Critical energies in random palindrome models

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Abstract

We investigate the occurrence of critical energies – where the Lyapunov exponent vanishes – in random Schrödinger operators when the potential have some local order, which we call *random palindrome models.* We give necessary and sufficient conditions for the presence of such critical energies: the commutativity of finite word elliptic transfer matrices. Finally, we perform some numerical calculations of the Lyapunov exponents showing their behaviour near the critical energies and the respective time evolution of an initially localized wave packet, obtaining the exponent ruling the algebraic growth of the second momentum. We also consider special random palindrome models with one-letter bounded gap property; the transport effects of such long range order are showed numerically.

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1 Introduction

This paper is concerned with the random tight-binding Hamiltonian H_v on $l^2(\mathbb{Z})$

$$(H_v\psi)_n = v_n\psi_n + \psi_{n+1} + \psi_{n-1}, \tag{1}$$

where $v = (v_n)$ is a finitely valued sequence with some local correlation. We address the problem of occurrence of *critical energies* in the spectrum, which are those where the *Lyapunov exponent* $\gamma(E)$ vanishes. For Schrödinger operators in the one dimensional lattice, $\gamma(E)$ is defined when the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n(E)\cdots T_1(E)\| \tag{2}$$

exists, where $T_i(E)$ is a sequence of unimodular matrices of the form

$$T_i(E) = \begin{pmatrix} E - v_i & -1 \\ 1 & 0 \end{pmatrix}.$$

This question has come to surface after Dunlap, Wu and Phillips' work on the random dimer [10], and has been addressed rigorously by some authors [4, 9, 13]. Although the spectrum is pure point [9], the behaviour of the Lyapunov exponent near the critical energies is responsible for dynamical delocalization, characterized by the positivity of the asymptotic growth exponent, α, of the moment

$$m_\beta(t) = \sum_{n\in\mathbb{Z}} |n|^\beta |\psi_n(t)|^2 \sim t^{\beta\alpha} \tag{3}$$

for a packet initially localized at the origin: $\psi_n(0) = \delta_{n,0}$. This is proven by Jitomirskaya, Schulz-Baldes and Stolz [13] about the time average of the above moment. In the vicinity of a critical energy, the same authors show that the Lyapunov exponent vanishes quadratically, a fact already analyzed by Bovier [4] for the dimer model with some assumptions on the density of the invariant measure $\lambda(E)$ of the one-dimensional dynamical system obtained from the eigenvalue equation $H_v\psi = E\psi$.

We take as our motivation this striking effect of local order on dynamical delocalization. Our contribution is on the conditions for the appearance of critical energies in random palindrome models: these are obtained as random concatenations of palindrome words (of length greater than 2). Recall that a *palindrome* is a word that reads the same from left to right or from right to left. Since the potential assumes a finite number of values, it is useful to think of a bijective nonconstant real function φ, such that $\varphi^{-1} : [-\Delta/2, \Delta/2] \subset \mathbb{R} \to \mathcal{B}$, where $\mathcal{B}$ is some finite alphabet, and identify clusters with words in $\mathcal{B}$.

We are able to tackle the problem of local order fairly generally with what we call *cluster maps.* These are injective maps $\mathfrak{C} : \mathcal{A} \to \mathcal{B}^*$, taking each letter a of an alphabet $\mathcal{A}$ to a cluster $\mathfrak{C}(a) \in \mathcal{B}^*$, where $\mathcal{B}^*$ is the set of finite words in the alphabet $\mathcal{B}$ and $\mathcal{B} \supset \mathcal{A}$. Thus we think of random palindrome models as images of random sequences under a cluster map for which $\mathfrak{C}(a)$ are palindromes for all $a \in \mathcal{A}$. We make some additional assumptions on these maps in Section 2 in order to control the extent of the correlations introduced on $\mathfrak{C}(v)$ for a random v. We stress that the palindromes have appeared quite naturally, as indicated in the discussion in Subsection 3.4.

The critical energies in random palindrome models occur when the transfer matrices associated with each palindrome in the potential commute and are elliptic. We get a recipe for obtaining a prescribed minimum number of critical energies, in contrast with just two in the random dimer. This result is interesting in its failure to produce an interval of critical energies, which would be an absolutely continuous component of the spectrum. An absolutely continuous spectral component is (already) prohibited by Kotani's theory [14] for almost every H_v in some ergodic structure, but the reasoning which lead us to this provides further understanding of potentials assuming a finite number of values. Our second result will tell us that, apart from a set of codimension 1 in the potential intensities, $\gamma(E) > 0$ for almost every E different from the above constructed critical energies, cf. [9].

The analysis of the invariant measure $\lambda(E)$ [4, 15], which could yield the Taylor series of $\gamma(E)$ near these critical energies is not possible to carry over, because the palindromes need not have the same length. To check the behaviour of the Lyapunov exponent near the critical energies, we perform numerical calculations of $\gamma(E)$ using the Thouless formula, or rather its approximation in finite dimension. We also provide some numerics on the mean square displacement, for which $\beta = 2$ in eq.(3) and obtain, due to the larger number of critical energies, a superdiffusive behaviour with an exponent α larger than 3/4 for small disorder, as in the random dimer [10]. For sufficiently large disorder, m_2 is bounded and dynamical localization sets in.

Iterating cluster maps $\mathfrak{C} : \mathcal{A} \to \mathcal{A}^*$ such that for any $a \in \mathcal{A}$, $\mathfrak{C}^k(a)$ contains all letters of $\mathcal{A}$, i.e., *primitive* cluster maps, provides a sequence of potentials $u^{(n)} = \mathfrak{C}^{nk}(v)$, for any $v \in \mathcal{A}^{\mathbb{Z}}$ which, under some additional hypotheses, will converge to a substitution sequence. Here we do not specialize the clusters to palindromes. This is stated and proven in Subsection 3.6. For primitive cluster maps, for any given potential v, any letter in $\mathcal{A}$ occurs in $\mathfrak{C}(v)$ with bounded gaps. This long range order is not generally present in random palindrome model and one applied reason for its proposal is that it yields an interesting boost in transport, measured by the second moment m_2.

The paper is organized as follows: in Section 2 we give the definition of a cluster map, specialize it to give random palindrome models and state most of our analytic results. We prove them in Section 3, where we also discuss the ergodic structure of the models and a simple particular example. In Subsection 3.6 we discuss some random models generated by primitive cluster maps. In Section 4 we illustrate the numerical results with plots of $\gamma(E)$ and $m_2(t)$.

2 The model and statement of results

We are given a random sequence in $t = (t_j) \in \mathcal{A}^{\mathbb{Z}}$ and a *cluster map*

$$\begin{aligned} \mathfrak{C} : \mathcal{A} &\to \mathcal{B}^* \\ a &\mapsto \mathfrak{C}(a), \end{aligned}$$

where $\mathcal{B}^*$ is the set of finite words in an alphabet $\mathcal{B} \supset \mathcal{A}$. We consider only cluster maps for which the whole word $\mathfrak{C}(a)$ does not occur in $\mathfrak{C}(b)$, for all $a \neq b \in \mathcal{A}$. This prevents the introduction of spurious long range correlations in $\mathfrak{C}(t)$ even if $t \in \mathcal{A}^{\mathbb{Z}}$ is random (see, however, Section 3.6).

To extend $\mathfrak{C}$ to $t \in \mathcal{A}^{\mathbb{Z}}$ by concatenation, we declare the position of the first letter of $\mathfrak{C}(t_0)$ to be -1 (this will be convenient in Section 3.6):

$$r \overset{\text{def}}{=} \mathfrak{C}(t), \quad r_{-1} r_0 \cdots r_{|\mathfrak{C}(t_0)|-2} = \mathfrak{C}(t_0).$$

When $\mathcal{B} = \mathcal{A}$, $\mathfrak{C}$ is also a *substitution.*

The range of the function φ defined in the Introduction defines the *disorder* parameter Δ. Denoting the potential sequences by $v = \varphi(t)$ and $w = \varphi(\mathfrak{C}(t))$, we are interested in finding conditions for the vanishing of the Lyapunov exponent γ for some E in the spectrum of H_w, albeit γ in the spectrum of H_v is positive for almost every $v \in \mathcal{A}^{\mathbb{Z}}$ [7]. With the exception of Subsection 3.6, we restrict our attention to cluster maps for which $\mathfrak{C}(a)$ is a palindrome, for each $a \in \mathcal{A}$. The potential $w = \mathfrak{C}(v)$ for $v \in \mathcal{A}^{\mathbb{Z}}$ is then called a *random palindrome model.* In this setting, we can get a rather complete answer to the problem of making $\gamma(E)$ vanish in the spectrum of H_w.

