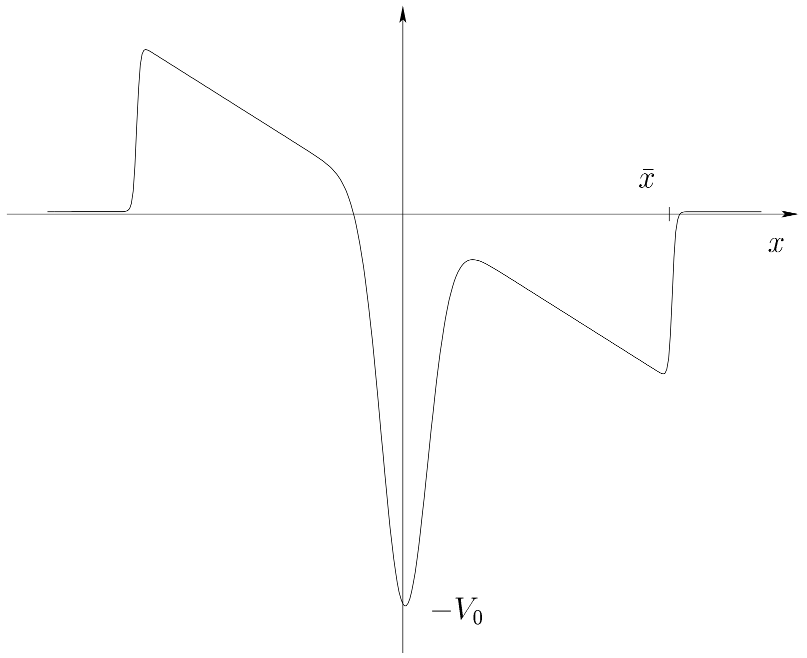

Figure 1: *The* $x-$*section for the potential of* $H_2(F)$ *satisfying condition* (3.1) *for a negative Gaussian potential.*

1. *For each* $e_\alpha \in \sigma(H(0))$ *there is a family of* $\lambda_\alpha(F) \in \sigma(H_2(F))$ *such that* $\lambda_\alpha(F) \to e_\alpha$ *for* $F \to 0$.
2. *Let* $P_\Delta(F)$ *respectively* $P_\Delta(0)$ *be the eigenprojector of* $H_2(F)$ *respectively* $H(0)$ *on the interval* Δ. *Then* $\|P_\Delta(F) - P_\Delta(0)\| \to 0$ *as* $F \to 0$.
3. $\sigma_{ess}(H_2(F)) = \sigma_{ess}(H_L) = \{(2n+1)B; n \in \mathbb{N}\}$
4. *For each* $e_\alpha \in \sigma_d(H(0))$ *there exists a constant* c *such that*

$$\lambda_\alpha(F) \in [e_\alpha - cF^\varepsilon, e_\alpha + cF^\varepsilon]$$

Proof. We have

$$\begin{aligned} \|(H(0)-z)^{-1} - (H_2-z)^{-1}\| &= \|(H_2-z)^{-1}[H_2 - H(0)](H(0)-z)^{-1}\| \\ &\leq \|(H_2-z)^{-1}\|\|(H_2-H(0))\|\|(H(0)-z)^{-1}\| \\ &\leq \frac{1}{|\Im z|^2}\|Fxh_F(x)\chi_A(y)\| \to 0 \qquad (3.3) \end{aligned}$$

as $F \to 0$ due to the choice of h_F. Thus $H_2(F) \to H(0)$ in the norm resolvent sense. The Statement 1. and 2. of the Lemma now follows from [Ka, Thm. VIII.1.14] and [RS, Thm. VIII.23]. Statement 3. follows from the fact that W_F and V are H_L-compact, see proof of Lemma 3.1 below. Finally the estimate

$$\|Fxh_F(x)\chi_A(y)\| \leq F\|xh_F(x)\|_\infty \leq cF^\varepsilon \tag{3.4}$$

yields Statement 4. $\square$

We now show that the spectrum of $H_2(F)$ is not affected by the transformation $U(ib)$:

Lemma 3.1. *Under the assumptions of Proposition 3.1 $\{H_2(F,\theta) : \theta \in \mathcal{D}_\theta\}$ forms a self-adjoint holomorphic family of type A. Moreover, for each $ib \in \mathcal{D}_\theta$ one has*

$$\begin{aligned} \sigma_{ess}(H_2(F,ib)) &= \sigma_{ess}(H_2(F)) \\ \sigma_d(H_2(F,ib)) &= \sigma_d(H_2(F)) \end{aligned}$$

Proof. To prove that $\{H_2(F,\theta) : \theta \in \mathcal{D}_\theta\}$ forms a self-adjoint holomorphic family we have show that $H_2(F,\theta)$ is holomorphic w.r.t. $\theta \in \mathcal{D}_\theta$ and that its domain is independent of θ, see [Ka, pp. 375, 385]. First claim follows from the assumptions on V and from the explicit form of W_F. The boundedness of V, W_F then implies the $\theta-$independence of the domain. For the the stability of essential spectrum we recall [HS, Thm. 18.8], which tells us that it is enough to prove that $W_F(x+ib,y)(H_L+i)^{-1}$ and $V(x+ib,y)(H_L+i)^{-1}$ are compact. We first observe that

$$h_F(x+ib) = \frac{e^{2\gamma_F\bar{x}} - e^{-2\gamma_F\bar{x}}}{e^{2\gamma_F\bar{x}} + e^{-2\gamma_F\bar{x}} + e^{2\gamma_F(x+ib)} + e^{-2\gamma_F(x+ib)}}.$$

Thus

$$|h_F(x+ib)| \leq \frac{e^{2\gamma_F\bar{x}}}{[e^{2\gamma_F x} + e^{-2\gamma_F x}]\cos(2\gamma_F b) + [e^{2\gamma_F\bar{x}} + e^{-2\gamma_F\bar{x}}]}$$

From the latter estimate we deduce that $\lim_{x\to\pm\infty} |W_F(x+ib,y)| = 0$ and that $|W_F(x+ib,y)|$ is uniformly bounded. Since χ_A has compact support, $W_F(ib) \in L^2(\mathbb{R}^2)$. Then

$$\begin{aligned} \|W_F(ib)(H_L+i)^{-1}\|^2_{HS} &= \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}|W_F(x+ib,y)|^2 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}'|G_L(\mathbf{x},\mathbf{x}';i)|^2 \\ &= \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}|W_F(x+ib,y)|^2 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{u}|G_L(\mathbf{u};i)|^2 < \infty \end{aligned} \tag{3.5}$$

where $|G_L(\mathbf{x},\mathbf{x}';i)| = |G_L(\mathbf{x}-\mathbf{x}';i)| = |G_L(\mathbf{u};i)| \in L^2(\mathbb{R}^2)$ is the integral kernel of $(H_L+i)^{-1}$, see for example [CN]. Hence $W_F(ib)(H_L+i)^{-1}$ is compact. Same argument shows that also $V(ib)(H_L+i)^{-1}$ is compact.

Finally the stability of the discrete spectrum follows from a standard analyticity argument [RS, Prob. XIII.76]. $\square$

We now give a result on the norm of $R_2(z; ib)$, which will be used later in the proof of our main theorem. [2]

Lemma 3.2. *Let* $z \in \mathbb{C}$ *such that* $(2q-1)B+\delta < \Re z < (2q+1)B-\delta$ $(\delta > 0)$ *for some* $q \in \mathbb{N}$. *Then there exists a natural number* $0 < s < \infty$, *such that*

$$\|R_2(z; ib)\| \le \mathcal{C}\,|\Im z|^{-s},$$

holds true provided F *is small enough.*

Proof. We introduce the operator $A(ib)$ by

$$A(ib) = H_2(ib) - H_2 \tag{3.6}$$

(here we note $H_2(ib) \equiv H_2(F, ib)$ and $H_2 \equiv H_2(F)$). ¿From the definition of $H_2(ib)$ it easily follows that there exists certain constant A_0 such that for $b = b_0F^{\alpha}$

$$\|A(ib)\| \le A_0F^{\alpha-1+\varepsilon}(1+\mathcal{O}(F^{\alpha}))$$

We need a preliminary result. A standard perturbation argument now shows that if

$$\operatorname{dist}(\sigma(H_2(F)), \xi) = d_0F^{\varepsilon}$$

then

$$\|R_2(\xi; ib)\| \le \frac{\|R_2(\xi; 0)\|}{1-\|A(ib)R_2(\xi; 0)\|} = F^{-\varepsilon}\frac{1}{d_0 - F^{\alpha-1}A_0} \tag{3.7}$$

whenever $d_0 > F^{\alpha-1}A_0$, i.e. whenever F is small enough. To continue let e_α be the eigenvalue of $H(0)$ which minimises $|z-(e_\alpha \pm cF^{\varepsilon})|$. We define a circle $\tilde{\Gamma} \equiv \{\xi \in \mathbb{C} : |\xi - e_\alpha| = \Gamma_0F^{\varepsilon}\}$ enclosing only the eigenvalues of $H_2(F)$ converging to e_α for given e_α. Let $P_2^{\tilde{\Gamma}}(ib)$ the projector onto $\overline{\operatorname{Int}\tilde{\Gamma}}$ associated to $H_2(ib)$

$$P_2^{\tilde{\Gamma}}(ib) \equiv P_2(ib) = \frac{1}{2\pi i}\oint_{\tilde{\Gamma}} R_2(z; ib)\,\mathrm{d}z$$

Since $P_2(ib)$ is a projector, applying [Ka, Thm.III.6.17], the resolvent of $H_2(ib)$ decomposes as follows

$$R_2(z; ib) = R_2'(z; ib) + R_2''(z; ib)$$

where

$$\begin{aligned} R_2'(z; ib) &= P_2(ib)R_2'(z; ib) = R_2'(z; ib)P_2(ib) && (3.8)\\ R_2''(z; ib) &= [1-P_2(ib)]R_2'(z; ib) = R_2'(z; ib)[1-P_2(ib)] && (3.9) \end{aligned}$$

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[2] Henceforth the symbol $\mathcal{C}$ denotes a strictly positive real number independent of F.

