C^*-algebras of anisotropic Schrödinger operators on trees

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Abstract

We study a C^*-algebra generated by "differential" operators on a tree. We give a complete description of its quotient with respect to the compact operators. This allows us to compute the essential spectrum of self-adjoint operators affiliated to the algebra. Results cover Schrödinger operators with highly anisotropic, possibly unbounded, potentials.

1 Introduction

Given a ν-fold tree Γ of origin e with its canonical metric d, we denote by $x \sim y$ when x and y are connected by an edge and we set $|x| = d(x,e)$. For each $x \in \Gamma \setminus \{e\}$, we denote by $x' \equiv x^{(1)}$ the unique element $y \sim x$ such that $|y| = |x| - 1$ and we set $x^{(p)} = (x^{(p-1)})'$ for $1 \leq p \leq |x|$. Let $x\Gamma = \{y \in \Gamma \mid y^{(|y|-|x|)} = x\}$.

On $\ell^2(\Gamma)$ we define the bounded operator ∂ given by $(\partial f)(x) = \sum_{y'=x} f(y)$. Its adjoint is given by $(\partial^* f)(e) = 0$ and $(\partial^* f)(x) = f(x')$ for $|x| \geq 1$. Let $\mathscr{D}$ be the C^*-algebra generated by ∂.

In order to obtain our algebra of potentials, we consider the "hyperbolic" compactification $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$ of Γ constructed as follows. An element x of the boundary at infinity $\partial\Gamma$ is a Γ-valued sequence $x = (x_n)_{n\in\mathbb{N}}$ such that $|x_n| = n$ and $x_{n+1} \sim x_n$ for all $n \in \mathbb{N}$. We set $|x| = \infty$ for $x \in \partial\Gamma$. The space $\widehat{\Gamma}$ is equipped with a natural ultrametric space structure. For $x \in \partial\Gamma$ and $(y_n)_{n\in\mathbb{N}}$ a sequence in Γ we have $\lim_{n\to\infty} y_n = x$ if for each $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for each $n \geq N$ we have $y_n \in x_m\Gamma$. We denote by $C(\widehat{\Gamma})$ the set of complex-valued continuous functions defined on $\widehat{\Gamma}$. Since Γ is dense in $\widehat{\Gamma}$, we can view $C(\widehat{\Gamma})$ as a C^*-subalgebra of $C_b(\Gamma)$, the algebra of bounded complex-valued functions defined on Γ. For $V \in C(\widehat{\Gamma})$, we denote by $V(Q)$ the operator of multiplication by V in $\ell^2(\Gamma)$.

Let us now denote by $\mathscr{C}(\widehat{\Gamma})$ the C^*-algebra generated by $\mathscr{D}$ and $C(\widehat{\Gamma})$. It contains the compact operators of $\ell^2(\Gamma)$. Following the strategy exposed in [GI], we shall first compute its quotient with respect to the ideal of compact operators. Then, we shall use the following theorem in order to calculate the essential spectrum of self-adjoint operators related to $\mathscr{C}(\widehat{\Gamma})$. In this introduction we restrict ourselves to the most important case when $\nu > 1$.

Theorem 1.1 *Let* $\nu > 1$. *There is a unique morphism* $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ *such that* $\Phi(D) = D \otimes 1$ *for all* $D \in \mathscr{D}$ *and* $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. *This morphism is surjective and its kernel is* $\mathbb{K}(\Gamma)$.

We now give examples. Let $\nu > 1$ and let $H = \sum_{\alpha,\beta} a_{\alpha,\beta}(Q)\partial^{*\alpha}\partial^{\beta} + K$, where $a_{\alpha,\beta} \in C(\widehat{\Gamma})$, $a_{\alpha,\beta} = 0$ for almost all $(\alpha,\beta) \in \mathbb{N}^2$ and K is a compact operator. Clearly $H \in \mathscr{C}(\widehat{\Gamma})$ and $\Phi(H) = \sum_{\alpha,\beta} \partial^{*\alpha}\partial^{\beta} \otimes (a_{\alpha,\beta})|_{\partial\Gamma}$. As a consequence of the Theorem 1.1, if H self-adjoint, its essential spectrum is given by:

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{\gamma\in\partial\Gamma} \sigma\big(\sum_{\alpha,\beta} a_{\alpha,\beta}(\gamma)\partial^{*\alpha}\partial^{\beta}\big).$$

This result can made quite explicit in the particular case of a Shrödinger operator $H = \Delta + V(Q)$ with potential V in $C(\widehat{\Gamma})$. We recall that Δ is a bounded operator on $\ell^2(\Gamma)$ defined by $(\Delta f)(x) = \sum_{y\sim x}(f(y) - f(x))$. It belongs to $\mathscr{C}(\widehat{\Gamma})$. We then set $\Delta_0 = \partial + \partial^* - \nu\mathrm{Id}$ (which belongs to $\mathscr{D}$) and notice that $\Delta - \Delta_0$ is compact. We get (see [AF] for instance):

$$\sigma_{ess}(\partial + \partial^*) = \sigma_{ac}(\partial + \partial^*) = \sigma(\partial + \partial^*) = [-2\sqrt{\nu}, 2\sqrt{\nu}],$$

where $\sigma_{ac}(T)$ denotes the absolute continuous part of the spectrum of a given self-adjoint operator T. Notice that Theorem 1.1 gives us directly that

$\sigma_{\mathrm{ess}}(\partial^* + \partial) = \sigma(\partial^* + \partial)$. Finally, we obtain

$$\begin{aligned}\sigma_{ess}(\Delta + V(Q)) &= \sigma(\Delta_0) + V(\partial\Gamma)\\ &= [-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\partial\Gamma).\end{aligned}$$

In fact this result holds (and is trivial) in the case of $\nu = 1$, when $\Gamma = \mathbb{N}$.