Consider first the case $\mathcal{A} = \{a, b\}$. Let C_a and C_b be general palindromes with sizes $|C_a| = n$ and $|C_b| = k$; and let $A = \Phi(C_a)$ and $B = \Phi(C_b)$ be their associated transfer matrices. The set of values of the potentials $\{v_i : 1 \leq i \leq n + k\}$ in both clusters defines an ordered s-tuple $\mathbf{v}$ taken from some open set $\mathcal{V} \subset \mathbb{R}^s$, where $s = \lceil \frac{n}{2} \rceil + \lceil \frac{k}{2} \rceil$.

Proposition 1. *Let $T_i \in \{A, B\}, i \in \mathbb{Z}$, be a Bernoulli trial with A, B transfer matrices of palindromes. Then, if $[A, B] \neq 0$,*

$$\Gamma = \lim_{n\to\infty} \frac{1}{n} ||T_n \cdots T_1|| \tag{4}$$

exists and is positive for almost every trial, and except for a set of codimension greater than 1 in $\mathcal{V}$.

In this excluded set of potential intensities, critical energies do occur and have appeared before in Section 3 of [9]. There is a simple proportional relation between Γ, eq.(4), and γ, eq.(2), when both exist; see Corollary 2.

In proving Proposition 1, we will use Furstenberg and Kesten theorem, thus the statement is restricted to a set of full measure μ, with μ being the product or Bernoulli measure in $\mathcal{A}^{\mathbb{Z}}$.

Define the left-shift by $(\tau(u))_n = u_{n+1}$. The left-shift translations of $\mathfrak{C}(u)$, $u \in \mathcal{A}^{\mathbb{Z}}$, generate an ergodic system $\Lambda \subset \mathcal{B}^{\mathbb{Z}}$ with a Markov measure ν. We will discuss this point in detail in Subsection 3.1.

Given a random sequence $u = (u_n)_{n\in\mathbb{Z}}$ in a finite alphabet $\mathcal{A}$, we assume that each letter occurring in u does so with positive probability.

Definition 1. *Let $\mathfrak{C}$ be a cluster map defined on the alphabet $\mathcal{A}$, carrying each letter a_i to a palindrome $p_i = \mathfrak{C}(a_i)$. Then we say the sequence $v = \mathfrak{C}(u)$ is* tessellated by palindromes.

A transversality argument together with Proposition 1 lead to the following

Theorem 1. *There is a set $Y \subset \Lambda$ of aperiodic sequences tessellated by palindromes, with $\nu(Y) = 1$ such that for all $v \in Y$ the critical energies in $\sigma(H_v)$ is at most countable.*

Theorem 1 is a special case of Kotani's results [14] which rule out the absolutely continuous spectrum from potentials assuming finitely many values. However, the sets of realizations which one and the other refer are not directly related. In order to properly make a comparison, the above mentioned ergodic structure (Λ, τ, ν) is needed. We settle this question in Subsection 3.1. Finally, we do not have reasons to exclude the possibility of a singular continuous spectrum in random palindrome (cluster) models, particularly with long range order.

We analyze a simple generalization of the random dimer, where $\mathcal{A} = \{a, b\}$. If we restrict C_b to a constant word of length two, $C_b = bb$, we get an open condition on the word C_a, if C_a is a palindrome of odd length, $C_a = t_0 t_1 \cdots t_k \cdots t_1 t_0$, for small disorder. We set $v_i = \varphi(t_i)$, $v_a = \varphi(a)$, $v_b = \varphi(b)$ and

$$\Delta = \max\{|v_i - v_b|, 0 \leq i \leq k\}.$$

Theorem 2. *Let $C_b = bb$ and C_a be a palindrome of length $2k + 1$, $k \geq 1$, with potentials v_i such that $|v_i - v_a| < \epsilon < \Delta$, $0 \leq i \leq k$. Then*

i. for small enough disorder $\Delta > 0$, any realization of such random palindrome model has at least $2k + 1$ critical energies;

ii. for $0 < \Delta \leq 2$, the random palindrome model has at least k critical energies.

iii. for $2 < \Delta < 4$, if k is sufficiently large, the random palindrome model has at least one critical energy.

We build on the proof by De Bièvre and Germinet [9] on the existence of critical energies at $E = \pm v$ if $v \leq 1$ for the random dimer. The steps are elementary and the conclusion is based on Taylor series expansions. Bovier mentioned some plans along these lines in his paper on the random dimer [4]. Some generalizations of the random dimer were studied earlier [15].

Applying Furstenberg and Kesten's theorem we have

Corollary 1. *For C_a and C_b as in Theorem 2 and sufficiently small ϵ, there exists $0 < z_{\max} < 2$ such that, for disorder $\Delta > 2+z_{\max}$, the Lyapunov exponent $\gamma(E)$ is positive, whenever it exists.*

We will say more about $z_{\max}$ for $\epsilon = 0$ from the details of the proof of Theorem 2.

3 Proofs

3.1 Ergodic structure

We are given two finite alphabets $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{B} \supset \mathcal{A}$, $\mathcal{A}$ having more than one element and a cluster map $\mathfrak{C} : \mathcal{A} \to \mathcal{B}^*$. We construct the Kakutani tower [18] from $\mathcal{A}^{\mathbb{Z}}$ as follows: $S_a = \{x \in \mathcal{A}^{\mathbb{Z}} : x_0 = a\} \times \{1, \cdots, n_a\}$, where $n_a \overset{\text{def}}{=} |\mathfrak{C}(a)|$, for each $a \in \mathcal{A}$. Write the disjoint union $\Omega = \dot{\cup}_{a\in\mathcal{A}} S_a$ endowed with the metric

$$\rho((x,i),(y,j)) = \sum_{n\in\mathbb{Z}} \frac{d(x_n,y_n)}{2^{|n|}} + d(i,j) \tag{5}$$

where d is the discrete metric. Then Ω is compact. On Ω, we have the shift transformation

$$\hat{\tau}(x,i) = \begin{cases} (x, i+1) & \text{if } i < n_a, \\ (\tau(x), 1) & \text{if } i = n_a \end{cases}$$

where τ is the shift on $\mathcal{A}^{\mathbb{Z}}$. We extend $\mathfrak{C}$ to Ω, defining $\hat{\mathfrak{C}}$ so that the diagram below commutes

$$\begin{tikzcd} \Omega \arrow[r, "\hat{\mathfrak{C}}"] \arrow[d, "\hat{\tau}"'] & \mathcal{B}^{\mathbb{Z}} \arrow[d, "\tau"] \\ \Omega \arrow[r, "\hat{\mathfrak{C}}"] & \mathcal{B}^{\mathbb{Z}} \end{tikzcd}$$

and $\hat{\mathfrak{C}}(x,1) = \mathfrak{C}(x)$.

A simple averaging over the components of Ω gives an ergodic measure [13] which, however, does not account for the systematic local repetitions of letters in $\hat{\mathfrak{C}}(\Omega)$. Therefore the push-forward of this ergodic measure does not give a suitable ergodic structure. We have instead

Proposition 2. *Let Ω be as above and $\Lambda = \hat{\mathfrak{C}}(\Omega) \subset \mathcal{B}^{\mathbb{Z}}$ be its image under the cluster map. Then Λ is a subshift of finite type.*

Let us introduce some terminology. If v is a finite word in $\mathcal{B}^*$, define

$$Y_v = \{x \in \mathcal{B}^{\mathbb{Z}} : v \text{ does not occur in } x\}.$$

We are to prove that Λ is the intersection of finitely many Y_v's. We say that a (finite) word is *acceptable* (in Λ) if it occurs in some $x \in \Lambda$; otherwise we call it *unacceptable*. We shall use *block*, cluster and word interchangeably. If the word u is contained in w, then $Y_u \subset Y_w$.

In our context, note that a word w is acceptable, i.e., $w \in \hat{\mathfrak{C}}(\Omega)$, if, and only if, w is contained in some concatenation of clusters. If an acceptable word v ends with a block and u is an acceptable word beginning with a block, then the concatenation vu is acceptable in $\hat{\mathfrak{C}}(\Omega)$.