Let H' be the restriction of $H_2(ib)$ on $M' \equiv \operatorname{Ran} P_2(ib)$ and H'' the restriction of $H_2(ib)$ on $M'' \equiv \operatorname{Ran}[1 - P_2(ib)]$. From [Ka, Thm.III.6.17] it follows that $R_2'(z; ib)$ coincides with $(z - H')^{-1}$ on M' and vanishes on M''. Similarly $R_2''(z; ib)$ coincides with $(z - H'')^{-1}$ on M'' and vanishes on M'. Since $\operatorname{dist}(\sigma(H''), z)$ is bounded from below by a constant we can use (3.7) to get

$$\|R_2''(z; ib)\| \leq \mathcal{C}$$

Let us denote $r_0 = \dim P_2(ib)$. We can then write

$$R_2'(z; ib) = \sum_{h=1}^{r_0} \left[(z - \zeta_h)^{-1} P_h + (z - \zeta_h)^{-1} \sum_{n=1}^{m_h - 1} (z - \zeta_h)^{-n} D_h^n \right]$$

where $\zeta_h \equiv \lambda_{\alpha,h} \in \mathbb{R}$ are the eigenvalues of H', P_h the corresponding projectors, $m_h = \dim P_h$ and D_h denotes the nilpotent associated to ζ_h, see [Ka, Chap.I]. So we can always find some $s \in \mathbb{N}$ $(1 \leq s \leq \max_h m_h \leq r_0)$, such that

$$\|R_2'(z; ib)\| \leq \mathcal{C} \operatorname{dist}(z, \sigma(H'))^{-s} \leq \mathcal{C}\, |\Im z|^{-s},$$

which concludes the proof. □

4 Setup of a decoupling scheme

As already mentioned in the Introduction, the eigenfunctions of $H(0)$ "feel" the electric field only locally and the properties of the Hamiltonian $H(F)$ can be derived on the basis of those of the "local field" Hamiltonian $H_2(F)$ described above. To make this idea work we use the geometric resolvent perturbation theory in the form developed in [BG] (see also [BCD], [HS]). It consists of dividing the configuration space $\mathbb{R}^2$ in different regions and study of Hamiltonians H_i with associated potentials V_i which are in the considered regions close to that of the full Hamiltonian $H(F)$.

We introduce the following functions that give a decoupling along the $x-$axis.

$$\begin{aligned}
J_-(x) &= \tfrac{1}{2}\left[1 + \tanh(\gamma_F(x - x_2))\right] \\
\tilde{J}_-(x) &= \tfrac{1}{2}\left[1 + \tanh(\gamma_F(x - x_0))\right] \\
J_0(x) &= \tfrac{1}{2}\left[\tanh(\gamma_F(x + x_1)) - \tanh(\gamma_F(x - x_1))\right] \\
\tilde{J}_0(x) &= \tfrac{1}{2}\left[\tanh(\gamma_F(x + x_0)) - \tanh(\gamma_F(x - x_0))\right] \\
J_+(x) &= \tfrac{1}{2}\left[1 - \tanh(\gamma_F(x + x_2))\right] \\
\tilde{J}_+(x) &= \tfrac{1}{2}\left[1 - \tanh(\gamma_F(x + x_0))\right]
\end{aligned} \tag{4.1}$$

where $0 < x_2 = \frac{C_2}{F^{1-\varepsilon}} < x_0 = \frac{C_0}{F^{1-\varepsilon}} < x_1 = \frac{C_1}{F^{1-\varepsilon}} < \bar{x}$. Along the $y-$axis we use three bounded $C^\infty(\mathbb{R})$ functions

$$J_<(y) = \begin{cases} 1 & \text{if } y \le -y_0 + \frac{1}{F^\tau} \\ 0 & \text{if } y \ge -y_2 \end{cases} \qquad J_c(y) = \begin{cases} 1 & \text{if } |y| \le y_0 + \frac{1}{F^\tau} \\ 0 & \text{if } |y| \ge y_1 \end{cases}$$

$$J_>(y) = \begin{cases} 1 & \text{if } y \ge y_0 - \frac{1}{F^\tau} \\ 0 & \text{if } y \le y_2 \end{cases} \tag{4.2}$$

where $0 < y_2 = a_1 + 1$, $y_0 = y_2 + \frac{1}{F^\tau} + 1$, $y_1 = y_0 + \frac{1}{F^\tau} + 1$, where $\tau > \alpha + 2$. We will also assume that $\|J_i'\|_\infty$, $\|J_i''\|_\infty < \infty$, $i \in \{<, >, c\}$.

Note that for the $x-$cut the dependence on F of x_0, x_1, x_2 is the optimal choice to get the desired results, while in the $y-$cut the dependence on F, i.e. the factor $F^{-\tau}$, is such that τ can be chosen as large as we need.

The system is then cut in five parts according to the following "full" decoupling functions (see Figure 2):

$$\begin{cases} J_1(x,y) &= J_-(x) J_c(y) \\ \tilde{J}_1(x,y) &= \tilde{J}_-(x) \tilde{J}_c(y) \end{cases} \qquad \begin{cases} J_2(x,y) &= J_0(x) J_c(y) \\ \tilde{J}_2(x,y) &= \tilde{J}_0(x) \tilde{J}_c(y) \end{cases}$$

$$\begin{cases} J_3(x,y) &= J_>(y) \\ \tilde{J}_3(x,y) &= \tilde{J}_>(y) \end{cases} \qquad \begin{cases} J_4(x,y) &= J_<(y) \\ \tilde{J}_4(x,y) &= \tilde{J}_<(y) \end{cases} \qquad \begin{cases} J_5(x,y) &= J_+(x) J_c(y) \\ \tilde{J}_5(x,y) &= \tilde{J}_+(x) \tilde{J}_c(y) \end{cases}$$

with

$$\tilde{J}_<(y) = \chi_{(-\infty,-y_0]}(y), \quad \tilde{J}_c(y) = \chi_{[-y_0,y_0]}(y), \quad \tilde{J}_>(y) = \chi_{[y_0,\infty)}(y)$$

We remark that all these functions have an analytic continuation in the x variable ($x \to x + ib$) if $ib \in \mathcal{D}_\theta$.

We are now ready to establish the decoupling scheme. We introduce the following auxiliary Hamiltonians: $H_3 = H_4 = H_5 = H_1 = H_L - Fx$ and $H_2(F) \equiv H_2$ treated in the previous paragraph. For simplicity we write H for $H(F)$.

Note that

$$HJ_1 = H_1 J_1 + V J_1, \quad HJ_5 = H_5 J_5 + V J_5, \quad HJ_3 = H_3 J_3, \quad HJ_4 = H_4 J_4$$

and, using $\chi_A(y) J_c(y) = J_c(y)$,

$$HJ_2 = H_2 J_2 - Fx(1 - h_F)(x) J_2$$

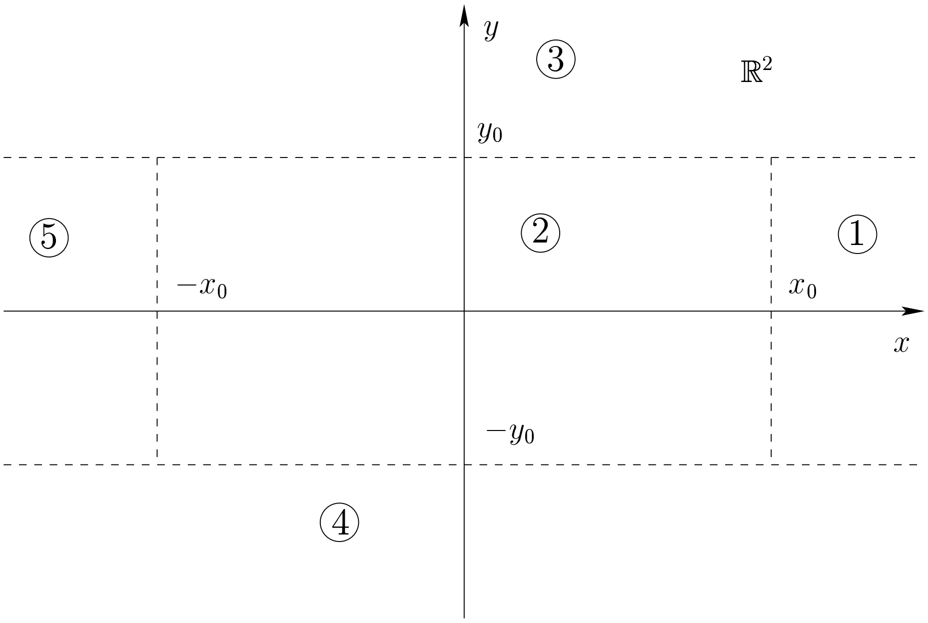

Figure 2: *Schematic representation of the decoupling scheme. In region 2 the total potential* $V(x,y)-Fx$ *is close to the local potential of the auxiliary Hamiltonian* $H_2(F)$*, while in the others it is close to the electric potential* $-Fx$*.*

thus

$$(z-H)\sum_{i=1}^{5} J_i R_i(z)\tilde{J}_i = \sum_{i=1}^{5}(z-H_i)J_i R_i(z)\tilde{J}_i + A_1 + A_5 + A_2 = 1 - K(z) \tag{4.3}$$

where $A_1 = VJ_1R_1(z)\tilde{J}_1$, $A_5 = VJ_5R_5(z)\tilde{J}_5$, $A_2 = -Fx(1-h_F)(x)J_2R_2(z)\tilde{J}_2$ and

$$K(z) = \sum_{i=1}^{5}[H_L, J_i]R_i(z)\tilde{J}_i + \left(\sum_{i=1}^{5} J_i\tilde{J}_i - 1\right) - A_1 - A_5 - A_2$$

From (4.3) we deduce the decoupling formula

$$R(z) = \left(\sum_{i=1}^{5} J_i R_i(z)\tilde{J}_i\right)(1-K(z))^{-1}\ . \tag{4.4}$$

which is now to be transformed by the translation group $U(ib)$:

$$R(z;ib) = \left(\sum_{i=1}^{5} J_i(ib)R_i(z;ib)\tilde{J}_i(ib)\right)(1-K(z;ib))^{-1} \tag{4.5}$$

To prove that the eigenvalues of $H(F,ib)$ are at distance $\mathcal{O}\left(\exp\left(-1/F^{2(1-\varepsilon)}\right)\right)$ from those of $H_2(F,ib)$, we have to show that the norm of $K(z;ib)$ becomes smaller than 1 as $\operatorname{dist}(\sigma(H_2(F)), z)$ becomes Gaussian small. We will write $K(z; ib)$ as

$$K(z; ib) = \sum_{j=1}^{5} K_j(z; ib) + M(z; ib) \tag{4.6}$$

where

$$K_j(z; ib) = [H_L, J_j(ib)] R_j(z; ib) \tilde{J}_j(ib)$$

and

$$M(z; ib) = \left(\sum_{j=1}^{5} J_j(ib) \tilde{J}_j(ib) - 1 \right) - A_1(ib) - A_5(ib) - A_2(ib)$$

In Appendix A we estimate the norm of each term in the definition of $K(z; ib)$ separately. Our strategy is the following. Each of $K_j(z; ib)$ can be viewed as an integral operator with the corresponding kernel of the form $f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}'; z) h(\mathbf{x}')$, where $G(\mathbf{x}, \mathbf{x}'; z)$ is the Green function of H_1. Typically, the overlap of the functions $f(x)$ and $h(x')$ decreases as $F \to 0$. Fact, which together with the Gaussian decay of $G(\mathbf{x}, \mathbf{x}'; z)$ at large distances, see Appendix A, assures that the norm of each of $K_j(z; ib)$ will tend to zero in the limit $F \to 0$. As for the operator $M(z; ib)$, we will see that for small values of F its norm can be made arbitrarily small by a proper choice of the parameters of the decoupling functions.

