Periodic solutions of nonlinear wave equations with general nonlinearities

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Abstract: We prove the existence of small amplitude periodic solutions, with strongly irrational frequency ω close to one, for completely resonant nonlinear wave equations. We provide multiplicity results for both monotone and nonmonotone nonlinearities. For ω close to one we prove the existence of a large number N_ω of $2\pi/\omega$-periodic in time solutions $u_1, \ldots, u_n, \ldots, u_N$: $N_\omega \to +\infty$ as $\omega \to 1$. The minimal period of the n-th solution u_n is proved to be $2\pi/n\omega$. The proofs are based on a Lyapunov-Schmidt reduction and variational arguments. [1]

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1 Introduction

In this paper we look for periodic solutions of the completely resonant nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(u) = 0, \\ u(t,0) = u(t,\pi) = 0. \end{cases} \tag{1.1}$$

The wave equation (1.1) is said completely resonant if $f(0) = f'(0) = 0$. Precisely, we assume that the nonlinearity f belongs to $\mathcal{C}^r(\mathbf{R}, \mathbf{R})$, for some large r, and satisfies

- $(F1)$ $f(0) = f'(0) = \ldots = f^{(p-1)}(0) = 0$, $f^{(p)}(0) := ap! \neq 0$ for some $p \in \mathbf{N}$, $p \geq 2$.

Equation (1.1) can be viewed as an *infinite dimensional Hamiltonian system* with Hamiltonian

$$\mathcal{H}(u,p) := \int_0^\pi dx \; \frac{p^2}{2} + \frac{u_x^2}{2} + F(u) \tag{1.2}$$

where $F(u)$ is a primitive of f.

For finite dimensional Hamiltonian systems existence of periodic orbits has been widely investigated starting from the pioneering work of Poincaré [26]. The first result on the existence of periodic orbits close to an elliptic equilibrium is the well-known Lyapunov Center Theorem, see [22]. Assume that a Hamiltonian vector field possesses at 0 an elliptic equilibrium and let $\{\omega_j\}_{j=1,\ldots,n}$ be the frequencies of the linear oscillations close to 0. Lyapunov's Theorem states that, if one frequency, say ω_i, is non resonant with the other ones, i.e. $\omega_i l - \omega_j \neq 0$, $\forall l \in \mathbf{N}, j \neq i$, then the periodic solutions of frequency ω_i of the linearized system at 0 can be continued into small amplitude periodic solutions of the non-linear system with frequencies ω close to ω_i. This result is proved using bifurcation theory (see for example [4], pag 145 and follows) which can be applied thanks to the previous non-resonance condition, yielding the existence of smooth families of periodic solutions.

The existence of periodic orbits close to an elliptic equibrium whose linear frequencies $\{\omega_j\}_{j=1,\ldots,n}$ are possibly resonant, but only at an order greater than 4, i.e. $\sum_{j=1}^n \omega_j k_j \neq 0$, $\forall 0 < |k| = \sum_{j=1}^n |k_j| \leq 4$, has been later proved in the famous Birkhoof-Lewis Theorem [11], see also [24]. The proof is based on a normal form argument, an Implicit Function Theorem and variational arguments.

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Existence of small amplitude periodic solutions of Hamiltonian systems with a finite number of degrees of freedom, in case of general resonances between the linear frequencies, has been proved much later in the celebrated Weinstein Theorem [32] (see also [25] and [9]). The Weinstein Theorem ensures, that, if the Hamiltonian H possesses at 0 a non-degenerate minimum, then, on each small energy level, there exist at least n periodic orbits geometrically distinct. This Theorem is proved through a Lyapunov-Schmidt reduction: one first partially solves the above problem by the techniques of bifurcation theory and then restricts the action functional to the pseudo-solutions so obtained. It turns out that the restricted functional is defined on a manifold homeomorphic to[2] S^{2n-1}/S^1, and admits, for topological reasons, at least n distinct critical points giving rise to periodic solutions of the system. In general such periodic solutions will be no more organized into smooth families. Existence of periodic orbits with fixed period close to an elliptic resonant equilibrium have been later proved by Fadell and Rabinowitz in [19]. The result of [19] is still based on a Lyapunov-Schmidt reduction together with a cohomological index to estimate the number of critical points of a reduced action functional (and hence of small amplitude periodic solutions with fixed period). No information on the minimal periods of these solutions is given.

It is natural to try to extend such type of results to Hamiltonian PDE's as the wave equation, the plate equation, the nonlinear Schrödinger equation, etc... . The main difficulty in carrying over this program is the fact that, in infinite dimensional systems, "small denominator problems" appear, whereas in finite dimensional systems "small denominators" appear only in the construction of quasi-periodic solutions. This is simply explained as follows. The frequencies $\{\omega_j\}_{j\in\mathbf{N}}$ of the linear oscillations close to the equilibrium typically grow as $\omega_j \approx j^d$ for some $d > 0$ and then the sequence $\{\omega_i l - \omega_j\}$, $l \in \mathbf{N}, j \in \mathbf{N}, j \neq i$, accumulates to 0.

Despite this difficulty, Kuksin [20] and Wayne [31], were able to find, extending in a suitable way KAM techniques, periodic (and even quasi-periodic) solutions in some Hamiltonian PDE's in one spatial dimension under Dirichlet boundary conditions. As usual in KAM-type results, the periods of such persistent solutions satisfy a strong irrationality condition, as the classical Diophantine condition, so that these orbits exist only on energy levels belonging to some Cantor set of positive measure; see for example [21] for a complete and updated exposition of the KAM approach. The main limitation of this method is the fact that standard KAM-techniques require the linear frequencies ω_j to be well separated (non resonance between the linear frequencies).

To overcome such difficulty a new method for proving the existence of small amplitude periodic solutions, based on the Lyapunov-Schmidt reduction, has been developed in [17]. Rather than attempting to make a series of canonical transformations which bring the Hamiltonian into some normal form, the solution is constructed directly. Making the ansatz that a periodic solution exists one writes this solution as a Fourier series and substitutes that series into the partial differential equation. In this way one is reduced to solve two equations: the so called (P) equation, which is infinite dimensional, where small denominators appear, and the finite dimensional (Q) equation, which corresponds to resonances. Due to the presence of small divisors the (P) equation is solved by a Nash-Moser Implicit Function Theorem. Later on, this method has been improved by Bourgain to show the persistence of periodic solutions in higher spatial dimensions [13] and of quasi-periodic solutions in Hamiltonian perturbations of 2D linear Schrödinger equations [14].

A simple proof of the Lyapunov center Theorem in PDE's, still based on the Lyapunov Schmidt procedure, has been given more recently in [6], where, making use of a non-resonance condition on the frequency stronger than the classical Diophantine one, the (P) equation is solved by means of the standard Contraction Mapping Theorem. Apart its simplicity, [6] has also the advantage of avoiding some further condition present in [17]. Other related results on the existence of periodic solutions in nonlinear plate equations in higher dimension have been given in [8].

We remark that in all the previous cases the (Q) equation is still finite dimensional since one handles cases with only a finite number of resonances among the linear frequencies.

The first results on the existence of small amplitude periodic solutions for some *completely resonant* PDE's as (1.1) have been given in [23], for the specific nonlinearity $f(u) = u^3$, and in [7] when $f(u) = u^3 +$ h.o.t.. One looks for small amplitude periodic solutions of (1.1) bifurcating from the infinite dimensional

[2]Here $S^{2n-1} := \{x \in \mathbf{R}^{2n} \mid |x| = 1\}$.

linear space of solutions

$$V := \Big\{ v_{tt} - v_{xx} = 0,\ v(t,0) = v(t,\pi) = 0,\ v(t+2\pi, x) = v(t,x) \Big\}.$$

Here the linear frequencies are $\omega_j = j \in \mathbf{N}$. The approach of [7] is still based on the Lyapunov-Schmidt reduction. The (P) equation is solved, for the strongly irrational frequencies $\omega \in W_\gamma$, where

$$W_\gamma := \Big\{ \omega \in \mathbf{R} \ \Big|\ |\omega l - j| \geq \frac{\gamma}{l},\ \forall l \neq j \Big\},$$

through the Contraction Mapping Theorem (it is proved in [7] that, for $0 < \gamma < 1/3$, the set W_γ is uncountable and accumulates to $\omega = 1$ both from the left and from the right). Next, the (Q) equation, infinite dimensional, is solved by looking for non degenerate critical points of a suitable functional (see remark 3.1) and continuing them, by means of the Implicit Function Theorem, into families of periodic solutions of the nonlinear equation. Clearly the main difficulty in applying this method is to check that the fore mentioned non-degeneracy condition holds. This is verified in [7] for the particular case $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$ for which informations of the minimal periods of the solutions are also given.

The aim of the present paper is to prove, assuming only that the nonlinearity f satisfies $(F1)$, the existence of a large number of small amplitude periodic solutions of (1.1) with fixed period, see Theorems 4.1, 4.2, 4.3, 4.4. Roughly, we show that equation (1.1) has a large number N_ω of $2\pi/\omega$ periodic solutions $u_1, \ldots, u_n, \ldots, u_N$, where $N_\omega \approx \sqrt{\gamma^\tau/(\omega - 1)} \to +\infty$ as $\omega \to 1$, for some $\tau \in [1,2]$. We expect the number of small amplitude periodic solutions of frequency ω to be in general $O(\sqrt{1/(\omega-1)})$, see remark 4.5. We also prove that the minimal period of the n-th periodic solution u_n is $2\pi/n\omega$.

These Theorems can also be considered a generalization of the results of [19] (which hold for finite dimensional Hamiltonian systems) for the nonlinear wave equation (1.1) satisfying $(F1)$.

Theorems 4.1, 4.2, 4.3 and 4.4 are proved using the Lyapunov-Schmidt procedure which leads to solve the equations (P) and (Q). We overcome the small denominator problem arising in the (P) equation, by imposing on the frequency ω the same strong irrationality condition as in [7]. On the contrary, the infinite dimensional (Q) equation is solved by means of a variational principle which provides periodic solutions of (1.1) with fixed frequency: this is the main novelty of our approach with respect to [7]. This variational principle is inspired by the perturbation method in critical point theory introduced in [1]-[2]. With respect to these last two papers, the new feature of our approach is the presence of an infinite dimensional (Q) equation.

Let us give a more precise outiline of our arguments. Critical points of the action functional $\Psi(u) := \int_0^{2\pi} dt \int_0^\pi dx\ (\omega^2/2)u_t^2 - (1/2)u_x^2 - F(u)$ defined in a suitable space of functions 2π periodic in time and vanishing at $x = 0$ and $x = \pi$ are (weak) solutions of our problem. First, for any $v \in V$, we find as in [7], a unique $w(v) \in V^\perp$, $w(v) = O(||v||^p)$, which solves the (P) equation. Then, according to the variational procedure of [1]-[2], see also [19], all the critical points of the action functional Ψ close to the origin can be obtained as critical points of the reduced action functional $\Phi(v) := \Psi(v + w(v))$, defined on the linear space V by

$$\Phi(v) := \Phi_\varepsilon(v) = \frac{\varepsilon}{2}||v||^2 - \int_0^{2\pi} dt \int_0^\pi dx\ F(v + w(v)) - \frac{1}{2} f(v + w(v)) w(v) \qquad \text{where} \qquad \varepsilon := \frac{\omega^2 - 1}{2},$$

see lemma 2.4 and remark 2.1. By $(F1)$, Φ_ε possesses a local minimum at the origin. To show that Φ_ε satisfies the geometrical hypotheses of the Mountain Pass Theorem [5] the nonlinearity f plays its role. Here we have to make a different study whether the first term in the Taylor expansion at 0 of the nonlinearity $f(u) = au^p + h.o.t.$ is an odd power of u, or an even power of u. In the first case one has that

$$\Phi_\varepsilon(v) = \frac{\varepsilon}{2}||v||^2 - \frac{a}{p+1} \int_0^{2\pi} \int_0^\pi v^{p+1} + \text{ h.o.t.}$$

and hence Φ_ε (or $-\Phi_\varepsilon$) satisfies the mountain pass geometry provided that sign ε=sign a. In other words, when $a > 0$ (resp. $a < 0$) we will find solutions for $\omega > 1$ (resp. $\omega < 1$) and the bifurcation is supercritical (subcritical).

The second case (p even) is more difficult since $\int_0^{2\pi}\int_0^{\pi} v^{p+1} \equiv 0$ and then one must look more carefully in the expansion of Φ_ε. We point out that this difficulty is similar to the one arising in the Yamabe problem, see in [3]-[10], where the first order expansion term in the corresponding reduced functional identically vanishes and one needs a sharper analysis for finding its critical points. Let us consider for simplicity $f(u) = u^p$ with p even. Then it is possible to show that

$$\Phi_\varepsilon(v) = \frac{\varepsilon}{2}||v||^2 - \frac{a^2}{2}\int_\Omega v^p L^{-1} v^p + \text{ h.o.t.}$$

where $z := L^{-1}v^p$ is a solution of $z_{tt} - z_{xx} = v^p$, $z(t,0) = z(t,\pi) = 0$, which is 2π periodic in time, see formulas (4.9)-(4.10). Now our existence proof requires a careful study of the nonquadratic term $(-a^2/2)$ $\int_\Omega v^p L^{-1} v^p$, see lemmas 4.7, 4.8, 4.9. In particular, we show in lemma 4.9 that $(-a^2/2)\int_\Omega v^p L^{-1} v^p \geq 0$, and this implies that $-\Phi_\varepsilon$ satisfies the Mountain pass geometry for $\varepsilon < 0$, i.e. $\omega < 1$.

Next, we prove that the Palais-Smale sequences at the mountain pass critical level converge to a critical point which lies inside the domain of definition of the functional (where the Lyapunov-Schmidt reduction is defined) provided $|\varepsilon|$ is small enough, i.e. ω is sufficiently close to 1. This is done by means of suitable estimates on the mountain pass critical level.

Furthermore, in order to find multiple critical points of Φ_ε, we do not use Ljusternik-Schnirelmann theory as for the symmetric Mountain pass Theorem [5] (remark that Φ_ε is even). Instead, we look for "local" mountain-pass critical points of Φ_ε restricted to the subspaces $V_n \subset V$ consisting by the functions of V which are $2\pi/n$-periodic w.r.t. time. Since each V_n is invariant for the gradient flow of Φ_ε, any critical point of $\Phi_{\varepsilon|V_n}$ is a critical point of Φ_ε on the whole V. This method also provides precise informations on the minimal periods, on the norm and on the energies of the solutions, see also remark 4.4. It also follows that, as the frequency ω tends to 1, the number of small amplitude solutions we find increases to $+\infty$.

Let us further point out that the preceding arguments would also improve the results of [19] for finite dimensional Hamiltonian systems, giving informations on the minimal periods of the solutions.

Finally we prove that the weak solutions found are indeed classical C^2 solutions of (1.1).

We underline that our results do not require any monotonicity assumption on the nonlinearity. The major difficulty outlined above for proving the existence of periodic solutions when the first term in the Taylor expansion of the nonlinearity f is an even power of u reflects an intrinsic difficulty of the problem and it is encountered by all the known methods when dealing with non monotone nonlinearities. Up to our knowledge the only existence result (in any case quite different in nature from ours) for periodic solutions without the assumption of monotonicity of the nonlinearity is given in [16] where the existence of one, not small, periodic solution with rational frequency and symmetry properties, is proved through global variational techniques.

Before concluding this introduction we would like to mention that the first breakthrough in the study of bifurcation of periodic solutions for nonlinear wave equations was the work of Rabinowitz [28] (see also [18]) where the existence of forced periodic solutions under a monotonicity assumption on the nonlinearity is proved. For a global bifurcation result see [27]. Moreover, large free vibrations have been found by Rabinowitz in [29] (see also [15]) using global variational methods. Both in [29] and [15] the nonlinearity is assumed to be (superquardatic and) monotone. In these papers the period of the solutions of (1.1) is a rational multiple of π. Actually this condition introduces in the problem strong compactness properties, while in the case of irrational periods, "small-denominator phenomenona", similar to the ones described above, would appear into the proof.

