# Perturbative Analysis of Dynamical Localisation

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**Abstract.** In this paper we extend previous results on convergent perturbative solutions of the Schrödinger equation of a class of periodically time-dependent two-level systems. The situation treated here is particularly suited for the investigation of two-level systems exhibiting the phenomenon of (approximate) dynamical localisation. We also present a convergent perturbative expansion for the secular frequency and discuss in detail the particular case of monochromatic interactions (ac-dc fields), providing a complete perturbative solution for that case. Our method is based on a "renormalisation" procedure, which we develop in a more systematic way here. For being free of secular terms and uniformly convergent in time, our expansions allow a rigorous study of the long-time behaviour of such systems and are also well-suited for numerical computations, as we briefly discuss, leading to very accurate calculations of quantities like transition probabilities for very long times compared to the cycles of the external field.

## 1 General Description and Previous Results

The study of periodically or quasi-periodically time-dependent two-level systems is of basic importance for many physical applications, ranging from condensed matter physics to quantum optics, as in problems of the theory of spin resonance, in problems of quantum tunnelling or in the semi-classical theory of the laser. They can be used, for instance, to describe the behaviour of a spin 1/2 system in a time-dependent magnetic field, in which case the corresponding Schrödinger equation takes the form (we adopt $\hbar = 1$)

$$i\partial_t \Psi \;=\; H(t)\Psi\,, \qquad \text{with} \qquad H(t) \;=\; -\frac{1}{2}\vec{B}(t)\cdot\vec{\sigma}\,, \tag{1.1}$$

where $\Psi(t) = \begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} \in \mathbb{C}^2$, $\vec{B}(t) = (B_1(t),\, B_2(t),\, B_3(t))$ and $\vec{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$ are the Pauli matrices.

Systems like this have been analysed by many authors in various approximations, as in the pioneering works of Rabi [8], of Bloch and Siegert [9] and of

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Autler and Townes [10] (see also [5, 6] for more recent discussions). We should remark, however, that the interest in the solutions of (1.1) is not restricted to the investigation of quantum systems. As first pointed by Feynman, Vernon and Hellwarth [11] (see also the recent discussion in [4]), the quantum system (1.1) is equivalent to the classical Hamiltonian system describing a classical gyromagnet precessing in a magnetic field: $\frac{d}{dt}\vec{\mathcal{S}} = -\vec{B}(t) \times \vec{\mathcal{S}}$, where $\vec{\mathcal{S}}$ is a unit vector.

Of particular interest is the situation where the Schrödinger equation takes the form

$$i\partial_t \Psi(t) \;=\; H_1(t)\Psi(t), \qquad \text{with} \qquad H_1(t) \;:=\; \epsilon\sigma_3 - f(t)\sigma_1, \tag{1.2}$$

where $f(t)$ is a function of time $t$ and $\epsilon \in \mathbb{R}$ is constant. By a time-independent unitary transformation, representing a rotation of $\pi/2$ around the 2-axis, we get the equivalent system

$$i\partial_t \Phi(t) \;=\; H_2(t)\Phi(t), \qquad \text{with} \qquad H_2(t) \;:=\; \epsilon\sigma_1 + f(t)\sigma_3, \tag{1.3}$$

where $\Phi(t) := \exp(-i\pi\sigma_2/4)\,\Psi(t)$ and $H_2(t) := \exp(-i\pi\sigma_2/4)\; H_1(t)\, \exp(i\pi\sigma_2/4)$.

One can either interpret the system (1.2) as describing a spin 1/2 system as (1.1) under a magnetic field $\vec{B} = (2f(t),\, 0,\, -2\epsilon)$, or as a system with an unperturbed diagonal Hamiltonian $H_0 := \epsilon\sigma_3$, representing a two-level system with energy levels $\pm\epsilon$, subjected to a time-dependent perturbation $H_I(t) := -f(t)\sigma_1$, inducing a time-depending transition between the unperturbed eigenstates of $H_0$. The equivalent system (1.3), in turn, represents either a spin 1/2 system as (1.1) under a magnetic field $\vec{B} = (-2\epsilon,\, 0,\, -2f(t))$, or a two-level system composed by two uncoupled (for $\epsilon = 0$) orthogonal time-dependent states $\exp\left(-i\int_0^t f(\tau)d\tau\right)\binom{1}{0}$ and $\exp\left(+i\int_0^t f(\tau)d\tau\right)\binom{0}{1}$, subjected to a constant perturbation $\epsilon\sigma_1$ inducing a transition between them.

To explain the purpose of the present paper, we have to describe some of our previous results. In [1] and [2] we studied the system described by (1.2) or (1.3) in the situation where $f$ is a periodic or quasi-periodic function of time and $\epsilon$ is "small". It is well know that the usual perturbative approach, based, f.i., on the Dyson series, leads to difficulties involving secular terms (i.e., polynomials in $t$ that appear order by order in perturbation theory and spoil the uniform convergence (in $t$) of the perturbative series) and, for quasi-periodic interactions, small denominators. This last problem is typical of perturbative approximations for solutions of differential equations with quasi-periodic coefficients and is well-known as one of the main sources of problems in the mathematically precise treatment of such equations.

In [1] and [2], a special perturbative expansion (power series expansion in $\epsilon$) was developed, whose main virtue is to be free of secular terms. The algorithm employed involves an inductive "renormalization" of a sort of effective field introduced through an exponential Ansatz (the function $g$, to be introduced below). For the sake of the reader we will shortly recall our method of elimination of secular

terms in Section 2. In the general case where $f$ is *quasi-periodic*, it was established in [1] that the coefficients of the expansion are also well-defined quasi-periodic functions of time but, due mainly to the presence of small denominators, we were not able to prove convergence of our $\epsilon$-expansion. Actually, a convergent power expansion in $\epsilon$ is not expected without further assumptions (for a detailed analysis of these issues in related systems, see [12]).

Less problematic is the situation where $f$ is a *periodic* function, when the obstacle represented by the small denominators is naturally absent. In [2], we showed how the difficulties analysed in [1] can be circumvented in the case of periodic $f$ and we were able to establish the convergence of our perturbative $\epsilon$-expansion uniformly in $t \in \mathbb{R}$.

As discussed in [2], our method not only recovers the Floquet form of the solution of the time-depending Schrödinger equation (see (1.8)-(1.9) below), but also allows the computation of the secular frequency and of the Fourier coefficients in terms of explicit *convergent* $\epsilon$-expansions, what constitutes a feature of our algorithm, compared to other expansion methods.

Due to the technical difficulties involved, we restricted our analysis in [2] to two classes of periodic functions, namely those satisfying the conditions (I) or (II) presented below (see [2]). Our purpose in the present paper is to extend the results of [2] to an additional class of periodic functions. The inclusion of this additional class leads to a essentially complete perturbative solution for some simple periodic functions, as $f(t) = F_0 + \varphi \cos(\omega t)$, representing the important case of a monochromatic interaction (also known as ac-dc fields).

The situation we treat here is also relevant for the rigorous discussion of the phenomenon of dynamical localisation, also known (less properly) as coherent destruction of tunnelling. This phenomenon, first pointed in [14], indicates the possibility to (approximately) freeze the initial state of a quantum system through the action of a suitable external time-dependent interaction. This effect has been the object of various recent investigations. In [7], for instance, a rigorous general criterion for the occurrence of dynamical localisation was established and applied to interesting situations, like the ac-dc field and the bichromatic field. Some of the conclusions of [7] on the ac-dc field are indirectly reproduced in Section 5, below. We refer the reader to [3] and [7] for more references on this subject.

The main result of [2] can be captured in the next theorem, for whose statement we need a definition we will repeatedly use in this work: for an almost periodic function $h$ we denote by $M(h)$ the "mean value" of $h$, defined as

$$M(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(t)\, dt\,. \tag{1.4}$$

We remark that the limit in (1.4) is always well defined for any quasi-periodic function $h$. The mean value $M(h)$ equals the constant term in the Fourier expansion of $h$. Details can be found in [15, 16].

**Theorem 1.1** *Let $f$ be a real $T_\omega$-periodic function of time ($T_\omega := 2\pi/\omega$ with $\omega > 0$) whose Fourier decomposition $f(t) = \sum_{n\in\mathbb{Z}} F_n e^{in\omega t}$, contains only a finite number of terms, i.e., the set of integers $\{n \in \mathbb{Z} \big| F_n \neq 0\}$ is a finite set. Let*

$$\Phi(t) = \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} = U(t)\Phi(0) = U(t,\,0)\Phi(0) \tag{1.5}$$

*be the solution of the Schrödinger equation (1.3). Consider the two following distinct conditions on $f$:*

(I) $M(\mathcal{Q}_0) \neq 0$.

(II) $M(\mathcal{Q}_0) = 0$ *but* $M(\mathcal{Q}_1) \neq 0$, *where*

$$q(t) := \exp\left(i\int_0^t f(\tau)d\tau\right), \quad \mathcal{Q}_0(t) := q(t)^2 \;=\; \exp\left(2i\int_0^t f(\tau)d\tau\right) \tag{1.6}$$

*and*

$$\mathcal{Q}_1(t) \;:=\; \mathcal{Q}_0(t)\int_0^t \left(\mathcal{Q}_0(\tau)^{-1} - M\left(\mathcal{Q}_0^{-1}\right)\right)d\tau. \tag{1.7}$$

*Then, for each $f$ as above, satisfying condition (I) or (II), there exists a constant $K > 0$ (depending on the Fourier coefficients $\{F_n,\ n \in \mathbb{Z}\ , n \neq 0\}$ and on $\omega$) so that, for each $\epsilon$ with $|\epsilon| < K$, there are $\Omega \in \mathbb{R}$ and $T_\omega$-periodic functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ such that the propagator $U(t)$ of (1.5) can be written as*

$$U(t) \;=\; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ \\ U_{21}(t) & U_{22}(t) \end{pmatrix} \;=\; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix},$$

*with*

$$U_{11}(t) \;=\; e^{-i\Omega t}\,u_{11}^-(t) + e^{i\Omega t}\,u_{11}^+(t), \quad U_{12}(t) \;=\; e^{-i\Omega t}\,u_{12}^-(t) + e^{i\Omega t}\,u_{12}^+(t). \tag{1.8}$$

*The functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ have absolutely and uniformly converging Fourier expansions*

$$u_{11}^{\pm}(t) \;=\; \sum_{n\in\mathbb{Z}} \mathcal{U}_{11}^{\pm}(n)e^{in\omega t}, \qquad u_{12}^{\pm}(t) \;=\; \sum_{n\in\mathbb{Z}} \mathcal{U}_{12}^{\pm}(n)e^{in\omega t}. \tag{1.9}$$

*Moreover, under the same assumptions, $\Omega$ and the Fourier coefficients $\mathcal{U}_{11}^{\pm}(n)$ and $\mathcal{U}_{12}^{\pm}(n)$ can be expressed in terms of absolutely converging power series on $\epsilon$.* □

Let us now discuss the conditions (I) and (II) of Theorem 1.1. Writing the Fourier decomposition of $f$ as $f(t) = F_0 + \sum_{n=1}^{J}\left[\varphi_1^{(n)}\cos(n\omega t) + \varphi_2^{(n)}\sin(n\omega t)\right]$ the set $\mathfrak{F}_{J,\,\omega}$ of all possible functions $f$ with a given $J$ and $\omega$ can be identified

with the parameter space $\mathbb{R}^{2J+1}$ of all real coefficients $F_0$, $\varphi_{1,\,2}^{(n)}$, $1 \leq n \leq J$. The (complex) condition $M(\mathcal{Q}_0) = 0$ determines a $(2J)$ or $(2J-1)$-dimensional subset of $\mathfrak{F}_{J,\,\omega}$, where condition (II) eventually applies. It is also on this subset that the more restrictive condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$ should hold, restricting the parameter space of $f$ to a $(2J-1)$, $(2J-2)$ or $(2J-3)$-dimensional subset, if it is non-trivial. One should, therefore, expect that successive conditions like (I) and (II) would eventually exhaust completely the set $\mathfrak{F}_{J,\,\omega}$.

To illustrate all this, let us consider the simplest example, when $f$ represents a monochromatic interaction: $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$, with $(\varphi_1,\, \varphi_2) \in \mathbb{R}^2$. A simple computation shows that $M(\mathcal{Q}_0) = e^{2i\gamma_f} J_0\left(\frac{2\varphi_0}{\omega}\right)$, where $\varphi_0 := \sqrt{\varphi_1^2 + \varphi_2^2}$, $J_0$ is the Bessel function of first kind and order zero and, in this case, $\gamma_f = \varphi_2/\omega$. Moreover, $M(\mathcal{Q}_1) = 0$ for <u>all</u> $(\varphi_1,\, \varphi_2) \in \mathbb{R}^2$. (See [2] for details). Hence, condition (I) is satisfied for all $(\varphi_1,\, \varphi_2) \in \mathbb{R}^2$, except in the circles defined by $\varphi_0 = \omega x_a/2$, $a = 1,\, 2, \ldots$, where $x_a$ if the $a$-th zero of $J_0$ in $\mathbb{R}_+$. Condition (II), however, is never fulfilled in this case. To achieve a complete solution we have, therefore, to extend Theorem 1.1 to include further conditions beyond (I) and (II), holding on the circles $\varphi_0 = \omega x_a/2$.

The purposes of this paper are to identify the first condition following (I) and (II), which we call condition (III), to show that the method of elimination of secular terms holds in this case as well and, for periodic interactions, to show that the expansion (2.5) converges for $|\epsilon|$ sufficiently small, uniformly for $t \in \mathbb{R}$. As we will discuss, this leads to a complete perturbative solution for the monochromatic interaction. As we will see, the identification of condition (III) and the application of the method of elimination of secular terms to it are highly non-trivial tasks.

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This paper is organised as follows. In Section 2 we describe the general strategy employed to eliminate the secular terms and present our main theorems. In Section 3 we prove some interesting and useful mathematical results on the mean value of some quasi-periodic functions, introduce the "renormalisation" operation, important to organise our procedures, and introduce some notations we will use throughout the paper. In Section 4, which is the technically central piece or this work, we apply our strategy of elimination of the secular terms to the situation we wish to analyse. In Section 5 we apply our results to case of monochromatic interactions and discuss the issue of dynamical localisation in this case. In Section 6 we briefly describe some numerical calculations based on our results. Sections 5 and 6 contain some of the physical applications of our work. In Appendix A we treat several results used in Section 4, whose proofs unfortunately involve too many algebraic computations to be included in the main text. Appendix B is dedicated to the proof of convergence of our expansions for the periodic case. Finally, Appendix C contains some comments on the Fourier coefficients of the wave functions.

### 1.1 Comments on the Notation

In this paper, $\mathbb{Z}_+$ will denote the set of all non-negative integers (zero included) and $\mathbb{Z}_*$ the set of all integers, excluding zero. $\mathbb{Z}_{*+}$ is the set of all positive integers. These notations are also applied to $\mathbb{Z}^A$ and to the real line $\mathbb{R}$.

Vectors in $\mathbb{Z}^A$ (or $\mathbb{R}^A$) will be written as $\underline{v}$. The operation $\underline{v} \cdot \underline{u}$ will denote the scalar product in $\mathbb{Z}^A$ (or $\mathbb{R}^A$), defined as $\underline{v} \cdot \underline{u} := v_1 u_1 + \cdots + v_A u_A$.

For a quasi-periodic function $h : \mathbb{R} \to \mathbb{C}$, we write its Fourier decomposition as $h(t) = \sum_{\underline{m} \in \mathbb{Z}^A} H_{\underline{m}} e^{i \underline{m} \cdot \underline{\omega}_h t}$, where $A$ is some positive integer and $\underline{\omega}_h \in \mathbb{R}_+^A$ (see, e.g., [15, 16]). The Fourier coefficient $H_{\underline{0}}$ will be denoted simply as $H_0$.

For $m \in \mathbb{Z}$ we denote by $\ll m \gg$ the following function:

$$\ll m \gg := \begin{cases} |m|, & \text{for } m \neq 0 \\ 1, & \text{for } m = 0 \end{cases} . \tag{1.10}$$

Beyond the functions $\mathcal{Q}_0$ and $\mathcal{Q}_1$ defined in (1.6) and (1.7) we will frequently use the following functions

$$\mathcal{Q}_2(t) := \mathcal{Q}_0(t) \int_0^t \left( \mathcal{Q}_0(\tau) - M\left(\mathcal{Q}_0\right) \right) d\tau, \tag{1.11}$$

$$\mathcal{Q}_3(t) := \mathcal{Q}_0(t) \int_0^t \left( \mathcal{Q}_1(\tau) - M\left(\mathcal{Q}_1\right) \right) d\tau. \tag{1.12}$$

Note that, by their definitions, the functions $\mathcal{Q}_i$, $i = 0, \ldots, 3$, are quasi-periodic if $f$ is quasi-periodic. We will have more to say about their properties below.

## 2 Elimination of Secular Terms. The Main Results

We recall in this section some of the methods and techniques developed in the previous works [1] and [2]. A key result for our method is the theorem below, proven in [1], which presents the solution of the Schrödinger equation (1.3) in terms of particular solutions of a generalised Riccati equation.

**Theorem 2.1** *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{C}$, $g \in C^1(\mathbb{R})$, be a particular solution of the generalised Riccati equation*

$$\dot{G} - iG^2 - 2ifG + i\epsilon^2 = 0. \tag{2.1}$$

*Then, the function $\Phi : \mathbb{R} \to \mathbb{C}^2$ given by $\Phi(t) = \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} = U(t)\Phi(0) = U(t, 0)\Phi(0)$, where*

$$U(t) := \begin{pmatrix} R(t)\left(1 + ig(0)S(t)\right) & -i\epsilon R(t) S(t) \\ -i\epsilon \overline{R(t)}\, \overline{S(t)} & \overline{R(t)}\left(1 - i\, \overline{g(0)}\, \overline{S(t)}\right) \end{pmatrix}, \tag{2.2}$$

*with*

$$R(t) \;:=\; \exp\left(-i\int_0^t (f(\tau)+g(\tau))\,d\tau\right) \qquad \textit{and} \qquad S(t) \;:=\; \int_0^t R(\tau)^{-2}\,d\tau,$$

*is a solution of (1.3) with initial value* $\Phi(0) = \left(\begin{smallmatrix}\phi_+(0)\\ \phi_-(0)\end{smallmatrix}\right) \in \mathbb{C}^2$. □

Let us briefly describe some of the ideas leading to Theorem 2.1 and to other results of [1]. As we saw in [1], the solutions of the Schrödinger equation (1.3) can be studied in terms of the solutions of a particular complex version of Hill's equation:

$$\ddot{\phi}(t) + \left(i\dot{f}(t) + \epsilon^2 + f(t)^2\right)\phi(t) \;=\; 0. \tag{2.3}$$

In fact, a simple computation shows that the components $\Phi(t)$ satisfy $\ddot{\phi}_\pm + (\pm i\dot{f} + \epsilon^2 + f^2)\phi_+ = 0$. If we attempt to solve (2.3) using the Ansatz

$$\phi(t) = \exp\left(-i\int_0^t (f(\tau)+g(\tau))d\tau\right), \tag{2.4}$$

it follows that $g$ has to satisfy the generalised Riccati equation (2.1). We then try to find solutions for $g$ in terms of a power expansion in $\epsilon$ (vanishing for $\epsilon = 0$) like

$$g(t) \;=\; q(t)\sum_{n=1}^{\infty} v_n(t)\,\epsilon^n\,, \tag{2.5}$$

where the function $q$ was defined in (1.6) and is of central importance in this work.