Proof of Proposition 2:
$\hat{\mathfrak{C}}$ is clearly continuous, therefore Λ is compact, and shift invariant by $\hat{\mathfrak{C}}$'s definition. Then Λ is a subshift [16].

Note that $\mathcal{B}^*$ is partially ordered with word inclusion and the corresponding length. Fix $w = w_0 \cdots w_k$ an arbitrary unacceptable word and let v be a word contained in w formed by blocks $\mathfrak{C}(a)$ which cannot be enlarged with the concatenation of another such block and still be contained in w. Finally let $\bar{v} = w_i \cdots w_j$ an acceptable word satisfying $v \subset \bar{v} \subset w$ and which cannot be enlarged with the concatenation of any other letter of w. Put $n = \max\limits_{a \in \mathcal{A}} |\mathfrak{C}(a)|$.

We prove that there is some unacceptable u occurring in w, with uniformly bounded length.

Case (i) v is empty.
Then $|\bar{v}| \leq n - 1$. And any word u of length n occurring in w is unacceptable.

Case (ii) v is a single cluster.
Then $|\bar{v}| \leq n + 2(n-1)$ and hence a word $u \supset \bar{v}$, whose maximum length is $3n - 1$, occurring in w is unacceptable.

Case (iii) v is the concatenation of various clusters.
Suppose, without loss, that $j < k$ and let $p + 1$ be such that w_{p+1} is the first letter of the rightmost cluster in v. Let $s = w_{p+1} \cdots w_{j+1}$ and $r = w_i \cdots w_p$. So, s cannot be acceptable, since otherwise we could append r to its left producing an acceptable word larger than $\bar{v}$. Note that $|s| \leq 2n$. ☐

On Λ, τ is a Markov shift [16] and we have the Markovian measure ν. Consider the projection $\pi : \Omega \to \mathcal{A}^{\mathbb{Z}}$. It is clear that to each set $Y \subset \mathcal{A}^{\mathbb{Z}}$ invariant under the shift, there corresponds an invariant set $\hat{\mathfrak{C}}(\pi^{-1}(Y)) \subset \Lambda$.

Kotani's theorem [14] says there is a set $W \subset \Lambda$ with $\nu(W) = 1$ such that $\sigma_{ac}(H_v) = \emptyset$ for all $v \in W$. Proposition 1 tells that there is some $Y \subset \mathcal{A}^{\mathbb{Z}}$ with $\mu(Y) = 1$, μ the product measure, such that for $u \in Y$, then $\sigma_{ac}(H_{\mathfrak{C}(u)}) = \emptyset$. Since $v = \hat{\mathfrak{C}}(t)$ for some t, there is a j such that $\tau^j \hat{\mathfrak{C}}(t) = \mathfrak{C}(u)$, $u \in \mathcal{A}^{\mathbb{Z}}$. The spectrum is the same for points in the same orbit, hence $W \subset \hat{\mathfrak{C}}(\pi^{-1}(Y))$. Therefore $\nu\big(\hat{\mathfrak{C}}(\pi^{-1}(Y))\big) = 1$. We also have immediately the two useful corollaries:

Corollary 2. *Let* $Y = \{u \in \mathcal{A}^{\mathbb{Z}} : \Gamma \text{ exists}\}$ *and* $W = \{v \in \Lambda : \gamma(E) \text{ exists}\}$. *Then* $W = \hat{\mathfrak{C}}(\pi^{-1}(Y))$.

Proof. Denote by p_a the probability of the letter a appearing in a Bernoulli trial on $\mathcal{A} = \{a, b\}$ referred to in Proposition 1. p_a is the probability of $T_i = A$ in eq.(4), then if γ exists, by the law of large numbers:

$$\Gamma = \lim_{n \to \infty} \frac{1}{n} ||T_n \cdots T_1|| = (p_a |\mathfrak{C}(a)| + (1 - p_a)|\mathfrak{C}(b)|)\gamma.$$

The reciprocal is analogous. Note also that this can be extended to the case when $\mathcal{A}$ has more than two letters. ☐

Corollary 3. *Let* $Y_1 \subset \mathcal{A}^{\mathbb{Z}}$ *be an invariant set such that* $\mu(Y_1) = 1$*, then* $\nu\big(\hat{\mathfrak{C}}(\pi^{-1}(Y_1))\big) = 1$.

Proof. Let Y and W be the sets of Corollary 2. Consider $Y_2 = Y_1 \cap Y$. Y_2 is invariant and non-empty, since $\mu(Y_1) = 1$. Now $\hat{\mathfrak{C}}(\pi^{-1}(Y_2)) \cap W$, is invariant and clearly non-empty, therefore $\nu\big(\hat{\mathfrak{C}}(\pi^{-1}(Y_2))\big) = 1$. ☐

3.2 Some Preliminaries

In this subsection, we establish conditions for the existence of critical energies on such locally correlated disordered Hamiltonians. They are preliminaries for the proofs of the main results.

Let $\rho(T)$ denote the spectral radius of the square matrix T. We begin with the following observations

Lemma 1. *Suppose that the matrices* A *and* B *commute,* $[A, B] = 0$*, and* $\rho(A) = \rho(B) = 1$*. If* $T_i \in \{A, B\}, \forall i$*, then the Lyapunov exponent* γ *vanishes.*

Proof. If n_a is the frequency of A in $T_n \cdots T_1$, then by the law of large numbers $n_a \to p_a$, and similarly $n_b \to (1 - p_a)$. Hence, using the definition of the Lyapunov exponent eq.(2), $\gamma = \log\big(\rho(A)^{p_a}\rho(B)^{1-p_a}\big) = 0$. ☐

Lemma 2. *Let* A, B *be* 2×2 *real matrices, and* A *not a multiple of the identity matrix* I.
(i) If $[A, B] = 0$*, then there exist* $\theta, \lambda \in \mathbb{R}$ *such that* $B = \theta I + \lambda A$.
(ii) If moreover $\det A = \det B = 1$*, then*

$$\theta^2 + \lambda^2 + \lambda\theta \mathrm{tr}\ A = 1.$$

In this case, we denote by $B(\lambda)$ *the pencil of matrices commuting with* A*. If furthermore* $\rho(A) = 1$*, then* $\rho(B(\lambda)) = 1$.

Proof. (i) and the first part of (ii) follow by direct calculation. To show (ii), note first that when $|\mathrm{tr}\ A| < 2$, the pair (λ, θ) lies in an ellipse. The extrema of $\mathrm{tr}\ B = 2\theta + \lambda \mathrm{tr}\ A$ along this ellipse occur at $\lambda = 0$. ☐

Let A denote the transfer matrix corresponding to the cluster C_a and B the transfer matrix of the cluster C_b. For a letter t_i with corresponding potential $v_i = \varphi(t_i)$, recall that

$$T_{v_i} = \begin{pmatrix} E - v_i & -1 \\ 1 & 0 \end{pmatrix}$$

and for the word $C_a = t_0 t_1 \cdots t_n$, $A = T_{v_n} \cdots T_{v_1} T_{v_0}$ and correspondingly for B. Suppose we can find v_i such that, for some E, they commute. We can then write $(a_{ij}) = A(E)$ and $\big(b_{ij}(\lambda)\big) = B(E, \lambda)$, $\lambda \in \mathbb{R}$, with

$$\begin{aligned} b_{12} &= \lambda a_{12}, \\ b_{21} &= \lambda a_{21}, \\ b_{11} - b_{22} &= \lambda(a_{11} - a_{22}). \end{aligned} \tag{6}$$

Note that this system is equivalent to $[A, B] = 0$, in case A is not a multiple of the identity matrix.

Let us write $\Phi(w)$ for the transfer matrix associated with the finite word w. So the notation for the basic transfer matrices of our lattice is

$$A = \Phi(C_a) \text{ and } B = \Phi(C_b).$$

If t_i is the letter at position i, we have that $\Phi(t_i)$ is a function of $x_i = E - \varphi(t_i)$. The following is well known and can be checked by induction:

Lemma 3. *For any word $w = t_0 \cdots t_n$, the transfer matrix $\Phi(w)$ has the following form*

$$\Phi(w) = \begin{pmatrix} P_{n+1}(x_0, \cdots, x_n) & -P_n(x_1, \cdots, x_n) \\ P_n(x_0, \cdots, x_{n-1}) & -P_{n-1}(x_1, \cdots, x_{n-1}) \end{pmatrix},$$

where $P_{n+1}(x_0, \cdots, x_n) = P_n(x_0, \cdots, x_{n-1})x_n - P_{n-1}(x_0, \cdots, x_{n-2})$, $n \geq 0$, with the understanding that $P_{-1} = 0$ and $P_0 = 1$ (P_n is a polynomial of E of degree n).