The results of Appendix A yield the following estimate on the norm of $K(z; ib)$

$$\begin{aligned} \|K(z; ib)\| \;&\leq\; \mathcal{C}\, F^{-\mathcal{C}}\, \beta(z)^{-\sigma(\Re z)} \left(e^{-\frac{\beta(z)}{F^{\tau}}} + e^{-B \frac{\mathcal{C}'(B, \Re z)}{F^{2(1-\varepsilon)}}} \right) (1 + \|R_2(z; ib)\|) \\ &+\; \mathcal{C} e^{-\frac{\tilde{\mathcal{C}}}{F^{2(1-\varepsilon)}}} (\|R_1(z; ib)\| + \|R_2(z; ib)\| + 1) \end{aligned} \tag{4.7}$$

with $\mathcal{C}'(B, \Re z) = Bc(\Re z) \to 0$ as $\Re z \to \infty$, $\tilde{\mathcal{C}}$ depending on the decoupling scheme (in particular we can set $\tilde{\mathcal{C}} = B\tilde{c}$), $\beta(z) = \frac{\Im z + bF}{2F}$ and $\sigma(\Re z) \geq 1$ ($\sigma(\Re z) \to \infty$ as $\Re z \to \infty$). We remark that for $F < 1$ we have $\beta(z) \leq \operatorname{dist}(\sigma(H_1(ib)), z)$. Using the inequality

$$\|R_1(z; ib)\| \leq \frac{1}{\operatorname{dist}(z, \overline{\Theta(H_1(ib))})} = \frac{1}{\operatorname{dist}(z, \mathbb{R} - ibF)}, \tag{4.8}$$

where $\Theta(H_1(ib))$ is the numerical range of $H_1(ib)$, see [HS, Prop. 19.7], we can rewrite (4.7) as in the following Lemma:

Lemma 4.1. *Let F be small enough. Then for a given $z \in \mathbb{C}$ there exist positive numbers $\mathcal{C}_1$, $\mathcal{C}_2$, $\sigma(\Re z) \geq 1$ and $\mathcal{C}(B, \Re z) > 0$, with $\mathcal{C}(B, \Re z) = Bc(\Re z) \to 0$ as $\Re z \to \infty$, such* *that*

$$\begin{aligned} \|K(z;ib)\| &\leq \mathcal{C}_1\, F^{-\mathcal{C}_2}\ \mathrm{dist}(\sigma(H_1(ib)),z)^{-\sigma(\Re z)} \left(e^{-\frac{\mathrm{dist}(\sigma(H_1(ib)),z)}{F^{\tau}}} + e^{-\frac{\mathcal{C}(B,\Re z)}{F^{2(1-\varepsilon)}}} \right) \\ &\quad \times (1+\|R_2(z;ib)\|)\,. \end{aligned} \tag{4.9}$$

5 Main result

Armed with Lemma 4.1 we are ready to prove an estimate on the difference between the spectral projectors of $H(F,ib)$ and $H_2(F,ib)$.

Let $\Gamma(e_\alpha)$ the path in the complex plane enclosing the eigenvalue $e_\alpha \in \sigma(H(0))$ at finite distance to the Landau levels (see Figure 3). More precisely

$$\begin{aligned} \Gamma(e_\alpha) &:= \Gamma_1(e_\alpha)\cup\Gamma_2(e_\alpha)\cup\Gamma_3(e_\alpha)\cup\Gamma_4(e_\alpha) \\ \Gamma_1(e_\alpha) &:= \{\xi\in\mathbb{C} :\, \Re\xi = e_\alpha - cF^{\varepsilon/2},\ |\Im\xi|\leq\rho\} \\ \Gamma_2(e_\alpha) &:= \{\xi\in\mathbb{C} :\, \Re\xi = e_\alpha + cF^{\varepsilon/2},\ |\Im\xi|\leq\rho\} \\ \Gamma_3(e_\alpha) &:= \{\xi\in\mathbb{C} :\, e_\alpha - cF^{\varepsilon/2}\ \leq\Re\xi\leq\ e_\alpha + cF^{\varepsilon/2},\ \Im\xi=\rho\} \\ \Gamma_4(e_\alpha) &:= \{\xi\in\mathbb{C} :\, e_\alpha - cF^{\varepsilon/2}\ \leq\Re\xi\leq\ e_\alpha + cF^{\varepsilon/2},\ \Im\xi=-\rho\}\,. \end{aligned} \tag{5.1}$$

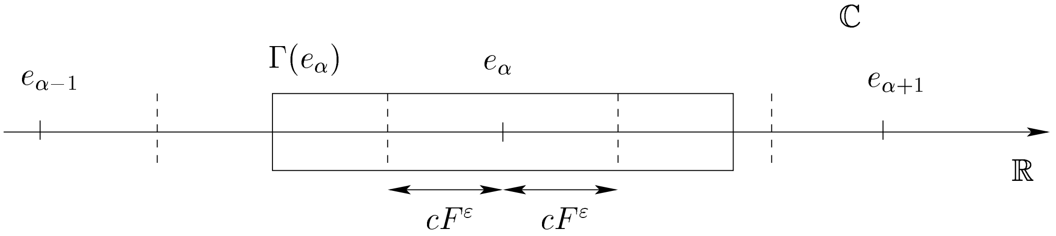

Figure 3: *The path $\Gamma(e_\alpha)$ in the complex plane. The spectrum of $H_2(F,ib)$ is localised in the vicinity of e_α, represented by the dashed vertical lines.* (*Proposition 3.1*).

For F sufficiently small this construction can be made in such a way that the spectrum of $H_2(F,ib)$ enclosed by $\Gamma(e_\alpha)$ consists only of the eigenvalues $\lambda_{\alpha,i}(F)\to e_\alpha$, where i denote the degeneracy index of the eigenvalue e_α $(1\leq i\leq r_\alpha)$, see Proposition 3.1. Moreover for $z\in\Gamma(e_\alpha)$ holds by Lemma 3.2

$$\|R_2(z;ib)\| \leq \mathcal{C}\rho^{-s}\,. \tag{5.2}$$

To control the inverse $(1-K(z,ib))^{-1}$ we need $\|K(z;ib)\|<1$ for $z\in\Gamma(e_\alpha)$. In particular we want $\|K(z;ib)\|\to 0$ as $F\to 0$. Looking at Lemma 4.1, together with (5.2) we see that the above requirement on the norm of $K(z;ib)$ is satisfied at best taking

$$\rho = e^{-\frac{\rho_0}{F^{2(1-\varepsilon)}}} \quad \text{with } s\rho_0 < \mathcal{C}(B, \Re z) \tag{5.3}$$

We point out that the Gaussian smallness of ρ is the optimal choice to get the eigenprojectors convergence. From the decoupling formula (4.5) we have

$$\begin{aligned} R(z;ib) - R_2(z;ib) = \left(\sum_{i=1}^{5} J_i(ib) R_i(z;ib) \tilde{J}_i(ib)\right) \sum_{n=1}^{\infty} K(z;ib)^n - (1 - J_2(ib)) R_2(z;ib) \\ - J_2(ib) R_2(z;ib)(1 - \tilde{J}_2(ib)) + \sum_{i \in \{1,3,4,5\}} J_i(ib) R_i(z;ib) \tilde{J}_i(ib) \ . \end{aligned} \tag{5.4}$$

Because of $\sigma(H_i(ib)) = \mathbb{R} - ibF$ (see (2.2)), $R_i(z;ib)$, $i \neq 2$, have no poles in $\Gamma(e_\alpha)$. Moreover the only poles of $R_2(z;ib)$ are precisely $\lambda_{\alpha,i}(F)$ $(1 \leq i \leq r_\alpha)$. Thus integrating (5.4) along the path $\Gamma(e_\alpha) \equiv \Gamma$

$$\begin{aligned} P^\Gamma(ib) - P_2^\Gamma(ib) &= \frac{1}{2\pi i} \oint_\Gamma \left(\sum_{i=1}^{5} J_i(ib) R_i(z;ib) \tilde{J}_i(ib)\right) \sum_{n=1}^{\infty} K(z;ib)^n \, \mathrm{d}z \\ &- J_2(ib) P_2^\Gamma(ib)(1 - \tilde{J}_2(ib)) - (1 - J_2(ib)) P_2^\Gamma(ib) \ . \end{aligned} \tag{5.5}$$

where $P_2^\Gamma(ib)$ is the spectral projector of $H_2(ib)$ onto $\overline{\mathrm{Int}\,\Gamma}$ and

$$P^\Gamma(ib) = \frac{1}{2\pi i} \oint_\Gamma (z - H(ib))^{-1} \, \mathrm{d}z$$

We estimate the norms of the three contributions on the r.h.s. of (5.5). If ρ_0 in the definition of $\Gamma(e_\alpha)$ satisfies a bit stronger condition than the bound in (5.3), the norm of the first term is smaller than

$$\mathcal{C}\left(\sum_{i=1}^{5} \sup_{z \in \Gamma} \|R_i(z;ib)\|\right) \frac{\sup_{z\in\Gamma} \|K(z;ib)\|}{1 - \sup_{z\in\Gamma} \|K(z;ib)\|} \leq g(F) \to 0 \quad \text{as } F \to 0 \ . \tag{5.6}$$

Indeed, for $i = 2$, by (5.2) and (5.3) there exists a smooth function $g(F)$ such that

$$\|R_2(z;ib)\| \|K(z;ib)\| \leq \mathcal{C} g(F)$$

for each $z \in \Gamma(e_\alpha)$ and $\lim_{F\to 0} g(F) = 0$ provided $2s\rho_0 < \mathcal{C}(B, \Re z)$. For $i \neq 2$ remembering that $b = b_0 F^\alpha$, by (4.8) we have $\sup_{z\in\Gamma} \|R_i(z;ib)\| \leq \frac{\mathcal{C}}{F^{\alpha+1}}$, and the result follows.

To estimate the second term in (5.5) we write

$$\begin{aligned}\|J_2(ib)P_2^\Gamma(ib)(1-\tilde J_2(ib))\| &\le \|J_2(ib)\|_\infty \|P_2^\Gamma(ib)(1-\tilde J_2(ib))\| \\ &\le \|[P_2^\Gamma(ib)-P_2^\Gamma(0)](1-\tilde J_2(ib))\| \\ &+ \|[P_2^\Gamma(0)-P^\Gamma](1-\tilde J_2(ib))\| + \|P^\Gamma(1-\tilde J_2(ib))\| \\ &\le \left(\|P_2^\Gamma(ib)-P_2^\Gamma(0)\| + \|P_2^\Gamma(0)-P^\Gamma\|\right)\|(1-\tilde J_2(ib))\|_\infty \\ &+ \sum_{i=1}^{r_\alpha} |(1-\tilde J_2(ib),\phi_0^i)| \qquad (5.7)\end{aligned}$$

where P^Γ is the spectral projector of $H(0)$ onto the eigenfunctions ϕ_0^i $(i=1,\ldots,r_\alpha)$ corresponding to the eigenvalue e_α. In order to control the term $\|P_2^\Gamma(ib)-P_2^\Gamma(0)\|$ we define a circle $\tilde\Gamma \equiv \{\xi\in\mathbb{C} : |\xi-e_\alpha| = \Gamma_0 F^\varepsilon\}$. Then for F small enough holds

$$\begin{aligned}\|P_2^\Gamma(ib)-P_2^\Gamma(0)\| &\le (2\pi)^{-1}\oint_{\tilde\Gamma} \|R_2(\xi;ib)A(ib)R_2(\xi;0)\|\,|\,\mathrm{d}\xi| \\ &\le \mathcal{C}\,F^{\alpha-1} \qquad (5.8)\end{aligned}$$

where $A(ib)$ is defined in (3.6) and the second inequality follows form (3.7). By Proposition 3.1 $\|P_2^\Gamma(0)-P^\Gamma\|\to 0$ as $F\to 0$. Thus for $F\to 0$ the two terms above are infinitesimally small. The last term can be easily estimated using the result of [CN, Thm. 4.2], which says that for any at least gaussian decaying potential one has the estimate

$$|\phi(\mathbf{x})| \le \mathcal{C}e^{-\mu|\mathbf{x}|^2},$$

where ϕ is associated to a discrete eigenvalue of $H(0)$. Using this result and a bound on $|1-\tilde J_2(ib)|$ similar to that of (A.3) we get

$$\|J_2(ib)P_2^\Gamma(ib)(1-\tilde J_2(ib))\| \to 0 \qquad \text{as } F\to 0 \qquad (5.9)$$