The heuristic idea behind the Ansätze (2.4) and (2.5) is the following. For $\epsilon \equiv 0$ a solution for (2.3) is given by $\exp\left(-i\int_0^t f(\tau)d\tau\right)$. Thus, in (2.4) and (2.5) we are searching for solutions in terms of an "effective external field" of the form $f+g$, with $g$ given in terms of a convergent power series expansion in $\epsilon$, vanishing for $\epsilon = 0$. A solution of the form (2.4) leads to one of the two independent solutions of (2.3). The full solution of (1.3) in terms of solutions of the generalised Riccati equation (2.1) is that described in Theorem 2.1 (see the discussion of [1]).

We then proceed inserting (2.5) into (2.1). The result is a set of recursive first order linear differential equations for the functions $v_n$ which can be easily integrated. The solutions of these equations are

$$v_1(t) \;=\; \kappa_1 q(t)\,, \tag{2.6}$$

$$v_2(t) \;=\; q(t)\left[i\int_0^t (\kappa_1^2\mathcal{Q}_0(\tau) - \mathcal{Q}_0(\tau)^{-1})\,d\tau + \kappa_2\right], \tag{2.7}$$

$$v_n(t) \;=\; q(t)\left[i\left(\int_0^t \sum_{p=1}^{n-1} v_p(\tau)v_{n-p}(\tau)\,d\tau\right) + \kappa_n\right], \quad \text{for } n \geq 3\,, \tag{2.8}$$

where the $\kappa_n$'s above, $n = 1, 2, \ldots$, are arbitrary integration constants. Defining

$$\mathcal{I}_2(t) \;:=\; \kappa_1^2 \mathcal{Q}_0(t) - \mathcal{Q}_0(t)^{-1}, \qquad \mathcal{I}_n(t) \;:=\; \sum_{p=1}^{n-1} v_p(t) v_{n-p}(t), \quad n \geq 3.$$

we can write (2.6)-(2.8) as

$$v_1(t) \;=\; \kappa_1 q(t), \qquad v_n(t) \;=\; iq(t) \int_0^t \mathcal{I}_n(\tau) d\tau + \kappa_n q(t), \quad n \geq 2.$$

Observe that, in particular, we could just set all the $\kappa_n$'s equal to zero. However, this is not a clever choice, since it would result in polynomial terms on $t$ (the so-called *Secular Terms*) for the series expansion (2.5) of $g$. This, of course, would restrict the convergence of the series just for small values of time. As noticed in [1], there is a choice of the constants $\kappa_n$ for which one can eliminate completely all the polynomial terms on $t$ that would eventually appear in $g$. This procedure, which we call the *Elimination of Secular Terms*, will be briefly described now.

First of all, assuming that the function $f : \mathbb{R} \to \mathbb{R}$ is quasi-periodic, it was proven in Appendix B of [1] that $q$, defined in (2.5), is also quasi-periodic. Hence, $v_1$ in (2.6) is quasi-periodic. The same is true for the integrand $\mathcal{I}_2$ which appears in $v_2$, equation (2.7). Recalling that $\mathcal{I}_2$ depends on the free integration constant $\kappa_1$, the key idea is to fix $\kappa_1$ in such a way that the mean value of $\mathcal{I}_2$ is equal to zero, that is $M(\mathcal{I}_2) = M(\kappa_1^2 \mathcal{Q}_0 - \mathcal{Q}_0^{-1}) = 0$. Since $\mathcal{Q}_0$ is a quasi-periodic function, it readily follows from this that

$$\kappa_1^2 \;=\; \frac{\overline{M(\mathcal{Q}_0)}}{M(\mathcal{Q}_0)}\,. \tag{2.9}$$

With this choice of $\kappa_1$ one guarantees the absence of a constant term in the Fourier expansion of $\mathcal{I}_2$. Since $\mathcal{I}_2$ is being integrated in time, this would imply the absence of a linear term on $t$ in the final expression for $v_1$. An important remark is that (2.9) will only make sense if we assume $M(\mathcal{Q}_0) \neq 0$.

Under this assumption we can now proceed and fix recursively all integration constants $\kappa_m$'s by imposing a zero mean value for the integrands $\mathcal{I}_n$'s, $n = 3,\ 4,\ \ldots$, which appear in (2.8). This procedure removes, order by order in $\epsilon$, the presence of the secular terms in the series expansion (2.5) for $g$ and recursively implies that all functions $v_n$ are quasi-periodic. Once all secular terms have been removed, one can write the Fourier expansion for the functions $v_n$ as

$$v_n(t) \;=\; \sum_{\underline{m} \in \mathbb{Z}^A} V_{\underline{m}}^{(n)} e^{i\underline{m} \cdot \underline{\omega} t}\,, \tag{2.10}$$

provided the sum converges absolutely. It was shown in [1] that this is indeed true. The proof of this fact was performed in the following way: first it was shown that the Fourier coefficients $Q_{\underline{m}}$ of the function $q$ satisfies the bound $|Q_{\underline{m}}| \leq \mathcal{Q} e^{-\chi |\underline{m}|}$,

for some $\chi > 0$. Then, the method of elimination of secular terms described above was applied to fix the integration constants $\kappa_n$ leading to inductive bounds of the form $|V_{\underline{m}}^{(n)}| \leq \mathcal{K}_n e^{-(\chi-\delta_n)|\underline{m}|}$ for the Fourier coefficients of the function $v_n$, where $0 < \delta_n < \chi$, for all $n = 1, 2, \ldots$. This exponential decay is enough to proof the convergence of the sum in (2.10) and to establish by induction the quasi-periodicity of all the functions $v_n$. Unfortunately, due to the bad behaviour in $n$ of the constants $\mathcal{K}_n$, it was not possible prove the convergence of the $\epsilon$-expansion (2.5), hence (2.5) has to be seen as a formal quasi-periodic power series solution of the generalised Riccati equation (2.1).

The reason for the bad behaviour of $\mathcal{K}_n$ is related to the presence of convolutions and to the *small denominators* appearing in the recursive relations for the coefficients $V_{\underline{m}}^{(n)}$. A general discussion of these problems is found in [1]. However, in the situation where $f$ is a *periodic* function, stronger results are possible. In [2], where this situation was studied, it was possible to prove the convergence of the power series (2.5) and uniform convergence of the Fourier series involved in the computation of the wave functions. Moreover, absolute convergence of the $\epsilon$-expansions leading to the *secular frequency* and to the coefficients of the Fourier expansion of the wave functions was also proven.

All the work done in [1] and [2] was restricted to one of the mutually exclusive conditions (I) and (II) of Theorem 1.1. These conditions are consequences of the method of elimination of secular terms. Clearly, (I) is vital for (2.9). When (I) is not satisfied we have to apply condition (II). Both cases were studied in [1] and [2], where the method of elimination of secular terms has been applied and equivalent results concerning the solution $g$ were obtained. In the present work, we apply the method of elimination of secular terms to study a more restrictive condition than those represented by (I) and (II). Namely, we are concerned here with the situation where $M(\mathcal{Q}_0) = 0$ and also $M(\mathcal{Q}_1) = 0$.

In this case, the complexity of the calculations involved to find the right choice of constants $\kappa_n$'s grows enormously, in contrast with those needed in the cases (I) and (II), already studied. The reason for that is quite simple: due to the hypothesis $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$ one needs to work explicitly with higher order terms involved in the expansion (2.5). This will become more clear in Section 4.

In Section 5 we will discuss an important example where conditions (I) and (II) are not satisfied. To solve it, we have to apply the solution (free of secular terms) obtained here. We present in Section 6 numerical calculations on this particular example and obtained some interesting results.

We are ready now to state two main theorems of this work.

**Theorem 2.2** *Let $f : \mathbb{R} \to \mathbb{R}$ be a real quasi-periodic function satisfying*

$$\text{(III)} \quad M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0, \text{ but } M(\mathcal{Q}_3) \neq 0.$$

*Then, there are constants $\kappa_n$, $n \geq 1$, such that all functions $v_n$ given in (2.6)-(2.8) are quasi-periodic. The explicit recursive expressions for the constants $\kappa_n$ are found in (4.36)-(4.38).* □

The proof of this theorem is the main content of Section 4. It states that the procedure of elimination of secular terms outlined above also works under condition (III). When $f$ is quasi-periodic this does not imply, however, that the formal solution (2.5) of the generalised Riccati equation (2.1) converges, since we have the same difficulties discussed in detail in [1].

For periodic $f$, the situation is different and stronger results can be proven. Let $f(t) = \sum_{m\in\mathbb{Z}} F_m e^{im\omega t}$ be a real periodic function with frequency $\omega$. If $F_0 = M(f) = 0$, $q$ and $\mathcal{Q}_0 = q^2$ are also periodic and their spectra of frequencies are subsets of $\{n\omega,\ n \in \mathbb{Z}\}$. Following the notation employed in [2], we write the Fourier expansions of $q$ and $\mathcal{Q}_0$ as

$$q(t) \ = \ \sum_{m\in\mathbb{Z}} Q_m e^{im\omega t}, \qquad \mathcal{Q}_0(t) \ = \ q(t)^2 \ = \ \sum_{m\in\mathbb{Z}} Q_m^{(2)} e^{im\omega t}. \tag{2.11}$$

By relations (2.6)-(2.8) and with the choice of constants $\kappa_n$ mentioned in Theorem 2.2 (see (4.36)-(4.38)), the functions $v_n$ are also periodic and their spectra of frequencies are also subsets of $\{n\omega,\ n \in \mathbb{Z}\}$. We write their Fourier expansions as

$$v_n(t) \ = \ \sum_{m\in\mathbb{Z}} V_m^{(n)} e^{im\omega t}. \tag{2.12}$$

In Appendix B we prove the theorem below, which justifies our whole procedure for the case of periodic interactions and establishes convergence of (2.5).

**Theorem 2.3** *Let $f(t)$ be as above with $F_0 = M(f) = 0$ and such that condition (III) of Theorem 2.2 is satisfied. Moreover, assume that the coefficients $Q_m$ and $Q_m^{(2)}$ above satisfy the following: for any $\chi > 0$ there is a positive constant $\mathcal{Q} \equiv \mathcal{Q}(\chi)$ such that*

$$|Q_m| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m\gg^2} \quad \text{and} \quad \left|Q_m^{(2)}\right| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m\gg^2}, \tag{2.13}$$

*for all $m \in \mathbb{Z}$, where the symbol $\ll m\gg$ was defined in (1.10). Then, with the constants $\kappa_n$ fixed as in Theorem 2.2, the Fourier coefficients of the functions $v_n$ given in (2.6)-(2.8) satisfy*

$$\left|V_m^{(n)}\right| \ \leq \ M_0\,(M_1)^n\,\frac{e^{-\chi|m|}}{\ll m\gg^2}$$

*for all $m \in \mathbb{Z}$ and all $n \geq 1$, for some positive constants $M_0$ and $M_1$. As a consequence, the power series expansion (2.5), representing a solution of the generalised Riccati equation (2.1), converges uniformly for $t \in \mathbb{R}$, provided $|\epsilon| < 1/M_1$.* □

**Remark 2.4** *In [1], (2.13) was established for $f$ is periodic and represented by a finite Fourier series. The condition $F_0 = M(f) = 0$ above is not crucial and can be eliminated following the procedure described in [2].*

It follows from this theorem that the main consequences of Theorem 1.1 are valid under condition (III) as well. In particular, the Floquet form (1.8) holds and the secular frequency $\Omega$ and the Fourier coefficients of (1.9) are analytic functions of $\epsilon$ for $|\epsilon|$ small enough. See Appendix C for a comment on this.

### 2.1 The Secular Frequency

A feature of our method is that it allows to present the complete $\epsilon$-expansion for the secular frequency $\Omega$ (also known as Rabi frequency) associated to the solutions of (1.2)-(1.3) (see (1.8)). One has (see [1, 2, 3])

$$\Omega \;=\; M(f) + M(g) = F_0 + \sum_{n=1}^{\infty} \epsilon^n M\left(q v_n\right) \,.$$

By Theorem 2.3 above, this expansion is convergent for $|\epsilon|$ small enough. The knowledge of the complete expansion is particularly important for the qualitative investigation of the large-time behaviour of that solutions. After some simple calculations using (2.6)-(2.8) one gets

$$\begin{aligned} \Omega \;\;=\;\; & F_0 + \epsilon\kappa_1 M(\mathcal{Q}_0) + \epsilon^2 \left[ i\kappa_1^2 M(\mathcal{Q}_2) - iM(\mathcal{Q}_1) + \kappa_2 M(\mathcal{Q}_0) \right] \\ & + \epsilon^3 \left[ 2\kappa_1 M(\mathcal{Q}_3) + \kappa_3 M(\mathcal{Q}_0) \right] + O(\epsilon^4) \,. \end{aligned} \tag{2.14}$$

As we will show in Corollary 3.2 below, $M(\mathcal{Q}_0) = 0$ implies $M(\mathcal{Q}_2) = 0$. Hence, for case (II),

$$\Omega \;=\; F_0 - i\epsilon^2 M(\mathcal{Q}_1) + 2\kappa_1 \epsilon^3 M(\mathcal{Q}_3) + O(\epsilon^4) \,, \tag{2.15}$$

and for case (III),

$$\Omega \;=\; F_0 + 2\kappa_1 \epsilon^3 M(\mathcal{Q}_3) + O(\epsilon^4) \,. \tag{2.16}$$

Actually, after fixing $\kappa_1$ in Section 4.1, we will see that $\Omega = F_0 + 2\epsilon^3 \left| M(\mathcal{Q}_3) \right| + O(\epsilon^4)$.

In this case (III), if one additionally has $F_0 = 0$, then $\Omega = O(\epsilon^3)$, a fact first pointed in [3]. This implies long transition times for certain probability amplitudes, a phenomenon known as (approximate) dynamical localisation (see [7] and other references therein). In Sections 5 and 6 we discuss this situation for $f$ describing a monochromatic interaction.

## 3 Properties of the Mean Value and the Renormalisation Operator

Let us introduce some notations that will be very useful. Since expressions like $f(\xi) \int_0^\xi d\xi' \, g(\xi')$ will often appear throughout our calculations, we define a short-

hand notation

$$(f\,|\,g)_\xi \;:=\; f(\xi)\int_0^\xi d\xi'\,g(\xi')\,. \tag{3.1}$$

Moreover, if $(f\,|\,g)_t$ is quasi-periodic function on $t$, then $M\,(f\,|\,g)$ will denote its mean value. We also define

$$(f\,|\,g\,|\,h\,)_\xi \;:=\; \Big(f\;\Big|\;(g\,|\,h)_{\xi'}\Big)_\xi \;=\; f(\xi)\int_0^\xi d\xi'\,g(\xi')\int_0^{\xi'} d\xi''\,h(\xi'')\,. \tag{3.2}$$

Further compositions like $(f_1\,|f_2\,|\,\cdots\,|\,f_n)_\xi$ are defined in analogous way, so that, for $n>2$,

$$(f_1\,|f_2\,|\,\cdots\,|\,f_n)_\xi \;:=\; \Big(f_1\;\Big|\;(f_2\,|\,\cdots\,|\,f_n)_{\xi'}\Big)_\xi\,.$$

Note, finally, the trivial fact that

$$f_0(\xi)(f_1\,|f_2\,|\,\cdots\,|\,f_n)_\xi \;=\; (f_0 f_1\,|f_2\,|\,\cdots\,|\,f_n)_\xi\,. \tag{3.3}$$

## 3.1 Properties of the Mean Value

The following general results on the mean value of some quasi-periodic functions (see definition (1.4)) will be used for many purposes in the present work.

**Proposition 3.1** *Let $a:\mathbb{R}\to\mathbb{C}$, $b:\mathbb{R}\to\mathbb{C}$ and $c:\mathbb{R}\to\mathbb{C}$ be quasi-periodic functions with Fourier components denoted by $A_{\underline{m}}$, $B_{\underline{m}}$ and $C_{\underline{m}}$, $\underline{m}\in\mathbb{Z}^A$, respectively. We have the following statements:*

1. *If $M(c)=0$, then the Fourier components of the function $h(t):=(b\,|\,c)_t$ are given by*

$$H_{\underline{n}} \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{i\,C_{\underline{m}}\left(B_{\underline{n}}-B_{\underline{n}-\underline{m}}\right)}{\underline{m}\cdot\underline{\omega}}\,, \tag{3.4}$$

   *for all $\underline{n}\in\mathbb{Z}^A$. Moreover, if $M(b)=0$ and $C_{\underline{m}}B_{-\underline{m}}=C_{-\underline{m}}B_{\underline{m}}$ for all $\underline{m}\in\mathbb{Z}^A_*$, then $M(h)=0$.*

2. *If $M(b)=0$ and $M(c)=0$, then $M\,(b\,|\,c)=-M\,(c\,|\,b)$.*

3. *If $M(c)=0$, then $M\,(c\,|\,c)=0$.*

4. *If $M(a)=M(c)=M\,(b\,|\,c)=0$, then*

$$M\,(a\,|\,b\,|\,c\,) \;=\; -M\left[b(t)\left(\int_0^t a(\tau)\,d\tau\right)\left(\int_0^t c(\tau)\,d\tau\right)\right]\,. \tag{3.5}$$

   *Moreover, if also $M\,(b\,|\,a)=0$, then $M\,(a\,|\,b\,|\,c\,)=M\,(c\,|\,b\,|\,a\,)$.*

5. *If* $M(a) = M(b) = M(c) = 0$ *and* $M(b\,|\,c) = M(c\,|\,a) = M(a\,|\,b) = 0$, *then*

$$M(a\,|\,b\,|\,c) + M(b\,|\,c\,|\,a) + M(c\,|\,a\,|\,b) \;=\; 0\,. \tag{3.6}$$

6. *If* $M(a) = 0$, *it follows from (3) and (5) that* $M(a\,|\,a\,|\,a) = 0$. □

The identity (3.6) is very remarkable. We have noticed that some non-trivial relations follow from it, specially if one write (3.6) in terms of the Fourier coefficients of $a$, $b$ and $c$.