Note that $\Phi(bb)$ has off-diagonal elements with the property $b_{21} = -b_{12}$. An inductive argument shows that A also has this property if C_a is a palindrome:

Lemma 4. *If C_a is a palindrome, the $A = \Phi(C_a)$ has off-diagonal elements with the property $a_{12} = -a_{21}$.*

Proof. If C_a is a dimer, we already know A satisfies $a_{12} = -a_{21}$. If it has length $n \geq 3$, then $C_a = t_0 P t_0$, where P is a palindrome of length $n - 2$, and we can write

$$A = \begin{pmatrix} x_0 & -1 \\ 1 & 0 \end{pmatrix} M \begin{pmatrix} x_0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $M = \left(\begin{smallmatrix} a'_{11} & a'_{12} \\ -a'_{12} & a'_{22} \end{smallmatrix} \right)$, by the induction hypothesis; from this we get $a_{21} = -a_{12}$. □

Lemma 5. *If C_a and C_b are palindromes with $|C_a| > |C_b| \geq 2$ and C_b does not occur in C_a, then the system (6) has a finite number of solutions.*

Proof. Put $C_a = a_0 \dots a_n$ and $C_b = b_0 \dots b_k$, with $1 < k < n$. We will make an induction on the size of the palindromes $|C_a| = n + 1$ and $|C_b| = k + 1$. Rewriting system (6) using Lemmas 3 and 4, we get

$$\begin{aligned} P_k(y_1, \cdots, y_k) &= \lambda P_n(x_1, \cdots, x_n), \\ P_{k+1}(y_0, \cdots, y_k) &+ P_{k-1}(y_1, \cdots, y_{k-1}) \\ &= \lambda\big(P_{n+1}(x_0, \cdots, x_n) + P_{n-1}(x_1, \cdots, x_{n-1})\big). \end{aligned}$$

where $y_i = E - \varphi(b_i)$ and $x_i = E - \varphi(a_i)$. Eliminating λ in the system above, we get the following equation

$$\begin{aligned} P_k(y_1, \cdots, y_k)\big(P_{n+1}(x_0, \cdots, x_n) + P_{n-1}(x_1, \cdots, x_{n-1})\big) - \\ P_n(x_1, \cdots, x_n)\big(P_{k+1}(y_0, \cdots, y_k) + P_{k-1}(y_1, \cdots, y_{k-1})\big) = 0 \end{aligned} \tag{7}$$

If there is an infinite number of energies E_i such that eq.(7) holds, then it is a *polynomial equality*, meaning that on the left each polynomial coefficient is zero.

For $k = 1$ and $k = 2$, from this polynomial equation and the recursion relation given in Lemma 3, we can prove directly that C_b occurs in C_a, contrary to our hypothesis. We reduce the problem to the following

Lemma 6. *Let* $|C_a| = n+1$ *and* $|C_b| = k+1$, $n > k \geq 3$, C_a *and* C_b *palindromes. Let* $Q(j)$ *denote the property described in items (a) and (b) below:*

$$\begin{aligned}
\text{(a)} \quad & P_{k-j}(y_1, \cdots, y_{k-j}) P_{n-j-1}(x_1, \cdots, x_{n-j-1}) - \\
& \qquad P_{k-j}(y_{j+1}, \cdots, y_k) P_{n-j-1}(x_{j+2}, \cdots, x_n) \equiv \\
& P_{n-j}(x_1, \cdots, x_{n-j}) P_{k-j-1}(y_1, \cdots, y_{k-j-1}) - \\
& \qquad P_{n-j}(x_{j+1}, \cdots, x_n) P_{k-j-1}(y_{j+2}, \cdots, y_k) \\
\text{(b)} \quad & x_0 = y_0 = x_n = y_k, \cdots, x_j = y_j = x_{n-j} = y_{k-j}
\end{aligned}$$

for some $0 \leq j \leq k-2$, *then* $Q(j+2)$ *holds. In particular* $x_{j+1} = y_{j+1} = x_{n-j-1} = y_{k-j-1}$ *and* $x_{j+2} = y_{j+2} = x_{n-j-2} = y_{k-j-2}$.

Proof of Lemma 6: We use the recursion relation of lemma 3 and

$$P_{n+1}(x_0, \cdots, x_n) = x_0 P_n(x_1, \cdots, x_n) - P_{n-1}(x_2, \cdots, x_n).$$

Properties $Q(0)$ and $Q(1)$ are immediate from eq.(7) and the recursion relations.

From point (a) of $Q(j)$, expanding the occurrences of P_{k-j} and P_{n-j}

$$\begin{aligned}
& y_{k-j} P_{k-j-1}(y_1, \cdots, y_{k-j-1}) P_{n-j-1}(x_1, \cdots, x_{n-j-1}) - \\
& \qquad P_{k-j-2}(y_1, \cdots, y_{k-j-2}) P_{n-j-1}(x_1, \cdots, x_{n-j-1}) - \\
& \qquad y_{j+1} P_{k-j-1}(y_{j+2}, \cdots, y_k) P_{n-j-1}(x_{j+2}, \cdots, x_n) + \\
& \qquad P_{k-j-2}(y_j + 3, \cdots, y_k) P_{n-j-1}(x_{j+2}, \cdots, x_n) = \\
& \qquad x_{n-j} P_{n-j-1}(x_1, \cdots, x_{n-j-1}) P_{k-j-1}(y_1, \cdots, y_{k-j-1}) - \\
& \qquad P_{n-j-2}(x_1, \cdots, x_{n-j-2}) P_{k-j-1}(y_1, \cdots, y_{k-j-1}) - \\
& \qquad x_{j+1} P_{n-j-1}(x_{j+2}, \cdots, x_n) P_{k-j-1}(y_{j+2}, \cdots, y_k) + \\
& \qquad\qquad P_{n-j-2}(x_{j+3}, \cdots, x_n) P_{k-j-1}(y_{j+2}, \cdots, y_k).
\end{aligned}$$

Since we know that $x_{n-j} = y_{n-j}$ already, the first term and the first term after the equality sign cancel out. The terms containing the highest power in E are the third before and after the equality sign. The other terms do not affect the leading coefficient. Therefore, we have $x_{j+1} = y_{j+1}$, which are also equal to x_{n-j-1} and y_{n-j-1} by palindromicity.

We now expand $P_{k-j-1}(y_{j+2}, \cdots, y_k)$ and $P_{n-j-1}(x_{j+2}, \cdots, x_n)$ to conclude that $x_{j+2} = y_{j+2}$. The polynomials remaining give the equality (a) of $Q(j+2)$. ☐

$Q(k)$ contradicts the hypothesis that C_b does not occur in C_a. ☐

3.3 Proofs of Proposition 1 and Theorem 1

Proof of Proposition 1:
We follow de Bièvre and Germinet [9] closely. We first note that the group generated by A and B is not compact and, resorting to Theorem I.4.4 of [3], we will prove that the orbit of any direction $x \in \mathbb{P}(\mathbb{R}^2)$ under this group contains at least three elements. It is sufficient to consider the three cases: (a) A and B elliptic, (b) A parabolic and (c) A hyperbolic, as the other cases will follow by interchanging the roles of A and B.

The non-compactness of the group is clear in cases (b) and (c). Since A and B are palindromic transfer matrices, they have the form (using Lemma 4):

$$A = \begin{pmatrix} a_1 + t_A & -a_2 \\ a_2 & -a_1 \end{pmatrix} \ , \ B = \begin{pmatrix} b_1 + t_B & -b_2 \\ b_2 & -b_1 \end{pmatrix}$$

where $t_A = \text{tr } A$ and $t_B = \text{tr } B$. For A and B palindromes, $[A, B] \neq 0 \Leftrightarrow [A, B]$ is invertible. Hence using Theorem 7.39.2 of [2] for A and B elliptic, the non-commutativity implies tr $(ABA^{-1}B^{-1}) > 2$. This shows that also in this case the group generated by A and B is not compact.
(a) If both $|t_A| < 2$ and $|t_B| < 2$, then from $A^2 = t_A A - I$, one sees that A^2 or B^2 are elliptic, unless both $t_A = t_B = 0$. In the first case, the set $\{x, Ax, A^2x, Bx, B^2x\}$ contains at least three different directions. The later case, viewed as a geometric condition, is the intersection set of zeroes of two transverse polynomials t_A and t_B (by the same analysis of leading order coefficients given in Lemma 5). This intersection is empty unless $\gcd(t_A(E), t_B(E)) \neq 1$, that is, if these polynomials have a common divisor.