Given a continuous function on $\partial\Gamma$, the Tietze theorem allows us to extend it to a continuous function on $\widehat{\Gamma}$, so one may construct a large class of Hamiltonians with given essential spectra. Nevertheless we are able to point out a concrete class of non-trivial potentials $V \in C(\widehat{\Gamma})$ with uniform behaviour at infinity which form a dense family of $C(\widehat{\Gamma})$. Namely, for each bounded function $f : \Gamma \to \mathbb{R}$ and each real $\alpha > 1$ let

$$V(x) = \sum_{k=1}^{|x|} \frac{f(x_k)}{k^\alpha}, \tag{1.1}$$

where $x_k = x^{|x|-k}$ for $x \in \Gamma$ (V belongs to $C(\widehat{\Gamma})$ because of Lemma 2.3).

Concerning finer spectral features, based mainly on the Mourre estimate, we mention that in the case $H = \Delta + V(Q)$, with V as in (1.1) where $\alpha \geq 3$ and such that $V(\partial\Gamma) = 0$, the results of [AF] can be applied (the hypotheses of the Lemmas 6 and 7 from [AF] are verified since $V(x) = O(|x|^{-\alpha+1})$ when $|x| \to \infty$). The aim of our work in preparation [Go] is to prove that the Mourre estimate holds for more general classes of Hamiltonians affiliated to $\mathscr{C}(\widehat{\Gamma})$ and to develop a scattering theory for them.

The preceding results on trees allow us to treat more general graphs. We recall that a graph is said to be *connected* if two of its elements can be joined by a sequence of neighbours. Let G be a finite disjoint union of Γ_i, each Γ_i being a ν_i-fold branching tree with $\nu_i \geq 1$ and of G_0 a compact connected graph:

$$G = \bigcup_{i=1}^{n} \Gamma_i \bigcup G_0.$$

We endow G with a connected graph structure that respects the graph structure of Γ_i and the one of G_0 and such that Γ_i is connected to Γ_j $(i \neq j)$ only through G_0 and such that Γ_i is connected to G_0 only through e_i, the origin of Γ_i. The graph G is hyperbolic and its boundary at infinity ∂G is exactly the disjoint union $\cup_{i=1}^{n} \partial\Gamma_i$. We now choose $V \in C(G \cup \partial G)$. One has

$V|_{\widehat{\Gamma}_i} \in C(\widehat{\Gamma}_i)$ for all $i = 1, \ldots, n$ and we easily obtain the next result:

$$\sigma_{ess}(\Delta + V(Q)) = \bigcup_{i=1}^{n} \big([-\nu_i - 2\sqrt{\nu_i}, -\nu_i + 2\sqrt{\nu_i}\,] + V(\partial\Gamma_i) \big).$$

This covers in particular the case of the Cayley graph of a free group with finite system of generators. We recall that the Cayley graph of a group G with a system of generators S is the graph defined on the set G with the relation $x \sim y$ if $xy^{-1} \in S$ or $yx^{-1} \in S$. Let G be a free group with a system of generators S such that $S = S^{-1}$. We denote by e its neutral element and we set $|S| = \nu + 1$. One may associate the restriction of the Cayley graph to the set of words starting with a given generator with a ν-fold branching tree having as origin the generator. Hence, the Cayley graph of G will be $\cup_{i=1}^{\nu} \Gamma_i \cup \{e\}$ where Γ_i is a ν-fold branching tree with the above graph structure.

We now go further by taking $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ such that $V(\Gamma) \subset \mathbb{R}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the Alexandrov compactification of $\mathbb{R}$. More precisely, $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ if and only if for each $\gamma \in \partial\Gamma$ we have either $\lim_{x \to \gamma} V(x) = l$ where $l \in \mathbb{R}$ or for each $M \geq 0$ there is $N \in \mathbb{N}$ such that $|V(x)| \geq M$ for all $n \geq N$ and $x \in \gamma_n\Gamma$ (see Lemma 2.3). We set

$$D(V) = \{f \in \ell^2(\Gamma) \mid \|V(Q)f\|^2 < \infty\}.$$

Let $T \in \mathscr{D}$ and $T_0 = \Phi(T)$. Since T is bounded, the operator $H = T + V(Q)$ with domain $D(V)$ is self-adjoint and it is affiliated to $\mathscr{C}(\widehat{\Gamma})$ (i.e. its resolvent belongs to $\mathscr{C}(\widehat{\Gamma})$). Indeed, for each $z \in \mathbb{C} \setminus \mathbb{R}$ we have $(V(Q) + z)^{-1} \in C(\widehat{\Gamma})$ and for large $z \in \mathbb{C} \setminus \mathbb{R}$ one has

$$(H + z)^{-1} = (V(Q) + z)^{-1} \sum_{n \geq 0} (T(V(Q) + z)^{-1})^n,$$

where the series is norm convergent. Now, with the same z, we use the Theorem 1.1 and the fact that $\mathscr{D} \otimes C(\partial\Gamma) \simeq C(\partial\Gamma, \mathscr{D})$ to obtain

$$\Phi_\gamma((H + z)^{-1}) \equiv \Phi((H + z)^{-1})(\gamma) = (V(\gamma) + z)^{-1} \sum_{n \geq 0} (T_0(V(\gamma) + z)^{-1})^n.$$

Note that $(V(\gamma) + z)^{-1} = 0$ if $V(\gamma) = \infty$. By analytic continuation we get that for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$\Phi_\gamma(T + V(Q) + z)^{-1} = (T_0 + V(\gamma) + z)^{-1},$$

with the convention that $(T_0 + V(\gamma) + z)^{-1} = 0$ if $V(\gamma) = \infty$.