**Proof.** We can demonstrate (1) by explicitly computing the Fourier decomposition of $h(t)$. Thus,

$$\begin{aligned} h(t) &= \sum_{\underline{n}\in\mathbb{Z}^A} B_{\underline{n}} e^{i\underline{n}\cdot\underline{\omega}t} \left[ \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{C_{\underline{m}}}{i\underline{m}\cdot\underline{\omega}} \left(e^{i\underline{m}\cdot\underline{\omega}t} - 1\right) \right] \\ &= \sum_{\underline{n}\in\mathbb{Z}^A} \left[ \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{i\,C_{\underline{m}}\left(B_{\underline{n}} - B_{\underline{n}-\underline{m}}\right)}{\underline{m}\cdot\underline{\omega}} \right] e^{i\underline{n}\cdot\underline{\omega}t}\,, \end{aligned}$$

proving the first statement of (1). Now, if $M(b) = B_0 = 0$ and $C_{\underline{m}}B_{-\underline{m}} = C_{-\underline{m}}B_{\underline{m}}$ for all $\underline{m} \in \mathbb{Z}^A_{*+}$, then

$$H_0 \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,C_{\underline{m}}B_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}} \;=\; \frac{-i}{2} \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{(C_{\underline{m}}B_{-\underline{m}} - C_{-\underline{m}}B_{\underline{m}})}{\underline{m}\cdot\underline{\omega}} \;=\; 0\,. \tag{3.7}$$

Since $M(h) = H_0$, we completed the proof of (1). To demonstrate (2) we simply use the first equality of (3.7) to write

$$M(b\,|\,c) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,C_{\underline{m}}B_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}} \qquad \text{and} \qquad M(c\,|\,b) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,B_{\underline{m}}C_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}}\,.$$

Changing $\underline{m} \to -\underline{m}$ in the second equation above, we get the desired claim.

Statement (3) is a mere consequence of (2) when we take $b = c$. Statement (4) can be proven using (2). Indeed, let $h(t) := (b\,|\,c)_t$. Since $M(h) = M(b\,|\,c) = 0$ and $M(a) = 0$, by (2), we can write

$$\begin{aligned} M(a\,|\,h) \;=\; -M(h\,|\,a) \;&=\; -M\left(h(t)\int_0^t a(\tau)\,d\tau\right) \\ &=\; -M\left[b(t)\int_0^t c(\tau)\,d\tau\left(\int_0^t a(\tau)\,d\tau\right)\right], \end{aligned}$$

which proves the first claim. To prove the second one, all we need to do is to interchange the the roles of $c$ and $a$ in the last equality (note that since $M(b\,|\,a) =$

0, the mean value of $(c \mid b \mid a)_t$ is well defined). Finally, statement (5) can be easily obtained as follows. Using (3.5), writing $b(t) = \frac{d}{dt}\int_0^t b(\tau)d\tau$, using definition (1.4) and integration by parts, one gets

$$\begin{aligned} M\,(a \mid b \mid c\,) \;\; &= \;\; -\lim_{T\to\infty}\frac{1}{2T}\left(\int_0^t b(\tau)\,d\tau\right)\left(\int_0^t a(\tau)\,d\tau\right)\left(\int_0^t c(\tau)\,d\tau\right)\Big|_{-T}^{T} \\ &\quad +M\left(\left(\int_0^t b(\tau)\,d\tau\right)a(t)\,\left(\int_0^t c(\tau)\,d\tau\right)\right) \\ &\quad +M\left(\left(\int_0^t b(\tau)\,d\tau\right)\left(\int_0^t a(\tau)\,d\tau\right)\,c(t)\right)\,. \end{aligned}$$

Now, since $M(a) = M(b) = M(c) = 0$, the integrals in the first line of (3.8) are all bounded. Thus, the limit $T \to \infty$ is zero because of the division by $2T$. Applying (3.5) to the remaining terms we obtain (5). ■

The following trivial corollary is of crucial importance for some of our calculations:

**Corollary 3.2** *For $M(\mathcal{Q}_0) = 0$ one always has $M(\mathcal{Q}_2) = 0$.* □

**Proof.** If $M(\mathcal{Q}_0) = 0$ then, by (1.11), $\mathcal{Q}_2(t) = (\mathcal{Q}_0 \mid \mathcal{Q}_0)_t$. Hence, from statement (3) of Proposition 3.1, it follows that $M(\mathcal{Q}_2) = 0$. ■

## 3.2 The Renormalisation Operator

For general quasi-periodic functions $a_1, \ldots, a_n$, the function $(a_1 \mid \cdots \mid a_n)_t$, defined above, is not generally quasi-periodic, since an integration performed on a quasi-periodic function with a non-zero mean value would produce a (linear in $t$) secular term, which would eventually become a higher degree polynomial after further integrations. We will here describe an operation designed to produce a quasi-periodic function out of $(a_1 \mid \cdots \mid a_n)_t$ through interactive subtractions of the mean value of the functions being integrated, a procedure we call "renormalisation" due to the analogy to the procedure of perturbative renormalization in quantum field theory. We will use this procedure of renormalisation in the following sections and here we present its definition and basic properties.

Let $a_1, \ldots, a_n$ be quasi-periodic functions. We define inductively the *renormalisation operator* $\mathfrak{R}_n$ acting on $(a_1 \mid \cdots \mid a_n)$ by

$$\begin{aligned} \mathfrak{R}_1 a_1(t) \;\; &:= \;\; a_1(t), \\ \mathfrak{R}_2\,(a_1 \mid a_2\,)_t \;\; &:= \;\; (a_1 \mid \mathfrak{R}_1(a_2) - M(\,\mathfrak{R}_1(a_2))\,)_t \;=\; (a_1 \mid a_2 - M(a_2)\,)_t\,, \\ \mathfrak{R}_n(a_1 \mid \cdots \mid a_n)_t \;\; &:= \;\; (a_1 \mid \;\mathfrak{R}_{n-1}(a_2 \mid \cdots \mid a_n) - M(\,\mathfrak{R}_{n-1}(a_2 \mid \cdots \mid a_n))\,)_t\,, \end{aligned}$$

for $n > 2$. We will now prove some elementary facts on $\mathfrak{R}_n$ which will be used below. The first important observation is that if $a_1, \ldots, a_n$ are quasi-periodic

functions, then $\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n)$ is also quasi-periodic. This can be easily seen by induction, through the obvious remark that the mean value of $\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n) - M(\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n))$ is zero. Note also that, trivially

$$a_0\, \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_n(a_0 a_1 \,|\, \cdots \,|\, a_n)\,. \tag{3.8}$$

The following proposition is a trivial but useful restatement of the definition of the $\mathfrak{R}_n$'s:

**Proposition 3.3** *For all $n \geq 2$ the following statement holds: if $a_1, \ldots, a_n$ are quasi-periodic functions, then*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_2\,(a_1 \mid \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n))\,.$$

*Consequently, for $n > 2$,*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_2\,(a_1 \mid \mathfrak{R}_2\,(a_2 \mid \mathfrak{R}_2(\cdots \,|\, \mathfrak{R}_2\,(a_{n-1} \,|\, a_n))\cdots)\,)\,. \tag{3.9}$$

□

**Proof.** By the definition of $\mathfrak{R}_2$,

$$\begin{aligned}
&\mathfrak{R}_2(a_1 \,|\, \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n)) = \\
&\qquad = \quad (a_1 \mid\; \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n) - M(\,\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n))\;) \\
&\qquad = \quad \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n).
\end{aligned}$$

■

Relation (3.9) shows that the operation $\mathfrak{R}_n$ can be obtained by iteration of the operation $\mathfrak{R}_2$. One also has the following useful

**Proposition 3.4** *For all $n \geq 1$ the following statement holds: if $a_1, \ldots, a_n$ are quasi-periodic functions and $a_n = \mathfrak{R}_2\,(b \,|\, c)$ for quasi-periodic functions $b$, $c$, then*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n) \;=\; \mathfrak{R}_{n+1}(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c).$$ □

**Proof.** For $n = 1$, let $a_1 = \mathfrak{R}_2\,(b \,|\, c)$. Then $\mathfrak{R}_1 a_1 = a_1 = \mathfrak{R}_2\,(b \,|\, c)$, trivially. For $n = 2$, let $a_2 = \mathfrak{R}_2\,(b \,|\, c)$. Then, $\mathfrak{R}_2\,(a_1 \,|\, a_2\,) = (a_1 \,|\, a_2 - M(a_2)) = (a_1 \mid \mathfrak{R}_2\,(b \,|\, c) - M(\,\mathfrak{R}_2\,(b \,|\, c)))$, but, by definition,

$$\mathfrak{R}_3\,(a_1 \,|\, b \,|\, c\,) = (a_1 \mid \mathfrak{R}_2\,(b \,|\, c) - M(\,\mathfrak{R}_2\,(b \,|\, c)))$$

and the statement holds again. For $n > 2$, let $a_n = \mathfrak{R}_2\,(b \,|\, c)$. Then, by induction,

$$\begin{aligned}
&\mathfrak{R}_{n+1}(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c) = \\
&\qquad = \quad (a_1 \mid\; \mathfrak{R}_n(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c) - M(\,\mathfrak{R}_n(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c))\;) \\
&\qquad = \quad (a_1 \mid\; \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n) - M(\,\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n)\;) \\
&\qquad = \quad \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n).
\end{aligned}$$

■

If $a$ and $b_1, \ldots, b_m$ are quasi-periodic, a function like $\sum_{k=1}^m (a \,|\, b_k)$ may not be a sum of quasi-periodic functions, even when $(a \,|\, \sum_{k=1}^m b_k) = \sum_{k=1}^m (a \,|\, b_k)$ is quasi-periodic, since we are not assuming that $M(b_k) = 0$ for each individual $k$. This fact notwithstanding, the following simple statement holds and will be repeatedly used:

**Proposition 3.5** *Let $a$ and $b_1, \ldots, b_m$ be quasi-periodic functions. Then*

$$\mathfrak{R}_2 \left( a \,\middle|\, \sum_{k=1}^m b_k \right) \;=\; \sum_{k=1}^m \mathfrak{R}_2 \,(a \,|\, b_k)\,. \tag{3.10}$$

*Consequently,*

$$\mathfrak{R}_n \left( a_1 \,\middle|\, \cdots \,\middle|\, a_{n-1} \,\middle|\, \sum_{k=1}^m b_k \right) \;=\; \sum_{k=1}^m \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b_k) \tag{3.11}$$

*for quasi-periodic functions $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_m$.* □

**Proof.** We have

$$\begin{aligned} \mathfrak{R}_2 \left( a \,\middle|\, \sum_{k=1}^m b_k \right) \;&=\; \left( a \,\middle|\, \sum_{k=1}^m b_k - M\left( \sum_{k=1}^m b_k \right) \right) \;=\; \left( a \,\middle|\, \sum_{k=1}^m \left( b_k - M\,(b_k) \right) \right) \\ &=\; \sum_{k=1}^m (a \,|\, b_k - M(b_k)) \;=\; \sum_{k=1}^m \mathfrak{R}_2 \,(a \,|\, b_k)\,. \end{aligned}$$

Note, in the third equality, that the mean value of $b_k - M(b_k)$ is zero and, hence, $(a \,|\, b_k - M(b_k))$ are quasi-periodic. Relation (3.11) follows from (3.9)-(3.10). ■

The following corollary follows from Propositions 3.4 and 3.5.

**Corollary 3.6** *Let $a_1, \ldots, a_{n-1}, b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ be quasi-periodic functions. Then,*

$$\mathfrak{R}_n \left( a_1 \,\middle|\, \cdots \,\middle|\, a_{n-1} \,\middle|\, \sum_{k=1}^m \mathfrak{R}_2 \,(b_k \,|\, c_k) \right) \;=\; \sum_{k=1}^m \mathfrak{R}_{n+1}(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b_k \,|\, c_k). \qquad \square$$

**Remark 3.7** *The reader should be warned of the following fact: If $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ are quasi-periodic functions and $(a_1 \,|\, \cdots \,|\, a_n) + (b_1 \,|\, \cdots \,|\, b_m)$ is also quasi-periodic, it is not always true that $(a_1 \,|\, \cdots \,|\, a_n) + (b_1 \,|\, \cdots \,|\, b_m)$ equals*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) + \mathfrak{R}_m(b_1 \,|\, \cdots \,|\, b_m).$$

*For a counter-example, take $a_1(t) = 1$, $a_2(t) = e^{it}$, $a_3(t) = 1$ $b_1(t) = ie^{it}$ and $b_2(t) = 1$. One has $(a_1 \,|\, a_2 \,|\, a_3) + (b_1 \,|\, b_2) = e^{it} - 1$, but $\mathfrak{R}_3\,(a_1 \,|\, a_2 \,|\, a_3) + \mathfrak{R}_2\,(b_1 \,|\, b_2) = 0$. Hence, the renormalisation operations are not additive in this sense.*

### 3.3 More on the Notation and Some Definitions

With the shorthand notation introduced in (3.1) and the definition of $\mathcal{Q}_0$ in (1.6), we see from (1.7), (1.11) and (1.12) that

$$\mathcal{Q}_1(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \,\middle|\, \overline{\mathcal{Q}_0}\right)_t, \quad \mathcal{Q}_2(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \,|\, \mathcal{Q}_0\right)_t, \quad \mathcal{Q}_3(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \,|\, \mathcal{Q}_1\right)_t .$$

Below, we will often use the following compact notation

$$(i\,|\,j)_t := (\mathcal{Q}_i \,|\, \mathcal{Q}_j)_t, \quad \left(i \,\middle|\, \overline{j}\right)_t := \left(\mathcal{Q}_i \,\middle|\, \overline{\mathcal{Q}_j}\right)_t, \quad (i\,|\,j\,|\,k\,)_t := (\mathcal{Q}_i \,|\, \mathcal{Q}_j \,|\, \mathcal{Q}_k\,)_t,$$

etc, for $i,\ j,\ k = 0, \ldots, 3$. In other words, we simply use the index $n$ of $\mathcal{Q}_n$ to denote $\mathcal{Q}_n$ itself. Moreover, by $M\,(i\,|\,j\,)$ we will denote the mean value of $(i\,|\,j\,)_t$, etc.

Note that for $M(\mathcal{Q}_0) = 0$ one has with this notation $\mathcal{Q}_1(t) = \left(0 \,\middle|\, \overline{0}\right)_t$ and $\mathcal{Q}_2(t) = (0\,|\,0\,)_t$ and for $M(\mathcal{Q}_1) = 0$ one has

$$\mathcal{Q}_3(t) \;=\; (0\,|\,1\,)_t .$$

We will often write $\mathcal{Q}_3$ this way.

For $M(\mathcal{Q}_0) = 0$, other identities will also be at hand. For instance, one has

$$(0\,|\,0\,|\,0\,)_t \;=\; (0\,|\,(0\,|\,0\,)_{t'}\,)_t \;=\; (0\,|\,2)_t \qquad \text{and} \qquad (1\,|\,0)_t \;=\; \left(2 \,\middle|\, \overline{0}\right)_t , \tag{3.12}$$

since $M\,(0\,|\,0\,) = 0$, by item (3) of Proposition 3.1. Relations like these will be often employed.

# 4 The Case $M(\mathcal{Q}_0) = 0$ and $M(\mathcal{Q}_1) = 0$

In this Section we will prove the Theorem 2.2. Our interest is to study the situation complementary to cases (I) and (II), i.e., the situation where one has condition

$$(\mathrm{III}_0) \quad M(\mathcal{Q}_0) = 0 \text{ and } M(\mathcal{Q}_1) = 0.$$

To remove the secular terms from $g$, applying the method described in the previous section, we will be forced to add a further restriction to $(\mathrm{III}_0)$, namely the condition $M\,(0\,|\,1\,) \neq 0$.

Recall that the functions $q$ and $\mathcal{Q}_1$ depend primordially on the interaction $f$ (see the definitions given in (1.6) and (1.7)), so conditions (I), (II) or $(\mathrm{III}_0)$ apply upon the properties of $f$. As we already saw in Section 1, the function $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$ only satisfies condition (I) or $(\mathrm{III}_0)$, depending on the particular choice of the parameters $\varphi_1, \varphi_2$. We will have more to say about this example latter on in Section 5. Now, let us work with the expansion for $g$ in order to remove all of its secular terms.

Again, our Ansatz to solve the generalised Riccati equation (2.1) is (2.5). The explicit solutions for the coefficients $v_n$ are given in (2.6), (2.7) and (2.8). One

sees immediately from condition ($\mathrm{III}_0$) that $v_1$ and $v_2$ do not suffer from secular terms. Indeed, $q$ in quasi-periodic and since $M(\mathcal{Q}_0^{-1}) = \overline{M(\mathcal{Q}_0)} = 0$, we conclude that the mean value of the integrand $\mathcal{I}_2$ occurring in (2.7) is zero. Therefore, the integration occurring in the definition of $v_2$ in (2.7) does not produce a linear term in $t$. These fats imply that $v_1$ and $v_2$ are quasi-periodic under ($\mathrm{III}_0$). From these considerations we see that the condition $M(\mathcal{I}_n) = 0$, $n \geq 3$, becomes recursively identical to $\sum_{p=1}^{n-1} M(v_p v_{n-p})$, $n \geq 3$, since the $v_n$'s become successively quasi-periodic when the interactive procedure is run.

If this is achieved, i.e., if we succeed in fixing $M(\mathcal{I}_n) = 0$ for all $n$, we can rewrite (2.6)-(2.8) in a "renormalised" form:

$$
\begin{aligned}
v_1(t) &= \kappa_1 q(t), && (4.1)\\
v_2(t) &= q(t)^{-1}\left(i\kappa_1^2 \mathcal{Q}_2(t) - i\mathcal{Q}_1(t) + \kappa_2 \mathcal{Q}_0(t)\right), && (4.2)\\
v_n(t) &= q(t)^{-1}\left\{ i\sum_{p=1}^{n-1} \mathfrak{R}_2\left(0 \mid v_p v_{n-p}\right)_t + \kappa_n \mathcal{Q}_0(t)\right\}, \quad n \geq 3, && (4.3)
\end{aligned}
$$

where $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ were defined in (1.6), (1.7) and (1.11), respectively.