For the third term in (5.5) we obtain the same estimate, since $\|A^*\| = \|A\|$. In conclusion we arrive at

Proposition 5.1. *Let* $\Gamma(e_\alpha)$ *be as in* (5.1)*, then*

$$\|P^\Gamma(ib)-P_2^\Gamma(ib)\|\to 0, \qquad F\to 0$$

In other words, $\dim\operatorname{Ran}P^\Gamma(ib) = \dim\operatorname{Ran}P_2^\Gamma(ib)$ *for* F *sufficiently small.*

Propositions 5.1 and 3.1 yield

Theorem 5.1. *Assume V satisfies (a), (b) and let e_α be the eigenvalue of $H(0)$ of multiplicity r_α. Then near e_α there are eigenvalues $E_{\alpha,i}$ of $H(F,ib), (1 \le i \le r_\alpha)$, repeated according to their multiplicity, and*

$$E_{\alpha,i} \to e_\alpha \quad \text{as} \quad F \to 0.$$

Now we can formulate our main result.

Theorem 5.2. *Assume V satisfies (a) and (b). Let e_α and $E_{\alpha,i}$ be the eigenvalues defined in Theorem 5.1. Then there exist some positive constants $\mathcal{C}$ and $R_\alpha(B)$, such that for F small enough the following inequality holds true*

$$|\Im E_{\alpha,i}| \le \mathcal{C}\, e^{-\frac{R_\alpha(B)}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0,$$

where ε can be made arbitrarily small and $R_\alpha(B) = B\tilde{R}_\alpha$.

Proof. Consider the path $\Gamma(e_\alpha)$ defined through (5.1), with $\rho_0 = R_\alpha(B)$. We have proved in Proposition 5.1 that if

$$2s\, R_\alpha(B) < \mathcal{C}(B, e_\alpha), \tag{5.10}$$

with $\mathcal{C}(B, e_\alpha)$ defined in Lemma 4.1, then $\dim \operatorname{Ran} P^\Gamma(ib) = \dim \operatorname{Ran} P_2^\Gamma(ib)$ and the only eigenvalues of $H(F,ib)$ in $\overline{\operatorname{Int}\Gamma}$ are the eigenvalues $E_{\alpha,i}$. By construction their imaginary parts satisfy the announced upper bound. The linear dependence on B follows from the linear dependence of $\mathcal{C}(B, e_\alpha)$ on B. ☐

Remark 5.1. *The behaviour of $\tilde{R}_\alpha$ w.r.t. α is not uniform. Indeed $\tilde{R}_\alpha \to 0$ as $e_\alpha \to \infty$, because $\mathcal{C}(B, \Re z) \to 0$ as $\Re z \to \infty$.*

As already mentioned at the end of Section 2 the resonance widths are given by the imaginary parts of the eigenvalues of $H(F,ib)$, and the lifetime by the inverse of the resonance width. Since ε is arbitrarily small, we thus get a lower bound on the life-times:

Corollary 5.1. *The life-times of the resonant states satisfy:*

$$\tau_\alpha = \tfrac{1}{2} \sup_{\varepsilon>0} |\Im E_{\alpha,i}|^{-1} \ge 1/\mathcal{C}\ \exp\left(\frac{B\tilde{R}_\alpha}{F^2}\right).$$

Conclusion

Theorem 5.2 gives a partial generalisation of the result obtained in [GM]. As expected, the fact that the lower bound on the resonance life-times is Gaussian in F^{-1} and not exponential is due to the presence of the magnetic field. However, further comparison with the purely electric Stark effect shows much larger restriction on the class of admissible potentials, in particular the condition on the Gaussian decay of $V(x,y)$. Let us now briefly discuss the issue of Gaussian versus exponential behaviour. As follows from the analysis of the Stark resonances, [Op] [HaSi] [Sig], the exponential law for the resonant states is in that case directly connected with the exponential decay of the eigenfunctions of a “free” Hamiltonian, i.e. without electric field. If we suppose that the same connection exists also in the magnetic case, then our result should hold whenever the eigenfunctions of $H(0) = H_L + V$, associated with the discrete spectrum, fall off as a Gaussian. Sufficient condition for the latter is the Gaussian decay of $V(x,y)$, see [CN], which is compatible with our assumption (b). Up to now, the optimal condition is known only for the ground state, in which case a sort of exponential decay of $V(x,y)$ is shown to be sufficient and necessary for Gaussian behaviour of the corresponding eigenfunctions at infinity, [Er].

Such a restriction is in contrast with the non magnetic Schrödinger operator, whose eigenfunctions decrease exponentially in the classically forbidden region independently on the rate at which $V(x,y)$ tends to zero at infinity. This might indicate a principal difference between the behaviour of resonant states in the presence respectively absence of magnetic field.

A Estimate of $\|K(z;ib)\|$

Here we estimate the norm of each term in the definition of $K(z;ib)$ separately. Since the calculations are often analogous, we skip the details in many places.

Norm of $M(z;ib)$

Terms $\|A_1(ib)\|$ and $\|A_5(ib)\|$:

$$\begin{aligned} \|A_1(ib)\| &\leq \|V(ib)J_1(ib)\|_\infty \|R_1(z;ib)\| \|\tilde{J}_1(ib)\| \\ &\leq \mathcal{C}\|V(ib)J_1(ib)\|_\infty \|R_1(z;ib)\| \qquad \text{(A.1)} \end{aligned}$$

and for F sufficiently small

$$\begin{aligned} \|V(ib)J_1(ib)\|_\infty &= \sup_{(x,y)} |V(x+ib,y)||J_-(x+ib)||J_c(y)| \\ &\leq \sup_x |V(x+ib,\hat{y})| \frac{e^{2\gamma(x-x_2)}}{\left(e^{4\gamma(x-x_2)}+1\right)^{1/2}} \end{aligned}$$

We estimate this term as $\max\{a,b,c\}$ where a, b, c are

$$\begin{aligned} a &= \sup_{|x|<a_0} |V(x+ib,\hat{y})|e^{2\gamma(x-x_2)} \leq V_0 e^{2\gamma(a_0-x_2)} \leq V_0 e^{\frac{2\gamma_0 a_0}{F^{1-\varepsilon}}} e^{-\frac{2\gamma_0 C_2}{F^{2(1-\varepsilon)}}} \\ b &= \sup_{a_0\leq|x|\leq a_0+\delta} V_0 e^{-\nu x^2} e^{2\gamma(x-x_2)} \leq V_0 e^{-\nu a_0^2} e^{2\gamma(a_0+\delta-x_2)} \leq V_0 e^{\frac{2\gamma_0 a_0}{F^{1-\varepsilon}}} e^{-\frac{2\gamma_0(C_2-\delta_0)}{F^{2(1-\varepsilon)}}} \\ c &= \sup_{|x|>a_0+\delta} V_0 e^{-\nu x^2} \leq V_0 e^{-\frac{{\delta_0}^2}{F^{2(1-\varepsilon)}}} \end{aligned}$$

and $\delta = \delta_0 F^{-(1-\varepsilon)} < x_2$. This leads to

$$\|A_1(ib)\| \leq \mathcal{C} e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \|R_1(z;ib)\|$$

In the same way we prove the estimate for $\|A_5(ib)\|$.

Term $\|A_2(ib)\|$:

$$\begin{aligned} \|A_2(ib)\| &\leq F\|(x+ib)(1-h_F(x+ib))J_2(ib)\|_\infty \|R_2(z;ib)\| \|\tilde{J}_2(ib)\| \\ &\leq \mathcal{C}F\|(x+ib)(1-h_F(x+ib))J_0(x+ib)\|_\infty \|R_2(z;ib)\| \end{aligned} \qquad \text{(A.2)}$$

We can easily found the following bounds

$$|J_0(x+ib)| \leq \frac{1}{\cos(2\gamma b)} \begin{cases} e^{2\gamma(x+x_1)} & \text{if } x<0 \\ e^{-2\gamma(x-x_1)} & \text{if } x>0 \end{cases} \qquad \text{(A.3)}$$

and

$$|1-h_F(x+ib)| \leq \left(e^{-4\gamma(x-\bar{x})}+1\right)^{-1/2} + \left(e^{4\gamma(x+\bar{x})}+1\right)^{-1/2} \equiv h_1+h_2 \qquad \text{(A.4)}$$

For $x > \frac{\bar{x}+x_1}{2} > 0$

$$|h_1|^2|J_0(x+ib)|^2 \leq \mathcal{C}\frac{e^{-4\gamma(x-x_1)}}{e^{-4\gamma(x-\bar{x})}+1} \leq \mathcal{C}\frac{e^{-4\gamma(x-\frac{\bar{x}+x_1}{2})}}{e^{-2\gamma(x_1-\bar{x})}}$$

the last inequality follows after multiplication by $(e^{2\gamma(\bar{x}-x_1)})/(e^{2\gamma(\bar{x}-x_1)})$. Now, $y = x - (\bar{x}+x_1)/2$, yields

$$\begin{aligned} \sup_{x>\frac{\bar{x}+x_1}{2}} F|x||h_1 J_0(x+ib)| &\le \mathcal{C}F \sup_y \left(|y| + |\bar{x}+x_1|/2\right) e^{-\gamma(\bar{x}-x_1)} e^{-2\gamma|y|} \\ &\le \mathcal{C}(F+F^{\varepsilon}) e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \end{aligned} \tag{A.5}$$

For $x < -\frac{\bar{x}+x_1}{2} < 0$ we get in the same way the upper bound (A.5). Finally, for $|x| \le \frac{\bar{x}+x_1}{2}$ obviously $\sup_x |x| = \frac{\bar{x}+x_1}{2}$ and

$$|h_1 J_0(x+ib)| \le e^{-2\gamma(\bar{x}-x_1)}$$

which gives a similar estimate as (A.5).
A similar argument holds for $|h_2 J_0(x+ib)|$ that leads to

$$\|A_2(ib)\| \le \mathcal{C} e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \|R_2(z;ib)\| \tag{A.6}$$