We now compute the essential spectrum of H. If $V(\gamma) = \infty$, we have $\sigma(\Phi_\gamma(H)) = \emptyset$. Otherwise, one has $\sigma(\Phi_\gamma(H)) = \sigma(T_0 + V(\gamma)) = \sigma(T_0) + V(\gamma)$. Hence we obtain:

$$\sigma_{\text{ess}}(T + V(Q)) = \sigma(T_0) + V(\partial\Gamma_0),$$

where $\partial\Gamma_0$ is the set of $\gamma \in \partial\Gamma$ such that $V(\gamma) \in \mathbb{R}$.

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2 Trees and related objects

2.1. The free monoïd Γ. Let $\mathscr{A}$ be a finite set consisting of ν objects. Let Γ be the free monoïd over $\mathscr{A}$; its elements are *words* and those of $\mathscr{A}$ *letters*. We refer to [Bo, Chapter I, §7] for a detailed discussion of these notions, but we recall that a word x is an $\mathscr{A}$-valued map defined on a set of the form $[\![1, n]\!]$ with $n \in \mathbb{N}$ (we use the notation[1] $[\![1, n]\!] = [1, n] \cap \mathbb{N}$), $x(i)$ being the i-th letter of the word x. The integer n (the number of letters of x) is the length of the word and will be denoted $|x|$. There is a unique word e of length 0, its domain being the empty set. This is the neutral element of $\mathscr{A}$. We will also identify $\mathscr{A}$ with the set of words of length 1.

The monoïd Γ will be endowed with the discrete topology. If $x \in \Gamma$, we denote $x\Gamma$ and Γx the right and left ideals generated by x. We have on Γ a canonical order relation which is by definition:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

We say that x *covers* y if $x \leq y$ and $|y| = |x| + 1$. Each element $x \in \Gamma \setminus \{e\}$ is covered by a unique element x', its *father* and each element $x \in \Gamma$ covers ν elements, its *sons*. Observe that the set of sons of x is $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. We notice that:

$$x \text{ covers } y \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}.$$

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[1] We stress that $\mathbb{N}$ is the set of integers ≥ 0 and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $|x| \geq n$, we define $x^{(n)}$ inductively by setting $x^{(0)} = x$ and $x^{(m+1)} = (x^{(m)})'$ for $m \leq n-1$. One may notice that:

$$|x^{(\alpha)}| = |x| - \alpha, \text{ if } \alpha \leq |x|$$

and for $\alpha \leq |ab|$

$$\begin{aligned} (ab)^{(\alpha)} &= ab^{(\alpha)}, \text{ if } \alpha \leq |b| \\ &= a^{(\alpha - |b|)}, \text{ if } \alpha \geq |b|. \end{aligned}$$

We remark that if $\nu = 1$ then $\Gamma = \mathbb{N}$ and if $\nu > 1$ then Γ is the set of monoms in ν non-commutative variables.

2.2. The tree Γ. A graph is a couple $G = (V, E)$, where V is a set (of *vertices*) and E is a set of pairs of elements of V (the *edges*). If x and y are joined by an edge, we say that they are *neighbours* and we abbreviate $x \sim y$. The graph structure allows one to endow V with a canonical metric d, where $d(x, y)$ is the length of the shortest path in G joining x to y.

The graph G_Γ associated to the monoïd Γ is defined as follows: $V = \Gamma$ and $x \sim y$ if x covers y or y covers x. It is usual to identify Γ and G_Γ, the so-called ν-fold branching tree. For all $x \in \Gamma$, we have $|x| = d(e, x)$. We will set $B(x, r) = \{y \in \Gamma \mid d(x, y) < r\}$ and $S^n = \{x \in \Gamma \mid |x| = n\}$.

2.3. Extended tree associated to $\mathscr{A}$. We shall define the extended tree by mimicking the definition of the free monoïd over $\mathscr{A}$. For each integer r, we set $\mathbb{Z}_r = \{i \in \mathbb{Z} \mid i \leq r\}$. The *extended tree* $\widetilde{\Gamma}$ associated to $\mathscr{A}$ is the set of $\mathscr{A}$-valued maps defined on sets of the form $\mathbb{Z}_r$. Hence $x \in \widetilde{\Gamma}$ if and only if there is $r \in \mathbb{Z}$ such that x is a map $\mathbb{Z}_r \to \mathscr{A}$; this uniquely defined integer r will be denoted $|x|$ and will be called *length* of x.

In order to embed Γ into $\widetilde{\Gamma}$, we choose $o \in \mathscr{A}$; this element will be fixed from now on. We set $\Gamma_0 = \{x \mid |x| \geq 0 \text{ and } x(i) = o \text{ if } i \leq 0\}$ and we identify Γ with Γ_0. More precisely, if $x \in \Gamma$ then the corresponding element of Γ_0 is defined on $\mathbb{Z}_{|x|}$ by extending x with $x(i) = o$ if $i \leq 0$. The element e will be identified with the map $e \in \widetilde{\Gamma}$ such that $|e| = 0$ and $e(i) = o$, $\forall i \leq 0$. Notice that the two notions of length are consistent on Γ.