The paper is organized as follows. In section 2 we perform the Lyapunov-Schmidt reduction and we define the variational formulation. In section 3 we prove a general abstract result on existence of critical points. In section 4 we apply the former result to the nonlinear wave equations. In section 5 we prove the regularity of the solutions.

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2 The variational setting

We look for periodic solutions of (1.1) with frequency ω as $u(t,x) = q(\omega t, x)$, where $q(\cdot, x)$ is 2π periodic. We are then led to solve the problem (still denoting $q \equiv u$)

$$\omega^2 u_{tt} - u_{xx} + f(u) = 0, \qquad u(t,0) = u(t,\pi) = 0, \qquad u(t+2\pi, x) = u(t,x). \tag{2.1}$$

We will look for solutions of (2.1) $u : \Omega \to \mathbf{R}$ where $\Omega := \mathbf{R}/2\pi\mathbf{Z} \times (0,\pi)$, in the following Banach space

$$X := \Big\{ u \in H^1(\Omega, \mathbf{R}) \cap L^\infty(\Omega, \mathbf{R}) \mid u(t,0) = u(t,\pi) = 0,\ u(-t,x) = u(t,x) \Big\}$$

endowed, for $\omega \in [1/2, 3/2]$, $\omega \neq 1$, with the norm

$$|u|_\omega := |u|_\infty + |\omega - 1|^{1/2} ||u||_{H^1}. \tag{2.2}$$

Note that we can restrict to the space X of functions even in time because equation (1.1) is reversible.

Any $u \in X$ can be developed in Fourier series as

$$u(t,x) = \sum_{l \geq 0, j \geq 1} u_{lj} \cos lt \sin jx,$$

and its H^1 norm and scalar product are written as

$$||u||^2 := ||u||^2_{H^1} := \int_0^{2\pi} dt \int_0^\pi dx\ u_t^2 + u_x^2 = \frac{\pi^2}{2} \sum_{l \geq 0, j \geq 1} u_{lj}^2 (j^2 + l^2),$$

$$(u,w) := \int_\Omega dt\ dx\ u_t w_t + u_x w_x = \frac{\pi^2}{2} \sum_{l \geq 0, j \geq 1} u_{lj} w_{lj} (l^2 + j^2) \qquad \forall u, w \in X. \tag{2.3}$$

We will also denote with $|u|_{L^2} := (\int_\Omega |u|^2)^{1/2} = (\pi/\sqrt{2})(\sum_{l \geq 0, j \geq 1} u_{lj}^2)^{1/2}$ the L^2-norm of $u \in$ and with $(u,w)_{L^2} := \int_\Omega uw$ the L^2-scalar product.

We need the following preliminary lemma proved in the Appendix.

Lemma 2.1 *The Nemitski operator $u \to f(u)$ is well defined and C^1 on X. Its differential at u is given by $h \to Df(u)[h] = f'(u)h$. Moreover there is $\rho_0 > 0$ and positive constants C_1, C_2, depending only on f, such that, $\forall u \in X$ with $|u|_\omega < \rho_0$,*

$$|f(u)|_\omega \leq C_1 |u|_\infty^{p-1} |u|_\omega \leq C_1 |u|_\omega^p, \tag{2.4}$$

$$|f'(u)h|_\omega \leq C_2 |u|_\infty^{p-2} |u|_\omega |h|_\omega \leq C_2 |u|_\omega^{p-1} |h|_\omega. \tag{2.5}$$

The solutions in $H_0^1(\Omega, \mathbf{R})$ of the linear equation $v_{tt} - v_{xx} = 0$ that are even in time are the elements of

$$V := \Big\{ v(t,x) = \sum_{j \geq 1} \xi_j \cos(jt) \sin jx \ \Big|\ \xi_j \in \mathbf{R},\ \sum_{j \geq 1} j^2 \xi_j^2 < +\infty \Big\}.$$

The space V can be also written as

$$V := \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \ \Big|\ \eta(\cdot) \ \in H^1(\mathbf{T}),\ \eta \text{ odd} \Big\}$$

where $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ is the one dimensional torus and $\eta(s) = \sum_{j \geq 1} (\xi_j/2) \sin(js)$. Note that $\langle \eta \rangle := (1/2\pi) \int_\mathbf{T} \eta(s)\ ds = 0$.

V, endowed with the H^1-norm $||\ ||$, is a Hilbert space, continuously embedded in X, since, for $v = \sum \xi_j \cos(jt) \sin(jx) \in V$, we have $|v|_\infty \leq \sum_{j \geq 1} |\xi_j| \leq C||v||$ by the Cauchy-Schwarz inequality. Moreover the embedding $(V, ||\cdot||) \to (V, |\cdot|_\infty)$ is compact.

In order to get multiplicity results we will consider in the sequel the subspaces of V

$$V_n := \Big\{ v \in V \ \Big| \ v \ \text{ is } \ \frac{2\pi}{n} - \text{periodic} \ \text{ w.r.t. } \ t \Big\}, \tag{2.6}$$

for $n \geq 1$. For $v = \eta(x+t) - \eta(x-t) \in V$ and any $n \in \mathbf{N}$ we define

$$(\mathcal{L}_n v)(t,x) := \eta(n(t+x)) - \eta(n(t-x)).$$

It is immediately realized that

$$V_n = \mathcal{L}_n V = \Big\{ v(t,x) = \eta(n(x+t)) - \eta(n(x-t)) \ \Big| \ \eta(\cdot) \in H^1(\mathbf{T}), \ \eta(\cdot) \text{ odd} \Big\}.$$

We next consider the Lagrangian action functional $\Psi : X \to \mathbf{R}$ defined by

$$\Psi(u) := \int_0^{2\pi} dt \int_0^{\pi} dx \ \frac{\omega^2}{2} u_t^2 - \frac{1}{2} u_x^2 - F(u) \tag{2.7}$$

where $F(\cdot)$ is the primitive of f such that $F(0) = 0$, i.e. $F(u) = \int_0^u f(s) \ ds$. It is easy to see that $\Psi \in C^1(X, \mathbf{R})$ and that

$$D\Psi(u)[h] = \int_0^{2\pi} dt \int_0^{\pi} dx \ \omega^2 u_t h_t - u_x h_x - f(u)h, \quad \forall h \in X. \tag{2.8}$$

Critical points of Ψ are weak solutions of (2.1).

In order to find critical points of Ψ we perform a Lyapunov-Schmidt reduction. The space X can be decomposed as $X = V \oplus W$ where

$$W := \Big\{ w \in X \ \Big| \ (w,v)_{L^2} = 0, \ \forall \ v \in V \Big\} = \Big\{ \sum_{l \geq 0, j \geq 1} w_{lj} \cos(lt) \sin jx \in X \ \Big| \ w_{jj} = 0 \ \forall j \geq 1 \Big\}.$$

In fact, if $u = \sum u_{lj} \cos(lt) \sin(jx) \in X$, then the H^1 norm of $\Pi_V(u) := \sum u_{jj} \cos(jt) \sin(jx)$ is finite, so that $\Pi_V(u) \in X$. As a consequence $\Pi_W(u) := u - \Pi_V(u) \in X$. Moreover the projectors $\Pi_V : X \to V$ and $\Pi_W : X \to W$ are continuous. W is also the H^1-orthogonal of V in X. Note that if $v = \Pi_V u$ has minimal period w.r.t time 2π then also u has minimal period w.r.t. time 2π.

Setting $u := v + w$ with $v \in V$ and $w \in W$, (2.1) is equivalent to the following system of two equations (called resp. the (Q) and the (P) equations)

$$(Q) \qquad -\omega^2 v_{tt} + v_{xx} = \Pi_V f(v+w),$$

$$(P) \qquad -\omega^2 w_{tt} + w_{xx} = \Pi_W f(v+w).$$

Note that w solves the (P) equation (in a weak sense) iff it is a critical point of the restricted functional $w \to \Psi(v+w) \in \mathbf{R}$, *i.e.* if and only if

$$D\Psi(v+w)[h] = \int_0^{2\pi} dt \int_0^{\pi} dx \ \omega^2 w_t h_t - w_x h_x - f(v+w)h = 0, \quad \forall h \in W. \tag{2.9}$$

We will first solve the (P) equation in lemma 2.3 through the standard Contraction Mapping Principle, assuming that $\omega \in \mathcal{W} := \cup_{\gamma > 0} W_\gamma$ where W_γ is the set of strongly non-resonant frequencies introduced in [7]

$$W_\gamma := \Big\{ \omega \in \mathbf{R} \ \Big| \ |\omega l - j| \geq \frac{\gamma}{l}, \quad \forall j \neq l \Big\}. \tag{2.10}$$

As proved in remark [2.4] of [7], for $\gamma < 1/3$, the set W_γ is uncountable, has zero measure and accumulates to $\omega = 1$ both from the left and from the right. It is also easy to show that $W_\gamma = \emptyset$ for $\gamma \geq 1$.

The proofs of the next lemmas 2.2 and 2.3 are given in the Appendix.

Lemma 2.2 *For $\omega \in W_\gamma$, let us define*

$$\varepsilon = \frac{\omega^2 - 1}{2} \quad (\varepsilon \sim \omega - 1 \quad \text{as} \quad \omega \to 1).$$

Then the operator $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx} : D(L_\omega) \subset W \to W$ has a bounded inverse $L_\omega^{-1} : W \to W$ which writes

$$L_\omega^{-1} h := \sum_{l \geq 0, j \geq 1, j \neq l} \frac{h_{lj}}{(\omega^2 l^2 - j^2)} \cos(lt) \sin(jx) \tag{2.11}$$

and satisfies, for a positive constant C_3 independent of γ and ω, $|L_\omega^{-1} \Pi_W u|_\omega \leq (C_3/\gamma)|u|_\omega$ for $u \in X$. Let $L^{-1} : W \to W$ be the inverse operator of $-\partial_{tt} + \partial_{xx}$. There exists $C_4 > 0$ such that $\forall r, s \in X$

$$\Big| \int_\Omega r(t,x) \Big(L_\omega^{-1} - L^{-1} \Big) (\Pi_W s(t,x)) \ dt \ dx \Big| \leq C_4 \frac{\varepsilon}{\gamma} |r|_\omega |s|_\omega. \tag{2.12}$$

Moreover, $\forall r, s \in X$

$$\Big| \int_\Omega r(t,x) L_\omega^{-1} (\Pi_W s(t,x)) \ dt \ dx \Big| \leq C_4 \Big(1 + \frac{\varepsilon}{\gamma} \Big) |r|_\omega |s|_\omega, \tag{2.13}$$

and finally, $\forall s \in X$,

$$|L_\omega^{-1} \Pi_W s|_{L^2} \leq C_4 \Big(1 + \frac{\sqrt{\varepsilon}}{\gamma} \Big) |s|_\omega. \tag{2.14}$$

Lemma 2.3 *There exists $\rho > 0$ such that, $\forall v \in \mathcal{D}_\rho := \{ v \in V \mid |v|_\omega^{p-1}/\gamma < \rho \}$ there exists a unique $w(v) \in W$ with $|w(v)|_\omega < |v|_\omega$ solving the (P) equation. Moreover, for some positive constant C_5, we have*

- *i)* $|w(v)|_\omega \leq C_5 |v|_\omega^p / \gamma$; $|w(v)|_{L^2} \leq C_5 (1 + \sqrt{\varepsilon}/\gamma) |v|_\omega^p$;
- *ii) for all $r \in X$,*

$$\Big| \int_\Omega \Big(w(v) - L_\omega^{-1} \Pi_W f(v) \Big) \ r \Big| \leq \frac{C_5}{\gamma} \Big(1 + \frac{\varepsilon}{\gamma} \Big) |v|_\omega^{2p-1} |r|_\omega;$$

- *iii)* $w(-v)(t,x) = w(v)(t + \pi, \pi - x)$;
- *iv)* $v \in V_n \Rightarrow w(v)$ *is $2\pi/n$ periodic w.r.t. t;*
- *v) the map $v \to w(v)$ is in $C^1(V, W)$.*

In order to solve the (Q) equation we consider the reduced Lagrangian action functional $\Phi_\varepsilon : \mathcal{D}_\rho \to \mathbf{R}$ defined by

$$\Phi_\varepsilon(v) := \Psi(v + w(v)) = \int_0^{2\pi} dt \int_0^\pi dx \ \frac{\omega^2}{2} (v_t + (w(v))_t)^2 - \frac{1}{2} (v_x + (w(v))_x)^2 - F(v + w(v)). \tag{2.15}$$

By (2.15) and formula (2.9) with $h = w(v)$ we can write the reduced action functional as

$$\Phi_\varepsilon(v) = \int_\Omega dt \ dx \ \frac{\omega^2}{2} v_t^2 - \frac{v_x^2}{2} - F(v + w(v)) + \frac{1}{2} f(v + w(v)) w(v), \tag{2.16}$$

Since $|v_t|_{L^2}^2 = |v_x|_{L^2}^2 = ||v||_{H^1}^2/2$, we have

$$\Phi_\varepsilon(v) = \frac{\varepsilon}{2} ||v||^2 + \int_\Omega dt \ dx \ \frac{1}{2} f(v + w(v)) w(v) - F(v + w(v)). \tag{2.17}$$

Defining the linear operator $\mathcal{I} : X \to X$ by $(\mathcal{I}u)(t,x) = u(t + \pi, \pi - x)$ it results that $\Psi \circ \mathcal{I} = \Psi$. Hence, by lemma 2.3-*iii*) and since $-v = \mathcal{I}v$, functional Φ_ε is even:

$$\Phi_\varepsilon(-v) = \Psi(-v + w(-v)) = \Psi(\mathcal{I}(v + w(v))) = \Psi(v + w(v)) = \Phi_\varepsilon(v), \qquad \forall v \in \mathcal{D}_\rho, \tag{2.18}$$

By lemma 2.3-v) functional Φ_ε is in $C^1(V,\mathbf{R})$. Arguing as in [1]-[2]-[19] we find that, $\forall h \in V$

$$D\Phi_\varepsilon(v)[h] = D\Psi(v+w(v))[h] = \int_\Omega dt\ dx\ \omega^2 v_t h_t - v_x h_x - f(v+w(v))h = \varepsilon(v,h) - \int_\Omega dt\ dx\ f(v+w(v))h. \tag{2.19}$$

Hence the following lemma holds

Lemma 2.4 *If $v \in V$ is a critical point of the reduced action functional $\Phi_\varepsilon : \mathcal{D}_\rho \to \mathbf{R}$ then $u = v + w(v)$ is a critical point of $\Phi_\varepsilon : X \to \mathbf{R}$.*

Remark 2.1 *On the other hand it is possible to prove that any critical point u of Φ_ε, sufficiently close to 0, can be written as $u = v + w$ with $w = w(v) \in W, v \in V$ and v is a critical point of Φ_ε.*

3 An abstract result

Let E be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $||\cdot||$. We will denote $B := \{v \in E \mid ||v|| \le 1\}$ the closed unit ball of E and $S := \{v \in E \mid ||v|| = 1\}$ the unit sphere.

Let us consider a C^1 functional $\Phi : B_{r_0} \subset E \to \mathbf{R}$, defined on the ball $B_{r_0} := \{v \in E \mid ||v|| < r_0\}$, by

$$\Phi(v) = \frac{\mu}{2}||v||^2 - G(v) + R(v) \tag{3.1}$$

where $\mu \in (0,+\infty)$ and

- $(H1)$ $G \in C^1(E,\mathbf{R})$ is positively homogeneous of degree $q+1$ with $q > 1$, i.e. $G(\lambda v) = \lambda^{q+1}G(v)$ $\forall \lambda \in \mathbf{R}_+$, and $DG : V \to V^*$ is compact;
- $(H2)$ $R \in C^1(B_{r_0},\mathbf{R})$ and $DR : B_{r_0} \to V^*$ is compact;
- $(H3)$ $R(0) = 0$ and $\exists \alpha > 0$ such that $|DR(v)[v]| \le \alpha\ ||v||^{q+1}$, $\forall v \in B_{r_0}$.