Now, if for a given s-tuple $\mathbf{v} \in \mathcal{V}$, $\gcd(t_A, t_B) \neq 1$, since t_A and t_B are also polynomials in $\mathbf{v}$, it is clear that to preserve this condition $\mathbf{v}$ has to stay in an algebraic variety in parameter space $\mathcal{V}$.
(b) If $|t_A| = 2$, then A has a single fixed point in $\mathbb{P}(\mathbb{R}^2)$, but it cannot be a fixed point of B since

$$[A, B] = ((2a_1 + t_A)a_2 - (2b_1 + t_B)b_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is invertible. The possibility $t_B = 0$ is analyzed in the same vein as above: $|t_A| = 2$ and $t_B = 0$ is empty in $\mathcal{V}$, except in some algebraic variety.
(c) Again it suffices to consider the orbit of the two eigenvectors of A. If $|t_B| \geq 2$, we are done, again because A and B do not have eigenvectors in common. But it could be the case that B, being elliptic, exchanged the eigendirections of A. These are $e_1 = (a_1 + \lambda, a_2)$ and $e_2 = (a_1 + \bar{\lambda}, a_2)$, where $\lambda = (t_A + \sqrt{t_A^2 - 4})/2$ and $\bar{\lambda} = 1/\lambda$.

From $Be_1 = ke_2$ and $Be_2 = -k^{-1}e_1$, it follows that $t_B = 0$. Moreover, from the system of equations

$$\begin{aligned} b_1(a_1 + \lambda) - b_2 a_2 &= k(a_1 + \bar{\lambda}) \\ b_2(a_1 + \lambda) - b_1 a_2 &= k a_2 \end{aligned}$$

one obtains the equation $2b_2a_2 = (2a_1 + t_A)b_1$. Now we have again two polynomials, whose sets of zeros have, by transversality, empty intersection. ☐

By Corollary 2, there is a set $W \subset \Lambda$ of measure $\nu(W) = 1$ which corresponds to the subset of $\mathcal{A}^{\mathbb{Z}}$ given in the proposition above. The closure of the orbit of each periodic sequence is a set of measure zero. The closure of the orbit of a sequence which is periodic except in a finite number of indices (that is, in the complement of a cylinder set [18]) is also a set of measure zero. Subtracting these sets from W we get a measurable set Y of full measure. For every $v \in Y$, $\gamma(E) > 0$ except possibly when $[A, B] = 0$.

Proof of Theorem 1:
Fix A and B transfer matrices of palindromes C_a and C_b. As noted in the proof of Proposition 1, eq. (6) is a system of two independent polynomials and two variables (λ, E), with a finite number of (real) roots.

A random palindrome model may have long range order, then we have to consider all possibilities of palindromes which tessellates the lattice potential. But since A, B are transfer matrices on factors in a finite alphabet, they belong to a denumerable set of unimodular matrices. Therefore the set of critical energies is at most countable. $\square$

Lemmas 1 and 2 tell us that to find critical energies for the tight binding Hamiltonian, we have to search for commuting transfer matrices with one of them with spectral radius equals to 1. From Lemma 2, the pencil of matrices commuting with a given one is compact in $SL(2, \mathbb{R})$. Hence, for each A, a choice of B from the intersection of a countable set and a compact set has generically a finite number of possibilities. This reasoning leads us to question whether the critical energies given by system (6) are real.

From Proposition 1, we also have

Corollary 4. *Consider Λ the set of all random palindrome models whose potentials are tessellated by palindromes $p_1, \cdots, p_n$ with p_i not contained in p_j, $i \neq j$. Let ν be the Markov measure invariant under the shift. For ν-almost every v, $\gamma(E) = 0$ implies that the transfer matrices $A_i = \Phi(p_i)$ are elliptic and pairwise commuting.*

3.4 A particular case

Now we illustrate the construction leading to critical energies in a simple situation: taking $C_a = t_0t_1t_2$ and $C_b = t_3t_4$, we shall determine transfer matrices for which the above setting applies. The potential $v_i = \varphi(t_i)$ are allowed to take on more than two values. Note that, being the transfer matrices unimodular, the condition on the determinant is automatically fulfilled. As above, denote by A the transfer matrix of the cluster C_a and B the transfer matrix of C_b:

$$A(E) = T_{v_2}T_{v_1}T_{v_0} \text{ and } B(E) = T_{v_4}T_{v_3}.$$

We shall use the abbreviation $x_i = E - v_i$ and drop the reference to E at our convenience in what follows. From the system of equations (6) we have the following

$$\begin{aligned} x_3 &= \lambda(x_0x_1 - 1) \\ x_4 &= \lambda(x_1x_2 - 1) \\ x_3x_4 &= \lambda(x_0x_1x_2 - x_0 - x_2 + x_1) \end{aligned}$$

and the condition $\rho(A) = 1 \;\Leftrightarrow\; |\text{tr } A| \leq 2$ reads

$$|x_0x_1x_2 - x_0 - x_1 - x_2| \leq 2.$$

Since the v_i are fixed values of the potential, we have 3 equations in 2 variables. We make the first two of these (algebraic) curves in the (E, λ) plane coincide by setting $v_4 = v_3$, and $v_2 = v_0$, but with v_0, v_1 and v_3 all different from each other. Notice that this makes C_a and C_b palindromes; that was the initial motivation for Theorem 2.

An obvious solution is $x_3 = 0$, that is, $E = v_3$, and $\lambda = 0$. We call this the *dimer* energy, because for it $B = -I$ and the above lemmas are satisfied trivially, except for the condition $\rho(A) = 1$, which remains to be verified for this energy to be critical. If $\lambda \neq 0$ we get the equation

$$x_3(x_0x_1 - 1) = x_0^2x_1 - x_0 + (v_0 - v_1),$$

with the two solutions (note that $x_i - x_j = v_j - v_i$ is a constant):

$$x_{0\pm} = -\frac{v_{01} \pm \sqrt{v_{01}^2 + 4 + 4v_{01}/v_{03}}}{2}$$

where $v_{ij} = v_i - v_j$.

The three energies $E = \{v_3, v_0 + x_{0+}, v_0 + x_{0-}\}$ will be critical energies if the following condition holds

$$|\text{tr } A| \leq 2 \;\Leftrightarrow\; |x_0^3 + v_{01}x_0^2 - 3x_0 - v_{01}| \leq 2.$$

One possible way to achieve this is by setting $v_{01} = 2v_{03}$ and $|v_{03}| \leq 1$, which is enough for the purpose of showing the existence of three energies where the Lyapunov exponent vanishes.

Remark 1. *(i) It may happen that* $A = \pm I$ *for transfer matrices, and obviously* $[B, A] = 0$ *for any* B*, but the equations (6) above would have the 'strange' solution* $\lambda = \infty$*. In this case* $\rho(B) = 1$ *alone implies* $\gamma = 0$*.*

(ii) From the above remark, or interchanging the roles of A *and* B *in system (6), we see that the natural domain for* λ *is the projective line* $(-\infty, \infty]$*.*

(iii) We fix $C_b = bb$ *from now on and will call the word* C_b *a* dimer*. For those* E^* *where* $A(E^*) = \pm I$*, we have* $\gamma(E^*) = 0$ *if* $\rho(B) = 1$*, that is, if* $|E^* - v_b| \leq 2$*.*

To proceed with the generation of critical energies, we choose to enlarge the cluster C_a. For $C_b = bb$, the system of equations (6) always keeps the trivial solution coming from the dimer: $E = v_b$ and $\lambda = 0$. Eliminating this solution, we get a polynomial of degree n in the variable E, which has $|C_a| - 1$ roots. However it is harder to anticipate if for these energies $|\text{tr } A| \leq 2$. Providing this, Lemmas 1 and 2 will yield $\gamma(E) = 0$, and since $\gamma(E) > 0$ outside the spectrum, $E \in \sigma(H_v)$.