There is a natural right action of Γ on $\widetilde{\Gamma}$ by concatenation, i.e. for $x \in \Gamma$ and $y \in \widetilde{\Gamma}$, yx will be the function z defined on $\mathbb{Z}_{|y|+|x|}$ such that $z|_{\mathbb{Z}_{|y|}}$ and $z(|y| + i) = x(i)$ for $i = 1, \ldots, |x|$. Then we equip $\widetilde{\Gamma}$ with an order relation by setting:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

Clearly x covers y if and only if $x \leq y$ and $|y| = |x| + 1$. Each element $x \in \widetilde{\Gamma}$ is covered by a unique element x', its *father*, and each element $x \in \Gamma$ covers ν elements, its *sons*, namely the elements of $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. We still have that:

$$x \text{ covers } y \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}.$$

Observe that $x' = x|_{\mathbb{Z}_{|x|-1}}$. We will set $x^{(\alpha)} = x|_{\mathbb{Z}_{|x|-\alpha}}$.

Let $G_{\widetilde{\Gamma}}$ be the graph such that $\widetilde{\Gamma}$ is the set of vertices and there is an edge between x, $y \in \widetilde{\Gamma}$ if x covers y or y covers x. As before, we shall not distinguish between $G_{\widetilde{\Gamma}}$ and $\widetilde{\Gamma}$.

2.4. The boundary at infinity of Γ. We shall see in the ending remark of this subsection that the boundary at infinity of Γ can be thought as the boundary of a 0-hyperbolic space in the sense of Gromov. We prefer, however, to give a simpler presentation that is closer to the theory of p-adic numbers (see [Ro] for instance). In fact, if ν is prime the boundary will be the set of ν-adic integers.

Definition 2.1 *The boundary at infinity of* Γ *is the set* $\partial\Gamma = \{x : \mathbb{N}^* \to \mathscr{A}\}$. *For* $x \in \partial\Gamma$, *we set* $|x| = \infty$.

Let $\widehat{\Gamma}$ be $\Gamma \cup \partial\Gamma$. For $x \in \widehat{\Gamma}$, we define the sequence $(x_n)_{n \leq |x|}$ with values in Γ by setting $x_0 = e$ and $x_n = x|_{\llbracket 1,n \rrbracket}$ for $n \geq 1$. Observe that the map $x \mapsto (x_n)_{n \leq |x|}$ is injective.

There is a natural left action of Γ on $\widehat{\Gamma}$. For $x \in \Gamma$ and $y \in \widehat{\Gamma}$, xy will be defined on $\llbracket 1, |x| + |y| \rrbracket$ by $x(i)$ for $i \leq |x|$ and by $y(i - |x|)$ for $i > |x|$ (we use the convention $\llbracket 1, \infty \rrbracket = ([1, \infty[\cap \mathbb{N}) \cup \{\infty\})$.

We will now equip $\widehat{\Gamma}$ with a structure of ultrametric space. We define a kind of valuation v on $\widehat{\Gamma} \times \widehat{\Gamma}$ by

$$v(x, y) = \max\{n \in \llbracket 0, \min(|x|, |y|) \rrbracket \mid x_n = y_n\} \tag{2.2}$$

with the convention that $v(x, x) = \infty$. If $x, y, z \in \widehat{\Gamma}$ it is easy to see that:

$$v(x, y) \geq \min(v(x, z), v(z, y)). \tag{2.3}$$

Let us set on $\widehat{\Gamma}$:

$$\widehat{d}(x, y) = \exp(-v(x, y)).$$

The relation (2.3) clearly implies that $(\widehat{\Gamma}, \widehat{d})$ is an ultrametric space, i.e. a metric space such that $\widehat{d}(x, y) \leq \max(\widehat{d}(x, z), \widehat{d}(z, y))$, for $x, y, z \in \widehat{\Gamma}$. We will denote by $\widehat{B}(x,r) = \{y \in \widehat{\Gamma} \mid \widehat{d}(x,y) < r\}$. Notice that ultrametricity implies that $\widehat{B}(x,r)$ is closed for all $x \in \widehat{\Gamma}$ and $r > 0$.

The topology induced by $\widehat{\Gamma}$ on Γ coincides with the initial topology of Γ, the discrete one. For $x \in \partial\Gamma$ and $n \in \mathbb{N}$, we have that

$$x_n\widehat{\Gamma} \;=\; \{y \in \widehat{\Gamma} \mid v(x,y) \geq n\} = \widehat{B}(x, \exp(-n+1))$$

is the closure of $x_n\Gamma$ in $\widehat{\Gamma}$. Hence $\{x_n\widehat{\Gamma}\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods in $\widehat{\Gamma}$ of x for $x \in \partial\Gamma$. Observe that $x\partial\Gamma = x\widehat{\Gamma} \cap \partial\Gamma$, for $x \in \Gamma$. We have the following property.

Proposition 2.2 Γ *and* $\partial\Gamma$ *are compact spaces.*

Proof: We have $\partial\Gamma = \mathscr{A}^{\mathbb{N}^*}$, so $\partial\Gamma$ endowed with the product topology is compact. In fact, this topology coincides with the one induced by the restriction of $\widehat{d}$ on $\partial\Gamma$. Indeed for $x \in \partial\Gamma$, the product topology gives us the same basis of neighbourhoods $\{x_n\partial\Gamma\}_{n\in\mathbb{N}}$ as $\widehat{d}|_{\partial\Gamma}$.