By the compactness of $DG : V \to V^*$, G maps the unit sphere S into a bounded subset of $\mathbf{R}$ (indeed $G(v) = \int_0^1 DG(sv)[v]ds$ and $DG(B)$ is bounded in V^*). It follows that the functional $U : E\backslash\{0\} \to \mathbf{R}$ defined by

$$U(v) := \frac{G(v)}{||v||^{q+1}},$$

which satisfies $U(\lambda v) = U(v)$, $\forall v \in E\backslash\{0\}$, $\forall \lambda \in \mathbf{R}_+$, is bounded. We shall assume the further condition

$$m := \sup_{v \in E\backslash\{0\}} U(v) > 0. \tag{3.2}$$

The following lemma, quite standard, ensures that U attains its maximum on S.

Lemma 3.1 $K_0 := \{v \in S \mid G(v) = m\} = \{v \in S \mid U(v) = m\}$ *is not empty and compact.*

PROOF. Since $DG(B)$ is compact, $\forall \delta > 0$ there is a finite dimensional subspace F_δ of E such that

$$\forall v \in B \quad \forall h \in F_\delta^\perp \quad |DG(v)[h]| \le \delta||h||.$$

As a result, calling P_δ the orthogonal projection onto F_δ, we obtain for $v \in B$

$$|G(v) - G(P_\delta v)| \le \int_0^1 |DG((1-s)P_\delta v + sv)[v - P_\delta v]|\ ds \le \int_0^1 \delta||v - P_\delta v||\ ds \le \delta||v - P_\delta v|| \le \delta.$$

One can derive from this property that $G_{|B}$ is continuous for the weak topology in E. Indeed $G_{|B}$ is the uniform limit, as $\delta \to 0$, of $G \circ P_\delta$ and each function $G \circ P_\delta$ is continuous in the weak topology since P_δ is a linear compact operator. Since B, the unit ball of E, is compact for the weak topology,

G attains its maximum on B at some point $v_0 \in B$. We have $G(v_0) = m > 0$, so that $v_0 \neq 0$, and $U(v_0) = m/||v_0||^{q+1} \geq m$. Hence, by the definition of m, $U(v_0) = m$ and $v_0 \in S$, *i.e.* $v_0 \in K_0$.

To prove that K_0 is compact, consider a sequence (v_k) with $v_k \in K_0$ for all k. Since (v_k) is bounded, we may assume that (up to a subsequence) (v_k) converges to some $v \in E$ for the weak topology. For all k, $v_k \in B$ hence, since B is closed for the weak topology, $v \in B$. Moreover, since $G_{|B}$ is continuous for the weak topology, $G(v) = m$. By the same argument as above, we get that $||v|| = 1$. Hence $(v_k) \rightharpoonup v$ and $(||v_k||) \to ||v||$ and we can conclude that $(v_k) \to v$ strongly. ■

The main result of this section -which will allow us in the next section to get our multiplicity results- is the following Proposition.

Proposition 3.1 *Assume that the functional $\Phi \in C^1(E, \mathbf{R})$ satisfies $(H1)$, $(H2)$, $(H3)$ and (3.2). There exists a small positive constant C_0, depending only on q, such that, if*

$$\frac{\alpha}{m} \leq C_0 \quad \text{and} \quad \Big(\frac{\mu}{m}\Big)^{\frac{1}{q-1}} \leq C_0 r_0, \tag{3.3}$$

functional Φ has a critical point $v \in B_{r_0}$ on a critical level

$$c = \frac{q-1}{2} m \Big(\frac{\mu}{(q+1)m}\Big)^{\frac{q+1}{q-1}} \Big[1 + O\Big(\frac{\alpha}{m}\Big)\Big]. \tag{3.4}$$

Moreover

$$v = \Big(\frac{\mu}{m(q+1)}\Big)^{\frac{1}{q-1}} y \qquad \text{with} \qquad \mathrm{dist}(y, K_0) \leq h\Big(\frac{\alpha}{m}\Big) \tag{3.5}$$

for some function h, depending only on G, which satisfies $\lim_{s\to 0} h(s) = 0$.

Remark 3.1 *The key difference with the approach of [7] is the following: instead of trying to show that the functional $(\mu/2)||v||^2 - G(v)$ possesses non-degenerate critical points (if ever true), and then continuing them through the Implicit Function Theorem as critical points of Φ, we find critical points of Φ showing that the nonlinear perturbation term R does not affect the mountain pass geometry of the functional $(\mu/2)||v||^2 - G(v)$. Actually, in [7], non degenerate critical points of G constrained on S are continued. This is equivalent to what said before since any critical point $\widetilde{v}$ of G constrained to S gives rise, by homogeneity, to a critical point $(\mu/(q+1)G(\widetilde{v}))^{1/q-1}\widetilde{v} \in V$ of the functional $(\mu/2)||v||^2 - G(v)$.*

The remainder of this section is devoted to the proof of Proposition 3.1. Let us define the positive constant $\delta := [2(q+1)]^{1/q-1}$. Assuming that

$$\Big(\frac{2\mu}{m}\Big)^{\frac{1}{q-1}} < r_0 \tag{3.6}$$

it results that the rescaled functional φ

$$\varphi(y) := \frac{1}{(q+1)m}\Big(\frac{(q+1)m}{\mu}\Big)^{\frac{q+1}{q-1}} \Phi\Big(\Big(\frac{\mu}{(q+1)m}\Big)^{1/q-1} y\Big) \tag{3.7}$$

is well defined on an open neighborhood of $\overline{B}_\delta$. We shall also assume that

$$\eta(\alpha, m) := \frac{\alpha}{(q+1)m} \leq \eta_1. \tag{3.8}$$

(3.6) and (3.8) stem from (3.3) if C_0 is chosen appropriately. We shall prove that, for η_1 small enough, φ has a critical point y on a critical level d such that

$$d = \Big(\frac{1}{2} - \frac{1}{q+1}\Big)(1 + O(\eta(\alpha, m))) \tag{3.9}$$

and

$$\mathrm{dist}(y,K_0)\le h\Big(\frac{\alpha}{m}\Big) \tag{3.10}$$

for some function h such that $\lim_{s\to 0}h(s)=0$. (3.9) and (3.10) imply resp. (3.4) and (3.5) since, by the definition of φ, $v:=\Big(\frac{\mu}{(q+1)m}\Big)^{1/q-1}y$ will then be a critical point of Φ at level $c=d(q+1)m$ $\Big(\frac{\mu}{(q+1)m}\Big)^{q+1/q-1}$. We can write

$$\varphi(y)=\frac{||y||^2}{2}-\frac{G(y)}{(q+1)m}+S(y), \tag{3.11}$$

where, by the properties of R,

$$\forall y\in B_\delta,\ \ |DS(y)[y]|\le\eta(\alpha,m)||y||^{q+1}\le\eta_1||y||^{q+1}. \tag{3.12}$$

As a consequence, since $S(y)=\int_0^1 dsDS(sy)[y]$,

$$|S(y)|\le\frac{\eta(\alpha,m)}{q+1}\,||y||^{q+1}\le\frac{\eta_1}{q+1}\,||y||^{q+1}. \tag{3.13}$$

We need the following technical lemma:

Lemma 3.2 *Let W be some (small) neighborhood of 0 in $\mathbf{R}$. There are constants $C_6,C_7>0$ such that, $\forall a\in[1/2,1]$, $\forall\ 0<\xi\le C_6, \forall\ 0<\eta\le C_6$, for all C^1 function $r:[0,\delta]\to\mathbf{R}$ with $\sup_{t\in[0,\delta]}|r(t)|+|r'(t)|\le\xi$, the map$f_{a,r}:[0,\delta]\to\mathbf{R}$ defined by $f_{a,r}(t):=(t^2/2)-a(t^{q+1}/q+1)+r(t)$, satisfies*

$$t\in[0,\delta]\backslash W\ ,\ |f'_{a,r}(t)|\le\eta\Longrightarrow|t-a^{-1/q-1}|\le C_7(\xi+\eta), \tag{3.14}$$

and

$$\inf_{t\in[0,\delta]\backslash W,|f'_{a,r}(t)|\le\eta}f_{a,r}(t)\ge\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\Big(\frac{1}{a}\Big)^{2/q-1}-C_7(\xi+\eta). \tag{3.15}$$

Proof. Consider first the function $f_a(t):=\frac{t^2}{2}-a\frac{t^{q+1}}{q+1}$. f'_a possesses outside W a unique zero $t_a:=a^{-\frac{1}{q-1}}$ at the level

$$f_a(t_a)=\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\Big(\frac{1}{a}\Big)^{\frac{2}{q-1}}.$$

$f''_a(t_a)=1-q<0$, and there are $\nu>0$, $\overline{\nu}>0$ such that, $\forall a\in[1/2,1]$,

$$\forall t\in[t_a-\nu,t_a+\nu]\ \ f''_a(t)\le-\nu/2\quad\text{and}\quad\forall t\in[0,\delta]\backslash(W\cup[t_a-\nu,t_a+\nu])\ \ |f'_a(t)|\ge\overline{\nu}. \tag{3.16}$$

Choose $C_6>0$ such that $2C_6<\overline{\nu}$. If $\sup_{t\in[0,\delta]}|r(t)|+|r'(t)|\le\xi\le C_6$ and $|f'_{a,r}(t)|\le\eta\le C_6$ for some $t\notin W$, then $|f'_a(t)|\le\xi+\eta<\overline{\nu}$ and therefore, by the second inequality in (3.16), $t\in[t_a-\nu,t_a+\nu]$. Moreover by (3.16) again, $|f'_a(t)|\ge(\nu/2)|t-t_a|$ for $t\in[t_a-\nu,t_a+\nu]$, and hence $|t-t_a|\le 2(\xi+\eta)/\nu$, which proves estimate (3.14). Finally, by the Taylor formula $f_a(t)=f_a(t_a)+O((\xi+\eta)^2)$, so that $f_{a,r}(t)=f_a(t_a)+O(\xi)+O((\xi+\eta)^2)$ and estimate (3.15) follows straightforwardly. ■

Let us define the open set

$$A:=\Big\{v\in E\backslash\{0\}\ \Big|\ U(v)>\frac{m}{2}\Big\},$$

the min-max class of path

$$\Gamma=\Big\{\gamma\in C([0,1],A\cup\{0\})\ \Big|\ \gamma(0)=0,\ ||\gamma(s)||\le\delta,\ \ ||\gamma(1)||=\delta\Big\}$$

and the "restricted-to-A" mountain pass level of φ

$$d:=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}\varphi(\gamma(s)).$$

Before proving that d is a critical level for φ, we shall check that, provided η_1 has been chosen small enough, it satisfies (3.9). For $y \in S_A := S \cap A$ we define $f_y : [0, \delta] \to \mathbf{R}$ by $f_y(t) = \varphi(ty)$. By (3.11) we have

$$f_y(t) = \frac{t^2}{2} - \frac{U(y)t^{q+1}}{(q+1)m} + r_y(t), \tag{3.17}$$

where, by (3.13) and (3.12),

$$|r_y(t)| = |S(ty)| \le \frac{\eta(\alpha, m)}{q+1} t^{q+1} \qquad \text{and} \qquad |r'_y(t)| \le \eta(\alpha, m) t^q. \tag{3.18}$$

We first prove an upper bound for d. For y_0 belonging to K_0 we clearly have $d \le \max_{t \in [0,\delta]} f_{y_0}(t)$ and hence, from (3.17) and (3.18),

$$d \le \max_{t \in [0,\delta]} \Big(\frac{t^2}{2} - \frac{t^{q+1}}{q+1} \Big) + \frac{\eta(\alpha, m)}{q+1} \delta^{q+1} = \Big(\frac{1}{2} - \frac{1}{q+1} \Big) + \frac{\eta(\alpha, m)}{q+1} \delta^{q+1}.$$

For the lower bound, we observe that there is a small neighborhood W of 0 in $\mathbf{R}$ such that $\forall y \in S_A$ and $\forall t \in W$, $f'_y(t) > 0$, whereas, by the choice of $\delta = [2(q+1)]^{1/q-1}$ and provided η_1 is small enough, $f'_y(\delta) < 0$. By the continuity of $D\varphi$, this implies that $\forall \gamma \in \Gamma$ there is $s \in [0, 1]$ such that $D\varphi(\gamma(s))[\gamma(s)] = 0$, $\gamma(s) \ne 0$. As a consequence,

$$d \ge \inf_{y \in S_A, f'_y(t)=0, t \in [0,\delta] \setminus W} f_y(t) \ge \inf_{y \in S_A, f'_y(t)=0, t \in [0,\delta] \setminus W} \frac{t^2}{2} - \frac{t^{q+1}}{(q+1)} + r_y(t)$$

and the desired lower bound follows by (3.18) and lemma 3.2.

In order to complete the proof of Proposition 3.1, we need the following

Lemma 3.3 *There exists a sequence (y_k), $y_k \in A \cap B_\delta$ such that either $\varphi(y_k) \to d$ and $D\varphi(y_k) \to 0$, or $\varphi(y_k) \to d$ and $U(y_k) \to m/2$ and $D\varphi(y_k)[y_k] \to 0$.*

Proof. Arguing by contradiction, assume that no sequence (y_k) satisfies the above properties. Then there exists $\nu > 0$ such that, for all $v \in A \cap B_\delta$,

$$|\varphi(v) - d| < \nu \quad \Longrightarrow \quad ||D\varphi(v)|| > \nu \tag{3.19}$$

and

$$|\varphi(v) - d| + \Big| U(v) - \frac{m}{2} \Big| < \nu \quad \Longrightarrow \quad |D\varphi(v)[v]| > \nu. \tag{3.20}$$

There exists a locally Lipschitz-continuous map $X : A_\nu := \{v \in A \cap B_\delta \mid ||D\varphi(v)|| > \nu\} \to E$ such that $||X(v)|| \le 2||D\varphi(v)||$ and $D\varphi(v)[X(v)] \ge ||D\varphi(v)||^2 \ge \nu^2$ for all $v \in A_\nu$ (X is called a pseudo gradient field for φ; for a proof of existence, see for example [30]).

Let $J : \mathbf{R} \to [0, 1]$ be a smooth function such that $J(s) = 0$ if $|s| \ge \nu/2$ and $J(s) = 1$ if $|s| \le \nu/4$.

Let β_1 and $\beta_2 < \delta$ be such that $\forall v \in A$ with $||v|| \le \beta_1$ or $||v|| \ge \beta_2$, $\varphi(v) < d - \nu$. β_2 does exists because $\forall y \in S_A$, by (3.17)-(3.18), $\varphi(\delta y) \le (\delta^2/2) - (\delta^{q+1}/2(q+1)) + O(\eta(\alpha, m)\delta^{q+1}) = \delta^2\ (-(1/2) + O(\eta(\alpha, m))) \le 0$, by the choice of δ provided η_1 is small enough. Let $K : \mathbf{R} \to [0, 1]$ be smooth and satisfy $K(s) = 0$ if $|s| \le \beta_1^2/4$ or $|s| \ge (\beta_2 + \delta)^2/4$ and $K(s) = 1$ if $\beta_1^2 \le |s| \le \beta_2^2$. We define

$$Y(v) = K(||v||^2) J(\varphi(v) - d) \Big[- \Big(1 - J\Big(U(v) - \frac{m}{2} \Big) \Big) X(v) - J\Big(U(v) - \frac{m}{2} \Big)\ (D\varphi(v)[v])\ v \Big].$$

The vectorfield Y is well defined on $A' := A \cup B_{\beta_1}$ and vanishes everywhere in $B_{\beta_1/2}$ and outside B_δ. It is locally Lipschitz continuous. Moreover it is bounded, and it is easy to see that its flow $\phi_r(v)$ is defined, for all time r, in A' (notice that in a neighbourhood of ∂A, $Y(v) \in \mathbf{R}v$, so that the flow leaves $U(v)$ invariant, and one does not reach ∂A); $r \mapsto \phi_r(v)$ is the unique solution of $(d/dr)\phi_r(v) = Y(\phi_r(v))$ with $\phi_0(v) = v$. We observe also that if $\gamma \in \Gamma$ then γ_r defined by $\gamma_r(s) = \phi_r(\gamma(s))$ belongs to Γ for all r because $\phi_r(y) = y$ for $||y|| \le \beta_1/2$ and $||y|| \ge \delta$.