Let us move on and analyse the third order term. According to (2.8) the integrand $\mathcal{I}_3$ which appears in the definition of $v_3$ is given by $2v_1v_2$. Using (4.1) and (4.2) we have

$$
v_1 v_2 \;=\; i\kappa_1^3 \mathcal{Q}_2 - i\kappa_1 \mathcal{Q}_1 + \kappa_1\kappa_2 \mathcal{Q}_0\,. \tag{4.4}
$$

From Corollary 3.2, we readily see that $M(\mathcal{I}_3) = 2M(v_1v_2) = 0$. This means that the integrand which appears in $v_3$ does not have a constant term in its Fourier expansion. Hence, $v_3$ is quasi-periodic.

Until now we have verified the absence of secular terms in the series expansion of $g$ up to order three in $\epsilon$. As we shall see next, for the same to be true up to order four, we have to make a especial choice for the value of the constant $\kappa_1$.

## 4.1 The Absence of Secular Terms in $v_4$. Fixing $\kappa_1$

As one sees from (2.8), the integrand in $v_4$ is $\mathcal{I}_4 \;:=\; 2v_1v_3 + v_2^2$. Since $v_1, v_2, v_3$ are quasi-periodic, the mean value of $\mathcal{I}_4$ is well defined. Let us explicitly evaluate $\mathcal{I}_4$ using (2.6)–(2.8):

$$
\begin{aligned}
\mathcal{I}_4 &= 2\kappa_1 \mathcal{Q}_0(t)\left(i\int_0^t 2v_1(\tau)v_2(\tau)\,d\tau + \kappa_3\right)\\
&\quad + \mathcal{Q}_0(t)\left(i\int_0^t \left(\kappa_1^2\mathcal{Q}_0(\tau) - \mathcal{Q}_0(\tau)^{-1}\right)\,d\tau + \kappa_2\right)^2\\
&= -4\kappa_1^4\left(0 \mid 2\right) + 4\kappa_1^2\left(0 \mid 1\right) + 6i\kappa_1^2\kappa_2\mathcal{Q}_2 + (2\kappa_1\kappa_3 + \kappa_2^2)\mathcal{Q}_0\\
&\quad -2i\kappa_2\mathcal{Q}_1 - \kappa_1^4\left(2 \mid 0\right) + 2\kappa_1^2\left(2 \,\middle|\, \overline{0}\right) - \left(1 \,\middle|\, \overline{0}\right)\,. \qquad (4.5)
\end{aligned}
$$

The functions appearing in the right-hand side of (4.5) are all quasi-periodic, since $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$. Therefore, we are allowed to take the mean value of each individual term above. The result is

$$M(\mathcal{I}_4) \;=\; -4\kappa_1^4 M(\,0\,|\,2\,) + 4\kappa_1^2 M(\,0\,|\,1\,) - \kappa_1^4 M(\,2\,|\,0\,) + 2\kappa_1^2 M(\,2|\,\overline{0}\,) - M(\,1\,|\,\overline{0}\,).$$

By statements (2) and (6) of Proposition 3.1 and by (3.12) we have

$$M\,(2\,|\,0\,) \;=\; -M\,(0\,|\,2\,) \;=\; -M\,(0\,|\,0\,|\,0\,) = 0\,, \tag{4.6}$$

$$M\,\big(2\,\big|\,\overline{0}\big) \;=\; M\,(1\,|\,0\,) \;=\; -M\,(0\,|\,1\,)\;. \tag{4.7}$$

Hence,

$$M(\mathcal{I}_4) \;=\; 2\kappa_1^2 M\,(0\,|\,1\,) - M\,\big(1\,\big|\,\overline{0}\big)\;. \tag{4.8}$$

We have the following

**Proposition 4.1** *Under $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$, one has $M(\,1\,|\,\overline{0}\,) = 2\overline{M(\,0\,|\,1\,)}$.* □

**Proof.** By the definition of $\mathcal{Q}_1$,

$$M\,(0\,|\,1\,) \;=\; M\,\big(0\,\big|\,0\,\big|\,\overline{0}\,\big) \tag{4.9}$$

and

$$M\,\big(1\;\big|\,\overline{0}\big) = M\left(\mathcal{Q}_0(t)\left(\int_0^t \mathcal{Q}_0(\tau)^{-1}\,d\tau\right)\left(\int_0^t \mathcal{Q}_0(\tau)^{-1}\,d\tau\right)\right) = -M\,\big(\overline{0}\,\big|\,0\,\big|\,\overline{0}\,\big)\;, \tag{4.10}$$

where, in the last equality, we have used statement (4) of Proposition 3.1. Now, by statement (5) of the same proposition, we can write

$$M\,\big(0\,\big|\,0\,\big|\,\overline{0}\,\big) + M\,\big(0\,\big|\,\overline{0}\,|\,0\,\big) + M\,\big(\overline{0}\,|\,0\,|\,0\,\big) \;=\; 0 \tag{4.11}$$

(recall that $M\,\big(0\,\big|\,\overline{0}\big) = M\,\big(\overline{0}\,|\,0\big) = M\,(0\,|\,0\,) = 0$). Once again, by statement (4) of Proposition 3.1, $M\,\big(0\,\big|\,0\,\big|\,\overline{0}\,\big) = M\,\big(\overline{0}\,|\,0\,|\,0\,\big)$. Thus, (4.11) reads $2M\,\big(0\,\big|\,0\,\big|\,\overline{0}\,\big) + M\,\big(0\,\big|\,\overline{0}\,|\,0\,\big) = 0$. Taking the complex conjugate, yields

$$2\overline{M\,\big(0\,\big|\,0\,\big|\,\overline{0}\,\big)} + M\,\big(\overline{0}\,\big|\,0\,\big|\,\overline{0}\,\big) = 0.$$

Finally, using (4.9) and (4.10), we get $M\,\big(1\,\big|\,\overline{0}\big) = 2\overline{M\,(0\,|\,1\,)}$. ■

We have just proven that $M(\mathcal{I}_4) \;=\; 2\left(\kappa_1^2\,M\,(0\,|\,1\,) - \overline{M\,(0\,|\,1\,)}\right)$. Now we impose $M(\mathcal{I}_4) = 0$. Of course, this will be the case if $M\,(0\,|\,1\,) = 0$ but, in the situation where $M\,(0\,|\,1\,) \neq 0$ this can be achieved by fixing $\kappa_1$ as (see (4.8))

$$\kappa_1 \;=\; \left(\frac{\overline{M\,(0\,|\,1\,)}}{M\,(0\,|\,1\,)}\right)^{1/2} \;=\; \left(\frac{\overline{M(\mathcal{Q}_3)}}{M(\mathcal{Q}_3)}\right)^{1/2}\;. \tag{4.12}$$

Thus, $\kappa_1$ is a phase: $|\kappa_1| = 1$. It will be henceforth assumed that $M(0\,|\,1) \neq 0$. If $M(0\,|\,1) = 0$, $\kappa_1$ has to be fixed by $M(\mathcal{I}_5) = 0$. We shall not treat this more restrictive case here.

So far, we have verified the absence of secular terms in the series expansion (2.5) for $g$ up to order three in $\epsilon$ and we have eliminated them from $v_4$ by making a especial choice for the value of $\kappa_1$ (given by (4.12)). At this point we would like to proceed recursively by imposing $M(\mathcal{I}_n) = 0$, for all $n \geq 5$. This would give the correct values for the constants $\kappa_p$, $p \geq 2$, and guarantee the absence of secular terms in all $v_n$, $n \geq 5$. This recursive procedure was used in [1] to eliminate the secular terms from $g$ in cases (I) and (II). Here, we still have to determine the constants $\kappa_2$ and $\kappa_3$ explicitly (not recursively) before running the recursive procedure.

## 4.2 The Absence of Secular Terms in $v_5$. Fixing $\kappa_2$

Let us begin by calculating the integrand $\mathcal{I}_5$ which appears in $v_5$. Taking $n = 5$ in (2.8) we get $\mathcal{I}_5 = 2v_1v_4 + 2v_2v_3$. We first evaluate $v_1v_4$ explicitly and then $v_2v_3$.

For $v_1v_4$, a lengthy computation (see Appendix A.1) shows that

$$\begin{aligned} v_1v_4 \;=\; & -6\kappa_1^3\kappa_2\;\mathfrak{R}_2\,(0\,|\,2) + 2\kappa_1\kappa_2\;\mathfrak{R}_2\,(0\,|\,1) + i\kappa_1(2\kappa_1\kappa_3 + \kappa_2^2)\,\mathcal{Q}_2 \\ & +\kappa_1\kappa_4\mathcal{Q}_0 + i\kappa_1\mathcal{A}_1\,, \end{aligned} \tag{4.13}$$

where $\mathcal{A}_1$ is the quasi-periodic function defined in (A.7) and depends only on the constant $\kappa_1$. Since we are working under the condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$, we can drop the symbol $\mathfrak{R}_2$ above. From this fact and from relation (4.6), we conclude that $M(v_1v_4) = 2\kappa_1\kappa_2\,M\,(0\,|\,1) + i\kappa_1 M(\mathcal{A}_1)$.

Let us now calculate the second term in $\mathcal{I}_5$, namely, $v_2v_3$. Another lengthy computation (see Appendix A.2) gives

$$\begin{aligned} v_2v_3 \;=\; & -2i\kappa_1^5\,\mathfrak{R}_2\,(2\,|\,2) + 2i\kappa_1^3\,\mathfrak{R}_2\,(2\,|\,1) - 2\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(2\,|\,0) + 2i\kappa_1^3\,\mathfrak{R}_2\,(1\,|\,2) \\ & -2i\kappa_1\,\mathfrak{R}_2\,(1\,|\,1) + 2\kappa_1\kappa_2\,\mathfrak{R}_2\,(1\,|\,0) - 2\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(0\,|\,2) \\ & +2\kappa_1\kappa_2\,\mathfrak{R}_2\,(0\,|\,1) + i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3\mathcal{Q}_1 + \kappa_2\kappa_3\mathcal{Q}_0\,. \end{aligned} \tag{4.14}$$

Clearly the right hand side of equation (4.14) is quasi-periodic. Since we are working under the condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$, we can drop the symbol $\mathfrak{R}_2$ above and reorder (4.14) in the form

$$\begin{aligned} v_2v_3 \;=\; & i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3\mathcal{Q}_1 + \kappa_2\kappa_3\mathcal{Q}_0 - 2i\kappa_1^5\,(2\,|\,2) - 2i\kappa_1\,(1\,|\,1) \\ & +2i\kappa_1^3\,[(2\,|\,1) + (1\,|\,2)] - 2\kappa_1^3\kappa_2\,[(2\,|\,0) + (0\,|\,2)] \\ & +2\kappa_1\kappa_2\,[(1\,|\,0) + (0\,|\,1)]\,. \end{aligned} \tag{4.15}$$

Above we also used $\mathfrak{R}_2\,(0\,|\,0) = \mathcal{Q}_2$. From (4.15), from the fact that $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and from statements (2) and (3) of Proposition 3.1, we see immediately that $M(v_2v_3) = 0$.

We are now ready to find the value of $\kappa_2$ by imposing the condition $M(\mathcal{I}_5) = 0$, which guarantees the absence of secular terms in $v_5$. Since $M(\mathcal{I}_5) = 2M(v_1v_4) + 2M(v_2v_3) = 4\kappa_1\kappa_2\, M\,(0\,|\,1) + 2i\kappa_1 M(\mathcal{A}_1)$, we conclude that

$$\kappa_2 \;=\; -\frac{iM(\mathcal{A}_1)}{2M\,(0\,|\,1)}\,. \tag{4.16}$$

Note that the right-hand side of (4.16) depends on the previously fixed $\kappa_1$.

## 4.3 The Absence of Secular Terms in $v_6$. Fixing $\kappa_3$

We still have to find $\kappa_3$ in order to fix recursively all $\kappa_n$'s for $n \geq 4$. $\kappa_3$ will be fixed by eliminating the secular terms from $v_6$, that is, by imposing $M(\mathcal{I}_6) = 0$. First of all, we need to write $\mathcal{I}_6$. Using relation (2.8) for $n = 6$ we find that $\mathcal{I}_6 = 2v_1v_5 + 2v_2v_4 + v_3^2$. Let us calculate $2v_1v_5$. Another lengthy computation (see Appendix A.3) gives

$$\begin{aligned} 2v_1v_5 \;&=\; \mathcal{A}_2 - 12\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(0\,|\,2) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(0\,|\,1) \\ &\quad +4i\kappa_1(\kappa_2\kappa_3 + \kappa_1\kappa_4)\mathcal{Q}_2 + 2\kappa_1\kappa_5\mathcal{Q}_0\,, \end{aligned} \tag{4.17}$$

where $\mathcal{A}_2$ is the quasi-periodic function defined in (A.11) and depends only on the constants $\kappa_1$ and $\kappa_2$. Since $2v_1v_5$ given above is a sum of quasi-periodic functions we can take the mean value of each individual term which appears in the right hand side of (4.17) and write

$$2M(v_1v_5) \;=\; M(\mathcal{A}_2) + 4\kappa_3\kappa_1 M\,(0\,|\,1)\,, \tag{4.18}$$

where, once again, we have used $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and the identity (4.6).

We will now calculate the second term in $\mathcal{I}_5$, namely, $2v_2v_4$. We have (see Appendix A.4)

$$\begin{aligned} 2v_2v_4 \;&=\; \mathcal{A}_3 - 4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0) \\ &\quad +2i(\kappa_1^2\kappa_4 + 2\kappa_1\kappa_2\kappa_3)\mathcal{Q}_2 - 2i\kappa_4\mathcal{Q}_1 + 2\kappa_2\kappa_4\mathcal{Q}_0\,, \end{aligned} \tag{4.19}$$

where $\mathcal{A}_3$ is defined in (A.12) and depends only on the constants $\kappa_1$ and $\kappa_2$.

We can now proceed and take the mean value of $2v_2v_4$ from (4.19). Using $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and (4.6), we get

$$2M(v_2v_4) \;=\; M(\mathcal{A}_3) + 4\kappa_1\kappa_3 M\,(0\,|\,1)\,. \tag{4.20}$$

We are almost through with the calculation of $\mathcal{I}_6$. We still need $v_3^2$. Using relations (4.1)–(4.4),

$$\begin{aligned} v_3 \;&=\; q^{-1}\left\{2i\,\mathfrak{R}_2\,(0\,|\,v_1v_2\,) + \kappa_3\mathcal{Q}_0\right\} \\ &=\; q^{-1}\left\{-2\kappa_1^3\,\mathfrak{R}_2\,(0\,|\,2) + 2\kappa_1\,\mathfrak{R}_2\,(0\,|\,1) + 2i\kappa_1\kappa_2\mathcal{Q}_2 + \kappa_3\mathcal{Q}_0\right\}. \end{aligned} \tag{4.21}$$

Hence,

$$v_3^2 \;=\; \kappa_3^2\mathcal{Q}_0 + 4\kappa_1\kappa_3\left[-\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,2\,) + \mathfrak{R}_2\,(0\,|\,1\,) + i\kappa_2\mathcal{Q}_2\right] - 4\mathcal{A}_4\,,$$

where

$$\mathcal{A}_4 \;:=\; \mathcal{Q}_0^{-1}\left[i\kappa_1^3\,\mathfrak{R}_2\,(0\,|\,2\,) - i\kappa_1\,\mathfrak{R}_2\,(0\,|\,1\,) + \kappa_1\kappa_2\mathcal{Q}_2\right]^2\,.$$

Again, $\mathcal{A}_4$ is quasi-periodic and depends only on the already fixed constants $\kappa_1$ and $\kappa_2$. Therefore, using once more $M(\mathcal{Q}_0) = M(\mathcal{Q}_2) = 0$ and (4.6), we get

$$M(v_3^2) \;=\; 4\kappa_3\kappa_1 M\,(0\,|\,1\,) - 4M(\mathcal{A}_4)\,. \tag{4.22}$$

Finally, imposing $M(\mathcal{I}_6) = 0$, i.e., $2M(v_1v_5) + 2M(v_2v_4) + M(v_3^2) = 0$, we obtain from our previous calculations (relations (4.18), (4.20) and (4.22)),

$$\kappa_3 \;=\; \frac{4M(\mathcal{A}_4) - M(\mathcal{A}_3) - M(\mathcal{A}_2)}{4\kappa_1 M\,(0\,|\,1\,)}\,. \tag{4.23}$$

Note that the right-hand side of (4.23) depends on the previously fixed $\kappa_1$ and $\kappa_2$.

## 4.4 The Absence of Secular Terms in $v_n$, $n \geq 7$. Fixing $\kappa_{n-3}$ Recursively

So far, we have fixed the constants $\kappa_1$, $\kappa_2$ and $\kappa_3$ individually. Now we proceed to fix recursively all other $\kappa_{n-3}$ for all $n \geq 7$. We have to impose

$$M(\mathcal{I}_n) \;=\; 0 \quad\Longrightarrow\quad M\left(\sum_{p=1}^{n-1} v_p v_{n-p}\right) \;=\; 0 \tag{4.24}$$

for all $n \geq 7$. Condition (4.24) guarantees the absence of secular terms in all $v_n$, $n \geq 7$.

The idea now is to use (4.24) to calculate recursively the constants $\kappa_{n-3}$, for all $n \geq 7$, that is, $\kappa_4,\ \kappa_5,\ \ldots$. Of course, we already have $\kappa_1$, $\kappa_2$ and $\kappa_3$ and, hence, we completely know $v_1$, $v_2$ and $v_3$. We also know that all functions, from $v_1$ to $v_6$, are quasi-periodic. From now on we will work inductively. Thus, it will be supposed for each $n \geq 7$ that we fixed $\kappa_1,\ \ldots,\ \kappa_{n-4}$ by imposing $M(\mathcal{I}_m) = 0$ for all $m = 2, \ldots, n-1$ and that, as a consequence, all functions $v_1,\ \ldots,\ v_{n-1}$ are quasi-periodic. Note that it is indeed true for $n = 7$.