It remains to show that $\widehat{\Gamma}$ is compact. In fact, since $\partial\Gamma$ is compact, it suffices to remark that $\cup_{x\in\partial\Gamma}\widehat{B}(x, \exp(-k)) = \{y\widehat{\Gamma} \mid |y| = k+1\}$ has a finite complementary in $\widehat{\Gamma}$, for all $k \in \mathbb{N}$. This ends our proof. $\square$

Since Γ is dense in $\widehat{\Gamma}$, $\widehat{\Gamma}$ is a compactification of Γ. One may also notice that $\partial\Gamma$ is a perfect topological space if $\nu > 1$.

The C^*-algebra $C(\widehat{\Gamma})$ of continuous complex-valued functions on $\widehat{\Gamma}$ plays an important rôle in our developments. Notice that the dense embedding $\Gamma \subset \widehat{\Gamma}$ gives a canonical inclusion $C(\widehat{\Gamma}) \subset C_b(\Gamma)$, where $C_b(\Gamma)$ is the space of bounded complex-valued functions on Γ. Moreover, we will have

$$C_0(\Gamma) = \{f \in C(\widehat{\Gamma}) \mid f|_{\partial\Gamma} = 0\}, \tag{2.4}$$

where

$$C_0(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \forall\varepsilon > 0,\, \exists M > 0 \mid |x| > M \Rightarrow |f(x)| < \varepsilon\}.$$

We shall often abbreviate $C_0(\Gamma)$ by C_0.

The following proposition gives us a better understanding of the functions in $C(\widehat{\Gamma})$.

Proposition 2.3 *Let E be a metrisable topological space. A function $V : \Gamma \to E$ extends to a continuous function $\widehat{V} : \widehat{\Gamma} \to E$ if and only if for each $x \in \partial\Gamma$ the limit of $V(y)$, when $y \in \Gamma$ converges to x, exists.*

Proof: Let $x \in \partial\Gamma$ and $\widehat{V}(x)$ be the above limit. Let F be a closed neighbourhood of $\widehat{V}(x)$ in E; there is k such that $V(x_k\Gamma) \subset F$. Then $x_k\widehat{\Gamma}$ is a neighbourhood of x in $\widehat{\Gamma}$ and, since F is closed, we have $\widehat{V}(x_k\widehat{\Gamma}) \subset F$. $\square$

Later on, we will need the next ultrametricity result. We will say that $\mathscr{U} = \{x_i\Gamma\}$ is a *covering of* $\partial\Gamma$ if $\widehat{\mathscr{U}} = \{x_i\widehat{\Gamma}\}$ is a covering of $\partial\Gamma$.

Proposition 2.4 *For each open covering $\{\mathscr{O}_i\}_{i\in I}$ of $\partial\Gamma$, there is a disjoint and finite covering $\{x_j\Gamma\}_{j\in J}$ of $\partial\Gamma$ such that for each $j \in J$ there is $i \in I$ such that $x_j\widehat{\Gamma} \subset \mathscr{O}_i$.*

Proof: For each $x \in \partial\Gamma$ there is i such that x belongs to the open set $\mathscr{O}_i$ and there is $n = n(x,i)$ such that $x_n\widehat{\Gamma} \subset \mathscr{O}_i$. From the compactness of $\partial\Gamma$, we obtain a finite covering of $\partial\Gamma$ by sets of the form $\{y_j\widehat{\Gamma}\}_{j=1,\dots,m}$ which is a sub-cover of $\{\mathscr{O}_i\}_{i\in I}$. Now since in ultrametric spaces two balls are either disjoint or one of them is included in the other one, and since $\{y_j\widehat{\Gamma}\}$ are balls, we get the result. One may also choose $\{y\widehat{\Gamma} \mid |y| = \max_{j=1,\dots,m}|y_j|\}$ as the required covering. $\square$

Remark: This section can be reread from the perspective of hyperbolicity in the sense of Gromov, see [An, Chapter V] (a deeper investigation can be found in [CDP] and [GH]). Let (M,d) be a metric space. For $x, y \in M$ and a given $O \in M$, we define the *Gromov product* as being:

$$(x,y)_O = \frac{1}{2}(d(O,x) + d(O,y) - d(x,y)). \tag{2.5}$$

The metric space (M,d) is said to be *δ-hyperbolic* if there is δ such that, for all $x, y, z, O \in M$,

$$(x,y)_O \geq \min((x,z)_O, (z,y)_O) - \delta. \tag{2.6}$$

A metric space is *hyperbolic* if it is δ-hyperbolic for a certain δ. In fact, if there is δ such that (2.6) holds for all $x, y, z \in M$ and a given O then (M,d) is 2δ-hyperbolic. Classical examples of 0-hyperbolic spaces are trees (connected graphs with no cycle) and real trees (see [GH] for this notion). Cartan-Hadamard manifolds, the Poincaré half-plane and, more generally, complete simply connected manifolds with sectional curvature bounded by $\kappa < 0$ are δ-hyperbolic spaces with $\delta > 0$.