3.5 Proof of Theorem 2

Let us begin with the idea of the proof. The strategy is first to analyze the simple case where $C_a = a^{2k+1}$ and then make use of perturbative arguments. Without loss of generality, we suppose $v_b > v_a$, so that the disorder strength is $\Delta = v_b - v_a$. We can show that there are $2k$ energies where $A = \pm I$ (actually this is an old known fact [6]), and v_b can be chosen as in Remark 1.(iii). *This imposes the bound on the disorder* $\Delta < 4$*, see below.* These energies are roots of a Chebyshev polynomial of degree $2k$, so they are simple. Then we consider the palindrome $C_a = a^kt'a^k$, meaning that only the central position has a different potential $\varphi(t') = v_a + \epsilon$. If $\epsilon > 0$ is small, we can prove that there are $2k$ perturbed energies satisfying $|\text{tr } A| < 2$. Since this is now an open condition on C_a, we will conclude the proof of the first part of the theorem. The second and third parts will be proven by an analysis of the sign of the second derivative of tr A in case $\Delta < 4$ is arbitrary.

Firstly, for $C_a = a^n$, we have

$$A = \begin{pmatrix} P_n(x_a) & -P_{n-1}(x_a) \\ P_{n-1}(x_a) & -P_{n-2}(x_a) \end{pmatrix}$$

where $x_a = E - v_a$. The recursion relation $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$ gives that $P_n(x) = U_n(x/2)$, where U_n are the Chebyshev polynomials of the second type, following the notation of Gradshteyn-Ryzhik [11]. The equations for commutativity of A and B read

$$x_b = \lambda P_{n-1}(x_a) \quad \text{and} \quad x_b^2 = \lambda\big(P_n(x_a) + P_{n-2}(x_a)\big),$$

with $\lambda \in (-\infty, \infty]$ as noted in remark 1.(iii). Since $x_a \neq x_b$, we have only two possibilities: $x_b = 0$ or $P_{n-1}(x_a) = 0$. In the latter case, tr $A = -2P_{n-2}(x_a)$. A calculation using the expression $U_n(\theta) = \frac{\sin{(n+1)\theta}}{\sin\theta}$, $\theta = \arccos(x/2)$ shows that at the (simple) roots of P_{n-1}, tr $A = \pm 2$.

So we have the partial conclusion that if we combine constant words of length n and dimers we get at least n critical energies for Δ small enough, namely, less than $2 + \min\{z\}$, where z are the roots of $P_{n-1}(x) = 0$, again using remark 1.(iii). In general, these roots z are critical energies if, and only if, $|\Delta - z| \leq 2$, for $\epsilon = 0$.

However, if $\Delta \geq 4$, since every root of $P_{n-1}(x)$ satisfies $|z| < 2$, one cannot have simultaneously $\rho(B) = 1$, so these critical energies cease to be so. Also, for the dimer energy $E = v_b$, a necessary condition for $|\text{tr } A| \leq 2$ is $\Delta \leq 2$. In other words, the dimer energy ceases to be critical when the disorder strength $\Delta > 2$. Note that this is the same bound for the original *random dimer model* [10], in particular, for $\Delta = 2$,

$$|\text{tr } A| = \big|U_n(-1) - U_{n-2}(-1)\big| = 2.$$

Now consider the case where $n = 2k + 1$ and $x_i = x_a$ except for $i = k$ where $x_k = x_a - \epsilon$. Let us make the abbreviation $x = x_a$ and replace $x_b = x - \Delta$. Now A reads

$$A = \begin{pmatrix} (x-\epsilon)P_k^2 - 2P_kP_{k-1} & -(x-\epsilon)P_kP_{k-1} + P_{k-1}^2 + P_kP_{k-2} \\ (x-\epsilon)P_kP_{k-1} - P_{k-1}^2 - P_kP_{k-2} & -(x-\epsilon)P_{k-1}^2 + 2P_{k-1}P_{k-2} \end{pmatrix}$$

Then for $\lambda \neq 0$,

$$\begin{aligned} &\text{tr } A = (P_k^2 - P_{k-1}^2)(x-\epsilon) + 2P_{k-1}(xP_{k-1} - 2P_k) \\ &\Delta(P_k^2 - P_{k-1}^2) = \epsilon(P_k^2 + P_{k-1}^2 - (x-\Delta)P_kP_{k-1}). \end{aligned} \tag{8}$$

We make Taylor expansions on equations (8) around the critical energies for $\epsilon = 0$, namely, around the zeroes of $P_{2k} = P_k^2 - P_{k-1}^2$. Let $\{z_i = 2\cos\frac{\pi i}{2k+1}\,; 1 \leq i \leq 2k\}$ be the set of roots of $P_k^2(x) = P_{k-1}^2(x)$. For $\epsilon = 0$, the fact that tr $A = \pm 2$ at these $2k$ roots yields the useful relation

$$P_k(z_i)\big(2P_k(z_i) - z_iP_{k-1}(z_i)\big) = 1. \tag{9}$$

For small ϵ, the roots of the second equation in (8) will be $\delta(z_i, \epsilon)$ away from z_i, for each $1 \leq i \leq 2k$. We fix i and write $\delta = \delta(z_i, \epsilon)$. Expanding the second equation in (8) one obtains

$$2\Delta\delta\left(\frac{2k+1}{4-z_i^2}\frac{P_{k-1}}{P_k}\right) = \epsilon(1 + \Delta P_kP_{k-1}) + O_2(\epsilon, \delta),$$

where $O_2(\epsilon,\delta)$ is a remainder of order 2 in ϵ and δ.

Now we compute the first terms in the power series of tr A starting from eq.(8.949.6) in [11]. To first order in δ and ϵ, $|\text{tr } A| = 2$. We omit the details of the calculation of second order terms, which yields

$$\text{tr } A = \tau_0 \left(1 - \frac{\delta^2}{2} p(z_i,k)\right) + O(\delta^3)$$

where

$$\begin{aligned} p(x,k) &= \frac{4x^2(2k+1)}{(4-x^2)^2} + f(\Delta)\frac{(2k+1)^2}{(4-x^2)^2} + \frac{(2k+1)^2}{4-x^2} \\ f(\Delta) &= \frac{2\tau_0\Delta}{1-\tau_0\Delta P_k^2(z_i)/2} \\ \tau_0 &= -\frac{2P_{k-1}(z_i)}{P_k(z_i)} \end{aligned}$$

The problem is reduced to evaluating the sign of $p(z_i,k)$. We remark that δ is independent of the disorder strength Δ and therefore can be made as small as one likes by diminishing ϵ. Hence

$$p(z_i,k) > 0 \Leftrightarrow -f(\Delta) < \frac{4z_i^2}{2k+1} + 4 - z_i^2 \tag{10}$$

and it is clear that, for small Δ, in particular $\Delta \leq \sin^2 \frac{\pi}{2k+1}$, the above holds for all $1 \leq i \leq 2k$. Part (i) is proved.

We have $z_i = 2\cos\left(\frac{\pi i}{2k+1}\right)$, $1 \leq i \leq 2k$, and for even i, $P_k(z_i) = -P_{k-1}(z_i)$ and consequently $\tau_0 = 2$, while for odd i, $\tau_0 = -2$. If $\tau_0 = 2$ and $i = 2j$, we note that $-f(\Delta) < 0$ for small Δ. Therefore, the above inequality (10) remains true for all z_i such that

$$1 - \Delta P_k^2(z_i) < 0 \;\Leftrightarrow\; \Delta < \frac{\sin^2\frac{\pi i}{2k+1}}{\sin^2\frac{\pi k i}{2k+1}} = 4\cos^2\left(\frac{\pi i}{2(2k+1)}\right).$$

For $\tau_0 = -2$ and $i = 2j+1$, we have

$$\frac{4\Delta}{1+\Delta P_k^2(z_i)} \leq 4\sin^2\frac{\pi i}{2k+1} \;\Leftrightarrow\; \Delta \leq \frac{\sin^2\frac{\pi i}{2k+1}}{\cos^2\frac{\pi k i}{2k+1}} = 4\cos^2\left(\frac{\pi i}{2(2k+1)}\right).$$

Note that the bound for Δ is a decreasing function of i for $1 \leq i \leq 2k$. To finish, note that for $\Delta < 2$, every z_i for $i \leq k$ satisfies the above inequality. For $\Delta < 4$, the critical energy z_1 remains so for large enough k. □

Remark 2. *From the condition that, for* $\epsilon = 0$, $|\Delta - z_i| \leq 2$ *if, and only if,* $E = v_a + z_i$ *is a critical energy, we obtain* $z_{\max} = 2\cos\frac{\pi}{2k+1}$. *This relates to the critical energy referred to in Theorem 2(iii). This also proves Corollary 1.*

3.6 Primitive cluster maps

In this subsection, we comment upon cluster maps of the form $\mathfrak{C} : \mathcal{A} \to \mathcal{A}^*$, i.e., *substitutions* [17]. Fixing one such cluster map, it can be iterated on random sequences and, providing conditions so that it is a contraction, its fixed point is a substitution sequence. Thus we have a sequence of random locally correlated potentials converging to a potential with long range order.