By our inductive hypothesis, we are allowed to take the summation out of the mean value $M$ in (4.24) and write

$$2\left[M(v_1v_{n-1}) + M(v_2v_{n-2}) + M(v_3v_{n-3})\right] + \sum_{p=4}^{n-4} M(v_pv_{n-p}) \;=\; 0\,, \tag{4.25}$$

where, by convention, $\sum_{p=4}^{n-4} M(v_p v_{n-p}) = 0$ for $n = 7$. Let us introduce now the following definition:

$$l_m(t) \;:=\; q(t)(v_m(t) - \kappa_m q(t))\,, \tag{4.26}$$

for all $m \geq 4$. Note that, by relation (2.8), the functions $l_m$'s above can also be written as

$$l_m(t) \;=\; i\,q(t)^2 \left( \int_0^t \sum_{p=1}^{m-1} v_p(\tau) v_{m-p}(\tau)\, d\tau \right) . \tag{4.27}$$

For $m < n$ we are allowed to write

$$\begin{aligned} l_m(t) \;&=\; i\,q(t)^2 \left( \int_0^t \sum_{p=1}^{m-1} \left[ v_p(\tau) v_{m-p}(\tau) - M\left(v_p v_{m-p}\right) \right] d\tau \right) \\ &=\; i \sum_{p=1}^{m-1} \mathfrak{R}_2\left(0 \mid v_p v_{m-p}\right)_t\,, \end{aligned} \tag{4.28}$$

since we assumed $M(\sum_{p=1}^{m-1} v_p v_{m-p}) = M(\mathcal{I}_m) = 0$ for all $m < n$, by the inductive hypothesis. Hence, $l_m$ are quasi-periodic for all $m < n$, by the inductive hypothesis.

Let us use the definition given in (4.26) and evaluate the first three terms which appear in (4.25). Beginning with the first one, we have

$$M(v_1 v_{n-1}) \;=\; M\left(v_1(q^{-1} l_{n-1} + \kappa_{n-1} q)\right) \;=\; \kappa_1 M(l_{n-1})\,, \tag{4.29}$$

where we have used (2.6) and the fact that $M(\mathcal{Q}_0) = 0$. Using (4.26), the second term of (4.25) can be evaluated as

$$M(v_2 v_{n-2}) \;=\; M\left(v_2(q^{-1} l_{n-2} + \kappa_{n-2} q)\right) \;=\; M(q^{-1} v_2 l_{n-2})\,, \tag{4.30}$$

where we have used (4.2) to express $q v_2$ and the fact that $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$. Finally, for the third term of (4.25), we have

$$M(v_3 v_{n-3}) \;=\; M\left(v_3(q^{-1} l_{n-3} + \kappa_{n-3} q)\right) \;=\; M(q^{-1} v_3 l_{n-3}) + \kappa_{n-3} M(q v_3)\,. \tag{4.31}$$

The product $q v_3$ can be obtained from (4.21), from which we conclude that $M(q v_3) = 2\kappa_1 M\left(0 \mid 1\right)$. Inserting this into (4.31), gives

$$M(v_3 v_{n-3}) \;=\; M(q^{-1} v_3 l_{n-3}) + 2\kappa_1 \kappa_{n-3} M\left(0 \mid 1\right)\,. \tag{4.32}$$

The substitution of (4.29), (4.30) and (4.32) into (4.25), gives us

$$2\left\{\underbrace{\kappa_1 M(l_{n-1})}_{\text{(i)}}+\underbrace{M(q^{-1}v_2 l_{n-2})}_{\text{(ii)}}+\underbrace{M(q^{-1}v_3 l_{n-3})}_{\text{(iii)}}+\underbrace{2\kappa_{n-3}\kappa_1 M\,(0\,|\,1\,)}_{\text{(iv)}}\right\} + \underbrace{\sum_{p=4}^{n-4} M(v_p v_{n-p})}_{\text{(v)}} \;=\; 0\,. \quad (4.33)$$

Before we proceed, let us make some comments on our strategy. Equation (4.33) is a direct consequence of (4.24) and, thus, is being imposed for each $n \geq 7$, leading to the values of $\kappa_4$, $\kappa_5$ and so on. By our induction hypothesis, we have fixed the constants $\kappa_1, \ldots, \kappa_{n-4}$ and, hence, we completely know $v_1, \ldots, v_{n-4}$. For this reason the terms (iii) and (v) are known by assumption (by (4.28), the evaluation of $l_{n-3}$ requires the knowledge of $v_1, \ldots, v_{n-4}$). Our aim is to use (4.33) as a condition to fix $\kappa_{n-3}$ and we, therefore, have to isolate the dependence of (4.33) on $\kappa_{n-3}$. The function $l_{n-2}$, in term (ii), depends implicitly on $v_{n-3}$ and, hence, on $\kappa_{n-3}$ (see, again, relation (4.28)). The term (i), however, depends implicitly on $\kappa_{n-2}$ and $\kappa_{n-3}$. This dependence on $\kappa_{n-2}$ could be a problem, since we are still working to fix $\kappa_{n-3}$. Nevertheless, as will be shown, the conditions $M(\mathcal{Q}_1) = 0$ and $M(\mathcal{Q}_2) = 0$ fortunately eliminate $\kappa_{n-2}$ from the final expression, and we will be led to a condition expressing $\kappa_{n-3}$ in terms of known quantities.

Let us now compute terms (i) and (ii). After a long computation, found in Appendix A.5, we get

$$\kappa_1 M(l_{n-1}) \;=\; 2\kappa_1\kappa_{n-3} M\,(0\,|\,1\,) + \mathcal{R}_n^{(1)}\,, \quad (4.34)$$

where $\mathcal{R}_n^{(1)}$ is defined in (A.18) and depends on constants $\kappa_1, \ldots, \kappa_{n-4}$ only. For term (ii) of (4.33) we get, after another long computation presented in Appendix A.6,

$$M(q^{-1}v_2 l_{n-2}) \;=\; -2\kappa_1\kappa_{n-3} M\,(0\,|\,1\,) + \mathcal{R}_n^{(2)}\,, \quad (4.35)$$

where $\mathcal{R}_n^{(2)}$ is defined in (A.19). We again stress that $\mathcal{R}_n^{(2)}$ depends on $\kappa_1, \ldots, \kappa_{n-4}$ only.

We are now ready to give the precise value of $\kappa_{n-3}$ in order to satisfy (4.24). Collecting (4.34) and (4.35) and inserting them into (4.33), we obtain

$$\kappa_{n-3} \;=\; \frac{-1}{4\kappa_1 M\,(0\,|\,1\,)}\left\{\sum_{p=4}^{n-4} M(v_p v_{n-p}) + 2\mathcal{R}_n^{(1)} + 2\mathcal{R}_n^{(2)} + 2M(q^{-1}v_3 l_{n-3})\right\}\,,$$

for all $n \geq 7$. Note that $\mathcal{R}_n^{(1)}$ and $\mathcal{R}_n^{(2)}$ can be completely computed in any order of recursion (see equations (A.18) and (A.19)). The function $l_{n-3}$ can also be computed in any order of recursion if one uses relation (4.28).

Summarising our conclusions, under conditions ($III_0$) and $M\,(0\,|\,1\,) \neq 0$, i.e., under condition (III) of Theorem 2.2, and with the constants $\kappa_n$ recursively chosen as

$$\kappa_1 \;=\; \left(\frac{\overline{M\,(0\,|\,1\,)}}{M\,(0\,|\,1\,)}\right)^{1/2}, \tag{4.36}$$

$$\kappa_2 \;=\; -\frac{iM\,(\mathcal{A}_1)}{2M\,(0\,|\,1\,)}\,, \qquad \kappa_3 \;=\; \frac{4M\,(\mathcal{A}_4) - M\,(\mathcal{A}_3) - M\,(\mathcal{A}_2)}{4\kappa_1 M\,(0\,|\,1\,)}\,, \tag{4.37}$$

$$\kappa_{n-3} \;=\; \frac{-1}{4\kappa_1 M\,(0\,|\,1\,)}\left\{\sum_{p=4}^{n-4} M\,(v_p v_{n-p}) + 2\mathcal{R}_n^{(1)} + 2\mathcal{R}_n^{(2)} + 2M\left(q^{-1}v_3 l_{n-3}\right)\right\}, \tag{4.38}$$

for $n \geq 7$, all secular terms are eliminated from the formal solution (2.5) of the generalised Riccati equation (2.1). Notice that hypothesis $M\,(0\,|\,1\,) \neq 0$ is the only additional restriction needed to (4.36)-(4.38). The proof of Theorem 2.2 is thus complete. ■

# 5 The Case of the Monochromatic Field

We illustrate our method and our results considering the simplest case of monochromatic interactions (ac-dc field)

$$f(t) \;=\; F_0 + \varphi\cos(\omega t), \tag{5.1}$$

which are of particular importance in physical applications. We want to show that with conditions (I)-(III) we obtain with our method convergent perturbative solutions of this problems for all parameters $F_0$ and $\varphi$, except perhaps for some spurious situations. For (5.1) one has

$$\mathcal{Q}_0(t) \;=\; \sum_{n\in\mathbb{Z}} J_n\,(\chi_1)\, e^{i(n+\chi_2)\omega t}\,,$$

where $J_n$ is the Bessel function of first kind and order $n$ and where we defined $\chi_1 := 2\varphi/\omega$, $\chi_2 := 2F_0/\omega$. Hence, condition (I) (treated in detail in [2]) holds provided $\chi_2 = -m$, with $m$ integer, and provided $\chi_1$ is not a zero of the Bessel function $J_m$. Here, we have by (2.14),

$$\Omega \;=\; -\frac{m\omega}{2} + \epsilon J_m(\chi_1) + O(\epsilon^2)\,.$$

See also the discussion in [7]. Let us consider the complementary situations.

**i.** Consider the case where $\chi_2 = -m$, a non-zero integer, and $\chi_1$ is a zero of $J_m$. One has $M(\mathcal{Q}_0) = 0$ and we have to look at $M(\mathcal{Q}_1)$. We get,

$$M(\mathcal{Q}_1) \;=\; \sum_{k\in\mathbb{Z}_*} \frac{J_{k+m}(\chi_1)^2}{ik\omega}\,.$$

Since $\chi_1$ is bound to be a zero of $J_m$, the r.h.s. does not vanish, except, perhaps, for spurious points. Hence we have now condition (II). Moreover, by (2.15),

$$\Omega \;=\; -\frac{m\omega}{2} - \epsilon^2 \sum_{k\in\mathbb{Z}_*} \frac{J_{k+m}(\chi_1)^2}{k\omega} + O(\epsilon^3)\,.$$

**ii.** Consider the case where $\chi_2$ is non-integer (see also the discussion in [7]). One has $M(\mathcal{Q}_0) = 0$ and we have to look at $M(\mathcal{Q}_1)$. We get,

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega\chi_2}\left[J_0(\chi_1)^2 + 2\chi_2^2 \sum_{k=1}^{\infty} \frac{J_k(\chi_1)^2}{\chi_2^2 - k^2}\right]\,.$$

Generally the r.h.s is non-zero and we have condition (II). Hence, by (2.15),

$$\Omega \;=\; F_0 + \frac{\epsilon^2}{\omega\chi_2}\left[J_0(\chi_1)^2 + 2\chi_2^2 \sum_{k=1}^{\infty} \frac{J_k(\chi_1)^2}{\chi_2^2 - k^2}\right] + O(\epsilon^3)\,. \tag{5.2}$$

Note, however, that on each interval $\chi_2 \in (k,\ k+1)$, $k = 1,\, 2, \ldots$, the terms $\frac{J_k(\chi_1)^2}{\chi_2^2-k^2} + \frac{J_{k+1}(\chi_1)^2}{\chi_2^2-(k+1)^2}$ vary continuously from $+\infty$ to $-\infty$. Hence, there is on each interval $(k,\ k+1)$, $k = 0,\, 1,\, 2, \ldots$, a special value $\chi_2^s$ of $\chi_2$ (depending on $\chi_1$) for which $M(\mathcal{Q}_1) = 0$, and we would be out of case (II). But when $2\chi_2$ is non-integer, one has $M(\mathcal{Q}_3) = 0$. Hence, except for the very unlikely case where $2\chi_2^s$ is an integer, we would be out of condition (III) as well, and $\Omega = F_0 + O(\epsilon^4)$. This special case may not be very interesting, because $\chi_2$ has to be chosen with precision. That could be difficult to fix it in some experimental setting.

Another special situation would occur when $\chi_2$ is chosen to satisfy $F_0 - i\epsilon^2 M(\mathcal{Q}_1) = 0$. By the argument above, this is possible, but $\chi_2$ will depend on $\epsilon$. It is therefore unclear if $\Omega$ will be just $O(\epsilon^4)$ or "small" (eventually leading to an even stronger dynamical localisation than we have in case (III)). It is not even clear that we will be in a situation where our series converge, and we left this other special situation without more comments.

It is interesting to compare the expressions for the secular frequency $\Omega$ in the three situations above (for $F_0 \neq 0$) with the situation where $\varphi = 0$, where the secular frequency $\Omega_0$ is $\Omega_0 := F_0\sqrt{1 + (\frac{\epsilon}{F_0})^2} = F_0 + \frac{\epsilon^2}{2F_0} + O(\epsilon^3)$. This reveals the effect of the ac-field $\varphi\cos(\omega t)$ on the secular frequency. Taking $\chi_1 \to 0$ in (5.2) we recover $\Omega_0$.

**iii.** Consider the case where $\chi_2 = 0$, i.e., $F_0 = 0$, and $\chi_1 = x_a$, the $a$-th zero zero of the Bessel function $J_0$ on the positive real axis. This case is interesting in connection with the issue of dynamical localisation, as discussed by many authors ([14]. For more references, see [3, 7]). Here $M(\mathcal{Q}_0) = 0$ and,

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega}\sum_{m=1}^{\infty}\left(\frac{J_m(\chi_1)^2 - J_{-m}(\chi_1)^2}{m}\right) \;=\; 0\,,$$

since $J_k(x) = (-1)^k J_{-k}(x)$. Thus, condition (II) does not apply and we have to look at $M\,(0\,|\,1)$. We obtain $M\,(0\,|\,1) = \frac{1}{\omega^2}\mathcal{T}\,(x_a)$, with

$$\mathcal{T}\,(x_a) \;:=\; -\sum_{n,m\in\mathbb{Z}_*} \frac{J_n\,(x_a)\,J_{n-m}\,(x_a)\,J_m\,(x_a)}{n\,m}\,.$$

We conclude that condition (III) will be valid, except perhaps for spurious zeros of $J_0$ for which $\mathcal{T}\,(x_a)$ eventually vanishes. Numerical computations, though, indicate that such zeros may not exist. We abstain from giving an analytical proof that $\mathcal{T}\,(x_a) \neq 0$ for all zeros of the $J_0$ function. Figure 1 shows the values of $|\mathcal{T}\,(x_a)|$ calculated numerically on the first fifteen zeros $x_a$ of $J_0$. We note an exponential decay of $|\mathcal{T}\,(x_a)|$ as $x_a$ increases, suggesting that $|\mathcal{T}\,(x_a)\,| \to 0$ only for $a \to \infty$.

We conclude that condition (III) is suitable for studying the monochromatic field when $\chi_1$ lies over the "resonant" points $x_a$, leading, together with condition (I), to a complete solution of (5.1) except, perhaps, for some rather spurious situations.

Note finally that in this case **iii** we have $\Omega = O(\epsilon^3)$ (see (2.16)). In fact, the first contribution to $\Omega$ is $2\frac{\epsilon^3}{\omega^2}\mathcal{T}\,(x_a)$. This weak dependence on $\epsilon$ implies long transition times for certain probability amplitudes (see below), an issue known as dynamical localisation (see [7]).

In order to test our algorithm and to extract more information from our solutions, we wrote a computer program to compute the matrix elements of the propagator $U(t)$ given in (1.8) for the case **iii** described above, where one has approximate dynamical localisation. The results are excellent and are briefly reported in the next section.

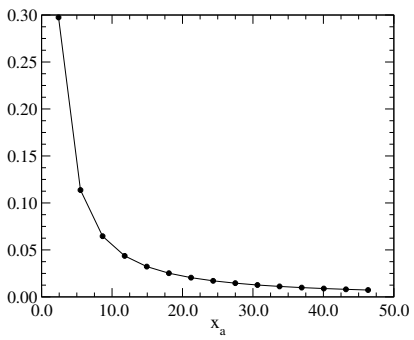


Figure 1: The filled circles represent the values of $|\mathcal{T}\,(x_a)|$ as a function of the $a$-th zero $x_a$ of $J_0$ in $\mathbb{R}_+$. We infer an exponential decay of $|M\,(0\,|\,1)\,|$ as $a$ increases.

**Remark 5.1** *We obtained for case* **iii**, $\Omega = 2\frac{\epsilon^3}{\omega^2}\mathcal{T}\,(x_a) + O(\epsilon^4)$. *Following another*

*path, we obtained in [3],*

$$\Omega = -2\frac{\epsilon^3}{\omega^2} \sum_{n_1,\, n_2 \in \mathbb{Z}} \frac{J_{2n_1+1}(x_a) J_{2n_2+1}(x_a) J_{-2(n_1+n_2+1)}(x_a)}{(2n_1+1)(2n_2+1)]} + O(\epsilon^4).$$

*Thus, we must have*

$$\sum_{n,m \in \mathbb{Z}_*} \frac{J_n(x_a)\, J_{n-m}(x_a)\, J_m(x_a)}{n\, m} = \sum_{n_1,\, n_2 \in \mathbb{Z}} \frac{J_{2n_1+1}(x_a) J_{2n_2+1}(x_a) J_{-2(n_1+n_2+1)}(x_a)}{(2n_1+1)(2n_2+1)}.$$

*Curiously, such an identity is not easy to prove by direct means. It can be established, however, with the use of statement (6) of Proposition 3.1, namely, from* $M(0\,|\,0\,|\,0) = 0$. *Actually, such an identity is not specific for Bessel functions, but is valid for sequences satisfying certain properties.*

# 6 Numerical Results

In this final section, we will briefly show some numerical results that were obtained using condition (III) to study the monochromatic interaction over the resonant points (see the discussion in the last section). Our goal here is to show how one can numerically use the results of Theorems 2.1, 2.2 and 2.3 to produce results of physical interest.

The computation of the wave function $\Phi(t)$ associated with the Schrödinger equation (1.3) can be performed by means of the propagator $U(t)$, given in (2.2), in terms of the solution of the generalised Riccati equation (2.1) given in Theorem 2.2. Due to the uniform convergence of $g(t)$ for all $t \in \mathbb{R}$ (see Theorem 2.3), one can use the proper expressions given in [2] to compute the elements of $U(t)$ via uniformly convergent power series in $\epsilon$. Although this task may not be so trivial, it can be implemented numerically with great success, as we are about to show now.