Note that if $J(\varphi(v)-d) \neq 0$ then $|\varphi(v)-d| < \nu/2$. In that case, either $|U(v)-m/2| < \nu/2$ and then by (3.20) $|D\varphi(v)[v]| \geq \nu$, or $|U(v)-m/2| \geq \nu/2$ and then $Y(v) = -K(||v||^2)J(\varphi(v)-d)X(v)$. In both cases, by (3.19) and the properties of $X(v)$,

$$D\varphi(v)[Y(v)] \leq -K(||v||^2)J(\varphi(v)-d)\nu^2 \leq 0. \tag{3.21}$$

Now, if $|\varphi(v)-d| < \nu/4$ then $J(\varphi(v)-d) = 1$ and by the definition of β_1, β_2, $\beta_1 \leq ||v|| \leq \beta_2$, so that $K(||v||^2) = 1$. Therefore, by (3.21), for all $v \in A' \cap \overline{B}_\delta$,

$$|\varphi(\phi_r(v)) - d| < \frac{\nu}{4} \quad \Longrightarrow \quad \frac{d}{dr}\varphi(\phi_r(v)) \leq -\nu^2 \tag{3.22}$$

Pick some $\gamma \in \Gamma$ which satisfies $\max_{s\in[0,1]} \varphi(\gamma(s)) \leq d + \nu/4$. By (3.22), $\max_{s\in[0,1]} \varphi(\phi_{1/2\nu}(\gamma(s))) \leq d - \nu/4$, which contradicts the definition of d. ■

We can now prove that d is a critical level. Let y_k be the sequence given by lemma 3.3. If $D\varphi(y_k)[y_k] \to 0$ and $U(y_k) \to m/2$ then, writing $y_k = t_k z_k$ with $z_k \in S$, we have $\lim_k f'_{z_k}(t_k) = 0$ and $U(z_k) \to m/2$ ($||y_k|| \geq \rho_0 > 0$ since $\varphi(y_k) \to d > 0$). Therefore in this case, by (3.17), (3.18) and lemma 3.2, if η_1 is small enough,

$$\liminf_k \varphi(y_k) \geq \Big(\frac{1}{2} - \frac{1}{q+1}\Big) 2^{2/q-1} + O(\eta_1) > d$$

by estimate (3.9). It follows that $D\varphi(y_k) \to 0$ and it is standard to prove (by the compactness properties of DG and DS) that, up to a subsequence, (y_k) converges strongly to a critical point y of φ on the level d (Palais Smale property, see e.g. [30]). Note that, since $\varphi(y_k) tod$ and $y_k \in A$, $y_k \in B_{\beta_2}$ for all k so that $y \in B_\delta$.

Let us now prove the last claim of Proposition 3.1. Writing $y = tz$ with $z \in S$ and $f'_z(t) = 0$, by lemma 3.2 we get

$$d \geq \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\Big(\frac{m}{G(z)}\Big)^{2/q-1} - C_7\eta(\alpha,m)(\delta^q + \delta^{q+1}/(q+1)),$$

and hence, by the upper estimate on d, $G(z) = m + O(\eta(\alpha,m))$. Moreover by (3.14) we have

$$t = \Big(\frac{m}{G(z)}\Big)^{1/q-1} + O(\eta(\alpha,m)) = 1 + O(\eta(\alpha,m)).$$

Hence $\mathrm{dist}(y, K_0) \leq g(C_8\eta(\alpha,m)) + C_8\eta(\alpha,m)$ for some positive constant C_8, where

$$g(s) := \sup_{v\in S, G(v)\geq m-s} \mathrm{dist}(v, K_0), \qquad \lim_{s\to 0^+} g(s) = 0.$$

The last claim of Proposition 3.1 follows with $h(s) = g(C_8 s) + C_8 s$.

4 Applications to nonlinear wave equations

In this section we prove existence and multiplicity results of periodic solutions for the nonlinear wave equation (1.1) satisfying $(F1)$. Precise informations on their minimal periods will be also given.

We start with some preliminary lemmas.

Lemma 4.1 *Let $m : \mathbf{R}^2 \to \mathbf{R}$ be locally integrable and 2π-periodic w.r.t. both variables. Then*

$$\int_0^{2\pi}\int_0^{\pi} m(t+x, t-x)\; dt\; dx = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} m(s_1, s_2)\; ds_1\; ds_2.$$

PROOF. Using new variables $(s_2, x) = (t - x, x)$ and the periodicity of m, we get

$$\int_0^{2\pi} \int_0^{\pi} m(t+x, t-x)\ dt\ dx = \int_0^{2\pi} \int_0^{\pi} m(s_2 + 2x, s_2)\ dx\ ds_2 = \int_0^{2\pi} ds_2 \int_0^{\pi} m(s_2 + 2x, s_2)\ dx.$$

By the periodicity of m, setting $s_1 := s_2 + 2x$, we get

$$\int_0^{\pi} m(s_2 + 2x, s_2)\ dx = \frac{1}{2} \int_0^{2\pi} m(s_1, s_2)\ ds_1,$$

which yields the result. ■

Lemma 4.2 *If $v \in V$ then $v^{2p} \in W$.*

PROOF. Writing $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ we get $\forall u(t,x) = q(t+x) - q(t-x) \in V$,

$$\int_0^{2\pi} \int_0^{\pi} v^{2p}(t,x) u(t,x)\ dx\ dt = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \Big(\eta(s_1) - \eta(s_2)\Big)^{2p} (q(s_1) - q(s_2))\ ds_1\ ds_2 = 0,$$

because $(s_1, s_2) \mapsto (\eta(s_1) - \eta(s_2))^{2p}(q(s_1) - q(s_2))$ is an odd function. This implies that $v^{2p} \in W$. ■

We now apply the arguments of Proposition 3.1 to functionals Φ of the form (3.1) defined on the Hilbert space V endowed with the H^1 norm.

Lemma 4.3 *Assume that $\Phi \in C^1(B_{r_0}, \mathbf{R})$ satisfies the hypotheses of Proposition 3.1 and furthermore that $\forall n \geq 2$ and $\forall v \in V$ for which $G(\mathcal{L}_n v) > 0$,*

$$G(\mathcal{L}_n v) < n^{q+1} G(v). \tag{4.1}$$

Then, there is a constant $C_9 \leq C_0$ depending only on G such that, provided $\alpha/m \leq C_9$ and $(\mu/m)^{1/(q-1)} \leq C_0 r_0$, the critical point v of Proposition 3.1 has minimal period (w.r.t. t) 2π.

PROOF. By Proposition 3.1 we know that $v = (\mu/m(p+1))^{1/p-1} y$ with $\text{dist}(y, K_0) \leq h(\alpha/m)$ and $\lim_{s \to 0} h(s) = 0$ (h depending on G only). The lemma is proved once we show that $\forall n \geq 2$, $V_n \cap K_0 = \emptyset$, i.e. any $z \in K_0$ has minimal period 2π. Indeed, if this is true, since K_0 is compact, there exists $\xi > 0$ such that any $y \in V$ with $\text{dist}(y, K_0) \leq \xi$ has minimal period 2π. Hence for α/m small enough so that $h(\alpha/m) < \xi$, the critical point v has minimal period 2π.

For $z(t,x) = \eta(t+x) - \eta(t-x) \in V$ we have

$$\begin{aligned}
||z||^2 &= \int_0^{2\pi} \int_0^{\pi} (\eta'(t+x) - \eta'(t-x))^2\ dt\ dx = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} (\eta'(s_1) - \eta'(s_2))^2\ ds_1\ ds_2 \\
&= \frac{1}{2} \Big[2\pi \int_0^{2\pi} \eta'(s_1)^2\ ds_1 - 2 \int_0^{2\pi} \eta'(s_1)\ ds_1 \int_0^{2\pi} \eta'(s_2)\ ds_2 + 2\pi \int_0^{2\pi} \eta'(s_2)^2\ ds_2 \Big] \\
&= 2\pi \int_0^{2\pi} \eta'(s)^2\ ds.
\end{aligned}$$

Hence $||\mathcal{L}_n z||^2 = 2\pi \int_0^{2\pi} n^2 \eta'(ns)^2\ ds = n^2 ||z||^2$, because η is 2π-periodic. As a consequence (for $z \neq 0$),

$$U(\mathcal{L}_n z) = \frac{G(\mathcal{L}_n z)}{||\mathcal{L}_n z||^{q+1}} = \frac{G(\mathcal{L}_n z)}{n^{p+1} ||z||^{q+1}} < U(z),$$

by (4.1). Hence for all $v \in V_n \backslash \{0\}$, $n \geq 2$, $U(v) < \max_{z \in V \backslash \{0\}} U(z) = m$ and therefore $V_n \cap K_0 = \emptyset$. ■

The next lemma provides a suitable approximation for the functional Φ_ε defined in (2.17).

Lemma 4.4 *The functional $\Phi_\varepsilon : \mathcal{D}_\rho \to \mathbf{R}$ defined in (2.17) can be developed as*

$$\Phi_\varepsilon(v) = \frac{\varepsilon}{2}||v||^2 - \int_\Omega F(v) - \frac{1}{2}\int_\Omega f(v)L^{-1}\Pi_W f(v) + R_{\omega,1}(v),$$

with

$$DR_{\omega,1}(v)[v] = O\Big(\frac{\varepsilon}{\gamma}|v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma}\Big).$$

Proof. We have, since $\int_\Omega f(v)L^{-1}\Pi_W f'(v)v = \int_\Omega f'(v)vL^{-1}\Pi_W f(v)$,

$$\begin{aligned}
DR_{\omega,1}(v)[v] &= \int_\Omega \Big[f(v) + f'(v)L^{-1}\Pi_W f(v) - f(v+w(v))\Big]v \\
&= \int_\Omega \Big[f(v) + f'(v)w(v) - f(v+w(v))\Big]v + \int_\Omega \Big[(L^{-1} - L_\omega^{-1})\Pi_W f(v)\Big]f'(v)v \\
&+ \int_\Omega \Big[L_\omega^{-1}\Pi_W f(v) - w(v)\Big]f'(v)v \\
&= O\Big(|v|_\omega^{p-1}|w(v)|_{L^2}^2\Big) + O\Big(\frac{\varepsilon}{\gamma}|v|_\omega^p|v|_\omega^p\Big) + O\Big(\frac{(\varepsilon+\gamma)|v|_\omega^{2p-1}}{\gamma^2}|v|_\omega^p\Big) \\
&= O\Big(\frac{\varepsilon}{\gamma^2}|v|_\omega^{3p-1} + |v|_\omega^{3p-1}\Big) + O\Big(\frac{\varepsilon}{\gamma}|v|_\omega^{2p}\Big) + O\Big(\frac{\gamma+\varepsilon}{\gamma^2}|v|_\omega^{3p-1}\Big) = O\Big(\frac{\varepsilon}{\gamma}|v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma}\Big)
\end{aligned}$$

by lemmas 2.1, 2.3, formula (2.12) and because $|v|_\omega^{p-1}/\gamma \le \rho$. ■

Critical points of the restriction of Φ_ε to the subspace $V_n \subset V$ are critical points of Φ_ε.

Lemma 4.5 *A critical point v_n of $\Phi_{\varepsilon|V_n} : V_n \cap \mathcal{D}_\rho \to \mathbf{R}$ is a critical point of $\Phi_\varepsilon : V \to \mathbf{R}$.*

Proof. We know that $D\Phi_\varepsilon(v_n)[h] = 0$, $\forall h \in V_n$ and we want to prove that $D\Phi_\varepsilon(v_n)[h] = 0$, $\forall h \in V_n^\perp \cap V$. By (2.19) it is sufficient to prove that $\forall h \in V_n^\perp \cap V$

$$\int_0^\pi dx \int_0^{2\pi} dt\ f(v(t,x) + w(v)(t,x))h(t,x) = 0.$$

This holds true because, by lemma 2.3-*iv*), $f(v(t,x)+w(v)(t,x))$ is $2\pi/n$ periodic w.r.t t and $t \to h(t,x) \in V_n^\perp \cap V$ does not contain any harmonic with a frequency multiple of n. ■

Let us define $\Phi_{\varepsilon,n} : \{v \in V \mid \mathcal{L}_n v \in \mathcal{D}_\rho\} \to \mathbf{R}$ by

$$\Phi_{\varepsilon,n}(v) := \Phi_\varepsilon(\mathcal{L}_n v).$$

By lemma 4.5, if v is a critical point of $\Phi_{\varepsilon,n}$ (of minimal period 2π), then $v_n := \mathcal{L}_n v$ is a critical point of Φ_ε (of minimal period $2\pi/n$). Hence, by lemma 2.4, $u = v_n + w(v_n)$ is a solution of (2.1) and it has minimal period with respect to time $2\pi/n$.

In the sequel we shall use the norm

$$|v|_{\omega,n} := |\mathcal{L}_n v|_\omega = |v|_\infty + \varepsilon^{1/2}n||v||. \tag{4.2}$$

Note that if $|\varepsilon n^2| < 1$, which we shall always assume in the sequel, then $|v|_{\omega,n} \le C||v||$. As a consequence there is $\rho_0 > 0$ such that $\mathcal{D} := \{v \in V \mid ||v||^{p-1}/\gamma \le \rho_0\}$ is included in the domain of definition of $\Phi_{\varepsilon,n}$.

Now we must distinguish between different cases depending whether the first term in the Taylor expansion of of the nonlinearity $f(u)$ at 0 is an odd power of u (like e.g. $f(u) = au^3 + h.o.t.$) or an even power of u (like e.g. $f(u) = au^2 + h.o.t.$). We consider first the easiest case of an odd dominant power for $f(u)$.

4.1 Case I: p odd

In this case, by lemma 4.4, the reduced action functional Φ_ε defined in (2.17) can be written in the form (3.1) with

$$\mu = \varepsilon = \frac{\omega^2 - 1}{2}, \qquad G(v) := a \int_\Omega \frac{v^{p+1}}{p+1}, \qquad a = \frac{f^{(p)}(0)}{p!} \neq 0 \tag{4.3}$$

(G is homogeneous of degree $p+1$) and

$$R(v) := R_{\omega,1}(v) - \frac{1}{2}\int_\Omega f(v) L^{-1}\Pi_W f(v) - \int_\Omega \Big(F(v) - a\frac{v^{p+1}}{p+1}\Big)$$

satisfies

$$|DR(v)[v]| \le C\Big(|v|_\omega^{p+3} + \frac{\varepsilon}{\gamma}|v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma}\Big). \tag{4.4}$$

In fact $\int f(v)v - av^{p+1} = O(|v|_\omega^{p+3})$ since, by lemma 4.2, $\int v^{p+2} = 0$. Moreover $\int f'(v)vL^{-1}\Pi_W f(v) = O(|f'(v)v|_{L^2}|f(v)|_{L^2}) = O(|v|_\omega^{2p}) = O(|v|_\omega^{p+3})$ since $p \ge 3$.

If $a > 0$, the functional $G(\cdot)$ defined in (4.3) is positive ($p+1$ is even) and then condition (3.2) is verified. On the contrary, if $a < 0$, we will look for critical points of $-\Phi_\varepsilon$ and we will take $\varepsilon < 0$, i.e. $\omega < 1$. For definiteness we will assume in the sequel that $a > 0$ and so $\varepsilon > 0$.