One extra requirement is that $\mathfrak{C}$ be *primitive*: there exists $k \geq 1$ such that for any $a \in \mathcal{A}$, $\mathfrak{C}^k(a)$ contains all letters of $\mathcal{A}$. Without loss of generality we assume $k = 1$. We call the extension of such $\mathfrak{C}$ to $\mathcal{A}^{\mathbb{Z}}$ a *primitive cluster map*. Note that $\mathfrak{C}(v)$, for any $v \in \mathcal{A}^{\mathbb{Z}}$, has also a long range order for primitive $\mathfrak{C}$: this is the bounded gap property for one-letter. Further iterations will give rise to larger words occurring in bounded gaps in $\mathfrak{C}^n(v)$.

Theorem 3. *Let $\mathfrak{C} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a primitive cluster map. If the second letter of $\mathfrak{C}(a)$ is the same for every $a \in \mathcal{A}$, then $\mathfrak{C}$ is a contraction.*

In this sense, given $u \in \mathcal{A}^{\mathbb{Z}}$, $\mathfrak{C}^n(u)$ provides a sequence of random cluster potentials converging to an almost periodic one.

Remarks. *(i) The assumption on the second letter is due to the definition of $\mathfrak{C}(u)$, $u = (u_n)$ so that the second letter of $\mathfrak{C}(u_0)$ sits at the zeroth position of $\mathfrak{C}(u)$. This assumption in turn is necessary for*

$$\lim_{n\to\infty} \mathfrak{C}^n(a)$$

to pertain to $\mathcal{A}^{\mathbb{Z}}$, for each $a \in \mathcal{A}$. In fact, here we also need $|\mathfrak{C}(a)| \geq 3$, but this can be accomplished by redefining the substitution $\mathfrak{C}$ on $\mathcal{A}$ as some power of itself.

(ii) It is known that, for primitive substitutions, the finite alphabet $\mathcal{A}$ splits into disjoint classes $\mathcal{A}_j, j = 1, \cdots, r$, such that, for some $k \geq 1$, for each $a \in \mathcal{A}_j$, $\mathfrak{C}^k(a)$ begins with a fixed $a' \in \mathcal{A}_j$. We can relax the assumption above to

for each $a \in \mathcal{A}_j$, the second letter in $\mathfrak{C}^k(a)$ also belongs to $\mathcal{A}_j$.

Thus we get r fixed points. The Rudin-Shapiro substitution is an example of this more general setting.

(iii) The hull of an almost periodic sequence u, $\Omega(u)$, is the set of $v \in \mathcal{A}^{\mathbb{Z}}$ such that any finite word occurring in v occurs in u. With this definition it is clear that the limit above yields a sequence in its hull.

(iv) The above assumption can be adjusted to a given substitution: for instance, for the Thue-Morse substitution: $\mathfrak{C}(a) = abbabaab$, $\mathfrak{C}(b) = baababba$, we may use the fourth letter in place of the second.

(v) If $\mathfrak{C}$ is primitive and palindromic, critical energies appearing in $\mathfrak{C}(u)$ for a random u will be given by Theorem 1. In $\mathfrak{C}^2(u)$, these critical energies are preserved, but others may appear as other tessellations by palindromes are generated.

4 Numerical Experiments

In these numerical experiments, we first analyze the behaviour of the Lyapunov exponent $\gamma(E)$ near a critical energy E^*. Fixing $\epsilon > 0$ we perform an average of $\gamma(E)$ over a number of realizations of the Schrödinger operator for a finite size lattice. From a fitting procedure, we extract the value of the exponent ν in the expression

$$\gamma(E) \sim C|E-E^*|^\nu \quad \text{for small} \quad |E-E^*| \ .$$

We use the finite basis approximation in the Thouless formula to calculate $\gamma(E)$, $E \in [-2+v_a, 2+v_b]$ (recall that $v_b > v_a$). Fluctuations in $\gamma(E)$, when it is small, spoil the calculation of the exponent ν, and this is why we perform an average over the hull. Since we do not have actual bounds on $|E-E^*|$, a numerical calculated ν is expected to be a rough approximation. Figure 1 illustrates the procedure to estimate the exponent ν.

For the random dimer potential, Bovier [4] has found that $\nu = 2$. A look at the calculations showing how the linear term vanishes in the expansion of $\gamma(E)$ and the expression of the leading (second) order term permit us to guess that ν does not increase for the random cluster model (compare [13]). For comparison, we present the numerical values of ν for the dimer case as well, calculated using the same procedure. However, Bovier's hypothesis that the invariant measure (its density, in fact) has a Taylor expansion in the vicinity of the critical energy is not necessarily true in these general models.

We then perform the time evolution of a packet initially located at the central position of a finite lattice. We integrate the time-dependent Schrödinger equation using a sympletic integrator, the details of which have been explained elsewhere [8].

We address the question whether propagation over the lattice is enhanced as an effect of the local order compared with the random dimer. We do this by computing the exponent α in the asymptotic regime of

$$[m_2(t)]^2 = \sum_{n\in\mathbb{L}} n^2|\psi_n(t)|^2 \sim Ct^{2\alpha},$$

where $\mathbb{L} = [-L+1, L] \cap \mathbb{Z}$ is the finite lattice (we used basis sizes up to 2^{16}). For the dimer, $\alpha = 3/4$ [10], and we obtain, for the palindrome C_a of size 5, that α is higher at the beginning, but settles down to around 3/4 at later times.

Table 1 summarizes the results on transport. For the extreme disorders considered, there is a clear transient behaviour. On the one hand, for small Δ it appears that the potential is more transparent, but around $t = 1000$, there is a change in slope in a curve $\log m_2$ vs $\log t$. This is why we consider the time span only for $\log t > 7$ until the packet reaches the border of the lattice, with probability greater than 10^{-8}. On the other hand, for large Δ, the lattice conducts poorly at the beginning, but after $\log t > 7$, m_2 also shows a power law behaviour. When Δ is increased α decreases. Furthermore, the disorder can be larger than 2 maintaining superdiffusive behaviour: from this we can say that conduction is enhanced in the random palindrome model as compared to the random dimer.

Table 1

Δ	ν	E^*	α	*time span t*
	(left branch)			$T_1<t<T_2$ $\ln T_1 - \ln T_2$
0.2	2.2	-1.720	0.78	7 – 9
0.6	0.8	-0.913	0.73	7 – 9
1.2	1.8	-1.197	0.75	0 – 9
2	1.1	-0.3855	0.68	0 – 9.5
3	1.1	0.1826	0.67	7 – 10

The reason for the higher α for short times and small disorder Δ stems from the extra critical energies, which forces $\gamma(E)$ to be small in a larger portion of the spectrum. This also explains why the ν column in Table 1 is not a decreasing function of the disorder. The leftmost critical energy for small Δ ceases to be so around $\Delta = 0.38$ for $k = 5$. For $\Delta = 0.6$, γ is still not growing sufficiently fast off the critical energy -0.913: it is as small as 0.036 at $E = -1.75$, just ten times its minimum value attained at 0.913. For long enough times, the exponential decay eventually sets in so only those states with localization lengths of the order of the lattice size or larger contribute to transport up until the packet reaches the border. Briefly, we have numerically found that α, the exponent measuring transport, does not increase if we have a finite number of contributing critical energies, all of them with the same exponent $\nu \leq 2$.

We can make the dimer critical energy and one of the critical energies due to the palindrome C_a for $\epsilon = 0$ coalesce, by taking $\Delta = 2\cos 2\pi/5$. It is reasonable to expect that $\gamma(E)$ will have a higher tangency at the corresponding critical energy, $E^* = \cos 2\pi/5$, or, more precisely, that $\nu(E^*)$ will be greater than 2. Even though this is apparently true (as we saw numerically), we did not observe an increase in α, which continued around 0.75.