The first step toward the computation of $U(t)$ is to evaluate the Fourier coefficients of $g(t)$ using the recursive relations given in (B.1)-(B.2). Of course, we need first to compute the constants $\kappa_n$. To this end, we have to use expressions (4.36) and (4.38). Those equations involves the mean value of some previously defined functions, which can be obtained trough the zero order terms of their Fourier decomposition. Equation (3.4) of Proposition 3.1 can be used to this end. Once $V_m^{(n)}$ are all computed, the $m$-th Fourier coefficient of order $n$ of $g(t)$ can be obtained via the convolution $G_m^{(n)} = \sum_{p \in \mathbb{Z}} Q_{m-p} V_p^{(n)}$ (see equation (2.5)). As a second and final step towards the computation of $U(t)$, we have to use the expressions given in [2]. Basically, these expressions convert the integral form of the elements of the propagator $U(t)$ (see Theorem 2.1) into convergent power series

in $\epsilon$ using the Fourier coefficients $G_m^{(n)}$ of $g(t)$. They also lead to the expression of $U(t)$ in terms of its *Floquet form* (see Theorem 1.1). A more detailed description of our numerical study will be postponed for a future publication [13]. For now, let us briefly show some of our results.

We have followed steps one and two above to numerically compute the propagator $U(t)$ associated with (1.3) when $f(t) = \varphi\cos(\omega t)$ in the situation where $M(\mathcal{Q}_0) = 0$, that is, over the "resonant" points defined by the condition $\varphi = \omega x_a/2$, $x_a$ being the $a$-th positive zero of $J_0$.

Let $\Phi_+ = \binom{1}{0}$ and $\Phi_- = \binom{0}{1}$ be two orthogonal states of a system described by (1.3). The probability for the transition from the initial state $\Phi_+$ to the final state $\Phi_-$ at time $t$ is $P(t) := |\langle \Phi_+, U(t)\Phi_-\rangle|^2 = |U_{12}(t)|^2$. We computed $P(t)$ numerically using our expansions. To estimate the accuracy of our calculations, we tested the unitarity of the time evolution and considered the quantity $N(t) := |U_{11}(t)|^2 + |U_{12}(t)|^2$, which should be identically equal to 1 for unitary $U(t)$.

In Figure 2**(a)**, we show $P(t)$ for $\varphi = \omega x_1/2$, $x_1$ being the first positive zero of $J_0$. We took $\omega = 10.0$, $\epsilon = 0.1$ and worked with a sixth order expansion in $\epsilon$. The time interval considered corresponds to $1.6 \times 10^6$ times the basic cycle $2\pi/\omega$ of $f$. We recall that in this case we have condition (III) and the *secular frequency*, which dominates the quasi-periodic evolution of the system, is of order $\epsilon^3/\omega^2$ (see [3] or the discussion in Section 2.1). This explains the long time ($\sim 10^6$ of the basic cycle of $f$) needed for the system to transit from $\Phi_+$ to $\Phi_-$.

In Figure 2**(b)**, we show the quantity $N(t) - 1$ as a function of time obtained from the same calculations leading to Figure 2**(a)**. By looking at the deviations of $N(t)$ from 1, we can infer that our perturbative solution produces errors of the order of only 0.3%. This excellent numerical precision for long times is a consequence of the uniform convergence in time of our expansions, i.e., of the elimination of the secular terms.

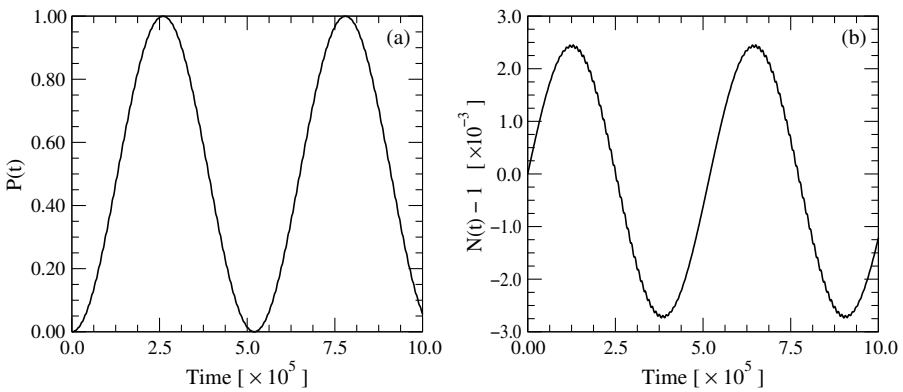


Figure 2: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.1$, $\omega = 10$ and $2\varphi_1/\omega = x_1$, the first zero of $J_0$. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

In Figure 3 we took $\omega = 1.0$, $\epsilon = 0.01$ and $\varphi_1 = \omega x_1/2$. From Figure 3**(b)** we infer errors of the order of only 0.05%. The time interval considered corresponds to $1.6 \times 10^6$ times the basic cycle $2\pi/\omega$ of $f$. In Figure 4 we took $\omega = 10.0$, $\epsilon = 0.01$ and again $\varphi_1 = \omega x_1/2$. From Figure 4**(b)** we infer errors of the order of only 0.003%. The time interval considered corresponds to $1.6 \times 10^9$ times the basic cycle $2\pi/\omega$ of $f$.

The numerical computations above involved the evaluation of the Fourier expansions of several functions, like $q$, $v_n$'s etc. We typically computed about 20 terms of that expansions, but since the coefficients decay very fast, even less terms are needed. More details about the numerical computations will be presented in a future publication [13].

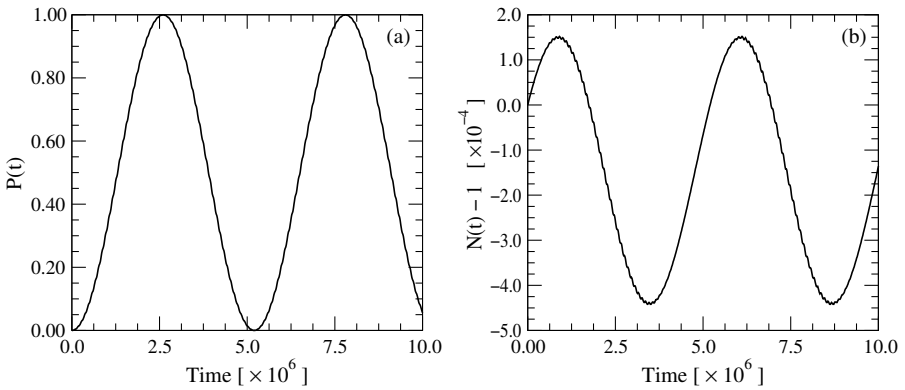


Figure 3: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.01$, $\omega = 1.0$ and $2\varphi_1/\omega = x_1$, the first zero of $J_0$. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

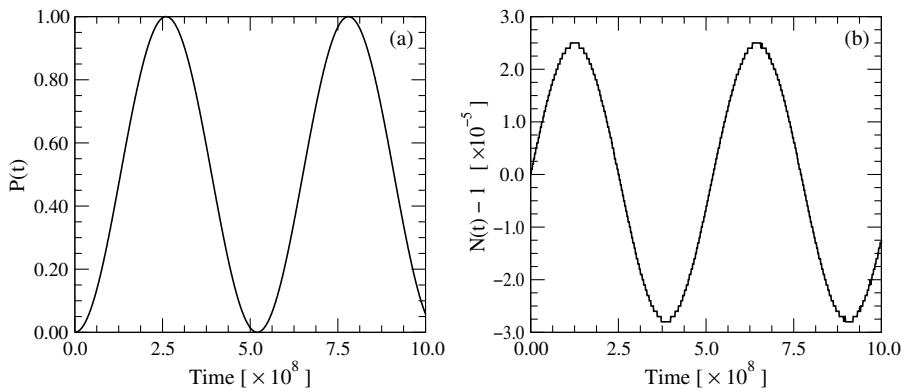


Figure 4: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.01$, $\omega = 10.0$ and $2\varphi_1/\omega = x_1$, the first zero of $J_0$. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

# Appendices

# A Some Special Relations

This appendix presents the proofs of some relations used in Section 4. Since they involve a somewhat large amount of algebraic manipulations we prefer to separate them from the main text. In the following we will (often without explicit mention) make repeated use of the propositions and corollaries proven in Sections 3.1 and 3.2.

## A.1 Obtaining Relation (4.13)

Since we have $M(\mathcal{I}_3) = 0$ and since with the choice of $\kappa_1$ in (4.12) we imposed $M(\mathcal{I}_4) = 0$, we have by the recursive relations (4.1)-(4.3)

$$v_1 v_3 = 2i\kappa_1 \,\mathfrak{R}_2\,(0\,|\,v_1 v_2\,) + \kappa_1\kappa_3 \mathcal{Q}_0\,, \tag{A.1}$$

$$v_1 v_4 = 2i\kappa_1 \,\mathfrak{R}_2\,(0\,|\,v_1 v_3\,) + i\kappa_1 \,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) + \kappa_1\kappa_4 \mathcal{Q}_0\,. \tag{A.2}$$

Inserting (4.4) into the r.h.s. of (A.1) we get

$$v_1 v_3 = -2\kappa_1^4\,\mathfrak{R}_2\,(0\,|\,2\,) + 2\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,1\,) + 2i\kappa_1^2\kappa_2\,\mathfrak{R}_2\,(0\,|\,0\,) + \kappa_1\kappa_3\,\mathcal{Q}_0\,. \tag{A.3}$$

Inserting this into the r.h.s. of (A.2) gives

$$\begin{aligned} v_1 v_4 = {} & -4i\kappa_1^5\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 4i\kappa_1^3\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 4\kappa_1^3\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) \\ & +2i\kappa_1^2\kappa_3\,\mathfrak{R}_2\,(0\,|\,0\,) + i\kappa_1\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) + \kappa_1\kappa_4\mathcal{Q}_0\,. \end{aligned} \tag{A.4}$$

Let us now compute $v_2^2$. Since $v_2 = \kappa_2 q + iq^{-1}(\kappa_1^2 \mathcal{Q}_2 - \mathcal{Q}_1)$, we have

$$v_2^2 \;=\; \kappa_2^2 \mathcal{Q}_0 + 2i\kappa_1^2 \kappa_2 \mathcal{Q}_2 - 2i\kappa_2 \mathcal{Q}_1 + s\,, \tag{A.5}$$

where $s := -\mathcal{Q}_0^{-1}(\kappa_1 \mathcal{Q}_2 - \mathcal{Q}_1)^2$. Note that $s$ contains $\kappa_1$ alone.

Now, we insert this into (A.4) and get

$$\begin{aligned} v_1 v_4 \;=\;& -4i\kappa_1^5\, \mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 4i\kappa_1^3\, \mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 6\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2\,) \\ & +2i\kappa_1^2\kappa_3\, \mathfrak{R}_2\,(0\,|\,0\,) + i\kappa_1\kappa_2^2\, \mathfrak{R}_2\,(0\,|\,0\,) + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) \\ & +i\kappa_1\, \mathfrak{R}_2\,(0\,|\,s\,) + \kappa_1\kappa_4 \mathcal{Q}_0\,, \end{aligned} \tag{A.6}$$

where we also used the fact that $\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) = \mathfrak{R}_2\,(0\,|\,2\,)$.

Collecting in (A.6) the terms depending only on $\kappa_1$, we define

$$\mathcal{A}_1 \;:=\; -4\kappa_1^4\, \mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 4\kappa_1^2\, \mathfrak{R}_3\,(0\,|\,0\,|\,1\,) + \mathfrak{R}_2\,(0\,|\,s\,) \tag{A.7}$$

and rewrite (A.6) as

$$v_1 v_4 = -6\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2\,) + i\kappa_1(2\kappa_1\kappa_3 + \kappa_2^2)\, \mathcal{Q}_2 + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) + \kappa_1\kappa_4 \mathcal{Q}_0 + i\kappa_1 \mathcal{A}_1,$$

which is expression (4.13), as desired. From definition (A.7), it is evident that $\mathcal{A}_1$ is quasi-periodic, since it is a sum of quasi-periodic functions.

## A.2 Obtaining Relation (4.14)

From relations (4.2)–(4.3)

$$v_2 v_3 \;=\; \mathcal{Q}_0^{-1}\left(i\kappa_1^2 \mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right)\, (2i\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) + \kappa_3 \mathcal{Q}_0)\,. \tag{A.8}$$

Now, by (3.8), we have $\mathcal{Q}_0^{-1}\mathcal{Q}_i\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) = \mathfrak{R}_2\,(i\,|\,v_1 v_2\,)$. Hence, (A.8) becomes

$$\begin{aligned} v_2 v_3 \;=\;& -2\kappa_1^2\, \mathfrak{R}_2\,(2\,|\,v_1 v_2\,) + 2\, \mathfrak{R}_2\,(1\,|\,v_1 v_2\,) + 2i\kappa_2\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) \\ & +\kappa_3\left(i\kappa_1^2 \mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right)\,. \end{aligned} \tag{A.9}$$

Inserting (4.4) into the r.h.s. of (A.9), it becomes

$$\begin{aligned} v_2 v_3 \;=\;& -2i\kappa_1^5\, \mathfrak{R}_2\,(2\,|\,2\,) + 2i\kappa_1^3\, \mathfrak{R}_2\,(2\,|\,1\,) - 2\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(2\,|\,0\,) + 2i\kappa_1^3\, \mathfrak{R}_2\,(1\,|\,2\,) \\ & -2i\kappa_1\, \mathfrak{R}_2\,(1\,|\,1\,) + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(1\,|\,0\,) - 2\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2\,) \\ & +2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) + i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3\mathcal{Q}_1 + \kappa_2\kappa_3\mathcal{Q}_0\,. \end{aligned}$$

This is (4.14), as desired.

## A.3 Obtaining Relation (4.17)

According to equations (4.1)–(4.3), we have

$$2v_1v_5 \;=\; 4i\kappa_1\,\Re_2\,(0\,|\,v_1v_4+v_2v_3\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,. \tag{A.10}$$

By (4.13) and (4.14)

$$\begin{aligned} v_1v_4+v_2v_3 \;=\;& -2i\kappa_1^5\,\Re_2\,(2\,|\,2\,)+2i\kappa_1^3\,\Re_2\,(2\,|\,1\,)-2\kappa_1^3\kappa_2\,\Re_2\,(2\,|\,0\,)\\ &+2i\kappa_1^3\,\Re_2\,(1\,|\,2\,)-2i\kappa_1\,\Re_2\,(1\,|\,1\,)+2\kappa_1\kappa_2\,\Re_2\,(1\,|\,0\,)\\ &-8\kappa_1^3\kappa_2\,\Re_2\,(0\,|\,2\,)+4\kappa_1\kappa_2\,\Re_2\,(0\,|\,1\,)+3i(\kappa_1^2\kappa_3+\kappa_1\kappa_2^2)\mathcal{Q}_2\\ &-i\kappa_3\mathcal{Q}_1+(\kappa_2\kappa_3+\kappa_1\kappa_4)\mathcal{Q}_0+i\kappa_1\mathcal{A}_1\,. \end{aligned}$$

Inserting this into (A.10), gives

$$\begin{aligned} 2v_1v_5 \;=\;& 8\kappa_1^6\,\Re_3\,(0\,|\,2\,|\,2\,)-8i\kappa_1^4\,\Re_3\,(0\,|\,2\,|\,1\,)-8i\kappa_1^4\kappa_2\,\Re_3\,(0\,|\,2\,|\,0\,)\\ &-8\kappa_1^4\,\Re_3\,(0\,|\,1\,|\,2\,)+8\kappa_1^2\,\Re_3\,(0\,|\,1\,|\,1\,)+8i\kappa_1^2\kappa_2\,\Re_3\,(0\,|\,1\,|\,0\,)\\ &-32i\kappa_1^4\kappa_2\,\Re_3\,(0\,|\,0\,|\,2\,)+16i\kappa_1^2\kappa_2\,\Re_3\,(0\,|\,0\,|\,1\,)\\ &-12(\kappa_1^3\kappa_3+\kappa_1^2\kappa_2^2)\,\Re_2\,(0\,|\,2\,)+4\kappa_1\kappa_3\,\Re_2\,(0\,|\,1\,)\\ &+4i\kappa_1(\kappa_2\kappa_3+\kappa_1\kappa_4)\,\Re_2\,(0\,|\,0\,)-4\kappa_1^2\,\Re_2\,(0\,|\,\mathcal{A}_1\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,. \end{aligned}$$

Let us now isolate all terms depending only on $\kappa_1$ and/or $\kappa_2$ in one single term, which we call $\mathcal{A}_2$:

$$\begin{aligned} \mathcal{A}_2 \;:=\;& 8\kappa_1^6\,\Re_3\,(0\,|\,2\,|\,2\,)-8i\kappa_1^4\,\Re_3\,(0\,|\,2\,|\,1\,)-8i\kappa_1^4\kappa_2\,\Re_3\,(0\,|\,2\,|\,0\,)\\ &-8\kappa_1^4\,\Re_3\,(0\,|\,1\,|\,2\,)+8\kappa_1^2\,\Re_3\,(0\,|\,1\,|\,1\,)+8i\kappa_1^2\kappa_2\,\Re_3\,(0\,|\,1\,|\,0\,)\\ &+16i\kappa_1^2\kappa_2\,\Re_3\,(0\,|\,0\,|\,1\,)-32i\kappa_1^4\kappa_2\,\Re_3\,(0\,|\,0\,|\,2\,)\\ &-12\kappa_1^2\kappa_2^2\,\Re_2\,(0\,|\,2\,)-4\kappa_1^2\,\Re_2\,(0\,|\,\mathcal{A}_1\,) \end{aligned} \tag{A.11}$$

and write

$$\begin{aligned} 2v_1v_5 \;=\;& \mathcal{A}_2-12\kappa_1^3\kappa_3\,\Re_2\,(0\,|\,2\,)+4\kappa_1\kappa_3\,\Re_2\,(0\,|\,1\,)\\ &+4i\kappa_1(\kappa_2\kappa_3+\kappa_1\kappa_4)\,\Re_2\,(0\,|\,0\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,. \end{aligned}$$

This is (4.17), as desired. Note that $\mathcal{A}_2$ is clearly quasi-periodic, being a sum of quasi-periodic functions.