We equip the set of sequences with values in M with the equivalence relation $(u_n)R(v_n)$ defined by the condition if $\lim_{(n,m)\to\infty}(u_n, v_m)_O = \infty$. The boundary at infinity ∂M is the set of equivalence classes. A basis of open set of ∂M is given by the sets

$$\widetilde{\mathscr{O}} = \{\gamma \in \partial M \mid \gamma \text{ is not associated with a sequence of } M \setminus \mathscr{O}\},$$

where $\mathscr{O}$ is an open of M. The boundary of 0-hyperbolic space is ultrametric.

In our context, if we drop the convention $v(x, x) = \infty$, our valuation (2.2) is exactly (2.5). Hence (2.3) implies that Γ is 0-hyperbolic. We define a *geodesic ray* as being $\gamma : \mathbb{N} \to \Gamma$ such that $|\gamma(n)| = n$ and $\gamma_{n+1} \sim \gamma_n$. Geodesic rays are representative elements of the above equivalence classes. The two notions of boundary at infinity are identified by setting $x_n = \gamma(n)$.

3 Operators in $\ell^2(\Gamma)$

3.1. Bounded and compact operators. We are interested in operators acting on the Hilbert space

$$\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \sum_{x\in\Gamma} |f(x)|^2 < \infty\}$$

endowed with the inner product: $\langle f, g\rangle = \sum_{x\in\Gamma} \overline{f(x)}g(x)$.

We embed $\Gamma \subset \ell^2(\Gamma)$ by identifying $x = \chi_{\{x\}}$, where χ_A is the characteristic function of the set A. Observe that Γ is the canonical orthonormal basis in $\ell^2(\Gamma)$.

We denote by $\mathbb{B}(\Gamma)$ (resp. $\mathbb{K}(\Gamma)$) the set of bounded (resp. compact) operators in $\ell^2(\Gamma)$. For $T \in \mathbb{B}(\Gamma)$, we will denote by T^* its adjoint. Given $A \subset \Gamma$ we denote by $\mathbf{1}_A$ the operator of multiplication by χ_A in $\ell^2(\Gamma)$ (this is an orthogonal projection). We have the following compacity criterion:

Proposition 3.1 *Let $T \in \mathbb{B}(\Gamma)$, then:*

$$T \in \mathbb{K}(\Gamma) \Leftrightarrow ||\mathbf{1}_{\geq r}T|| \underset{r\to\infty}{\longrightarrow} 0 \Leftrightarrow ||T\mathbf{1}_{\geq r}|| \underset{r\to\infty}{\longrightarrow} 0,$$

where $\mathbf{1}_{\geq r}$ is the orthogonal projection associated to the set $\{x \in \Gamma \mid |x| \geq r\}$.

Proof: The sequence of operators $\mathbf{1}_{\geq r}$ converges strongly to zero, so if $T \in \mathbb{K}(\Gamma)$ then $||\mathbf{1}_{\geq r}T|| \to 0$ and $||T\mathbf{1}_{\geq r}|| \to 0$. Reciprocally, if one has for example $||\mathbf{1}_{\geq r}T|| \to 0$, then T is the norm limit of the sequence of finite rank operators $\mathbf{1}_{B(e,r)}T$, hence is compact. $\square$

3.2. The operator ∂. We now extend $x \mapsto x'$ to a map $\ell^2(\Gamma) \to \ell^2(\Gamma)$. We set $e' = 0$. For $f \in \ell^2(\Gamma)$, we have $f = \sum_{x\in\Gamma} f(x)x$, and we define ∂f or f', the derivative of f, as follows:

$$(\partial f)(x) \equiv f'(x) = \sum_{x\in\Gamma} f(x)x' = \sum_{y'=x} f(y) = \sum_{y\in\widetilde{x}} f(y).$$

This gives a bounded operator $\partial : l^2(\Gamma) \to l^2(\Gamma)$. Indeed,

$$\begin{aligned} ||f'||^2 &= \sum_{x\in\Gamma} |f'(x)|^2 = \sum_{x\in\Gamma} |\sum_{y\in\widetilde{x}} f(y)|^2 \\ &\leq \nu \sum_{x\in\Gamma}\sum_{y\in\widetilde{x}} |f(y)|^2 \leq \nu \sum_{x\in\Gamma} |f(x)|^2 = \nu||f||^2. \end{aligned}$$

We prove that the adjoint ∂^* acts on $f \in \ell^2(\Gamma)$ as follows:

$$\partial^* f(x) = \chi_{\Gamma\setminus\{e\}}(x) f(x').$$

In fact

$$\begin{aligned} \langle \partial f, f\rangle &= \sum_{x\in\Gamma} \overline{(\partial f)(x)} f(x) = \sum_{x\in\Gamma} \overline{\sum_{y\in\widetilde{x}} f(y)} f(x) \\ &= \sum_{x\in\Gamma} \overline{f(x)\chi_{\Gamma\setminus\{e\}}(x)} f(x') = \langle f, \partial^* f\rangle. \end{aligned}$$

Observe that

$$\begin{aligned} ||\partial^* f||^2 &= \sum_{x\in\Gamma} \overline{(\partial^* f)(x)} \partial^* f(x) = \sum_{x\in\Gamma\setminus\{\emptyset\}} \overline{f(x')} f(x') \\ &= \nu \sum_{x\in\Gamma} \overline{f(x)} f(x) = \nu ||f||^2, \end{aligned}$$

hence $\partial^*/\sqrt{\nu}$ is isometric on $\ell^2(\Gamma)$, in other terms:

$$\partial\partial^* = \nu \mathrm{Id}. \tag{3.7}$$

This gives us $||\partial|| = ||\partial^*|| = \sqrt{\nu}$.