We also investigated the following interesting example: C_a a palindrome whose potential looks like a staircase:

$$-\frac{\varphi(C_a)}{\Delta/2} = \{0.4, 0.6, 0.8, 1, 0.8, 0.6, 0.4\}$$

and $C_b = bb$. We obtain qualitatively the same behaviour, as displayed in Figure 2, fixing $\Delta > 0$, and computing $\gamma(E)$ and $m_2(t)$. We also get apparent critical energies even for $\Delta > 4$, where $\gamma(E) \approx 0$ (however, this could be credited to the disorder definition, which is $\max\{\varphi(u) - v_b\ , u \in C_a\}$).

Finally, we consider a random palindrome model with a primitive cluster map, see subsection 3.6. The map $\mathfrak{C}(a) = abbba$ and $\mathfrak{C}(b) = aba$ for $\mathcal{A} = \{a, b\}$ has the one-letter bounded gap property on the image sequence. Since the second letter of $\mathfrak{C}(a)$ and $\mathfrak{C}(b)$ is the same, then independently of the initial sequence v in $\mathcal{A}^{\mathbb{Z}}$, $\mathfrak{C}^n(v)$ tends, for $n \to \infty$, to

$$\cdots abbbaa \cdot baabbba \cdots$$

which is the limit sequence of $\mathfrak{C}^n(b)$ (recall that the central dot marks, to its right, the zeroth position). Critical energies can be determined from the commutation of the transfer matrices $\Phi(\mathfrak{C}(a))$ and $\Phi(\mathfrak{C}(b))$.

For $\Delta \leq 1$ we have two critical energies, when $E = v_a$ and $E = v_b$, and only $E = v_a$ for $1 < \Delta \leq 2$. The commutation of the transfer matrices corresponding to each cluster is the sole responsible for the critical energies upon the first iteration of $\mathfrak{C}(u)$, for a random $u \in \mathcal{A}^{\mathbb{Z}}$, cf. Proposition 1. We considered however the random cluster $w = \mathfrak{C}^2(u)$, with u a random sequence in $\mathcal{A}^{\mathbb{Z}}$. The sequence w has the bounded gap property for the words $\mathfrak{C}(a)$ and $\mathfrak{C}(b)$, whose corresponding transfer matrices commute at the critical energies $E = 0.4$ and $E = -0.4$. There appears a new numerical behaviour in both $m_2(t)$ and $\gamma(E)$, with an apparent boost in transport $\alpha \approx 0.92$, as seen in Figure 3.

Acknowledgments

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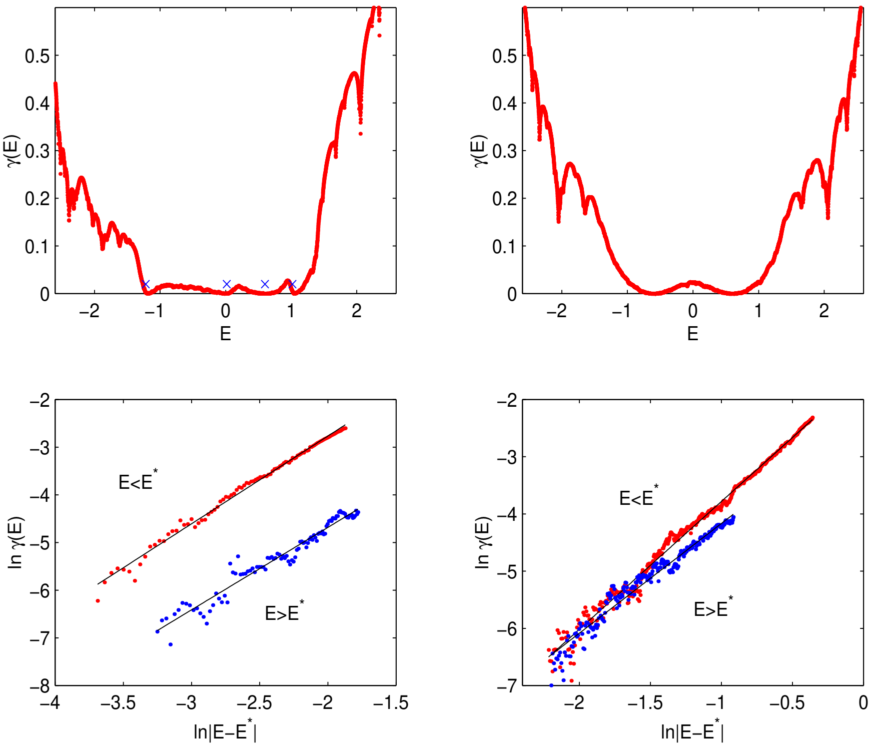

Figure 1: On top, the graphs of the Lyapunov exponent with disorder $\Delta = 1.2$, the *random palindrome model* on the left and the *random dimer* on the right. The bottom figures show examples of dilog fitting of the Lyapunov exponent: On the left, for the *random palindrome model*, near the critical energy $E^* = -1.197$, coming from the first minimum for $\epsilon = 0$ (the minima for $\epsilon = 0$ occur at $2\cos\frac{\pi i}{5} - 0.6$, $i = 1, 2, 3$ and at 0.6, positions marked with $\times$). As seen, the right branch $E < E^*$ approaches zero faster than the left branch $E > E^*$: the slopes of the fitting lines are 1.8 and 1.7, respectively. On the right, for the *random dimer* near $E^* = -0.6$, we get the slopes: 2.2 for the left branch and 1.9 for the right branch.

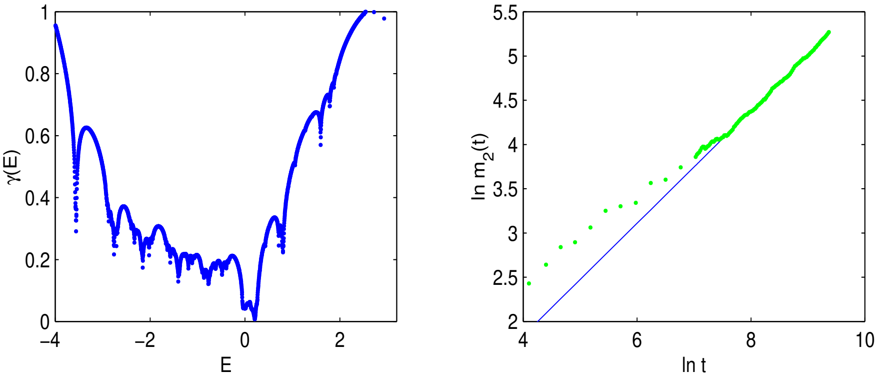

Figure 2: The Lyapunov exponent for disorder $\Delta = 4.2$ and palindrome $C_a = u_1u_2u_3u_4u_3u_2u_1$ of length 7, such that the values of the potential are $\varphi(u_i) = \frac{\Delta}{2}(1 - 0.2|i-4|)$. The approach to zero at $E \approx 0.2$ has $\nu = 0.5$ from the left and $\nu = 0.8$ from the right. On the right plot, the mean square displacement together with the linear fitting using data for $t > e^{7.5}$, from which we get the slope $\alpha = 0.64$.

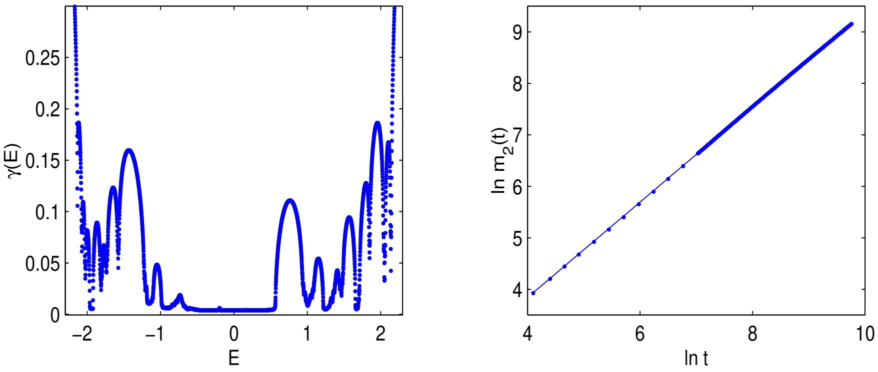

Figure 3: The Lyapunov exponent for the sequence $\mathfrak{C}^2(u)$, u a random sequence, and disorder $\Delta = 0.8$. Note the flat behaviour on an interval containing the critical energies -0.4 and 0.4: there $\gamma(E) \le 5 \times 10^{-3}$. On the right, the mean square displacement data are shown in dots and the line obtained by a fitting has angular coefficient $\alpha = 0.92$.