Term $\|\sum_{j=1}^5 J_j(ib)\tilde{J}_j(ib) - 1\|$:
First we remark that we can write $1 = \tilde{J}_c(y) + (1-\tilde{J}_c(y))$ and that $\sum_{i=3}^4 J_i(ib)\tilde{J}_i(ib) - (1-\tilde{J}_c) = 0$, thus it remains to estimate $\sum_{i\in\{1,2,5\}} J_i(ib)\tilde{J}_i(ib) - \tilde{J}_c$. We have

$$\begin{aligned} \sum_{i\in\{1,2,5\}} J_i(ib)\tilde{J}_i(ib) - \tilde{J}_c &= \Big[J_-(x+ib)\tilde{J}_-(x+ib) + J_0(x+ib)\tilde{J}_0(x+ib) \\ &+ J_+(x+ib)\tilde{J}_+(x+ib) - 1\Big] \tilde{J}_c(y) := \mathcal{X}(ib)\tilde{J}_c(y) \end{aligned}$$

Now $\|\tilde{J}_c(y)\|_\infty = 1$, and it remain to estimate

$$\|\mathcal{X}(ib)\|_\infty = \left\| \sum_{\alpha\in\{\pm,0\}} J_\alpha(x)\tilde{J}_\alpha(x) - 1 \right\|_\infty \tag{A.7}$$

This can be done by developing explicitly the functions in term of the exponentials and write the sum as fraction (denote by $\mathcal{K}$ the denominator). After a tedious straightforward computation we find out that each term in the sum

$$\sum_{\alpha\in\{\pm,0\}} J_\alpha(x+ib)\tilde{J}_\alpha(x+ib) - 1$$

can be bounded from above uniformly w.r.t. x by $\mathcal{C}e^{-\mathcal{C}F^{-(2-\varepsilon)}}$. For example

$$\left|\frac{e^{-2\gamma(2x+x_0+x_2)}}{\mathcal{K}}\right| \le \frac{e^{-2\gamma(2x+x_0+x_2)}}{\cos(4\gamma b)e^{4\gamma x}} = \frac{e^{-2\gamma(x_0+x_2)}}{\cos(4\gamma b)} \le \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}$$

for $F \to 0$ due to (3.2) and similarly in other cases. Therefore

$$\left\| \sum_{i=1}^{5} J_i(ib)\tilde{J}_i(ib) - 1 \right\|_\infty \leq \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}$$

Finally,

$$\|M(z;ib)\| \leq \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \left(\|R_1(z;ib)\| + \|R_2(z;ib)\| + 1\right)$$

Norm of $K_3(z;ib)$ and $K_4(z;ib)$

To control the operator norm we will use alternatively the Hilbert-Schmidt norm and the following inequality for the norm of an integral operator which can be found in [Ka, p. 144]

$$\|A\| \leq \max\left\{\sup_{\mathbf{x}} \int |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}; \sup_{\mathbf{x}'} \int |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}\right\} \tag{A.8}$$

Each integration that we need to evaluate is split in two parts according to $|x - x'| \geq 1$ and $|x - x'| < 1$:

Let φ such that $\|\varphi\| = 1$, and A an operator with integral kernel $A(\mathbf{x},\mathbf{x}')$, then

$$\begin{aligned} \|A\varphi\|^2 &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right|^2 \mathrm{d}\mathbf{x} && \text{(A.9)} \\ &\leq 2\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2:|x-x'|\geq 1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right|^2 \mathrm{d}\mathbf{x} \\ &\quad +2\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2:|x-x'|< 1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right|^2 \mathrm{d}\mathbf{x} =: 2(a+b)\,. && \text{(A.10)} \end{aligned}$$

We now treat the two terms separately. By the Schwartz inequality we have

$$a \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2:|x-x'|\geq 1} |A(\mathbf{x},\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\|\varphi\|^2 \leq \|A\|_{HS}^2\|\varphi\|^2$$

For b we proceed as follows, let

$$\psi(\mathbf{x}) \equiv \int_{\mathbb{R}^2:|x-x'|<1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}'$$

and

$$A(\mathbf{x}) = \int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}' \qquad A'(\mathbf{x}') = \int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}$$

we first remark that $\int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|/A(\mathbf{x})\,\mathrm{d}\mathbf{x}' = 1$, this implies by convexity, that

$$\left(\frac{|\psi(\mathbf{x})|}{A(\mathbf{x})}\right)^2 \leq \int_{\mathbb{R}^2:|x-x'|<1} \frac{|A(\mathbf{x},\mathbf{x}')|}{A(\mathbf{x})}|\varphi(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}'$$

and thus

$$
\begin{aligned}
b = \int_{\mathbb{R}^2} |\psi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} &\leq \sup_{\mathbf{x}} A(\mathbf{x}) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2 : |x-x'|<1} |A(\mathbf{x},\mathbf{x}')| |\varphi(\mathbf{x}')|^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
&= \sup_{\mathbf{x}} A(\mathbf{x}) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2 : |x-x'|<1} |A(\mathbf{x},\mathbf{x}')| |\varphi(\mathbf{x}')|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{x}' \\
&\leq \sup_{\mathbf{x}} A(\mathbf{x}) \sup_{\mathbf{x}'} A'(\mathbf{x}') \|\varphi\|^2 \\
&\leq \max \left\{ \sup_{\mathbf{x}} A(\mathbf{x}), \sup_{\mathbf{x}'} A'(\mathbf{x}') \right\}^2 \|\varphi\|^2 \qquad \text{(A.11)}
\end{aligned}
$$

Therefore, for $|x-x'| \geq 1$ we can use a Hilbert-Schmidt-like norm, while for $|x-x'| < 1$ we can use a (A.8) norm. We will need results on the behaviour of the Green function $G_1(\mathbf{x},\mathbf{x}';z)$ of $H_1(ib)$. We expect that at points $\mathbf{x},\mathbf{x}'$ with $|x-x'|$ large the Green function decay in the $x-$direction as a Gaussian due to the magnetic field, while in the $y-$direction (the drift direction of the classical particle) we expect only exponential decay. On the other we also expect integrable singularity at the origin. These properties are contained in the following two lemmas which are obtained in [FK].

Lemma A.1. *Let $|x-x'| \geq 1$ and let F be small enough. Then there exist some strictly positive constants G_0, $\omega(z)$ and $\sigma(z) \geq 1$ such that*

$$
|\partial^n_{x,y} G_1(\mathbf{x},\mathbf{x}';z)| \leq G_0 \, \beta(z)^{-\sigma(z)} \, e^{-\beta(z)|y'-y|} \, e^{-\omega(z)(x'-x)^2},
$$

where $n = 0, 1$ and $\beta(z) = \frac{\Im z + bF}{2F}$.

Lemma A.2. *For F small enough there exists some strictly positive constants G_0' and $\sigma(z)$, such that the following inequality holds true*

$$
\int_{\mathbb{R}} \int_{|x'-x|<1} |\partial^n_{x,y} G_1(\mathbf{x},\mathbf{x}';z)| e^{\frac{\beta(z)}{2}|y-y'|} \, \mathrm{d}x' \mathrm{d}y' \leq G_0' \, \beta(z)^{-\sigma(z)}, \qquad \text{(A.12)}
$$

where $n = 0, 1$ and $\beta(z) = \frac{\Im z + bF}{2F}$.

Since the integrands are positive functions, for $|x-x'| \geq 1$ we first substitute the integral kernels by their upper bounds and then integrate without any restriction.

Remark A.1. *In the Lemmas above the coefficient $\omega(z)$ depends only in $\Re z$ and decreases as $\Re z$ increases. Moreover, $\omega(z)$ is linear in B: $\omega(z) \sim B$. $\sigma(z) \geq 1$, and also depends only on $\Re z$ and diverges for $\Re z \to \infty$. For the sake of brevity we do not write z in the arguments of σ and ω.*

We now evaluate the norm of $K_3(z;ib)$. The terms in the commutator are

$$[p_y^2, J_3(ib)]R_3(z;ib)\tilde{J}_3(ib) = -2\partial_x J_3(ib)\partial_x R_3(z;ib)\tilde{J}_3(ib) - \partial_x^2 J_3(ib)R_3(z;ib)\tilde{J}_3(ib)$$

We use again inequality (A.8). Due to the upper bound on the Green function and its derivatives when $|x-x'| \geq 1$ the integration can be separated in two parts, which for F small enough gives us (for $n=1,2$)

$$\begin{aligned} & \sup_{\mathbf{x}} \int \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)||\tilde{J}_3(x'+ib,y')| \\ \leq\ & \mathcal{C} \sup_y \int dy' |\partial_y^n J_>(y)| \beta(z)^{-\sigma} e^{-\beta(z)|y-y'|} |\tilde{J}_>(y')| \\ \leq\ & \mathcal{C}\beta(z)^{-\sigma} \sup_{y \in \operatorname{supp} \partial_y^n J_>} \ \sup_{y' \in \operatorname{supp} \tilde{J}_>} e^{-\frac{\beta(z)}{2}|y-y'|} = \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}} \end{aligned}$$

and similarly for the second term. We now consider the situation $|x-x'| < 1$, let be the set $D = \{x' \in \mathbb{R} : |x-x'| < 1\} \times \mathbb{R}$

$$\begin{aligned} & \sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)||\tilde{J}_3(x'+ib,y')| \\ \leq\ & \sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)| e^{-\frac{\beta(z)}{2}|y-y'|} |\tilde{J}_3(x'+ib,y')||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)| e^{\frac{\beta(z)}{2}|y-y'|} \\ \leq\ & \sup_{y \in \operatorname{supp} \partial_y^n J_>} \ \sup_{y' \in \operatorname{supp} \tilde{J}_>} e^{-\frac{\beta(z)}{2}|y-y'|} \sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)| e^{\frac{\beta(z)}{2}|y-y'|} \\ \leq\ & \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}} \end{aligned}$$

Thus we can conclude that

$$||K_3(z;ib)|| \leq \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}}$$

In the same way we prove the estimate for $||K_4(z;ib)||$.

Norm of $K_1(z;ib)$ and $K_5(z;ib)$

Here below when we write $\|\cdot\|_{HS}$ for $|x-x'| \geq 1$ it is understood that part of the Hilbert-Schmidt, which corresponds to the integration over $\mathbb{R}^2$ with the restriction $|x-x'| \geq 1$. For the integral kernel of $R_1(z;ib)$ and $\partial_{x,y} R_1(z;ib)$ we then use the upper bounds of Lemma A.1.