For $\alpha \in \mathbb{N}$ we set $f^{(\alpha)} = \partial^\alpha f$. Thus, if $x \in \Gamma$ then $x^{(\alpha)}$ is well defined in $\ell^2(\Gamma)$ and $x^{(\alpha)} = 0 \Leftrightarrow \alpha > |x|$. For $|x| \geq \alpha$ the notation is consistent with our old definition.

3.3. C^*-algebras of energy observables related to Γ. We summarize the method used in [GI] to study the essential spectrum of large families of operators.

Let $\mathscr{H}$ be a Hilbert space and H a bounded self-adjoint operator on $\mathscr{H}$. The essential spectrum $\sigma_{ess}(H)$ of H is the set of $\lambda \in \sigma(H)$ such that either λ is not isolated from the rest of the spectrum or it is an eigenvalue of infinite multiplicity. Let $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ be the Calkin C^*-algebra. We denote by $S \mapsto \widehat{S}$ the canonical surjection of $B(\mathscr{H})$ onto $C(\mathscr{H})$ and we recall that $\sigma_{ess}(H) = \sigma(\widehat{H})$ (this is a version of Weyl Theorem). If $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathscr{H})$ which contains the compacts, then one has a canonical embedding $\mathfrak{C}/K(\mathscr{H}) \subset C(\mathscr{H})$. Thus in order to determine the essential spectrum of an operator $H \in \mathfrak{C}$ it suffices to give a good description of the quotient $\mathfrak{C}/K(\mathscr{H})$ and to compute $\widehat{H}$ as element of this quotient. In fact, as explained in [GI], we can go further by taking H as an unbounded operator over $\mathscr{H}$ such that $(H+i)^{-1} \in \mathfrak{C}$. We shall apply this strategy in our context.

Let $\mathscr{D}_{\text{alg}}$ be the $*$-algebra of operators in $\ell^2(\Gamma)$ generated by ∂ and $\mathscr{D}$ the C^*-algebra of operators in $\ell^2(\Gamma)$ generated by ∂. Because of (3.7), $\mathscr{D}_{\text{alg}}$ is unital. We will denote by $\varphi(Q)$ the operator on multiplication by φ on $\ell^2(\Gamma)$. If C is a C^*-algebra of bounded function on Γ (i.e. $C \subset \ell^\infty(\Gamma)$ C^*-subalgebra), then C is embedded in $\mathbb{B}(\Gamma)$ by $\varphi \mapsto \varphi(Q)$. Let $\langle \mathscr{D}, C \rangle$ be the C^*-algebra generated by $\mathscr{D} \cup C$. In this paper we shall take $\mathfrak{C} = \langle \mathscr{D}, C \rangle$. This algebra contains many Hamiltonians of physical interest, for instance Shrödinger operators with potential in C. We recall that given a graph G, the Laplace operator acts on $\ell^2(G)$ as follows:

$$(\Delta f)(x) = \sum_{y \sim x} (f(y) - f(x)).$$

In our case $\Delta = \partial + \partial^* - \nu \mathrm{Id} + \chi_{\{e\}}$. If $\nu > 1$ then $\mathscr{D}$ does not contain any compact operator (see below), so Δ does not belong to $\mathscr{D}$. On the other hand, if $C \supset C_0$ and $V \in C$ then the Schrödinger operator $\Delta + V(Q)$ clearly belongs to $\langle \mathscr{D}, C \rangle$.

We now give a new description of $\mathbb{K}(\Gamma)$.

Proposition 3.2 *Let $\mathscr{C}_0$ be the C^*-algebra generated by $\mathscr{D} \cdot C_0$. One has:*

$$\mathscr{C}_0 = \mathbb{K}(\Gamma). \tag{3.8}$$

Proof: We have $\mathscr{C}_0 \subset \mathbb{K}(\Gamma)$ because $\varphi(Q)$ is compact for $\varphi \in C_0$ (indeed, we have $\|\mathbf{1}_{\geq n}\varphi(Q)\| \to 0$ so we can use Proposition 3.1).

To show the $\mathscr{C}_0 \supset \mathbb{K}(\Gamma)$, we take $T \in \mathbb{K}(\Gamma)$ and we fix $\varepsilon > 0$. From Proposition (3.1), it follows that there is an operator T' with compactly supported kernel such that $||T - T'|| \leq \varepsilon$. Define $\delta_{x,y} \in \mathbb{K}(\Gamma)$ by $(\delta_{x,y}f)(z) = f(y)$ if $z = x$ and 0 elsewhere. We have $\delta_{x,x} = \chi_{\{x\}}(Q) \in C_0$. As T' is a linear combinaison of $\delta_{x,y}$, it suffices to show that $\delta_{x,y}$ is in $\mathscr{C}_0$. But we have $\delta_{x,y} = \delta_{x,x}(\partial^*)^{|x|}\partial^{|y|}\delta_{y,y}$, which finishes the proof. □

Thus, if C is a C^*-subalgebra of $\ell^\infty(\Gamma)$ that contain C_0, we obtain $\mathbb{K}(\Gamma) \subset \langle \mathscr{D}, C \rangle$. Hence, in order to apply the technique described above, it remains to give a sufficiently explicit description of

$$\langle \mathscr{D}, C \rangle \big/ \mathbb{K}(\Gamma).$$

In this paper, we shall concentrate on the case $C(\widehat{\Gamma})$ which is, geometrically speaking, the most interesting one (see the last Remark in 2.4).