## A.4 Obtaining Relation (4.19)

Using relations (4.2)–(4.3), and (3.8) (which implies $\mathcal{Q}_0^{-1}\mathcal{Q}_i\,\Re_2\,(0\,|\,a\,)=\Re_2\,(i\,|\,a\,)$) we have

$$\begin{aligned} 2v_2v_4 \;=\;& 2\mathcal{Q}_0^{-1}\left(i\kappa_1^2\mathcal{Q}_2-i\mathcal{Q}_1+\kappa_2\mathcal{Q}_0\right)\left[2i\,\Re_2\,(0\,|\,v_1v_3\,)+i\,\Re_2\left(0\,\middle|\,v_2^2\right)+\kappa_4\mathcal{Q}_0\right]\\ =\;& -4\kappa_1^2\,\Re_2\,(2\,|\,v_1v_3\,)+4\,\Re_2\,(1\,|\,v_1v_3\,)+4i\kappa_2\,\Re_2\,(0\,|\,v_1v_3\,)\\ &-2\kappa_1^2\,\Re_2\left(2\,\middle|\,v_2^2\right)+2\,\Re_2\left(1\,\middle|\,v_2^2\right)+2i\kappa_2\,\Re_2\left(0\,\middle|\,v_2^2\right)\\ &+2\kappa_4\left(i\kappa_1^2\mathcal{Q}_2-i\mathcal{Q}_1+\kappa_2\mathcal{Q}_0\right) \end{aligned}$$

Inserting above the expression for $v_1 v_3$ given in (A.3), we get

$$\begin{aligned} 2v_2v_4 &= 8\kappa_1^6\,\mathfrak{R}_3\,(2\,|\,0\,|\,2\,) - 8\kappa_1^4\,\mathfrak{R}_3\,(2\,|\,0\,|\,1\,) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(2\,|\,0\,|\,0\,) \\ &\quad -4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0) - 8\kappa_1^4\,\mathfrak{R}_3\,(1\,|\,0\,|\,2\,) + 8\kappa_1^2\,\mathfrak{R}_3\,(1\,|\,0\,|\,1\,) \\ &\quad +8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(1\,|\,0\,|\,0\,) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,) \\ &\quad +8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 8\kappa_1^2\kappa_2^2\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) + 4i\kappa_1\kappa_2\kappa_3\,\mathfrak{R}_2\,(0\,|\,0) \\ &\quad -2\kappa_1^2\,\mathfrak{R}_2\left(2\,\middle|\,v_2^2\right) + 2\,\mathfrak{R}_2\left(1\,\middle|\,v_2^2\right) + 2i\kappa_2\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) \\ &\quad +2\kappa_4\left(i\kappa_1^2\mathcal{Q}_2 - i\,\mathcal{Q}_1 + \kappa_2\mathcal{Q}_0\right)\,. \end{aligned}$$

We now isolate all terms depending only on $\kappa_1$ and/or $\kappa_2$ in one single term, which we call $\mathcal{A}_3$:

$$\begin{aligned} \mathcal{A}_3 &:= 8\kappa_1^6\,\mathfrak{R}_3\,(2\,|\,0\,|\,2\,) - 8\kappa_1^4\,\mathfrak{R}_3\,(2\,|\,0\,|\,1\,) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(2\,|\,0\,|\,0\,) \\ &\quad -8\kappa_1^4\,\mathfrak{R}_3\,(1\,|\,0\,|\,2\,) + 8\kappa_1^2\,\mathfrak{R}_3\,(1\,|\,0\,|\,1\,) + 8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(1\,|\,0\,|\,0\,) \\ &\quad -8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 8\kappa_1^2\kappa_2^2\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) \\ &\quad -2\kappa_1^2\,\mathfrak{R}_2\left(2\,\middle|\,v_2^2\right) + 2\,\mathfrak{R}_2\left(1\,\middle|\,v_2^2\right) + 2i\kappa_2\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) \end{aligned} \tag{A.12}$$

(note here that, by (A.5), $v_2^2$ depends only on $\kappa_1$ and $\kappa_2$). We can now write

$$\begin{aligned} 2v_2v_4 &= \mathcal{A}_3 - 4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0) \\ &\quad +2i(\kappa_1^2\kappa_4 + 2\kappa_1\kappa_2\kappa_3)\,\mathcal{Q}_2 - 2i\kappa_4\mathcal{Q}_1 + 2\kappa_2\kappa_4\mathcal{Q}_0\,. \end{aligned}$$

This is the desired relation (4.19). Again, note that $\mathcal{A}_3$ is quasi-periodic, for it is a sum of quasi-periodic functions.

## A.5 Obtaining Relation (4.34)

Let us explicitly write $\kappa_1 l_{n-1}$ in term (i) of (4.33) using equation (4.27). We have,

$$\kappa_1 l_{n-1}(t) \;=\; i\kappa_1 \sum_{p=1}^{n-2} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)_t \;=:\; A(t) + B(t)\,,$$

where

$$A(t) \;:=\; 2i\,\kappa_1\,\mathfrak{R}_2\,(0\,|\,v_1 v_{n-2}\,)\,, \quad B(t) \;:=\; i\kappa_1 \sum_{p=2}^{n-3} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)\,. \tag{A.13}$$

The above expressions for $A(t)$ and $B(t)$ will now be worked individually. Let us start with $A(t)$. By the inductive hypothesis, we are allowed to use (4.1) and (4.3). We write

$$\begin{aligned} A(t) &= 2\,i\,\kappa_1^2\,\mathfrak{R}_2\left(0\,\middle|\,i\sum_{p=1}^{n-3}\mathfrak{R}_2\,(0\,|\,v_p v_{n-2-p}\,) + \kappa_{n-2}\mathcal{Q}_0\right) \\ &= -2\,\kappa_1^2 \sum_{p=1}^{n-3} \mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-2-p}\,) + 2\,i\,\kappa_1^2\kappa_{n-2}\,\mathfrak{R}_2\,(0\,|\,0)\,. \end{aligned}$$

Note that $A(t)$ is implicitly dependent on $\kappa_{n-3}$, namely through $v_{n-3}$. To make this dependence explicit we have to split the sum containing $v_{n-3}$ and write $v_{n-3}$ with the use of (4.3):

$$\begin{aligned} A(t) &= -2\,\kappa_1^2 \sum_{p=2}^{n-4} \mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-2-p}\,) - 4\,\kappa_1^2\,\mathfrak{R}_3\,(0\,|\,0\,|\,v_1 v_{n-3}\,) \\ &\quad +2\,i\,\kappa_1^2\kappa_{n-2}\,\mathfrak{R}_2\,(0\,|\,0\,) \\ &= -2\,\kappa_1^2 \sum_{p=2}^{n-4} \mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-2-p}\,) - 4i\,\kappa_1^3 \sum_{p=1}^{n-4} \mathfrak{R}_4\,(0\,|\,0\,|\,0\,|\,v_p v_{n-3-p}\,) \\ &\quad -4\,\kappa_1^3\kappa_{n-3}\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) + 2\,i\,\kappa_1^2\kappa_{n-2}\,\mathfrak{R}_2\,(0\,|\,0\,)\ . \end{aligned} \tag{A.14}$$

We will now work on $B(t)$, equation (A.13). Using (4.1) and (4.3), we have

$$B(t) = 2i\kappa_1\,\mathfrak{R}_2\,(0\,|\,v_2 v_{n-3}\,) + i\kappa_1 \sum_{p=3}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)\ . \tag{A.15}$$

Next, we have to compute separately $v_2 v_{n-3}$. Using once more (4.2), (4.3) and (3.8) (which implies $\mathcal{Q}_0^{-1}\mathcal{Q}_i\,\mathfrak{R}_2\,(0\,|\,a\,) = \mathfrak{R}_2\,(i\,|\,a\,)$), we get

$$\begin{aligned} v_2 v_{n-3} &= \mathcal{Q}_0^{-1}\left(i\kappa_1^2\mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2\mathcal{Q}_0\right)\left\{ i\sum_{p=1}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-3-p}\,)_t + \kappa_{n-3}\mathcal{Q}_0(t)\right\} \\ &= -\kappa_1^2 \sum_{p=1}^{n-4} \mathfrak{R}_2\,(2\,|\,v_p v_{n-3-p}\,) + \sum_{p=1}^{n-4} \mathfrak{R}_2\,(1\,|\,v_p v_{n-3-p}\,) \\ &\quad +i\kappa_2 \sum_{p=1}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-3-p}\,) + \kappa_{n-3}\left(i\kappa_1^2\mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2\mathcal{Q}_0\right)\ . \end{aligned}$$

This expression for $v_2 v_{n-3}$ has to be introduced into the first term of (A.15). The result is

$$\begin{aligned} B(t) &= 2i\kappa_1 \sum_{p=1}^{n-4} \Big\{ -\kappa_1^2\,\mathfrak{R}_3\,(0\,|\,2\,|\,v_p v_{n-3-p}\,) + \mathfrak{R}_3\,(0\,|\,1\,|\,v_p v_{n-3-p}\,) \\ &\quad +i\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-3-p}\,)\Big\} + i\kappa_1 \sum_{p=3}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)\ . \\ &\quad +2i\kappa_1\kappa_{n-3}\left[i\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,2\,) - i\,\mathfrak{R}_2\,(0\,|\,1\,) + \kappa_2\,\mathfrak{R}_2\,(0\,|\,0\,)\right]\ . \end{aligned} \tag{A.16}$$

Since both $A$ and $B$ are quasi-periodic, we can now compute $\kappa_1 M(l_{n-1}) = M(A) + M(B)$. Using (A.14), (A.16), (4.6) and the already proven fact that $M\,(0\,|\,0\,) = 0$, we get

$$\kappa_1 M(l_{n-1}) = 2\kappa_1\kappa_{n-3} M\,(0\,|\,1\,) + \mathcal{R}_n^{(1)}\,, \tag{A.17}$$

where

$$\mathcal{R}_n^{(1)} := 2i\kappa_1 \sum_{p=1}^{n-4} \Big\{ -2\kappa_1^2 M(\,\mathfrak{R}_4\,(0\,|0\,|0\,|\,v_p v_{n-3-p}\,)) + M(\,\mathfrak{R}_3\,(0\,|1\,|\,v_p v_{n-3-p}\,)) \\ -\kappa_1^2 M(\,\mathfrak{R}_3\,(0\,|2\,|\,v_p v_{n-3-p}\,)) + i\kappa_2 M(\,\mathfrak{R}_3\,(0\,|0\,|\,v_p v_{n-3-p}\,)) \Big\} \\ -2\,\kappa_1^2 \sum_{p=2}^{n-4} M(\,\mathfrak{R}_3\,(0\,|0\,|\,v_p v_{n-2-p}\,)) + i\kappa_1 \sum_{p=3}^{n-4} M(\,\mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p})) \,. \tag{A.18}$$

This is the desired relation (4.34). By inspection, one verifies that $\mathcal{R}_n^{(1)}$ depends on the constants $\kappa_1, \ldots, \kappa_{n-4}$ only. Notice that the constant $\kappa_{n-2}$ disappeared completely when we took the mean value of $A(t) + B(t)$, due to the crucial fact that $M(\mathcal{Q}_2) = 0$. This is very important, otherwise we would have in (A.17) an equation for <u>two</u> unknowns $\kappa_{n-3}$ and $\kappa_{n-2}$.

## A.6 Obtaining Relation (4.35)

The main point is to make the $\kappa_{n-3}$ dependence of $l_{n-2}(t)$ explicit. Using (4.28) and (4.3) for $v_{n-3}(t)$, we can write

$$\begin{aligned} q^{-1}v_2 l_{n-2} &= iq^{-1}v_2 \sum_{p=1}^{n-3} \mathfrak{R}_2\,(0\,|\,v_p v_{n-2-p}) \;=\; i\sum_{p=1}^{n-3} \mathfrak{R}_2\,(qv_2\,|\,v_p v_{n-2-p}) \\ &= 2i\,\mathfrak{R}_2\,(qv_2\,|\,v_1 v_{n-3}) + i\sum_{p=2}^{n-4} \mathfrak{R}_2\,(qv_2\,|\,v_p v_{n-2-p}) \\ &= 2i\kappa_1\,\mathfrak{R}_2\left(qv_2 \;\middle|\; i\sum_{p=1}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-3-p}) + \kappa_{n-3}\mathcal{Q}_0\right) \\ &\quad + i\sum_{p=2}^{n-4} \mathfrak{R}_2\,(qv_2\,|\,v_p v_{n-2-p}) \\ &= -2\kappa_1 \sum_{p=1}^{n-4} \mathfrak{R}_3\,(qv_2\,|\,0\,|\,v_p v_{n-3-p}) + i\sum_{p=2}^{n-4} \mathfrak{R}_2\,(qv_2\,|\,v_p v_{n-2-p}) \\ &\quad + 2i\kappa_1\kappa_{n-3}\,\mathfrak{R}_2\,(qv_2\,|\,0)\,. \end{aligned}$$

According to (4.2),

$$\mathfrak{R}_2\,(qv_2\,|\,0) \;=\; i\kappa_1^2\,\mathfrak{R}_2\,(2\,|\,0) - i\,\mathfrak{R}_2\,(1\,|\,0) + \kappa_2\,\mathfrak{R}_2\,(0\,|\,0)$$

and, hence, $M(\,\mathfrak{R}_2\,(qv_2\,|\,0)) = -iM\,(1\,|\,0) = iM\,(0\,|\,1)$. Therefore,

$$M(q^{-1}v_2 l_{n-2}) \;=\; -2\kappa_1\kappa_{n-3} M\,(0\,|\,1) + \mathcal{R}_n^{(2)}\,,$$

where

$$\mathcal{R}_n^{(2)} \;:=\; -2\kappa_1 \sum_{p=1}^{n-4} M\left(\,\mathfrak{R}_3\left(qv_2 \mid 0 \mid v_p v_{n-3-p}\,\right)\right) + i \sum_{p=2}^{n-4} M\left(\,\mathfrak{R}_2\left(qv_2 \mid v_p v_{n-2-p}\,\right)\right). \tag{A.19}$$

This is the desired equation (4.35). By inspection, one verifies that $\mathcal{R}_n^{(2)}$ depends on the constants $\kappa_1, \ldots, \kappa_{n-4}$ only.

# B Proof of Convergence of the $\epsilon$ Expansion for Periodic $f$

Here we will present the proof of Theorem 2.3, i.e., the proof of convergence of the $\epsilon$ expansion of (2.5) for periodic $f$. It follows the ideas of [2], but technical adaptations are necessary. For the sake of simplification we shall consider here only the case where $F_0 = M(f) = 0$. The general case $F_0 \neq 0$ can be treated following the lines described in detail in [2].

In terms of the Fourier coefficients $Q_m$ and $Q_m^{(2)}$, appearing in (2.11), of the Fourier coefficients $V_m^{(n)}$ of (2.12) and of the constants $\kappa_n$, relations (4.1)-(4.3), become

$$V_m^{(1)} \;=\; \kappa_1 Q_m, \quad V_m^{(2)} \;=\; \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq 0}} \frac{Q_{m-n_1}\left(\kappa_1^2 Q_{n_1}^{(2)} - \overline{Q_{-n_1}^{(2)}}\right)}{n_1 \omega} + \kappa_2 Q_m, \tag{B.1}$$

$$V_m^{(n)} \;=\; \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1 + n_2 \neq 0}} \frac{Q_{m-(n_1+n_2)}}{(n_1+n_2)\omega} \left( \sum_{p=1}^{n-1} V_{n_1}^{(p)} V_{n_2}^{(n-p)} \right) + \kappa_n Q_m, \quad \text{for } n \geq 3. \tag{B.2}$$

Of course, due to the choices of the constants $\kappa_n$ described before, no secular terms appear.

By (2.13) and by an inductive argument, we will prove the following statement: for all $p \in \mathbb{N}$ and all $m \in \mathbb{Z}$ there are constants $K_p > 0$ such that

$$\left| V_m^{(p)} \right| \;\leq\; K_p \frac{e^{-\chi |m|}}{\ll m \gg^2}. \tag{B.3}$$

To show this, let us first recall the following result, proven in [2]:

**Lemma B.1** *For $\chi > 0$ and $m \in \mathbb{Z}$ define*

$$\mathcal{B}(m) \equiv \mathcal{B}(m,\, \chi) \;:=\; \sum_{n \in \mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m-n \gg^2 \ll n \gg^2}.$$

*Then one has*

$$\mathcal{B}(m) \;\le\; B_0 \frac{e^{-\chi|m|}}{\ll m \gg^2},$$

*for some constant* $B_0 \equiv B_0(\chi) > 0$ *and for all* $m \in \mathbb{Z}$. □

From (B.1) and (2.13), we have

$$\begin{aligned}
\left|V_m^{(1)}\right| &\le \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2}, \\
\left|V_m^{(2)}\right| &\le \frac{2\mathcal{Q}^2}{\omega} \sum_{n_1 \in \mathbb{Z}_*} \frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1 \gg^2 \ll n_1 \gg^2} \frac{1}{|n_1|} \;+\; |\kappa_2|\mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2},
\end{aligned}$$

where we used $|\kappa_1| = 1$. Now

$$\sum_{n_1 \in \mathbb{Z}_*} \frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1 \gg^2 \ll n_1 \gg^2} \frac{1}{|n_1|} \;\le\; \sum_{n_1 \in \mathbb{Z}} \frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1 \gg^2 \ll n_1 \gg^2} \;\le\; B_0 \frac{e^{-\chi|m|}}{\ll m \gg^2},$$

where the last inequality comes from Lemma B.1. Hence, we can write

$$\left|V_m^{(1)}\right| \;\le\; K_1 \frac{e^{-\chi|m|}}{\ll m \gg^2} \qquad \text{and} \qquad \left|V_m^{(2)}\right| \;\le\; K_2 \frac{e^{-\chi|m|}}{\ll m \gg^2},$$

for all $m \in \mathbb{Z}$, by choosing

$$K_1 \;:=\; \mathcal{Q} \qquad \text{and} \qquad K_2 \;:=\; \frac{2\mathcal{Q}^2 B_0}{\omega} + |\kappa_2|\mathcal{Q}.$$

To proceed, let us assume (B.3) for all $p = 1, \ldots, n-1$. By (B.2) and (2.13), we have

$$\begin{aligned}
\left|V_m^{(n)}\right| \;\le\; & \frac{\mathcal{Q}}{\omega} \left( \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1+n_2 \neq 0}} \frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2 \gg^2 \ll n_1 \gg^2 \ll n_2 \gg^2} \frac{1}{|n_1+n_2|} \right) \\
& \times \left( \sum_{p=1}^{n-1} K_p K_{n-p} \right) \;+\; |\kappa_n|\mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2}, \qquad \text{for } n \ge 3.
\end{aligned}$$

Again, applying twice Lemma B.1,

$$\begin{aligned}
&\sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1+n_2 \neq 0}} \frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2 \gg^2 \ll n_1 \gg^2 \ll n_2 \gg^2} \frac{1}{|n_1+n_2|} \le \\
&\qquad\qquad \sum_{n_1,\, n_2 \in \mathbb{Z}} \frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2 \gg^2 \ll n_1 \gg^2 \ll n_2 \gg^2} \;\le\; (B_0)^2 \frac{e^{-\chi|m|}}{\ll m \gg^2}.
\end{aligned}$$

Therefore,

$$\left|V_m^{(n)}\right| \;\leq\; K_n \frac{e^{-\chi|m|}}{\ll m \gg^2}$$

for all $m \in \mathbb{Z}$, by choosing

$$K_n \;:=\; \frac{(B_0)^2 \mathcal{Q}}{\omega} \left( \sum_{p=1}^{n-1} K_p K_{n-p} \right) + \kappa_n^0 \mathcal{Q}, \tag{B.4}$$

where $\kappa_n^0$ is some suitably chosen upper bound for $|\kappa_n|$. We now turn our attention to $|\kappa_n|$, for which we have to find estimates using again the inductive hypothesis (B.3) for all $p = 1, \ldots, n-1$. The constants $\kappa_1$, $\kappa_2$ and $\kappa_3$ are are fixed by (4.36) and $\kappa_n$, $n \geq 4$, are given by (4.38), from which we get

$$|\kappa_n| \leq \frac{1}{4|M\,(0\,|\,1)\,|} \left[ \underbrace{\sum_{p=4}^{n-1} |M\,(v_p v_{n+3-p})|}_{T_1} + 2 \underbrace{\left|\mathcal{R}_{n+3}^{(1)}\right|}_{T_2} + 2 \underbrace{\left|\mathcal{R}_{n+3}^{(2)}\right|}_{T_3} + 2 \underbrace{\left|M\,\left(q^{-1} v_3 l_n\right)\right|}_{T_4} \right], \tag{B.5}$$

for $n \geq 4$. We have to bound each of the terms $T_i$ indicated above. Let us start with $T_1$.