We can easily prove that

Lemma 4.6 *$DG : V \to V^*$ and $DR : \mathcal{D} \to V^*$ are compact maps.*

Proof. Let $||v_k||$ be bounded. Then, up to a subsequence, $v_k \rightharpoonup \overline{v} \in V$ weakly in H^1 and $v_k \to \overline{v}$ in $|\cdot|_\infty$. Moreover since $w_k := w(v_k)$ too is bounded we can also assume that $w_k \rightharpoonup \overline{w}$ weakly in H^1 and $w_k \to \overline{w}$ in $|\cdot|_{L^q}$ norm for all $q < \infty$. Since $DG(v)[h] = \int_\Omega av^p h$, it follows easily that $DG(v_k) \to DG(\overline{v})$ in V^*. We claim that $DR(v_k) \to \overline{R}$ where $\overline{R}[h] = \int_\Omega f(\overline{v} + \overline{w})h - a\overline{v}^p h = DR(\overline{v})[h]$. Indeed, since $w_k \to \overline{w}$ in $|\cdot|_{L^q}$, it converges (up to a subsequence) also *a.e.* We can deduce, by the Lebesgue dominated convergence theorem, that $f(v_k+w_k) \to f(\overline{v}+\overline{w})$ in L^2 since $f(v_k+w_k) \to f(\overline{v}+\overline{w})$ a.e. and $(f(v_k+w_k))$ is bounded in L^∞. Hence, since $DR(v_k)[h] = \int_\Omega f(v_k + w_k)h - av_k^p h$, $DR(v_k) \to \overline{R}$. ■

$G(v)$ is nonnegative for all $v \in V$ and it is easy to see that

$$G(\mathcal{L}_n(v)) = G(v), \quad \forall v \in V,\ n \in \mathbf{N}. \tag{4.5}$$

Hence

$$\Phi_{\varepsilon,n}(v) := \Phi_\varepsilon(\mathcal{L}_n v) = \frac{\varepsilon n^2}{2}||v||^2 - G(v) + R_n(v), \qquad \forall v \in \mathcal{D}$$

where $R_n(v) := R(\mathcal{L}_n v)$. Of course DR_n (as DR) is compact. Moreover, by (4.4) and (4.2)

$$|DR_n(v)[v]| = |DR(\mathcal{L}_n v)[\mathcal{L}_n v]| \le C\Big(|v|_{\omega,n}^{p+3} + \frac{\varepsilon}{\gamma}|v|_{\omega,n}^{2p} + \frac{|v|_{\omega,n}^{3p-1}}{\gamma}\Big) \le C\Big(||v||^{p+3} + \frac{\varepsilon}{\gamma}||v||^{2p} + \frac{||v||^{3p-1}}{\gamma}\Big). \tag{4.6}$$

We want to apply Proposition 3.1 in order to get the existence, for ω sufficiently close to 1, of a critical point of $\Phi_{\varepsilon,n}$. With the notations of Proposition 3.1, we set

$$r_0 = \frac{1}{C_0}\Big(\frac{\varepsilon n^2}{m}\Big)^{1/p-1}.$$

It is clear that $B_{r_0} \subset \mathcal{D}$ provided $\varepsilon n^2/\gamma$ is small.

Moreover, if $\varepsilon n^2/\gamma$ is small enough, the hypotheses of Proposition 3.1 are satisfied by $\Phi_{\varepsilon,n}$. In fact the first condition in (3.3) is satisfied, because, by (4.6),

$$\alpha = O\Big({r_0}^2 + \frac{\varepsilon}{\gamma}{r_0}^{p-1} + \frac{{r_0}^{2p-2}}{\gamma}\Big) = O\Big((\varepsilon n^2)^{2/(p-1)} + \Big(\frac{\varepsilon n^2}{\gamma}\Big)^2\Big)$$

(here $q = p$ and $\mu = \varepsilon n^2$). The second condition in (3.3) is trivially satisfied by the choice of r_0.

Let us define, for $\omega \in \mathcal{W} := \cup_{\gamma>0} W_\gamma$, $\gamma_\omega := \max\{\gamma \mid \omega \in W_\gamma\}$. By (4.5), condition (4.1) is satisfied. As a conclusion, using Proposition 3.1 and lemma 4.3 we obtain the following result (we obtain pairs of solutions because Φ_ε is even; if $u(t,x)$ is a solution of (1.1) then so is $u(t+\pi, \pi - x)$)

Theorem 4.1 *Let f satisfy $(F1)$ for an odd integer $p \geq 3$. Then there exists a positive constant $C_{10} := C_{10}(f)$ such that, $\forall \omega \in \mathcal{W}$ and $\forall n \in \mathbf{N} \backslash \{0\}$ satisfying*

$$\frac{|\omega - 1| n^2}{\gamma_\omega} \leq C_{10} \tag{4.7}$$

and $\omega > 1$ if $a > 0$ (resp. $\omega < 1$ if $a < 0$), equation (1.1) possesses at least one pair of even periodic (in time) classical C^2 solutions of minimal period $2\pi/(n\omega)$.

The regularity of the solutions u_n is proved in section 5. By condition (4.7) we can find $N_\omega \approx O(\gamma_\omega / \sqrt{|\omega - 1|})$ periodic solutions of (1.1) with period $2\pi/\omega$. For a discussion on the optimality of the estimate (4.7) see remark 4.5.

Remark 4.1 *The unique small amplitude 2π periodic in time solution of (1.1) is the equilibrium solution $u \equiv 0$. Indeed it must be a critical point of the functional $\Phi_0(v) = (a/p+1) \int v^{p+1} + R(v)$. Using (4.4) we see that its unique critical point is $v = 0$.*

4.2 Case II: p even

The case in which f satisfies $(F1)$ for some even p is more difficult to deal with since, by lemma 4.2, $\int_\Omega v^{p+1} = 0$, $\forall v \in V$. Then one must look more carefully for the dominant nonlinear term in the reduced functional Φ_ε. We must distinguish different cases. Let f satisfy $(F1)$ for some even p and

- $(N1)$ $f^{(d)}(0) = bd! \neq 0$ for some odd integer $d < 2p-1$ and $f^{(r)}(0) = 0$ for any odd integer $r < d$;
- $(N2)$ $f^{(r)}(0) = 0$ for any odd integer $r \leq 2p-1$;
- $(N3)$ $f^{(r)}(0) = 0$ for any odd integer $r < 2p-1$ and $f^{(2p-1)}(0) =: b(2p-1)! \neq 0$.

If $(N1)$ holds the dominant term in the reduced functional is supplied by the odd nonlinearity bu^d in the Taylor expansion of f at 0 and then one reduces immediately to the situation discussed in the previous subsection. The following Theorem holds

Theorem 4.2 *Let f satisfy $(F1)$ with p even and $(N1)$. Then there exists a positive constant $C_{11} := C_{11}(f)$ such that, $\forall \omega \in \mathcal{W}$ and $\forall n \in \mathbf{N} \backslash \{0\}$ satisfying*

$$\frac{(|\omega - 1| n^2)^{(p-1)/(d-1)}}{\gamma_\omega} \leq C_{11} \tag{4.8}$$

and $\omega > 1$ if $b > 0$ (resp. $\omega < 1$ if $b < 0$), equation (1.1) possesses at least one pair of even periodic (in time) classical C^2 solutions of minimal period $2\pi/(n\omega)$.

Proof. In this case the reduced functional Φ_ε can be written in the form (3.1) with $\mu = \varepsilon$, $G(v) := (b/d+1) \int_\Omega v^{d+1}$, where $b := f^{(d)}(0)/d!$, and by lemma 4.4 (since $\int_\Omega v^{d+2} = 0$)

$$|DR(v)[v]| \leq C\Big(|v|_\omega^{d+3} + \frac{\varepsilon}{\gamma} |v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma} \Big).$$

As before $G \geq 0$ and $G(\mathcal{L}_n v) = G(v)$. Applying the same arguments as in the previous subsection, with $q = d$, $r_0 = (1/C_0)(|\varepsilon| n^2/m)^{1/d-1}$, and

$$\alpha = O\Big({r_0}^2 + \frac{|\varepsilon|}{\gamma} {r_0}^{2p-1-d} + \frac{{r_0}^{3p-2-d}}{\gamma} \Big),$$

we obtain that condition (4.8) ensures that $B_{r_0} \subset \mathcal{D}$ and that conditions (3.3) hold, which implies the existence of a pair of critical points of $\Phi_{\varepsilon,n}$ (observe that $2p-1-d \geq 2$). ■

When $(N1)$ is not satisfied the situation is more delicate. We shall first consider the case in which $(N2)$ holds. We can write, since $\Pi_W v^p = v^p$,

$$\frac{1}{2}\int_\Omega f(v) L^{-1}\Pi_W f(v) = -G(v) + r(v),$$

with

$$G(v) = -\frac{a^2}{2}\int_\Omega v^p L^{-1} v^p \qquad \text{and} \qquad Dr(v)[v] = O(|v|_\omega^{2p+2}) \tag{4.9}$$

(recall that $f^{(p+1)}(0) = 0$ by $(N2)$). Hence, by lemma 4.4, we have $-\Phi_\varepsilon(v) = (-\varepsilon/2)||v||^2 - G(v) + R(v)$, with

$$DR(v)[v] = O\Big(|v|_\omega^{2p+2} + \frac{\varepsilon}{\gamma}|v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma}\Big). \tag{4.10}$$

$DG, DR : V \to V^*$ are still compact operators and now G is homogeneous of degree $q+1 := 2p$.

A difficulty arises for proving that G satisfies (4.1). The following lemmas will enable to get a suitable expression of $G(v)$.

Lemma 4.7 *Let $w(t,x) \in W$ be of the form $w(t,x) = m(t+x, t-x)$ for some $m : \mathbf{R}^2 \to \mathbf{R}$ 2π-periodic with respect to both variables. Then there exists $\widetilde{m}$, a $2-\pi$ periodic such that m can be decomposed as*

$$m(s_1,s_2) = \widetilde{m}(s_1,s_2) + a(s_1) + a(s_2) + \alpha, \tag{4.11}$$

where $\alpha \in \mathbf{R}$ ($\alpha = \langle m\rangle$), $\langle a\rangle = 0$, $\langle \widetilde{m}\rangle_{s_1}(s_2) = 0$, $\langle \widetilde{m}\rangle_{s_2}(s_1) = 0$. Moreover

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} L^{-1}(w)w \; dt\; dx \;&=\; -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} M(s_1,s_2)\widetilde{m}(s_1,s_2)\; ds_1\; ds_2 + 2\pi\int_0^{2\pi} M(s,s)a(s)\; ds \\ &+\; 2\pi\alpha\int_0^{2\pi} M(s,s)\; ds - 8\pi\int_0^{2\pi} A(s)^2\; ds - \frac{\alpha^2\pi^4}{6},\end{aligned}$$

where M and A are the 2π-periodic functions defined by

$$A'(s) = \frac{1}{4}a(s), \quad \langle A\rangle = 0, \quad \partial_{s_1}\partial_{s_2} M(s_1,s_2) = \frac{1}{4}\widetilde{m}(s_1,s_2), \quad \langle M\rangle_{s_1}(s_2) = 0, \quad \langle M\rangle_{s_2}(s_1) = 0. \tag{4.12}$$

PROOF. The proof is in the Appendix. ■

Lemma 4.8 *Let $m : \mathbf{R}^2 \to \mathbf{R}$ be 2π-periodic with respect to both variables and such that $w(t,x) = m(t+x, t-x) \in W$. For $n \in \mathbf{N}$ define $(\mathcal{L}_n w)(t,x) := m(n(t+x), n(t-x))$. Then*

$$\int_0^{2\pi}\int_0^{\pi} L^{-1}(\mathcal{L}_n w)\mathcal{L}_n w\; dx\; dt = -\frac{\pi^4}{6}\alpha^2 + \frac{1}{n^2}\Big(\int_0^{2\pi}\int_0^{\pi} wL^{-1}w\; dx\; dt + \frac{\pi^4}{6}\alpha^2\Big). \tag{4.13}$$

PROOF. We have $\mathcal{L}_n w(t,x) = m_n(t+x, t-x)$, with $m_n(s_1,s_2) := m(ns_1, ns_2)$. Using the decomposition given in the latter lemma, we can write $m_n(s_1,s_2) = \widetilde{m}_n(s_1,s_2) + a_n(s_1) + a_n(s_2) + \alpha$, where $\widetilde{m}_n := m(n\,\cdot)$ and $a_n := a(n\,\cdot)$. Therefore (using the abbreviation $w_n := \mathcal{L}_n w$)

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} L^{-1}(w_n)w_n\; dx\; dt \;&=\; -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} M_n(s_1,s_2)\widetilde{m}_n(s_1,s_2)\; ds_1\; ds_2 + 2\pi\int_0^{2\pi} M_n(s,s)a_n(s)\; ds \\ &+\; 2\pi\alpha\int_0^{2\pi} M_n(s,s)\; ds - 8\pi\int_0^{2\pi} A_n(s)^2\; ds - \frac{\alpha^2\pi^4}{6},\end{aligned}$$

where $M_n(s_1,s_2) := M(ns_1, ns_2)/n^2$ and $A_n(s) := A(ns)/n$. (4.13) follows straightforwardly. ■

Lemma 4.7 allows to obtain a handable expression for $G(v) = -(a^2/2)\int v^p L^{-1} v^p$ when $p = 2$. For $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ we have $v^2(t,x) = m(t+x, t-x) \in W$ with $m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^2$. We can then decompose m as in (4.11) with $\alpha = 2\langle \eta^2 \rangle$, $a(s) = \eta^2(s) - \langle \eta^2 \rangle$ and $\widetilde{m}(s_1, s_2) = -2\eta(s_1)\eta(s_2)$. Let us call P_1, resp. P_2, the primitive of η, resp. $\eta^2 - \langle \eta^2 \rangle$, of zero mean-value. The functions A, M defined in (4.12) are $A(s) := P_2(s)/4$ and $M(s_1, s_2) = -P_1(s_1)P_1(s_2)/2$. Hence, by lemma 4.7 we obtain the following expression for $G(v) = -(a^2/2)\int v^2 L^{-1} v^2$:

$$G(v) = \frac{a^2}{2}\Big[\pi \int_0^{2\pi} P_1(s)^2 \Big(\eta(s)^2 + \langle \eta^2 \rangle\Big)\, ds + \frac{\pi}{2}\int_0^{2\pi} P_2(s)^2\, ds + \frac{2\pi^4}{3}\langle \eta^2 \rangle^2\Big]. \tag{4.14}$$

From (4.14) we see that

$$\forall v \in V,\ G(v) > 0 \qquad \text{and} \qquad 0 \le G(\mathcal{L}_n v) \le G(v),\ \forall n \ge 2. \tag{4.15}$$

In general we can prove that

Lemma 4.9 *For any even p, $G(v) := -(a^2/2)\int_\Omega v^p L^{-1} v^p$ is non negative.*

PROOF. Writing $v^p(t,x) = m(t+x, t-x)$ with $m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^p$, we claim that $L^{-1}(v^p) \in -(1/8)M(t+x, t-x) + V$, where

$$M(s_1, s_2) := \int_{R_{s_1,s_2}} m(\xi, \nu)\, d\xi\, d\nu, \tag{4.16}$$

and $R_{s_1,s_2} := \{(\xi,\nu) \in \mathbf{R}^2 \mid s_1 \le \xi \le s_2 + 2\pi,\ s_2 \le \nu \le s_1\}$. In fact

$$\partial_{s_1} M(s_1, s_2) = \int_{s_1}^{s_2+2\pi} m(\xi, s_1)\, d\xi - \int_{s_2}^{s_1} m(s_1, \nu)\, d\nu.$$

Hence, since $m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^p$ with even p, and η is 2π-periodic,

$$\partial_{s_2}\partial_{s_1} M = m(s_2 + 2\pi, s_1) + m(s_1, s_2) = 2m(s_1, s_2).$$

Moreover, by the definition of M, $M(s_1, s_1) = M(s_1, s_1 - 2\pi) = 0$ and $M(s_1 + 2\pi, s_2 + 2\pi) = M(s_1, s_2)$ for all $(s_1, s_2) \in \mathbf{R}^2$, so that $(t,x) \mapsto M(t+x, t-x) \in X$. Hence $L[M(t+x, t-x)] = -4(\partial_{s_2}\partial_{s_1} M)(t+x, t-x) = -8v^p(t,x)$ and our claim follows. From (4.16) we see that M is non negative and as a result $-a^2 \int L^{-1}(v^p)v^p = (a^2/8)\int M v^p \ge 0$. ■