The first term in the commutator $[H_L, J_1(ib)]$ gives

$$[p_x^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib) = -2\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib) - \partial_x^2 J_1(ib)R_1(z;ib)\tilde{J}_1(ib) \tag{A.13}$$

In the case $|x-x'|\geq 1$ we estimate the "restricted" Hilbert-Schmidt norms term by term.

$$\begin{aligned}&\|\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib)\|_{HS}^2 =\\ =\ &\int_{\mathbb{R}^4} |J'_-(x+ib)J_c(y)|^2|\partial_x G_1(\mathbf{x},\mathbf{x}';z)|^2|\tilde{J}_-(x'+ib)\tilde{J}_c(y')|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\end{aligned}$$

As before due to the properties of the Green function for $|x-x'|\geq 1$ the integration can be separated it two parts. One can easily check that the integral with respect to y, y' gives the factor

$$\mathcal{C}\,F^{-2\tau}$$

The second part is bounded above by

$$\beta(z)^{-\sigma}\int_{\mathbb{R}} |J'_-(x+ib)|^2 f(x,x_0)\,\mathrm{d}x$$

where

$$f(x,x_0) := \int_{\mathbb{R}} e^{-\omega(x-x')^2}\frac{1}{1+e^{-4\gamma(x'-x_0)}}\,\mathrm{d}x'$$

Here we have used the fact that for F sufficiently small (see (3.2))

$$\begin{aligned}|\tilde{J}_-(x'+ib)|^2 &= \left(1+e^{-4\gamma(x'-x_0)}+2\cos(2\gamma b)e^{-2\gamma(x'-x_0)}\right)^{-1}\\ &\leq \frac{1}{1+e^{-4\gamma(x'-x_0)}}\end{aligned} \tag{A.14}$$

In the similar way we find out that

$$|J'_-(x+ib)|^2 \leq \mathcal{C}\,F^{-2}e^{-4\gamma|x-x_2|} \tag{A.15}$$

so that it suffices to look for an upper bound on the functional

$$\begin{aligned}\int_{\mathbb{R}} e^{-4\gamma|x-x_2|}f(x,x_0)\,\mathrm{d}x &= \int_{-\infty}^{x_2-\delta} e^{-4\gamma|x-x_2|}f(x,x_0)\,\mathrm{d}x\\ &+\int_{x_2+\delta}^{\infty} e^{-4\gamma|x-x_2|}f(x,x_0)\,\mathrm{d}x+\int_{x_2-\delta}^{x_2+\delta} e^{-4\gamma|x-x_2|}f(x,x_0)\,\mathrm{d}x\\ &= I_1+I_2+I_3\end{aligned} \tag{A.16}$$

where $\delta=\delta_0 F^{-1(1-\varepsilon)}$ such that $(x_2+\delta)<x_0$. As $f(x,x_0)$ is by definition strictly positive and bounded, the first two integrals on the r.h.s. of (A.16) can be easily estimated as follows

$$\begin{aligned}I_1+I_2 &\leq e^{-2\gamma\delta}\|f\|_\infty\left[\int_{-\infty}^{x_2-\delta} e^{2\gamma(x-x_2)}\,\mathrm{d}x+\int_{x_2+\delta}^{\infty} e^{-2\gamma(x-x_2)}\,\mathrm{d}x\right]\\ &\leq \gamma^{-1}\sqrt{\frac{\pi}{\omega}}e^{-2\gamma\delta}\end{aligned}$$

In order to control I_3 we have to look at the function $f(x,x_0)$ in more detail. First we note that

$$\begin{aligned} f(x,x_0) &= \int_{\mathbb{R}} e^{-\omega(x-x_0-t)^2}\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \\ &\le \int_0^\infty e^{-\omega(x-x_0-t)^2}\,\mathrm{d}t + \int_{-\infty}^0 e^{-\omega(x-x_0-t)^2+4\gamma t}\,\mathrm{d}t \end{aligned} \tag{A.17}$$

¿From [GR, p. 1064] (see also (A.36)) we then get the bound on $f(x,x_0)$ in the form

$$\begin{aligned} f(x,x_0) &\le \sqrt{\frac{1}{2\omega}}\, e^{-\omega(x-x_0)^2}\Bigg[e^{\frac{\omega(x-x_0)^2}{2}} D_{-1}(\sqrt{2\omega}\,(x_0-x)) \\ &+ e^{\frac{(2\omega(x-x_0)+4\gamma)^2}{8\omega}} D_{-1}\left(\frac{2\omega(x-x_0)+4\gamma}{\sqrt{2\omega}}\right)\Bigg] \end{aligned}$$

where $D_{-1}(\cdot)$ denotes the parabolic cylinder function. Using its asymptotic expansion [GR, p. 1065]

$$\begin{aligned} D_{-1}(z) &= e^{-z^2/4}z^{-1}(1-\mathcal{O}(z^{-2})), \quad z\to\infty \\ D_{-1}(z) &= e^{z^2/4}(1+\mathcal{O}(z^{-2})), \qquad z\to-\infty \end{aligned}$$

it is not difficult to verify that

$$f(x,x_0) \le \mathcal{C}e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}, \quad F\to 0$$

uniformly for any $x\in[x_2-\delta,x_2+\delta]$. Now we employ the mean value theorem of the integral calculus which tells us that there exists some $\tilde{x}\in[x_2-\delta,x_2+\delta]$ for which

$$I_3 = f(\tilde{x})\int_{x_2-\delta}^{x_2+\delta} e^{-4\gamma|x-x_2|}\,\mathrm{d}x = \frac{1}{2\gamma}\left(1-e^{-4\gamma\delta}\right) f(\tilde{x})$$

Let us remark that the second term of the commutator (A.13) can be bounded in the same way, since

$$|J''_{-}(x+ib)|^2 \le \mathcal{C}\,F^{-4}\,e^{-4\gamma|x-x_2|}, \quad F\to 0 \tag{A.18}$$

Moreover, due to the decoupling with respect to $y-$axis, the above procedure can be applied also to the second term in the commutator $[H_L, J_1(ib)]$, namely

$$[2Byp_x, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib) = -2By\partial_x J_1(ib)R_1(z;ib)\tilde{J}_1(ib)$$

This allows us to find some $c_1(V,B)>0$ such that the following holds true for $|x-x'|\ge 1$:

$$\left\|[(p_x+By)^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\right\|_{HS}^2 \le \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-c_1(B)F^{-2(1-\varepsilon)}} \tag{A.19}$$

where the constant $c_1(B)$ is proportional to B (since the factor ω is linear in B).
When $|x-x'|<1$ we use (A.11). As in the case $|x-x'|\geq 1$ all the term in the commutator $[H_L, J_1(ib)]$ involving $x-$derivatives are treated in the same way. For example for $\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib)$ we have

$$\begin{aligned}
&\sup_{\mathbf{x}}\int_{\mathbb{R}} \mathrm{d}y' \int_{x':|x-x'|<1} \mathrm{d}x' |J_-'(x+ib)J_c(y)||\partial_x G_1(\mathbf{x},\mathbf{x}';z)||\tilde{J}_-(x'+ib)\tilde{J}_c(y')| \\
\leq\ & \sup_{\mathbf{x}}\sup_{x':|x-x'|<1} |J_-'(x+ib)\tilde{J}_-(x'+ib)| \int_{\mathbb{R}} \mathrm{d}y' \int_{x':|x-x'|<1} \mathrm{d}x' |\partial_x G_1(\mathbf{x},\mathbf{x}';z)| \\
\leq\ & \mathcal{C}\beta(z)^{-\sigma} \sup_{x',x:|x-x'|<1} |J_-'(x+ib)\tilde{J}_-(x'+ib)| \qquad \text{(A.20)}
\end{aligned}$$

and similarly for $\mathbf{x}$ and $\mathbf{x}'$ interchanged. Now, using (A.14) and (A.15), we get

$$\sup_{x',x:|x-x'|<1} |J_-'(x+ib)\tilde{J}_-(x'+ib)| \leq \mathcal{C}F^{-1}\sup_x \frac{e^{-4\gamma|x-x_2|}}{1+e^{-4\gamma(x-x_0)}} \leq \mathcal{C}F^{-1}e^{-\mathcal{C}F^{-2(1-\varepsilon)}}$$

This with (A.19) leads to

$$\left\|[(p_x+By)^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\right\|^2 \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-c_2(B)F^{-2(1-\varepsilon)}}$$

for $c_2(B)>0$.
To control the operator norm of the last term in the commutator $[H_L, J_1(ib)]$, namely

$$[p_y^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)$$

we use again the inequality (A.8). When $|x-x'|\geq 1$, since both $f(x,x_0)$ and $f(x,x_2)$ are bounded as well as $J_-(x+ib)$, $\tilde{J}_-(x+ib)$, it suffices to estimate these parts in (A.8) which correspond to the integration w.r.t. y, y':

$$\begin{aligned}
&\sup_y |J_c'(y)| \int_{\mathbb{R}} e^{-\beta(z)|y-y'|}|\tilde{J}_c(y')|\ \mathrm{d}y' \leq \sup_y |J_c'(y)| \int_{-y_0}^{y_0} e^{-\beta(z)|y-y'|}\ \mathrm{d}y' \\
&\leq 2y_0\|J_c'\|_\infty\, e^{-\beta(z)F^{-\tau}} \qquad \text{(A.21)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sup_{y'} |\tilde{J}_c(y')| \int_{\mathbb{R}} e^{-\beta(z)|y-y'|}|J_c'(y)|\ \mathrm{d}y \leq \|\tilde{J}_c\|_\infty \sup_{y'\in[-y_0,y_0]} \int_{y_0+F^{-\tau}}^{y_0+F^{-\tau}+1} e^{-\beta(z)|y-y'|}\ \mathrm{d}y \\
&\leq \|\tilde{J}_c\|_\infty\, e^{-\beta(z)F^{-\tau}} \qquad \text{(A.22)}
\end{aligned}$$

and similarly for the terms with $J_c''(y)$. When $|x-x'|<1$ we proceed in a similar way as for the case $i=3$ and we get the desired result.

Thus we can conclude that

$$\|[p_y^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\| \le \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\frac{\beta(z)}{F^\tau}} \tag{A.23}$$

Finally,

$$\|K_1(z;ib)\| \le \mathcal{C}F^{-\mathcal{C}}\beta(z)^{-\sigma}\left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right)$$

The upper bound on the term $\|K_5(z;ib)\|$ is found in the same way.

Norm of $K_2(z;ib)$

The operator $K_2(z;ib)$ includes the resolvent $R_2(z;ib)$, which can be evaluated with respect to $R_1(z;ib)$

$$R_2(z;ib) = R_1(z;ib) - R_1(z;ib)[F(x+ib)(\chi_A^c + h_F^c(ib)\chi_A) + V(ib)]R_2(z;ib) \tag{A.24}$$

Obviously, the first term coming from (A.24) is to be treated in the same way as above. The second term $R_1(z;ib)[\cdots]R_2(z;ib)$ is estimated using

$$\|[H_L, J_2(ib)]R_1(z;ib)[\cdots]R_2(z;ib)\tilde{J}_2(ib)\| \le \|[H_L, J_2(ib)]R_1(z;ib)[\cdots]\|\|R_2(z;ib)\|\|\tilde{J}_2(ib)\|$$

Now, $\|\tilde{J}_2(ib)\|$ is bounded and for $\|R_2(z;ib)\|$ we use the result of Lemma 3.2. It then remains to estimate

$$\|[H_L, J_2(ib)]R_1(z;ib)[F(x+ib)(\chi_A^c + h_F^c(ib)\chi_A) + V(ib)]\| \tag{A.25}$$

Before we give the estimation of the different contribution to (A.25), we remind that

$$\begin{aligned} |J_0'(x+ib)| &\le \mathcal{C}\,F^{-1}\left\{e^{-2\gamma|x-x_1|} + e^{-2\gamma|x+x_1|}\right\} & \text{(A.26)} \\ |J_0''(x+ib)| &\le \mathcal{C}\,F^{-2}\left\{e^{-2\gamma|x-x_1|} + e^{-2\gamma|x+x_1|}\right\}, & \text{(A.27)} \end{aligned}$$

where we have used the similar bounds as in (A.15). In the estimations we will separate the two contributions coming from $\bar{J}_+$ and $\bar{J}_-$.