**Bound for $T_1$.** By (2.12), $M\,(v_p v_{n+3-p}) = \sum_{m\in\mathbb{Z}} V_m^{(p)} V_{-m}^{(n+3-p)}$. Hence, by the inductive hypothesis (B.3), assumed for $p = 1, \ldots, n-1$, we have

$$T_1 \;=\; \sum_{p=4}^{n-1} |M\,(v_p v_{n+3-p})| \;\leq\; \eta_1 \sum_{p=4}^{n-1} K_p K_{n+3-p}, \tag{B.6}$$

where $\eta_1 := \sum_{m\in\mathbb{Z}} \dfrac{e^{-2\chi|m|}}{\ll m \gg^2}$.

**Bound for $T_2$.** Expression (A.18) involves sums of the mean value of functions like $\mathfrak{R}_k\,(a_1\,|\cdots|\,a_k\,)$. Let us prove a general statement about such functions.

**Proposition B.2** *For $k \geq 2$, let $a_1, \ldots, a_k$ be periodic functions with the same frequency $\omega$, and such that $a_l(t) = \sum_{m\in\mathbb{Z}} A^{(l)}(m) e^{im\omega t}$, where the Fourier coefficients $A^{(l)}(m)$ satisfy*

$$\left|A^{(l)}(m)\right| \;\leq\; \alpha_l \frac{e^{-\chi|m|}}{\ll m \gg^2}, \tag{B.7}$$

*for all $m \in \mathbb{Z}$ and all $l = 1, \ldots, k$, where $\alpha_l > 0$ and $\chi > 0$. Then, there is a positive constant $\beta_k$ such that the Fourier coefficients $\mathfrak{R}_k\,(a_1\,|\cdots|\,a_k\,)\,(m)$, $m \in \mathbb{Z}$, of $\mathfrak{R}_k\,(a_1\,|\cdots|\,a_k\,)_t$ are bounded by*

$$|\,\mathfrak{R}_k\,(a_1\,|\cdots|\,a_k\,)\,(m)| \;\leq\; \beta_k\,\alpha_1 \cdots \alpha_k\, \frac{e^{-\chi|m|}}{\ll m \gg^2}\,. \tag{B.8}$$

□

**Proof.** Let us first consider the case $k = 2$. The Fourier coefficients of $\mathfrak{R}_2\,(a_1\ |\,a_2\,)_t$ are given by

$$\mathfrak{R}_2\,(a_1\ |\,a_2\,)\,(m)\ =\ \sum_{n\in\mathbb{Z}} A^{(1)}(m-n)\tilde{A}^{(2)}(n), \tag{B.9}$$

where

$$\tilde{A}^{(2)}(m)\ :=\ \begin{cases} \dfrac{A^{(2)}(m)}{im\omega}, & \text{for } m\neq 0, \\ -\dfrac{1}{i\omega}\displaystyle\sum_{k\in\mathbb{Z}_*}\frac{A^{(2)}(k)}{k}, & \text{for } m=0 \end{cases}.$$

From (B.7), it follows that

$$\left|A_I^{(2)}(m)\right|\ \leq\ \alpha_2\frac{D_0}{\omega}\frac{e^{-\chi|m|}}{\ll m\gg^2},$$

where $D_0 := \sum_{m\in\mathbb{Z}}\frac{e^{-\chi|m|}}{\ll m\gg^2}$. Therefore, from (B.9), by (B.7) and by Lemma B.1, one has

$$|\,\mathfrak{R}_2\,(a_1\ |\,a_2\,)\,(m)|\ \leq\ \alpha_1\alpha_2\frac{B_0D_0}{\omega}\ \frac{e^{-\chi|m|}}{\ll m\gg^2}.$$

This proves the statement for $k = 2$. The general case follows from (3.9), by induction. ∎

As a corollary, one sees from (2.13) that the functions $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have Fourier coefficients bounded as $|\mathcal{Q}_i(m)|\leq\gamma_i\frac{e^{-\chi|m|}}{\ll m\gg^2}$ for some positive $\gamma_i$. Moreover, by the inductive hypothesis (B.3) and by Lemma B.1, the Fourier coefficients $(v_pv_q)(m)$ of product functions like $v_p(t)v_q(t)$, with $p,\ q = 1,\ \dots\ ,\ n-1$, are also bounded as

$$|(v_pv_q)(m)|\ \leq\ B_0\,K_p\,K_q\,\frac{e^{-\chi|m|}}{\ll m\gg^2} \tag{B.10}$$

for all $m\in\mathbb{Z}$. The consequence of all this is that for indices $i_j\in\{0,\ 1,\ 2\}$ and $p,\ q = 1,\ \dots\ ,\ n-1$ one has

$$|\,\mathfrak{R}_k\,(i_1|i_2|\cdots|i_{k-1}|v_pv_q)\,(m)|\ \ \leq\ \ \gamma_{i_1,\,i_2,\,\cdots,\,i_{k-1}}\,K_p\,K_q\,\frac{e^{-\chi|m|}}{\ll m\gg^2},\qquad \forall m\in\mathbb{Z},$$

from some positive constants $\gamma_{i_1,\,i_2,\,\cdots,\,i_{k-1}}$, depending on the indices $i_j$.

Returning our attention to expression (A.18), we conclude that

$$\left|\mathcal{R}_{n+3}^{(1)}\right|\ \ \leq\ \ \eta_2\sum_{p=1}^{n-1}K_pK_{n-p}+\eta_3\sum_{p=2}^{n-1}K_pK_{n+1-p}+\eta_4\sum_{p=3}^{n-1}K_pK_{n+2-p}\,, \tag{B.11}$$

for certain positive constants $\eta_2,\ \eta_3,\ \eta_4$.

**Bound for $T_3$.** Since $qv_2$ is a linear combination of the functions $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, we conclude from (A.19) and from the previous arguments that

$$\left|\mathcal{R}^{(2)}_{n+3}\right| \;\leq\; \eta_5 \sum_{p=1}^{n-1} K_p K_{n-p} + \eta_6 \sum_{p=2}^{n-1} K_p K_{n+1-p}\,, \tag{B.12}$$

for certain positive constants $\eta_5,\ \eta_6$.

**Bound for $T_4$.** By (4.28), one has

$$M(q^{-1}v_3 l_n) \;=\; i \sum_{p=1}^{n-1} M(\,\mathfrak{R}_2\,(qv_3 \mid v_p v_{n-p}))\,.$$

From (4.21) and (B.10), we see that both $qv_3$ and $v_p v_{n-p}$ satisfy the conditions of Proposition B.2. Hence, by (B.8), $M(\,\mathfrak{R}_2\,(qv_3 \mid v_p v_{n-p})) \leq \eta_7 K_p K_{n-p}$ for some positive constant $\eta_7$ and

$$M(q^{-1}v_3 l_n) \;\leq\; \eta_7 \sum_{p=1}^{n-1} K_p K_{n-p}. \tag{B.13}$$

We are finished with the bounds for the terms $T_i$ of (B.5). If we collect (B.6), (B.11), (B.12) and (B.13) and return to (B.4), we conclude that there are positive constants $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ such that we can recursively define

$$\begin{aligned} K_n \;:=\; \Gamma_1 \sum_{p=1}^{n-1} K_p K_{n-p} + \Gamma_2 \sum_{p=2}^{n-1} K_p K_{n+1-p} \\ + \Gamma_3 \sum_{p=3}^{n-1} K_p K_{n+2-p} + \Gamma_4 \sum_{p=4}^{n-1} K_p K_{n+3-p}\,, \end{aligned}$$

for $n > 4$, after fixing the convenient values for $K_1,\ K_2,\ K_3$ and $K_4$. Note that we can choose $K_1 = K_2 = K_3 = K_4$ taking $K_i = \max\{K_1,\ K_2,\ K_3,\ K_4\}$ for all $i = 1, \ldots, 4$. Defining $\Gamma := \max\{\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4\}$, we can redefine the $K_n$ so as to have the more convenient choice

$$\begin{aligned} K_n \;:=\; \Gamma \Bigg[ \sum_{p=1}^{n-1} K_p K_{n-p} + \sum_{p=2}^{n-1} K_p K_{n+1-p} \\ + \sum_{p=3}^{n-1} K_p K_{n+2-p} + \sum_{p=4}^{n-1} K_p K_{n+3-p} \Bigg]\,. \end{aligned} \tag{B.14}$$

Expression (B.14) has an analogous one in [2], with the distinction that only the two first sums above occurred in the corresponding expression for $K_n$. From

now on, we follow closely the steps of [2]. The first one is to show that $K_n$ is a *non-decreasing* sequence. We have

$$K_{n+1} - K_n \;:=\; \Gamma \left[ \left( \sum_{a=1}^{4} K_a \right) K_n + \left( \sum_{a=1}^{3} K_a \right) (K_n - K_{n-a}) + \sum_{p=4}^{n-1} K_p (K_{n+4-p} - K_{n-p}) \right].$$

Therefore, assuming inductively $K_1 = K_2 = K_3 = K_4 \leq \ldots \leq K_n$ implies $K_n \leq K_{n+1}$, thus proving that the sequence is non-decreasing. Next, we write (B.14) as

$$K_n = \Gamma \left[ \sum_{a=1}^{3} \left( \sum_{b=a}^{3} K_b \right) K_{n-a} + \sum_{p=4}^{n-1} K_p \left( K_{n-p} + K_{n+1-p} + K_{n+2-p} + K_{n+3-p} \right) \right].$$

Since the sequence is non-decreasing, we have $K_{n-p}+K_{n+1-p}+K_{n+2-p}+K_{n+3-p} \leq 4K_{n+3-p}$ and $K_{n-a} \leq K_{n-1}$ for $a = 1, 2, 3$. Hence, we may say that

$$\begin{aligned} K_n \;&\leq\; \Gamma \sum_{a=1}^{3} \left( \sum_{b=a}^{3} K_b \right) K_{n-1} + 4\Gamma \sum_{p=4}^{n-1} K_p K_{n+3-p} \\ &=\; \tilde{\Gamma}\, K_{n-1} K_4 + 4\Gamma \sum_{p=4}^{n-1} K_p K_{n+3-p}\,, \end{aligned} \tag{B.15}$$

where $\tilde{\Gamma} := \dfrac{\Gamma}{K_4} \displaystyle\sum_{a=1}^{3} \left( \sum_{b=a}^{3} K_b \right)$ is a positive constant. Adding up the positive quantity $\tilde{\Gamma} \displaystyle\sum_{p=4}^{n-2} K_p K_{n+3-p}$ to (B.15) and setting $\Lambda := \max\{\tilde{\Gamma},\ 4\Gamma\}$, we get

$$K_n \;\leq\; \Lambda \sum_{p=4}^{n-1} K_p K_{n+3-p}\,. \tag{B.16}$$

Let us now define another auxiliary sequence $J_k$ such that $J_l = K_l$ for $l = 1,\ 2,\ 3,\ 4$ and

$$J_n \;:=\; \Lambda \sum_{p=4}^{n-1} J_p J_{n+3-p}\,.$$

for $n > 4$. It is a simple exercise to show from (B.16) that $K_n \leq J_n$ for all $n$. Now, let us consider the translated sequence $L_n = J_{n+2}$, $n \geq 1$. We have

$$L_n \;=\; \Lambda \sum_{p=4}^{n+1} J_p J_{n+5-p} \;=\; \Lambda \sum_{p=4}^{n+1} L_{p-2} L_{n+3-p} \;=\; \Lambda \sum_{p=2}^{n-1} L_p L_{n+1-p}\,. \tag{B.17}$$

The sequence $\mathbf{c}_n$ defined by $\mathbf{c}_n = \sum_{p=2}^{n-1} \mathbf{c}_p \mathbf{c}_{n-p+1}$ for $n \geq 3$, with $\mathbf{c}_1 = \mathbf{c}_2 = 1$, defines the so-called "Catalan numbers", which can be expressed in a closed form as

$$\mathbf{c}_n \;=\; \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2\,.$$

By Stirling's formula, the $\mathbf{c}_n$'s have the following asymptotic behaviour: $\mathbf{c}_n \approx \frac{1}{16\sqrt{\pi}} \frac{4^n}{n^{3/2}}$, for $n$ large. The existence of a connection between the Catalan numbers and the sequence $L_n$ is evident from (B.17). Two distinctions are the factor $\Lambda$ appearing in (B.17) and the fact that $L_1 = L_2 = K_3 = K_4$ are not necessarily equal to 1. One can, however, easily show that $L_n = (K_3)^{n-1}\, \Lambda^{n-2}\, \mathbf{c}_n$, $n \geq 2$. Hence, the following asymptotic behaviour can be established:

$$L_n \;\approx\; \frac{1}{16\sqrt{\pi} K_3 \Lambda^2}\; \frac{(4K_3\Lambda)^n}{n^{3/2}}, \qquad n \text{ large.}$$

Since $K_n \leq J_n = L_{n-2}$, we conclude that for $n$ large $K_n \leq M_0 (M_1)^n$, for some positive constants $M_0$, $M_1$. This completes the proof of Theorem 2.3. ■

# C Comments on the Fourier Coefficients

Relations (1.8) and (1.9) can be obtained in our case by repeating the analysis of [2]. One of the conclusions is that the Fourier coefficients of (1.9) are analytic functions of $\epsilon$, for $|\epsilon|$ small enough. There is, however, a point to be noticed here. If we follow the steps of [2] and compute the Fourier expansion of $R(t)^{-2}$ to obtain the Fourier expansion for $S(t)$, this last function will contain terms like $\int_0^t e^{2i\Omega t} dt$, which behave like $\Omega^{-1}$. In case (I), treated in [2], this is not problematic, since $\Omega = O(\epsilon)$ and since such terms are always multiplied by a factor $\epsilon$ (see (2.2)). In our case, however, $\Omega = O(\epsilon^3)$ and we have to look such terms more carefully.

Looking at the $\epsilon$-expansion for $U_{12}(t)$, we have

$$U_{12}(t) = -ie^{-i\gamma(\epsilon)} \left[\epsilon W_1(t) + \epsilon^2 W_2(t) + O(\epsilon^3)\right],$$

where, after a lengthy computation, we get

$$\epsilon W_1(t) \;=\; \epsilon \sum_{n,\,m} \frac{Q_n^{(2)}}{i(n\omega + 2\Omega)} \left[\overline{Q_{n-m}} e^{i(m\omega+\Omega)t} - \overline{Q_{-m}} e^{i(m\omega-\Omega)t}.\right]$$

The problematic term is that proportional to $\epsilon/\Omega$, which appears for $n = 0$. However, this term is proportional to $Q_0^{(2)} = M(\mathcal{Q}_0) = 0$, by hypothesis and, hence,

the Fourier coefficients of $\epsilon W_1$ are analytic on $\epsilon$. Let us look now at $\epsilon^2 W_2$. After another lengthy computation, one gets

$$\epsilon^2 W_2(t) = \frac{2\kappa_1\epsilon^2}{i\omega} \sum_{p,\, n,\, m\neq 0} \left[ \frac{Q^{(2)}_{n-m}\overline{Q_{n-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega+\Omega)t} - \frac{Q^{(2)}_{n-m}\overline{Q_{-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega-\Omega)t} \right]$$
$$+ \frac{\kappa_1\epsilon^2}{i\omega} \sum_{p,\, n,\, m\neq 0} \left[ \frac{Q^{(2)}_n\overline{Q_{m-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega-\Omega)t} - \frac{Q^{(2)}_n\overline{Q_{m+n-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega+\Omega)t} \right].$$

Again, the dangerous terms are those with $n = 0$, since the are proportional to $\epsilon^2/\Omega$. However, one sees by inspection that such terms vanish in the expression above. In fact, in the last two sums they appear proportional to $Q^{(2)}_0$, which is zero. In the first two sums, they appear proportional to $\sum_{m\neq 0} \frac{Q^{(2)}_{-m}Q^{(2)}_m}{m}$, which is also zero, as one sees by interchanging $m \to -m$.

Our conclusion is that all dangerous terms like $1/\Omega$ appear multiplied at least by factors $\epsilon^3$ and are, therefore, analytic.

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