We do not know whether (4.15) still holds true for $G(v) = -(a^2/2)\int v^p L^{-1} v^p$, $p \ge 4$. In any case, we can deduce from lemma 4.9, that $G_n(v) := G(\mathcal{L}_n v)$ satisfies (4.1) for $n \ge 2$. Indeed, by (4.13),

$$G_n(v) = G(\mathcal{L}_n v) = \frac{G(v)}{n^2} + \frac{a^2\pi^4\alpha^2}{12}\Big(1 - \frac{1}{n^2}\Big), \tag{4.17}$$

with $\alpha := (1/2\pi^2)\int_\Omega v^p$. Hence, since $G(v) \ge 0$, $\forall n, m \ge 2$, for all $v \in V\backslash\{0\}$,

$$G_n(\mathcal{L}_m v) = G(\mathcal{L}_{mn} v) = \frac{G(v)}{(nm)^2} + \frac{a^2\pi^4\alpha^2}{12}\Big(1 - \frac{1}{(nm)^2}\Big) \le \frac{G(v)}{n^2} + \frac{a^2\pi^4\alpha^2}{12} < m^{2p} G_n(v). \tag{4.18}$$

We may write for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$

$$-\Phi_{\varepsilon,n}(v) = \frac{-\varepsilon n^2}{2}||v||^2 - G_n(v) + R_n(v),$$

where $G_n(v) = G(\mathcal{L}_n v)$, $R_n(v) = R(\mathcal{L}_n v)$ satisfies, by (4.10),

$$DR_n(v)[v] = DR(\mathcal{L}_n v)[\mathcal{L}_n v] = O\Big(||v||^{2p+2} + \frac{\varepsilon}{\gamma}||v||^{2p} + \frac{||v||^{3p-1}}{\gamma}\Big). \tag{4.19}$$

We shall consider frequencies $\omega < 1$ so that $-\varepsilon > 0$. By (4.18), for $n \geq 2$, $G_n(v)$ satisfies (4.1). For $(|\varepsilon|n^2)^{1/2}/\gamma$ small enough we shall apply Proposition 3.1 and lemma 4.3 to $-\Phi_{\varepsilon,n} : B_{r_0} \to \mathbf{R}$ as before, with $q = 2p-1$, $\mu = -\varepsilon n^2$,

$$r_0 = \frac{1}{C_0}\Big(\frac{|\varepsilon|n^2}{m(G_n)}\Big)^{1/(2p-2)} \qquad \text{and} \qquad \alpha = O\Big({r_0}^2 + \frac{|\varepsilon|}{\gamma} + \frac{{r_0}^{p-1}}{\gamma}\Big).$$

Note that $m(G_n) \to m(\overline{G})$ as $n \to +\infty$, where $\overline{G}(v) := (a^2\pi^4/12)\langle v^p\rangle^2 = (a^2/48)(\int v^p)^2$, and the sequence $(m(G_n))$ is bounded from below by a positive constant. We derive the following Theorem, where the condition r_0^{p-1}/γ small, i.e. $(|\varepsilon|n^2)^{1/2}/\gamma$ small, is required.

Theorem 4.3 *Assume that f satisfies $(F1)$ and $(N2)$. Then there is a constant C_{12} depending only on f such that, $\forall \omega \in \mathcal{W}$, $\omega < 1$, $\forall n \geq 2$ such that*

$$\frac{(|\omega - 1|n^2)^{1/2}}{\gamma_\omega} \leq C_{12} \tag{4.20}$$

equation (1.1) possesses at least one pair of even periodic (in time) classical C^2 solutions of minimal period $2\pi/(n\omega)$. If $p = 2$ the existence result holds true for $n = 1$ as well.

Remark 4.2 *We think that the restriction $n \geq 2$ in the case $p \neq 2$ even is of purely technical nature.*

Finally we consider the case in which $(N3)$ holds. We first assume that $b < 0$. We have

$$-\Phi_\varepsilon(v) = \frac{-\varepsilon}{2}||v||^2 - G(v) + R(v),$$

where

$$G(v) := -\frac{b}{2p}\int_\Omega v^{2p} - \frac{a^2}{2}\int_\Omega v^p L^{-1} v^p > 0 \tag{4.21}$$

for all $v \neq 0$ and $R(v)$ satisfies estimate (4.10). Moreover, defining $G_n(v) := G(\mathcal{L}_n v)$, we still have $G_n(\mathcal{L}_m v) < m^{2p} G_n(v)$ for all $n, m \geq 2$ and $v \neq 0$. For $\varepsilon < 0$ and $|\varepsilon|n^2/\gamma$ small enough we obtain the same existence result as in Theorem 4.3.

Now assume that $b > 0$. As a consequence of lemma 4.8 we have

$$\int_\Omega \mathcal{L}_n w L^{-1}(\mathcal{L}_n w) = -\frac{\pi^4}{6}\langle m\rangle^2 + O\Big(\frac{|m|_{L^2}^2 + |w|_{L^2}^2}{n^2}\Big) = -\frac{\pi^4}{6}\langle w\rangle^2 + O\Big(\frac{|w|_{L^2}^2}{n^2}\Big). \tag{4.22}$$

By (4.22), we may write for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$

$$\Phi_{\varepsilon,n}(v) = \frac{\varepsilon n^2}{2}||v||^2 - \widetilde{G}(v) + R_n(v),$$

with

$$\widetilde{G}(v) = \frac{b}{2p}\int_\Omega v^{2p} - \frac{a^2}{48}\Big(\int_\Omega v^p\Big)^2 = \frac{b}{4p}\int_{\mathbf{T}^2}\Big(\eta(s_1) - \eta(s_2)\Big)^{2p} - \frac{a^2}{192}\Big(\int_{\mathbf{T}^2}(\eta(s_1) - \eta(s_2))^p\Big)^2$$

and

$$DR_n(v)[v] = DR(\mathcal{L}_n v)[\mathcal{L}_n v] + O\Big(\frac{|v|_{\omega,n}^{2p}}{n^2}\Big) = O\Big(||v||^{2p+2} + \frac{\varepsilon}{\gamma}||v||^{2p} + \frac{||v||^{3p-1}}{\gamma} + \frac{||v||^{2p}}{n^2}\Big). \tag{4.23}$$

We are no longer able to prove that the condition of lemma 4.1 is satisfied by G defined in (4.21) or by $-G$. That is why we introduce $\widetilde{G}$. Clearly $\widetilde{G}(\mathcal{L}_n v) = \widetilde{G}(v)$ for all $v \in V$ and $n \in \mathbf{N}\backslash\{0\}$. We have to specify the sign of $\widetilde{G}(v)$. We need to introduce

$$\kappa(p) := \sup_{v \in V\backslash\{0\}}\Big(\int_\Omega v^p\Big)^2\Big(\int_\Omega v^{2p}\Big)^{-1} = \sup_{\eta \in H^1(\mathbf{T}), \eta\ odd}\Big(\int_{\mathbf{T}^2}(\eta(s_1) - \eta(s_2))^p\Big)^2\Big(2\int_{\mathbf{T}^2}(\eta(s_1) - \eta(s_2))^{2p}\Big)^{-1}.$$

We can prove that $\kappa(p) = \pi^2$. Indeed, using that η is odd, we get

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^p \; ds_1 \; ds_2 = \sum_{k=0}^{p/2} C_p^{2k} \int_{\mathbf{T}} \eta^{2k} \; ds \int_{\mathbf{T}} \eta^{p-2k} \; ds \leq \sum_{k=0}^{p/2} C_p^{2k} \times 2\pi \int_{\mathbf{T}} \eta^p \; ds,$$

by Hölder inequality. Hence

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^p \; ds_1 \; ds_2 \leq 2^p \pi \int_{\mathbf{T}} \eta^p \; ds.$$

Now, by Cauchy-Schwarz inequality, for all $0 \leq k \leq p$,

$$\Big(\int_{\mathbf{T}} \eta^p \Big)^2 = \Big(\int_{\mathbf{T}} |\eta|^k |\eta|^{p-k} \Big)^2 \leq \int_{\mathbf{T}} \eta^{2k} \int_{\mathbf{T}} \eta^{2p-2k}.$$

Hence

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^{2p} \; ds_1 \; ds_2 = \sum_{k=0}^{p} C_{2p}^{2k} \int_{\mathbf{T}} \eta^{2k} \; ds \int_{\mathbf{T}} \eta^{2p-2k} \; ds \geq 2^{2p-1} \Big(\int_{\mathbf{T}} \eta^p \Big)^2.$$

As a result we obtain $\kappa(p) \leq \pi^2$. Choosing for η H^1-approximations of the 2π-periodic function that takes value -1 on $(-\pi, 0]$ and value 1 on $(0, \pi]$, we obtain the converse inequality. Hence $\kappa(p) = \pi^2$. Note also that

$$\inf_{v \in V \setminus \{0\}} \Big(\int_\Omega v^p \Big)^2 \Big(\int_\Omega v^{2p} \Big)^{-1} = 0.$$

In fact, for all $\delta > 0$ there is an odd smooth 2π-periodic η_δ such that $\int_0^{2\pi} \eta_\delta^{2p} = 1$ and $\int_0^{2\pi} \eta_\delta^{2q} \leq \delta$ for $1 \leq q < p$. One can see easily that for $v(t,x) = \eta_\delta(t+x) - \eta_\delta(t-x)$, $(\int_\Omega v^p)^2 (\int_\Omega v^{2p})^{-1} \leq \beta(\delta)$ with $\lim_{\delta \to 0} \beta(\delta) = 0$.

If $b \geq p\kappa(p) a^2/24 = p\pi^2 a^2/24$ then for all $v \in V$, $\widetilde{G}(v) \geq 0$ and $G \not\equiv 0$. In this case we obtain an existence result for $\omega > 1$.

If $0 < b < p\kappa(p) a^2/24 = p\pi^2 a^2/24$ then there is $v_1 \in V$ such that $\widetilde{G}(v_1) < 0$ and there is $v_2 \in V$ such that $\widetilde{G}(v_2) > 0$. In this case we obtain an existence result both for $\omega < 1$ and $\omega > 1$.

In both cases we apply the same arguments as before, for $(|\varepsilon| n^2)^{1/2}/\gamma$ small and n large enough, with $q = 2p-1$, $\mu = |\varepsilon| n^2$,

$$r_0 = \frac{1}{C_0} \Big(\frac{|\varepsilon| n^2}{m(\pm \widetilde{G})} \Big)^{1/(2p-2)} \qquad \text{and} \qquad \alpha = O\Big({r_0}^2 + \frac{\varepsilon}{\gamma} + \frac{{r_0}^{p-1}}{\gamma} + \frac{1}{n^2} \Big).$$

$m(-\widetilde{G})$ appears only in the second case when $\omega < 1$. As a conclusion we have

Theorem 4.4 *Assume that f satisfies $(F1)$ and $(N3)$. Define $J \subset \mathbf{R}$ as $J = (0,1)$ if $b < 0$, $J = (1,2)$ if $b \geq p\pi^2 a^2/24$ and $J = (0,1) \cup (1,2)$ if $0 < b < p\pi^2 a^2/24$. Then there is a constant C_{13} and an integer $N_0 \geq 1$ depending only on f (with $N_0 = 2$ if $b < 0$ and $p \geq 4$, $N_0 = 1$ if $b < 0$ and $p = 2$) such that, $\forall \omega \in \mathcal{W} \cap J$, $\forall n \geq N_0$ such that*

$$\frac{(|\omega - 1| n^2)^{1/2}}{\gamma_\omega} \leq C_{13} \tag{4.24}$$

equation (1.1) possesses at least one pair of even periodic (in time) classical C^2 solutions of minimal period $2\pi/(n\omega)$.

In the next remarks we specify the norms and the energies of the solutions u_n obtained in Theorems 4.1, 4.2, 4.3, 4.4.

Remark 4.3 *It is readily seen by the choice of r_0 in the different cases and the fact that $|\mathcal{L}_n v|_\omega = O(||v||) = O(r_0)$, for $|\varepsilon| n^2 < 1$, that the critical points $v_{\varepsilon,n}$ of Φ_ε that we obtain satisfy $|v_{\varepsilon,n}|_\omega = O((|\varepsilon| n^2)^{1/(q-1)})$ with $q = p$ if p is odd and $q = d$ in case $(N1)$, $q = 2p-1$ in cases $(N2)$-$(N3)$, if p is even.*

Remark 4.4 *Let $u_{\omega,n} := u_n$ be the sequence of solutions ($u_{\omega,n}$ has minimal period $2\pi/n\omega$) obtained from the above Theorems. Let $\mathcal{E}_{\omega,n} := \int_0^\pi (u_{\omega,n})_t^2/2 + (u_{\omega,n})_x^2/2 + F(u)\ dx$ be the energy of $u_{\omega,n}$. It is possible to prove, using Proposition (3.1), the following estimates:*

$$||u_{\omega,n}||_{H^1} = n\Big(\frac{|\omega-1|n^2}{m_n(q+1)}\Big)^{1/(q-1)}\Big(1+g_1\Big(\frac{(|\omega-1|n^2)^{(p-1)/(q-1)}}{\gamma_\omega}\Big)\Big) \tag{4.25}$$

$$\mathcal{E}_{\omega,n} = \frac{n^2}{2\pi}\Big(\frac{|\omega-1|n^2}{m_n(q+1)}\Big)^{2/(q-1)}\Big(1+g_2\Big(\frac{(|\omega-1|n^2)^{(p-1)/(q-1)}}{\gamma_\omega}\Big)\Big)$$

where $q = p$ if p is odd, $q = d$ if p is even and $d < 2p-1$ is the smallest order of the odd terms, $q = 2p-1$ if p is even and there is no odd term of order $< 2p-1$ and $\lim_{s\to 0} g_i(s) = 0$; $m_n := m(G_n) = m$ if p is odd and $m_n = m + O(1/n^2)$ if p is even, where $m > 0$ depends only on f. In the case when the smallest order of the odd terms is $2p-1$ and $0 < b < p\pi^2a^2/24$, we have two sequences of solutions which correspond to two (possibly) different values of m.

We finally discuss the optimality of the estimate on the number of solutions that we find.

Remark 4.5 *By conditions (4.7), (4.8), (4.20), (4.24) in Theorems 4.1, 4.2, 4.3, 4.4 we can find $N_\omega \approx O(\gamma_\omega^\tau/\sqrt{|\omega-1|})$ with $1 \le \tau \le 2$, periodic solutions of (1.1) with period $2\pi/\omega$. They have decreasing minimal periods $2\pi/\omega, \pi/\omega, \ldots, 2\pi/N_\omega\omega$ and increasing energies in n, see remark 4.4. We expect the estimate on the number N_ω of solutions found, with small $|\cdot|_\omega^{p-1}/\gamma_\omega$, to be optimal since we could have performed a finite dimensional Lyapunov-Schmidt reduction $u := v_N + w$ with $v_N := \sum_{j=1}^N \xi_j \cos jt \sin jx$ and $N^2|\omega-1| = O(\gamma_\omega)$. Indeed, in lemma 2.3, the "small denominators" are $|\omega^2 l^2 - j^2| \ge \gamma_\omega$, $\forall l \ne j$, and for $j = l \ge N$ are $|\omega^2 l^2 - j^2| = |\omega^2-1|j^2 \ge (|\omega-1|/2)N^2 = O(\gamma_\omega)$. In this way we reduce to study a even functional like Φ_ε on a finite dimensional space of dimension $N \approx O(\gamma_\omega/\sqrt{|\omega-1|})$. In general the best estimate regarding the number of its critical points is N.*

5 Regularity

In order to complete the proof of Theorems 4.1, 4.2, 4.3, 4.4 we now prove that the weak solutions u_n actually are classical $C^2(\Omega)$ solutions of equation (1.1).