Let us now look at the contribution to (A.25) which includes the potential $V(ib)$. We again begin with the Hilbert-Schmidt norm (case $|x-x'| \ge 1$) of the terms in the commutator involving the $x-$derivatives. After separation of variables we can write

$(n = 1, 2)$

$$\begin{aligned}
&\|\partial_x^n \bar{J}_+(x+ib) J_c(y) \partial_x^{(2-n)} R_1(z;ib) V(ib)\|_{HS}^2 \\
&\leq \mathcal{C}\, F^{-2\tau} \beta(z)^{-\sigma} \int_{\mathbb{R}} |\partial_x^n \bar{J}_+(x+ib)|^2 \,\mathrm{d}x \int_{\mathbb{R}} e^{-\omega(x-x')^2}\, |V(x'+ib,\hat{y})|^2 \,\mathrm{d}x' \\
&\leq \mathcal{C}\, F^{-2-2\tau} \beta(z)^{-\sigma} \int_{\mathbb{R}} e^{-4\gamma|x-x_1|} \left[\int_{|x'|\leq a_0} e^{-\omega(x-x')^2} \,\mathrm{d}x' + \int_{|x'|>a_0} e^{-\omega(x-x')^2} e^{-\nu x'^2} \,\mathrm{d}x' \right] \mathrm{d}x \\
&\leq \mathcal{C}\, F^{-2-2\tau} \beta(z)^{-\sigma} \int_{\mathbb{R}} e^{-4\gamma|x-x_1|} \left[g(x,a_0) + \sqrt{\frac{\pi}{\omega+\nu}}\, e^{-\frac{\omega\nu}{\omega+\nu}\, x^2} \right] \mathrm{d}x \qquad \text{(A.28)}
\end{aligned}$$

where we have defined

$$g(x,a_0) := \int_{|x'|\leq a_0} e^{-\omega(x-x')^2} \,\mathrm{d}x'$$

Now we can apply the same argument as in (A.16) and repeat it for $\|\partial_x^n \bar{J}_-(x+ib) J_c(y) \partial_x^{(2-n)} R_1(z;ib) V(ib)\|_{HS}^2$ to arrive at

$$\left\| [(p_x + By)^2, J_2(ib)] R_1(z;ib) V(ib) \right\|_{HS}^2 \leq \mathcal{C}\, \beta(z)^{-\sigma} F^{-\mathcal{C}}\, e^{-\mathcal{C}\, F^{-2(1-\varepsilon)}} \qquad \text{(A.29)}$$

For $|x-x'| < 1$ we proceed like in (A.20) evaluating separately the contributions coming from $\bar{J}_+$ and $\bar{J}_-$. For example, for $\partial_x^n \bar{J}_+(x+ib) J_c(y) \partial_x^{(2-n)} R_1(z;ib) V(ib)$ we get an upper bound of the form

$$\begin{aligned}
&\sup_{\mathbf{x}} \sup_{x':|x-x'|<1, y'} |\partial_x^n \bar{J}'_+(x) V(x'+ib,y')| \int_{\mathbb{R}} \mathrm{d}y' \int_{x':|x-x'|<1} \mathrm{d}x' |\partial_x^{2-n} G_1(\mathbf{x},\mathbf{x}';z)| \\
\leq\;\; &\mathcal{C}\beta(z)^{-\sigma} \sup_{x,x':|x-x'|<1,y'} |\partial_x^n \bar{J}'_+(x) V(x'+ib,y')| \leq \mathcal{C}\beta(z)^{-\sigma} F^{-\mathcal{C}} e^{-\mathcal{C}F^{-2(1-\varepsilon)}} \qquad \text{(A.30)}
\end{aligned}$$

The last term in the commutator (A.25) which includes $V(ib)$ is the following

$$[p_y^2, J_2(ib)] R_1(z;ib) V(ib)$$

For $|x-x'| \geq 1$, since both

$$J_0(x+ib) \int_{\mathbb{R}} e^{-\omega(x-x')^2} \,\mathrm{d}x', \qquad \int_{\mathbb{R}} e^{-\omega(x-x')^2} J_0(x'+ib) \,\mathrm{d}x'$$

are bounded as functions of x, we apply again (A.8) to find out that

$$\begin{aligned}
&\sup_y |J'_c(y)| V_0 \int_{-a_1}^{a_1} e^{-\beta(z)|y-y'|} \,\mathrm{d}y' \leq \|J'_c\|_\infty V_0\, 2a_1 \sup_{y\in \mathrm{supp}\, J'_c}\; \sup_{y'\in[-a_1,a_1]} e^{-\beta(z)|y-y'|} \\
&\leq \|J'_c\|_\infty\, 2a_1 V_0\, e^{-\beta(z)F^{-\tau}} \qquad \text{(A.31)}
\end{aligned}$$

and similarly the other way around

$$\sup_{y'} |V(x'+ib,y')| \int_{y_0+F^{-\tau}}^{y_0+F^{-\tau}+1} e^{-\beta(z)|y-y'|} |J'_c(y)| \,\mathrm{d}y \leq V_0 \|J'_c\|_\infty\, e^{-\beta(z)F^{-\tau}}$$

For $|x-x'|<1$ we proceed as for $i=3$. Summing all the above given inequalities we obtain

$$\|[H_L, J_2(ib)]R_1(z;ib)V(ib)\| \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right) \tag{A.32}$$

Remark A.2. *Note that the hypothesis on the Gaussian-like decay of V w.r.t. x is necessary in order to obtain* (A.32) *as one can see from* (A.29) *and* (A.30).

Next we analyse those terms of (A.25), which include the potential $F(x+ib)h_F^c(ib)\chi_A$. We start again with the case $|x-x'|\geq 1$ looking at the Hilbert-Schmidt norm of

$$[(p_x+By)^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A \tag{A.33}$$

Note that since we have the same upper bounds on $J_c'(x+ib)$, $J_c''(x+ib)$ and also on $R_1(z;ib)$, $\partial_x R_1(z;ib)$, all terms in (A.33) can be estimated in the same way. As for the previous term we separate the contributions of $\bar{J}_\pm$, moreover $h_F^c = 1-h_F = h_+ + h_-$ with $h_\pm(x) = \frac{1}{2}\left[1 \mp \tanh(\gamma_F(x \pm \bar{x}))\right]$, and thus we separate also the contributions of h_+ and h_-. We are left with four terms, each of them is estimated as follows ($n=1,2$):

$$\begin{aligned}
&\|\partial_x^n \bar{J}_+(x+ib)J_c(y)\partial_x^{(2-n)}R_1(z;ib)F(x+ib)h_-(ib)\chi_A\|_{HS}^2 \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\int_{\mathbb{R}}|\partial_x^n\bar{J}_+(x+ib)|^2\,\mathrm{d}x\int_{\mathbb{R}}e^{-\omega(x-x')^2}\,|F(x'+ib)h_-(x'+ib)|^2\,\mathrm{d}x' \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\int_{\mathbb{R}}e^{-4\gamma|x-x_1|}\,\mathrm{d}x\int_{\mathbb{R}}e^{-\omega(x-\bar{x}-t)^2}|t+\bar{x}+ib|^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}}
\end{aligned} \tag{A.34}$$

recalling that the integration w.r.t. y, y' gives again the factor of order $F^{-2\tau}$. To evaluate the integral with respect to t we write

$$\begin{aligned}
&\int_{\mathbb{R}}e^{-\omega(x-\bar{x}-t)^2}|t+\bar{x}+ib|^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \qquad \text{(A.35)}\\
&\leq \int_{-\infty}^{0}e^{-\omega(x-\bar{x}-t)^2+4\gamma t}(2t^2+2\bar{x}^2+b^2)\,\mathrm{d}t + \int_0^\infty e^{-\omega(x-\bar{x}-t)^2}(2t^2+2\bar{x}^2+b^2)\,\mathrm{d}t
\end{aligned}$$

and use the following general result which can be found in [GR, p. 1064],

$$\int_0^\infty t^{\mu-1}e^{-bt^2-ct}\,\mathrm{d}t = (2b)^{-\mu/2}\Gamma(\mu)\exp(c^2/8b)D_{-\mu}(c/\sqrt{2b}) \tag{A.36}$$

Here $D_{-\mu}(\cdot)$ is the parabolic cylinder function of order $-\mu$. Its asymptotic behaviour is given by [GR, p.1065]

$$\begin{aligned}
D_p(z) &\simeq e^{-z^2/4}z^p(1-\mathcal{O}(z^{-2})), \qquad z\to\infty \\
D_p(z) &\simeq e^{z^2/4}z^{-p-1}(1+\mathcal{O}(z^{-2})), \quad z\to-\infty
\end{aligned} \tag{A.37}$$

The asymptotic behaviour allows us to apply once more the argument used in (A.16). We can thus claim that

$$\left\|[(p_x+By)^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A\right\|_{HS}^2 \le \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}$$

Also for the case $|x-x'|<1$ all the terms are treated analogously. For example for $\partial_x^n\bar{J}_+(ib)J_c\partial_x^{2-n}R_1(z;ib)F(x+ib)h_-(ib)\chi_A$ we have

$$\begin{aligned}
& \sup_{\mathbf{x}}\int_{\mathbb{R}}\mathrm{d}y'\int_{|x-x'|<1}\mathrm{d}x'|\partial_x^n\bar{J}_+(x+ib)J_c(y)||\partial_x^{2-n}G_1(\mathbf{x},\mathbf{x}';z)|F|x'+ib||h_-(x'+ib)\chi_A(y')| \\
\le\; & \sup_{\mathbf{x}}\sup_{x':|x-x'|<1}|\partial_x^n\bar{J}_+(x+ib)h_-(x'+ib)|^{1/2}\times \qquad \text{(A.38)} \\
\times\; & \int_{\mathbb{R}}\mathrm{d}y'\int_{|x-x'|<1}\mathrm{d}x'|\partial_x^{2-n}G_1(\mathbf{x},\mathbf{x}';z)||\partial_x^n\bar{J}_+(x)|^{1/2}F|x'+ib| \le \mathcal{C}\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\mathcal{C}F^{-2(1-\varepsilon)}}
\end{aligned}$$

where we used the fact that $|x'|\le|x|+1$ and $|\partial_x^n\bar{J}_+(x)|^{1/2}|x|\le\mathcal{C}F^{-(1-\varepsilon)}$.
We are now left with the last term in the commutator:

$$\begin{aligned}
[p_y^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A &= -2J_0(x+ib)J_c'(y)\partial_yR_1(z;ib)\times \\
\times F(x+ib)h_F^c(ib)\chi_A &- J_0(x+ib)J_c''(y)R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A \qquad \text{(A.39)}
\end{aligned}$$