Define the norm

$$|u|_{\omega,3} := |u|_\infty + \sqrt{|\varepsilon|}\ ||u||_{H^1} + |\varepsilon|\ ||u||_{H^2} + |\varepsilon|^{3/2}\ ||u||_{H^3}.$$

It is easy to see, arguing as in lemma 2.1, and using the Sobolev continuous embedding $H^3(\Omega) \subset W^{1,\infty}(\Omega)$ that

$$|f(u)|_{\omega,3} \le C|u|_{\omega,3}^p. \tag{5.1}$$

The following lemma holds:

Lemma 5.1 *If $u \in X \cap H^3(\Omega)$ then $w := L_\omega^{-1}\Pi_W u := \sum_{l\ge 0, j\ge 1, j\ne l} w_{lj}\cos(lt)\sin(jx)$, where $w_{lj} := u_{lj}/(\omega^2 l^2 - j^2)$, satisfies $\sum_{l\ge 0,j\ge 1,j\ne l} |w_{lj}|^2(l^6+j^6) \le (C_{14}/\gamma)^2||u||_{H^3}^2$. In particular $w \in H^3(\Omega)$, $||w||_{H^3} = O(||u||_{H^3})/\gamma$ and $w_{xx}(t,0) = w_{xx}(t,\pi) = 0$. Moreover $w \in C^2(\overline{\Omega})$.*

Proof. We point out that $u \in H^3(\Omega)$ does not imply that $\sum_{l\ge 0,j\ge 1} |u_{lj}|^2 j^6$ is finite (it is right only if also $u_{xx}(t,0) = u_{xx}(t,\pi) = 0$; in general we have only $||u||_{H^3}^2 \le C\sum_{l\ge 0,j\ge 1}|u_{lj}|^2(l^6+j^6)$). However it is true that $\sum_{l\ge 0,j\ge 1,j\ne l} |u_{lj}|^2 l^6 = O(|\partial_{ttt}u|_{L^2}^2) = O(||u||_{H^3}^2)$. Now

$$\sum_{l\ge 0,j\ge 1,j\ne l} w_{lj}^2(l^6+j^6) = \sum_{l\ge 0,j\ge 1,j\ne l} \frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6+j^6) = S_1 + S_2 \tag{5.2}$$

where

$$S_1 := \sum_{j\geq 2\omega l} \frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6 + j^6) \qquad \text{and} \qquad S_2 := \sum_{j<2\omega l, j\neq l} \frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6 + j^6).$$

If $j \geq 2\omega l$ then $|\omega^2 l^2 - j^2| \geq 3j^2/4$, hence

$$S_1 \leq \sum_{j\geq 2\omega l} \frac{2u_{lj}^2}{j^4}\Big(j^6 + \frac{j^6}{(2\omega)^2}\Big) \leq \sum_{j\geq 2\omega l} 4|u_{lj}|^2 j^2 = O(|u_x|_{L^2}^2) = O(||u||_{H^1}^2). \tag{5.3}$$

Since $|\omega^2 l^2 - j^2| \geq \gamma/2$ for all $j \neq l$ and $\omega \leq 3/2$ we have

$$S_2 \leq \sum_{1\leq j<2\omega l} \frac{4|u_{lj}|^2}{\gamma^2}(l^6 + j^6) \leq \sum_{1\leq j<2\omega l} \frac{4|u_{lj}|^2}{\gamma^2}(l^6 + (3l)^6) \leq \frac{C}{\gamma^2}|u_{ttt}|_{L^2}^2 \leq \frac{C'}{\gamma^2}||u||_{H^3}^2. \tag{5.4}$$

By (5.2), (5.3) and (5.4) we get $\sum_{l\geq 0, j\geq 1, j\neq l} |w_{lj}|^2(l^6 + j^6) \leq (C/\gamma)^2 ||u||_{H^3}^2$.

Now we prove that

$$\sum |w_{lj}|(l^2 + j^2) < \infty, \tag{5.5}$$

which implies that $w \in C^2(\overline{\Omega})$. We have

$$\sum_{l\geq 0, j\geq 1, j\neq l} |w_{lj}|(l^2 + j^2) = \sum_{l\geq 0, j\geq 1, j\neq l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2) = S_1' + S_2' \tag{5.6}$$

where

$$S_1' := \sum_{j\geq 2\omega l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2) \qquad \text{and} \qquad S_2' := \sum_{j<2\omega l, j\neq l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2).$$

For $j \geq 2\omega l$, $|\omega^2 l^2 - j^2| \geq 3j^2/4$ and $l \leq j/2\omega \leq j$, hence

$$S_1' \leq C \sum_{j\geq 2\omega l} |u_{lj}| \leq C\Big(\sum_{l\geq 0, j\geq 1} |u_{lj}|^2(l^2 + j^2)^2\Big)^{1/2}\Big(\sum_{l\geq 0, j\geq 1} \frac{1}{(l^2 + j^2)^2}\Big)^{1/2} \leq C'||u||_{H^2}. \tag{5.7}$$

Note that $\sum_{l\geq 0, j\geq 1} |u_{lj}|^2(l^2 + j^2)^2 = O(||u||_{H^2}^2)$. We claim that

$$S_2' \leq \sum_{j<2\omega l, j\neq l} C\frac{|u_{lj}|l^2}{|\omega^2 l^2 - j^2|} < +\infty. \tag{5.8}$$

Indeed we know that

$$\sum_{j<2\omega l} (u_{lj} l^2)^2 l^2 < +\infty. \tag{5.9}$$

As in the proof of (6.2) in lemma 2.2, (5.9) implies that S_2' is finite. By and (5.6), (5.7) and (5.8) we get (5.5). The lemma is proved. ■

By lemma 5.1 $||L_\omega^{-1}\Pi_W||_{\omega,3} \leq C/\gamma$. Hence, by (5.1), the map $\mathcal{G}_v : W \to W$, defined in the proof of lemma 2.3 by $\mathcal{G}_v(w) := L_\omega^{-1}\Pi_W f(v + w)$, satisfies (if $v, w \in H^3(\Omega)$)

$$|\mathcal{G}_v(w)|_{\omega,3} \leq \frac{\widetilde{C}}{\gamma}\Big(|v|_{\omega,3}^p + |w|_{\omega,3}^p\Big), \tag{5.10}$$

where $\widetilde{C}$ is some positive constant. Define $\widetilde{\delta} := [\gamma/(2\widetilde{C})]^{1/(p-1)}$ and assume that $|v|_{\omega,3} \leq \widetilde{\delta}$. Then by (5.10), $\mathcal{G}_v(\widetilde{B}_{\widetilde{\delta}}) \subset \widetilde{B}_{\widetilde{\delta}}$, where

$$\widetilde{B}_{\widetilde{\delta}} := \Big\{ w \in W \cap H^3(\Omega) \;\Big|\; |w|_{\omega,3} \leq \widetilde{\delta} \Big\}.$$

As a result, if $|v|_{\omega,3} \le \widetilde{\delta}$ then $w_k := \mathcal{G}_v^k(0) \in \widetilde{B}_{\widetilde{\delta}}$ for all $k \ge 1$. It follows that (w_k) converges weakly in $H^3(\Omega)$ to some $\widetilde{w} \in \widetilde{B}_{\widetilde{\delta}}$.

Now, recall that $w(v)$ is the unique fixed point of $\mathcal{G}_v$ in a small ball around 0 for the norm $|\cdot|_\omega$ in W and that, by the Contraction Mapping Principle, $w(v)$ is the H^1 limit of (w_k). Hence $w(v) = \widetilde{w}$ and $w(v) \in H^3(\Omega)$. By formula (5.1) and lemma 5.1 we derive from the above considerations that if $|v|_{\omega,3} \le \widetilde{\delta}$ then $w(v) \in C^2(\overline{\Omega})$, and $v + w(v) \in C^2(\overline{\Omega})$ (because $V \cap H^3(\Omega) \subset C^2(\overline{\Omega})$).

There remains to check that when condition (4.7) or (4.8) or (4.20) or (4.24) is satisfied (according to the different cases) the critical points $v_{\varepsilon,n}$ of Φ_ε that we have obtained in section 4 satisfy $|v_{\varepsilon,n}|_{\omega,3}^{p-1} \le \gamma/(2\widetilde{C})$. Since the critical points $v := v_{\varepsilon,n}$ of Φ_ε, obtained in Theorems 4.1, 4.2, 4.3, 4.4 satisfy $v_{tt} = v_{xx}$, $2\varepsilon v_{tt} = \Pi_V f(v + w(v))$,

$$||v||_{H^3} = O\Big(\frac{1}{\varepsilon}||\Pi_V f(v+w(v))||\Big) = O\Big(\frac{1}{\varepsilon}||f(v+w(v))||\Big), \quad ||v||_{H^2} = O\Big(\frac{1}{\varepsilon}|f(v+w(v))|_{L^2}\Big).$$

Therefore by lemmas 2.1 and 2.2

$$|v|_{\omega,3} = O\Big(|v|_\omega + |f(v+w(v))|_\omega\Big) = O\Big(|v|_\omega + |v|_\omega^p\Big(1 + \frac{|v|_\omega^{p-1}}{\gamma}\Big)\Big) = O(|v|_\omega) \tag{5.11}$$

because $v \in \mathcal{D}_\rho$. Hence, by remark 4.3, $|v|_{\omega,3}^{p-1}/\gamma \le 1/2\widetilde{C}$ provided the constants C_{10}, C_{11}, C_{12}, C_{13} have been chosen small enough.

6 Appendix

Proof of lemma 2.1. By standard results, if $u \in L^\infty \cap H^1(\mathbf{T} \times (0,\pi))$, then $f(u) \in H^1(\mathbf{T} \times (0,\pi))$ and $(f(u))_x \equiv f'(u)u_x$, $(f(u))_t \equiv f'(u)u_t$. We get

$$|(f(u))_x|_{L^2} = \Big(\int_\Omega |(f(u))_x|^2\Big)^{1/2} = \Big(\int_\Omega |f'(u)|^2|u_x|^2\Big)^{1/2} \le |f'(u)|_\infty ||u||_{H^1} \le C|u|_\infty^{p-1}||u||_{H^1} \tag{6.1}$$

for $|u|_\infty$ small enough. An analogous estimate holds for $|(f(u))_t|_{L^2}$. Since $|f(u)| \le C|u|_\infty^p$, we obtain (2.4).

We now prove that $u \to f(u)$ is $C^1(X,X)$. It is easy to show that it is Gateaux differentiable and that $D_G f(u)[h] = f'(u)h$. Moreover with estimates similar as before we have that $u \to D_G f(u)$ is continuous. Hence the Nemitski operator $u \to f(u)$ is in $C^1(X,X)$ and its Frechet differential is $h \to Df(u)[h] = D_G f(u)[h] = f'(u)h$. Estimate (2.5) can be obtained in the same way as (2.4). ∎

Proof of lemma 2.2. Writing $w(t,x) = \sum_{l\ge 0, j\ge 1, j\ne l} w_{l,j}\cos(lt)\sin(jx) \in X$ we have that

$$L_\omega^{-1} w(t,x) = \sum_{l\ge 0, j\ge 1, j\ne l} \frac{w_{l,j}}{(\omega l - j)(\omega l + j)}\cos(lt)\sin(jx).$$

Since $\omega \in W_\gamma$, it is clear that $||L_\omega^{-1}w|| = O(||w||/\gamma)$.

We now prove that for all $w \in W$

$$S := \sum_{l\ge 0, j\ge 1, j\ne l} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)} \le C\Big(|w|_{L^2} + \frac{|\omega - 1|^{1/2}}{\gamma}||w||_{H^1}\Big). \tag{6.2}$$

Note that (6.2) implies

$$|L_\omega^{-1} w|_\infty \le S \le C\Big(|w|_{L^2} + \frac{|\omega - 1|^{1/2}}{\gamma}||w||_{H^1}\Big). \tag{6.3}$$

For $l \in \mathbf{N}$, let $e(l) \in \mathbf{N}$ be defined by $|e(l) - \omega l| = \min_{j\in\mathbf{N}} |j - \omega l|$. Since ω is not rational, $e(l)$ is the only integer e such that $|e - \omega l| < 1/2$. Let us define

$$S_1 = \sum_{l\ge 0, j\ge 1, j\ne l, j\ne e(l)} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)}$$

and

$$S_2 = \sum_{l\ge 0, e(l)\ne l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))}.$$

For $j \ne e(l)$ we have that

$$|\omega l - j| \ge \frac{|j-e(l)|}{2} \qquad \text{and} \qquad \omega l + j \ge \frac{|j-e(l)|+l}{4}. \tag{6.4}$$

Indeed, for $j \ne e(l)$ we have that $|j-\omega l| \ge |j-e(l)| - |e(l)-\omega l| \ge |j-e(l)| - 1/2 \ge |j-e(l)|/2$. Moreover, since $|e(l)-\omega l| < 1/2$, it is easy to see (remember that $\omega \ge 1/2$) that $e(l)+l \le 4\omega l$ and hence $|j-e(l)|+l \le j+e(l)+l \le 4(j+\omega l)$. Defining $w_{l,j}$ by $w_{l,j}=0$ if $j\le 0$ or $j=l$, we get from (6.4)

$$S_1 \le \sum_{l\ge 0, j\in\mathbf{Z}, j\ne e(l)} \frac{8|w_{l,j}|}{|j-e(l)|(|j-e(l)|+l)}.$$

Hence, by the Cauchy-Schwarz inequality, $S_1 \le 8R_1|w|_{L^2}$, where

$$R_1^2 = \sum_{l\ge 0, j\in\mathbf{Z}, j\ne e(l)} \frac{1}{(j-e(l))^2(|j-e(l)|+l)^2} = \sum_{l\ge 0, j\in\mathbf{Z}, j\ne 0} \frac{1}{j^2(|j|+l)^2} \le \sum_{l\ge 0, j\in\mathbf{Z}, j\ne 0} \frac{1}{j^2(1+l)^2} < \infty.$$

In order to find an upper bound for S_2, we observe that if $l < 1/(2|\omega-1|)$ then $|\omega l - l| < 1/2$ and then $e(l)=l$. Hence we can write

$$S_2 = \sum_{l\ge 1/2|\omega-1|, e(l)\ne l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))}.$$

By the properties of $\omega \in W_\gamma$, for $l \ne e(l)$, $|\omega l - e(l)||\omega l + e(l)| \ge \gamma(\omega l + e(l))l^{-1} \ge \gamma$. Hence, still by the Cauchy-Schwarz inequality,

$$S_2 \le \frac{1}{\gamma} \sum_{l\ge 1/2|\omega-1|, e(l)\ne l} |w_{l,e(l)}| \le \frac{2R_2}{\gamma}||w||_{H^1},$$

where

$$R_2^2 = \sum_{l\ge 1/2|\omega-1|} \frac{1}{l^2} \le 2|\omega-1|.$$

The estimates $|L_\omega^{-1}w|_\omega \le (C/\gamma)|w|_\omega$ $\forall w \in W$ and $|L_\omega^{-1}\Pi_W u|_\omega \le (C_3/\gamma)|u|_\omega$ $\forall u \in X$ are an immediate consequence of the bound of $||L_\omega^{-1}w||$ and (6.3), since $|\Pi_W u|_{L_2} \le |u|_{L_2}$ and $||\Pi_W u|| \le ||u||$ For the estimate of $|L_\omega^{-1}w|_{L^2}$ ($w \in W$), we use the decomposition $|L_\omega^{-1}w|_{L^2}^2 = S_1' + S_2'$, where

$$S_1' := \sum_{j>0, l\ge 0, j\ne l, j\ne e(l)} \frac{w_{l,j}^2}{(\omega l - j)^2(\omega l + j)^2}, \qquad S_2' := \sum_{l\ge 1/2|\omega-1|, e(l)\ne l} \frac{w_{l,e(l)}^2}{(\omega l - e(l))^2(\omega l + e(l))^2}.$$

Recall that if $l < 1/2|\omega-1|$ then $e(l)=l$. Arguing as before it is obvious that $S_1' \le C|w|_{L^2}^2$. We have

$$S_2' \le \sum_{l\ge 1/2|\omega-1|, e(l)\ne l} \frac{w_{l,e(l)}^2}{\gamma^2} \le \frac{4(\omega-1)^2}{\gamma^2} \sum_{l\ge 1/2|\omega-1|, e(l)\ne l} l^2 w_{l,e(l)}^2 = O\Big(\frac{\varepsilon^2}{\gamma^2}||w||_{H^1}^2\Big) = O\Big(\frac{\varepsilon}{\gamma^2}|w|_\omega^2\Big),$$

which yields estimate (2.14). We now prove (2.12). Writing $r(t,x) = \sum_{l\ge 0, j\ge 1} r_{l,j}\cos(lt)\sin(jx)$, $s(t,x) = \sum_{l\ge 0, j\ge 1} s_{l,j}\cos(lt)\sin(jx)$ we get

$$L_1^{-1}\Pi_W s = \sum_{j\ne l} \frac{s_{l,j}}{l^2-j^2}\cos(lt)\sin(jx), \qquad L_\omega^{-1}s = \sum_{j\ne l} \frac{s_{l,j}}{\omega^2 l^2 - j^2}\cos(lt)\sin(jx)$$

and

$$S := \int r(L_\omega^{-1} - L^{-1})(\Pi_W s)\ dt\ dx = \pi^2 \sum_{j\neq l} \frac{s_{l,j} r_{l,j}\ \varepsilon l^2}{(\omega^2 l^2 - j^2)(l^2 - j^2)} = \pi^2 (S_1 + S_2),$$

where

$$S_1 := \sum_{j\neq l, j\neq e(l)} \frac{s_{l,j} r_{l,j}\ \varepsilon l^2}{(\omega l - j)(\omega l + j)(l - j)(l + j)}, \qquad S_2 := \sum_{l \neq e(l)} \frac{s_{l,e(l)} r_{l,e(l)}\ \varepsilon l^2}{(\omega l - e(l))(\omega l + e(l))(l - e(l))(l + e(l))}.$$

For $j \neq e(l)$ and $j \neq l$ we have that $|\omega l - j| \geq |j - e(l)|/2 \geq 1/2$ hence $|(\omega l - j)(\omega l + j)(l - j)(l + j)| \geq (1/2)\omega l^2$. We obtain

$$|S_1| \leq \frac{2\varepsilon}{\omega} \sum_{j\neq l} |r_{l,j}||s_{l,j}| = O\Big(\frac{2\varepsilon}{\omega} |r|_{L^2} |s|_{L^2}\Big) = O\Big(\frac{2\varepsilon}{\omega} |r|_\omega |s|_\omega\Big).$$

As before, for $\omega \in W_\gamma$ and $l \neq e(l)$, we have $|\omega l - e(l)| \geq \gamma/l$ and so $|(\omega l - e(l))(\omega l + e(l))(l - e(l))(l + e(l)| \geq \gamma l$. Moreover $l \neq e(l)$ implies $l \geq 1/(2|\omega - 1|)$. Hence

$$|S_2| \leq C \frac{\varepsilon^2}{\gamma} \sum_{l \geq 1/(2\varepsilon)} |r_{l,e(l)}||s_{l,e(l)}| l^2 = O\Big(\frac{\varepsilon^2}{\gamma} ||r||_{H^1} ||s||_{H^1}\Big).$$

Estimate (2.12) follows straightforwardly from the above estimates of S_1 and S_2. Finally (2.13) is an immediate consequence of (2.12) since, for all $(r, s) \in X \times X$,

$$\int_\Omega r L^{-1}(\Pi_W s) = O(|r|_{L^2} |s|_{L^2}) = O(|r|_\omega |s|_\omega).$$

■

Proof of lemma 2.3. It is sufficient to show that the operator $\mathcal{G} : W \to W$ defined by $\mathcal{G}(w) := L_\omega^{-1} \Pi_W f(v + w)$ is a contraction on a ball $B_\delta \subset W$ for some δ if $|v|_\omega^{p-1}/\gamma$ is small enough. Since, by lemma 2.2, $||L_\omega^{-1} \Pi_W||_\omega \leq C_3/\gamma$ and by (2.4), we have that, for all $w \in B_\delta$

$$|\mathcal{G}(w)|_\omega \leq \frac{C_3}{\gamma} |f(v + w)|_\omega \leq \frac{C}{\gamma} |v + w|_\omega^p \leq \frac{C}{\gamma} \Big(|v|_\omega + \delta\Big)^p \leq \frac{\overline{C}}{\gamma} \Big(|v|_\omega^p + \delta^p\Big). \tag{6.5}$$

for some $\overline{C} > 0$. Moreover by lemma 2.1 the operator $\mathcal{G}$ is differentiable and $D\mathcal{G}(w)[h] = L_\omega^{-1} \Pi_W (f'(v + w)h)$. Hence by (6.3),

$$|D\mathcal{G}(w)[h]|_\omega \leq \frac{C_3}{\gamma} |f'(v + w)h|_\omega \leq \frac{C}{\gamma} |v + w|_\omega^{p-1} |h|_\omega \leq \frac{\widetilde{C}}{\gamma} \Big(|v|_\omega^{p-1} + \delta^{p-1}\Big) |h|_\omega \tag{6.6}$$

for some $\widetilde{C} > 0$. Let us choose $\delta := |v|_\omega$ and $\rho := (4 \max(\overline{C}, \widetilde{C}))^{-1}$. For $v \in \mathcal{D}_\rho$, $(\overline{C}/\gamma)(|v|_\omega^p + \delta^p) \leq \delta/2$ and $(\widetilde{C}/\gamma)(|v|_\omega^{p-1} + \delta^{p-1}) \leq 1/2$. Hence $\mathcal{G}$ maps B_δ into the closed ball $\overline{B}_{\delta/2}$ and is a contraction on B_δ. By the Contraction Mapping Theorem there is a unique $w(v)$ in B_δ such that $\mathcal{G}(w(v)) = w(v)$. We have

$$|w(v)|_\omega = |\mathcal{G}(w(v))|_\omega \leq \frac{2\overline{C}}{\gamma} |v|_\omega^p \tag{6.7}$$

by (6.5) and the choice of δ. Moreover

$$|w(v)|_{L^2} = |L_\omega^{-1} \Pi_W f(v + w(v))|_{L^2} \leq C\Big(1 + \frac{\sqrt{\varepsilon}}{\gamma}\Big) |f(v + w)|_\omega \leq C\Big(1 + \frac{\sqrt{\varepsilon}}{\gamma}\Big) |v|_\omega^p,$$

by (2.14) and (6.7). In addition

$$\begin{aligned}\Big|\int_\Omega r\Big(w(v)-L_\omega^{-1}\Pi_W f(v)\Big)\Big| &= \Big|\int_\Omega r\ L_\omega^{-1}\Pi_W(f(v+w(v))-f(v))\Big|\\ &\le C_4\Big(1+\frac{\varepsilon}{\gamma}\Big)|f(v+w(v))-f(v)|_\omega |r|_\omega\\ &\le C\Big(1+\frac{\varepsilon}{\gamma}\Big)(|v|_\omega^{p-1}|w(v)|_\omega)|r|_\omega\\ &\le \frac{C}{\gamma}\Big(1+\frac{\varepsilon}{\gamma}\Big)|v|_\omega^{2p-1}|r|_\omega,\end{aligned}$$

by (2.13) and the bound on $|w(v)|_\omega$ given in (6.7). This proves $ii)$.

In order to prove $iii)$, let us call $\mathcal{I}: X\to X$ the linear operator defined by $\mathcal{I}u(t,x)=u(t+\pi,\pi-x)$. It is easy to see that $\mathcal{I}$ and $L_\omega^{-1}\Pi_W$ commute. Hence, since $-v=\mathcal{I}v$,

$$w(v)=L_\omega^{-1}\Pi_W f(v+w(v)) \ \Rightarrow\ \mathcal{I}w(v)=L_\omega^{-1}\Pi_W\mathcal{I}f(v+w(v))=L_\omega^{-1}\Pi_W f(-v+\mathcal{I}w(v)).$$

This implies, by the uniqueness of the solution of $w=L_\omega^{-1}\Pi_W f(-v+w)$ in the ball of radius $|v|_\omega$, that $w(v)(t+\pi,\pi-x)=w(-v)(t,x)$.

For $iv)$, we remark that if $v\in V_n$, then $w(v)$ is the unique solution in an appropriate ball of the equation $\mathcal{G}(w)=w$, where the map $\mathcal{G}$ satisfies $\mathcal{G}(W_n)\subset W_n$ (W_n is the closed subspace of W containing $w\in W$ which are $2\pi/n$ periodic in time). Since $\mathcal{G}^k(0)\to w(v)$ in W, this implies that $w(v)\in W_n$.

Finally, the map $(v,w)\to\mathcal{G}(v;w):=w-L_\omega^{-1}\Pi_W f(v+w)$ is of class C^1 and its differential with respect to w is invertible at any point $(v,w(v))$ by lemma 2.1 and the previous bounds. As a consequence of the Implicit Function Theorem the map $v\to w(v)$ is in $C^1(\mathcal{D}_\rho,W)$. ■

Proof of lemma 4.7. We define α, a_1, a_2, $\widetilde{m}$ by $\alpha=\langle m\rangle$,

$$a_1(s_1):=\langle m(s_1,\cdot)-\alpha\rangle_{s_2}=\frac{1}{2\pi}\int_0^{2\pi}m(s_1,s_2)-\alpha\,ds_2,\ \ a_2(s_2):=\langle m(\cdot,s_2)-\alpha\rangle_{s_1}=\frac{1}{2\pi}\int_0^{2\pi}m(s_1,s_2)-\alpha\,ds_1$$

and $\widetilde{m}(s_1,s_2):=m(s_1,s_2)-a_1(s_1)-a_2(s_2)-\alpha$. It is straightforward to check that $\langle a_i\rangle=0$, $\langle\widetilde{m}\rangle_{s_2}(s_1)=0$, $\langle\widetilde{m}\rangle_{s_1}(s_2)=0$. We have to prove that $a_1=a_2$. Since $w\in W$, for all p odd and 2π periodic,

$$\int w(t,x)(p(t+x)-p(t-x))\ dt\ dx=\frac{1}{2}\int_{\mathbf{T}^2}m(s_1,s_2)(p(s_1)-p(s_2))\ ds_1\ ds_2=0.$$

By the definition of a_i, we obtain that for all p odd and 2π-periodic

$$\int_{\mathbf{T}}(a_1(s)-a_2(s))p(s)\ ds=0.$$

Therefore a_1-a_2 is even. Now $w(t,x)=w(-t,x)$ for all $t\in\mathbf{T}$ and for all $x\in[0,\pi]$, and this implies that $m(-s_2,-s_1)=m(s_1,s_2)$. As a consequence $a_2(s)=a_1(-s)$, so that a_1-a_2 is odd. Since a_1-a_2 is both odd and even, $a_1=a_2$.

We now turn to the expression of $\int wL^{-1}(w)$. Let us define β_1, β_2, β_3 on $[0,2\pi]^2$ by

$$\begin{aligned}\beta_1(s_1,s_2)&:=-M(s_1,s_2)+\tfrac{1}{2}(M(s_1,s_1)+M(s_2,s_2)),\\ \beta_2(s_1,s_2)&:=(s_1-s_2)(A(s_1)-A(s_2)),\\ \beta_3(s_1,s_2)&:=-\frac{\alpha}{8}(s_1-s_2)(2\pi-(s_1-s_2)),\end{aligned}$$

and B_1,B_2,B_3 on $[0,2\pi]\times[0,\pi]$ by $B_i(t,x):=\beta_i(t+x,t-x)$. As an immediate consequence of its definition, B_i is 2π-periodic w.r.t. t and for $x\in\{0,\pi\}$, $B_i(t,x)=0$. Hence $B_i\in X$. Moreover $-\partial_{s_1}\partial_{s_2}\beta_i=(1/4)\lambda_i$, where

$$\lambda_1(s_1,s_2):=\widetilde{m}(s_1,s_2),\qquad \lambda_2(s_1,s_2):=a(s_1)+a(s_2),\qquad \lambda_3(s_1,s_2):=\alpha.$$

This implies that $[-(\partial_t)^2+(\partial_x)^2]B_i = L_i$ where we have defined $L_i(t,x) := \lambda_i(t+x,t-x)$. As a result $B_1+B_2+B_3 \in L^{-1}w+V$ and

$$\int_{[0,2\pi]\times[0,\pi]} w(t,x)L^{-1}(w)(t,x)\ dt\ dx = \int_{[0,2\pi]\times[0,\pi]} (B_1+B_2+B_3)(t,x)(L_1+L_2+L_3)(t,x)\ dt\ dx.$$

We observe that, since B_i vanishes for $x=0$ and $x=\pi$, for all i,j,

$$\int B_iL_j\ dt\ dx = \int B_i\Big((B_j)_{tt}-(B_j)_{xx}\Big)\ dt\ dx = \int \Big((B_i)_{tt}-(B_i)_{xx}\Big)B_j\ dt\ dx = \int L_iB_j\ dt\ dx.$$

Hence

$$\int wL^{-1}w = \int_{[0,2\pi]\times[0,\pi]} B_1L_1 + 2\int B_1L_2 + 2\int B_1L_3 + \int B_2L_2 + 2\int B_2L_3 + \int B_3L_3.$$

We have

$$\int B_1L_1 = \frac{1}{2}\int_{\mathbf{T}^2}(-M(s_1,s_2)+\frac{1}{2}M(s_1,s_1)+\frac{1}{2}M(s_2,s_2))\widetilde{m}(s_1,s_2)\ ds_1\ ds_2 = -\frac{1}{2}\int_{\mathbf{T}^2} M(s_1,s_2)\widetilde{m}(s_1,s_2)\ ds_1\ ds_2$$

because $\langle\widetilde{m}\rangle_{s_i}=0$.

$$\int B_1L_2 = \frac{1}{2}\int_{\mathbf{T}^2}(-M(s_1,s_2)+\frac{1}{2}M(s_1,s_1)+\frac{1}{2}M(s_2,s_2))(a(s_1)+a(s_2))\ ds_1\ ds_2 = \pi\int_{\mathbf{T}} M(s,s)a(s)\ ds$$

because $\langle M\rangle_{s_i}=0$ and $\langle a\rangle = 0$.

$$\int B_1L_3 = \frac{\alpha}{2}\int_{\mathbf{T}^2}\Big(-M(s_1,s_2)+\frac{1}{2}M(s_1,s_1)+\frac{1}{2}M(s_2,s_2)\Big) = -\pi\alpha\int_{\mathbf{T}} M(s,s)\ ds$$

because $\langle M\rangle = 0$.

$$\begin{aligned}
\int B_2L_2 &= \int_0^{2\pi}\int_0^{\pi} 2x\Big(A(t+x)-A(t-x)\Big)\Big(a(t+x)+a(t-x)\Big)\ dt\ dx\\
&= \int_0^{\pi} dx\ 8x\int_0^{2\pi}\Big(A(t+x)-A(t-x)\Big)\Big(A'(t+x)+A'(t-x)\Big)\ dt\\
&= \int_0^{\pi} dx\ 8x\int_0^{2\pi}\frac{d}{dt}\Big[\frac{(A(t+x)-A(t-x))^2}{2}-A(t-x)^2\Big]+2A(t+x)A'(t-x)\ dt\\
&= \int_0^{\pi} dx\ 16x\int_0^{2\pi} A(t+x)A'(t-x)\ dt = \int_0^{\pi} dx\ 16x\int_0^{2\pi} A(s)A'(s-2x)\ ds\\
&= \int_0^{2\pi} ds\ A(s)\int_0^{\pi} 16xA'(s-2x)\ dx = \int_0^{2\pi} ds\ A(s)\Big[8xA(s-2x)\Big]_{x=\pi}^{x=0}\\
&= -8\pi\int_0^{2\pi} A(s)^2\ ds.
\end{aligned}$$

In the fore-last line, we have integrated by parts (w.r.t. x) and used the fact that $\langle A\rangle = 0$.

$$\int B_2L_3 = \alpha\int_0^{\pi} dx\ 2x\int_0^{2\pi}(A(t+x)-A(t-x))\ dt = 0,$$

still because $\langle A\rangle = 0$. At last

$$\int B_3L_3 = -\frac{\alpha}{8}\int_0^{2\pi}\int_0^{\pi} 2x(2\pi-2x)\ dt\ dx = -\frac{\alpha^2\pi^4}{6}$$

Summing up, we get (4.12). ■

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