When $|x-x'|\ge 1$ the Hilbert-Schmidt norm of these terms can estimated separately for $h_\pm$. We do that for h_-, for the term coming from h_+ a similar argument holds. For h_- the Hilbert-Schmidt norm is bounded above by a constant times $\beta(z)^{-\sigma}F^{-\tau}$ (coming from the integration w.r.t. y and y') times

$$\begin{aligned}
& \int_{\mathbb{R}}\mathrm{d}x|J_0(x+ib)|^2\int_{\mathbb{R}}e^{-\omega(x-x')^2}|x'|^2\frac{\mathrm{d}x'}{1+e^{-4\gamma(x'-\bar{x})}} \\
\le & \int_{\mathbb{R}}\mathrm{d}x|J_0(x+ib)|^2\int_{\mathbb{R}}e^{-\omega(x-\bar{x}-t)^2}(2t^2+2\bar{x}^2)\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \qquad \text{(A.40)}
\end{aligned}$$

The last integral can be again evaluated through (A.36) and (A.37) and estimated up to a constant from above by

$$F^{-\mathcal{C}}\,e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}, \qquad \text{(A.41)}$$

To control the first term in (A.40), which is proportional to t^2, we proceed in the same way as in (A.17) to write

$$\begin{aligned}
& \int_{\mathbb{R}}e^{-\omega(x-\bar{x}-t)^2}\,t^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \le \mathcal{C}\,e^{-\omega(x-\bar{x})^2}\Bigg[e^{\frac{\omega(x-\bar{x})^2}{2}}D_{-3}(\sqrt{2\omega}\,(\bar{x}-x)) \\
& + \; e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}}D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right)\Bigg] \qquad \text{(A.42)}
\end{aligned}$$

We will split (A.40) in three parts:

$$(-\infty, x_1+\delta], \quad [x_1+\delta, \bar{x}], \quad [\bar{x}, \infty) \tag{A.43}$$

where $\delta = \delta_0\, F^{-(1-\varepsilon)}$ and $(x_1+\delta) < \bar{x}$. For the first part we get

$$\begin{aligned} &\int_{\bar{x}}^{\infty} \mathrm{d}x e^{-4\gamma(x-x_1)}\, e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x)) \\ &\le e^{-4\gamma(\bar{x}-x_1)} \int_0^{\infty} e^{-4\gamma t-\omega t^2/2} D_{-3}(-\sqrt{2\omega}\, t)\,\mathrm{d}t \le \mathcal{C}\, e^{-4\gamma(\bar{x}-x_1)} \end{aligned} \tag{A.44}$$

since $e^{-4\gamma t-\omega t^2/2} D_{-3}(-\sqrt{2\omega}\, t)$ is clearly $L_1([0,\infty))$, see (A.37). The second part can be estimated as follows

$$\begin{aligned} &\int_{x_1+\delta}^{\bar{x}} \mathrm{d}x e^{-4\gamma(x-x_1)}\, e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \\ &\le e^{-4\gamma\delta} \int_{x_1+\delta}^{\bar{x}} e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \\ &\le e^{-4\gamma\delta}(\bar{x}-x_1-\delta) \sup_{x\in[x_1+\delta,\bar{x}]} D_{-3}(\sqrt{2\omega}(\bar{x}-x)) \le \mathcal{C}\, F^{-(1-\varepsilon)}\, e^{-4\gamma\delta}, \quad F\to 0 \end{aligned} \tag{A.45}$$

Finally, the third part is bounded above by

$$\begin{aligned} &\int_{-\infty}^{x_1+\delta} e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \le e^{-\omega\bar{x}^2/2} \int_{-\infty}^{0} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \\ &+ e^{-\omega(\bar{x}-x_1-\delta)^2/2} \int_0^{x_1+\delta} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \\ &\le \mathcal{C}\, e^{-\omega(\bar{x}-x_1-\delta)^2/2}, \quad F\to 0 \end{aligned} \tag{A.46}$$

where we have employed the asymptotic expansion (A.37).

The estimate of the second part of (A.42), which contains the function

$$D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \tag{A.47}$$

is a bit more subtle. After dividing the integration again in three parts according to (A.43) and substituting

$$t := \frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}} \tag{A.48}$$

one gets

$$\begin{aligned} &\int_{\bar{x}}^{\infty} \mathrm{d}x e^{-4\gamma(x-x_1)} e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \\ &\le e^{-4\gamma(\bar{x}-x_1)} \int_{4\gamma/\sqrt{2\omega}}^{\infty} \exp\left[-\frac{t^2}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\, t - \frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \\ &\le \mathcal{C}\, e^{-\mathcal{C}\, F^{-2(1-\varepsilon)}}, \quad F\to 0 \end{aligned} \tag{A.49}$$

provided

$$\omega(\bar{x} - x_1) > \gamma \tag{A.50}$$

this can be seen taking the maximum of the exponential function in the integral and the fact that $D_{-3}(t) \in L_1([0,\infty))$.

For $x \in (-\infty, x_1 + \delta]$ we have similarly

$$\begin{aligned} &\int_{-\infty}^{x_1+\delta} \mathrm{d}x e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \\ &\leq \int_{-\infty}^{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t - \frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \end{aligned} \tag{A.51}$$

Since

$$\exp\left[-\frac{t^2}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t\right] D_{-3}(t) \in L_1((-\infty,0]) \tag{A.52}$$

it suffices to estimate the integral for positive values of t. In this case we use the fact that

$$D_{-3}(z)e^{\xi z^2/4} \in L_1([0,\infty)),$$

for any $\xi < 1$. Then

$$\begin{aligned} &\int_{0}^{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2(1+\xi)}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t - \frac{4\gamma^2}{\omega}\right] e^{\xi t^2/4}\, D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \\ &\leq \mathcal{C}\, e^{-\mathcal{C}\, F^{-2(1-\varepsilon)}}, \quad F \to 0 \end{aligned} \tag{A.53}$$

whenever

$$1 > \xi > \frac{4\gamma^2 - \omega^2(x_1+\delta-\bar{x})^2}{4\gamma^2 + \omega^2(x_1+\delta-\bar{x})^2} = \frac{4\gamma_0^2 - \omega^2(C_1+\delta_0-\bar{C})^2}{4\gamma_0^2 + \omega^2(C_1+\delta_0-\bar{C})^2}$$

We are thus left with

$$\begin{aligned} &\int_{x_1+\delta}^{\bar{x}} \mathrm{d}x\, e^{-4\gamma(x-x_1)} e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \\ &\leq e^{-4\gamma\delta} \int_{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}}^{\frac{4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t - \frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \end{aligned} \tag{A.54}$$

Due to (A.52) it is enough to show that

$$\int_{0}^{\frac{4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t - \frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \leq \mathcal{C}\, F^{-(1-\varepsilon)} \tag{A.55}$$

This is however easily seen since

$$-\frac{t^2}{2} + \frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t - \frac{4\gamma^2}{\omega} \leq 0, \quad \forall\, t \in \left[0, \frac{4\gamma}{\sqrt{2\omega}}\right] \tag{A.56}$$

and

$$\sup_{t\in[0,\frac{4\gamma}{\sqrt{2\omega}}]} e^{t^2/4} D_{-3}(t) \leq \sup_{t\in[0,\infty)} e^{t^2/4} D_{-3}(t) \leq \mathcal{C}$$

To conclude we remark that the second term of (A.40), which leads to

$$\begin{aligned}&\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}\,\bar{x}^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \leq \mathcal{C}\,F^{-2(1-\varepsilon)}\,e^{-\omega(x-\bar{x})^2}\left[e^{\frac{\omega(x-\bar{x})^2}{2}} D_{-1}(\sqrt{2\omega}\,(\bar{x}-x))\right.\\ &+\quad \left. e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-1}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right)\right], \qquad \text{(A.57)}\end{aligned}$$

can be control in the same way, because the asymptotic behaviour (A.37) is again governed by $\exp[\pm t^2/4]$.

Finally, for the case $|x-x'|<1$ we follows the same method as in (A.38) where the decay come from the "infinitesimally small" overlap of h_F^c with J_0 the latter also "localise" $|x'|$, i.e. $|J_0(x+ib)|^{1/2}|(x'+ib)| \leq \mathcal{C}F^{-(1-\varepsilon)}$. Summing up all the contributions we have

$$\|[H_L, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A\| \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \qquad \text{(A.58)}$$

Let us next analyse the last term of (A.25), which includes the potential $F(x+ib)\chi_A^c$. When $|x-x'|\geq 1$, for the terms in the commutator involving the $x-$derivatives, the integration w.r.t. x and x' in the Hilbert-Schmidt norm gives a constant proportional to $F^{-2(1-\varepsilon)}$. We then obtain the estimate on the Hilbert-Schmidt norm

$$\begin{aligned}&\|\partial_x^n J_2(ib)\partial_x^{(2-n)}R_1(z;ib)F(x+ib)\chi_A^c\|_{HS}^2\\ &\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\int_{-y_1}^{y_1}\mathrm{d}y\int_{|y'|\geq y_1+F^{-\tau}} e^{-2\beta(z)|y-y'|}\,\mathrm{d}y'\\ &\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-\frac{\beta(z)}{F^\tau}}\int_{-\infty}^{\infty} e^{-\beta(z)\,|y-y'|}\,\mathrm{d}y'\\ &\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-\frac{\beta(z)}{F^\tau}} \qquad \text{(A.59)}\end{aligned}$$

When $|x-x'|<1$ the $x-$derivative "localises" the term $|x'+ib|$ and the decay comes from the decay of the Green function along y as for the case $i=3$.

For the term of the commutator which corresponds to

$$\partial_y^n J_2(ib)\partial_y^{(2-n)}R_1(z;ib)F(x+ib)\chi_A^c, \quad |x-x'|\geq 1, \quad n=1,2$$

we recall (A.8) to find out that

$$\begin{aligned}
&\sup_{\mathbf{x}} \int_{\mathbb{R}^2} |J_0(x+ib)\partial_y^n J_c(y)\partial_y^{(2-n)} G_1(\mathbf{x},\mathbf{x}';z)F(x'+ib)\chi_A^c(y')|\,\mathrm{d}\mathbf{x}' \\
&\leq \frac{\mathcal{C}}{F^{1-\varepsilon}}\,\beta(z)^{-\sigma} \sup_{y\in\operatorname{supp}\partial_y^n J_c} \int_{|y'|\geq y_1+F^{-\tau}} e^{-\beta(z)|y-y'|}\,\mathrm{d}y' \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}}\, e^{-\frac{\beta(z)}{F^\tau}} \qquad \text{(A.60)}
\end{aligned}$$

and similarly the other way around. Finally at short distances the same argument as in the previous case holds. Therefore

$$\|[H_L, J_2(ib)]R_1(z;ib)F(x+ib)\chi_A^c\| \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\frac{\beta(z)}{F^\tau}} \qquad \text{(A.61)}$$

Taking into account all the estimates (A.32), (A.58), (A.61) made above, we can claim that for F small enough

$$\|K_2(z;ib)\| \leq \mathcal{C}\,F^{-\mathcal{C}}\,\beta(z)^{-\sigma(z)}\left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right)(1+\|R_2(z;ib)\|) \qquad \text{(A.62)}$$

Inequality (A.62) plays an essential role in our estimates, because it tells us how close we can get to the spectrum of $H_2(F,ib) = H_2(F)$ and $H_1(F,ib)$ while keeping the resolvent of $H(F,ib)$ bounded.

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