Definition 3.3 *The C^*-algebra generated by ∂ and $C(\widehat{\Gamma})$ is denoted by $\mathscr{C}(\widehat{\Gamma})$ and the $*$-subalgebra generated by ∂ and $C(\widehat{\Gamma})$ is denoted by $\mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$.*

We will need the next fundamental property.

Proposition 3.4 $[\partial, C(\widehat{\Gamma})] \subset \mathbb{K}(\Gamma)$.

Proof: If $\varphi \in C(\widehat{\Gamma})$ then

$$([\partial, \varphi(Q)]f)(x) = \sum_{y'=x} (\varphi(y) - \varphi(x))f(y) = (\partial \circ \psi(Q)f)(x),$$

where ψ belongs to $C(\widehat{\Gamma})$ and is defined by $\psi(y) = \varphi(y) - \varphi(y')$ when $|y| \geq 1$ and $\psi(e) = 0$. Observe that for $\gamma \in \partial\Gamma$ we have $\psi(\gamma) = \varphi(\gamma) - \varphi(\gamma) = 0$. Hence the relation (2.4) gives us that $\psi \in C_0$ and the Proposition 3.2 that $\psi(Q)$ is a compact operator. □

Remark: The algebra $\mathscr{D}$ is the tree analog of the algebra generated by the momentum operator on the real line. However, these algebras are rather different. First, $\mathscr{D}$ is not commutative. Then, the spectrum and the essential spectrum of the operators from $\mathscr{D}$ is not connected in general. For instance, one has $\sigma(\partial^*\partial) = \sigma_{\text{ess}}(\partial^*\partial) = \{0, \nu\}$ if $\nu > 1$. Indeed, we remind that if A, B are elements of a Banach algebra we have $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$ and that, as noticed below, $\dim\ker(\partial)$ is infinite for $\nu > 1$.

3.4. Translations in $\ell^2(\Gamma)$. Γ acts on itself to the left and to the right: if $a \in \Gamma$ then $\lambda_a : \Gamma \to \Gamma$ is $x \mapsto ax$ and $\rho_a : \Gamma \to \Gamma$ is $x \mapsto xa$. We clearly have, for $a, b \in \Gamma$, the following properties:

$$\lambda_{ab} = \lambda_a\lambda_b,\ \lambda_e = \text{Id},\ \rho_{ab} = \rho_b\rho_a,\ \rho_e = \text{Id}$$

and

$$\lambda_a\rho_b = \rho_b\lambda_a.$$

For $x \in a\Gamma$, we define $a^{-1}x$ as being y where $x = ay$, and for $x \in \Gamma a$, xa^{-1} will be y where $x = ya$.

We now give the natural extensions of these translations to $\ell^2(\Gamma)$. Let $f = \sum_{x\in\Gamma} f(x)x$ in $\ell^2(\Gamma)$. The translation λ_a on f will be $\sum_{x\in\Gamma} f(x)ax$, more precisely

$$(\lambda_a f)(x) = \chi_{a\Gamma}(x) f(a^{-1}x).$$

By the same way, we obtain:

$$(\rho_a f)(x) = \chi_{\Gamma a}(x) f(xa^{-1}).$$

The operators λ_a and ρ_a are isometries on $\ell^2(\Gamma)$. For instance:

$$\begin{aligned} ||\lambda_a f||^2 &= \sum_{x\in\Gamma} |\chi_{a\Gamma}(x) f(a^{-1}x)|^2 \\ &= \sum_{x\in a\Gamma} |f(a^{-1}x)|^2 = \sum_{x\in\Gamma} |f(x)|^2 = ||f||^2. \end{aligned}$$

Thus, we obtain:

$$\lambda_a^*\lambda_a = \text{Id and } \rho_a^*\rho_a = \text{Id}. \tag{3.9}$$

We now compute their adjoints. They act as follows on $f \in \ell^2(\Gamma)$:

$$(\lambda_a^* f)(x) = f(ax),\ (\rho_a^* f)(x) = f(xa).$$

Indeed, for λ_a, one has:

$$\begin{aligned}\langle f, \lambda_a f\rangle &= \sum_{x\in\Gamma} \overline{f(x)}\chi_{a\Gamma}(x) f(a^{-1}x) = \sum_{x\in a\Gamma} \overline{f(x)} f(a^{-1}x)\\ &= \sum_{x\in\Gamma} \overline{f(ax)} f(x) = \langle \lambda_a^* f, f\rangle.\end{aligned}$$

Moreover,

$$||\lambda_a^* f||^2 = \sum_{x\in\Gamma} |f(ax)|^2 = \sum_{x\in a\Gamma} |f(x)|^2 = ||\mathbf{1}_{a\Gamma} f||^2.$$

Thus, we obtain:

$$\lambda_a\lambda_a^* = \mathbf{1}_{a\Gamma} \text{ and } \rho_a\rho_a^* = \mathbf{1}_{\Gamma a}. \tag{3.10}$$

It is easy to check that

$$\partial^* = \sum_{|a|=1} \rho_a \text{ and } \partial = \sum_{|a|=1} \rho_a^*.$$

3.5. Localizations at infinity. In order to study $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to define the localizations at infinity of $T \in \mathscr{C}(\widehat{\Gamma})$ by looking at the behavior of the translated operator $\lambda_a^* T \lambda_a$ as a converges to γ in $\widehat{\Gamma}$ (abbreviated $a \to \gamma$), for each $\gamma \in \partial\Gamma$.

Let $\gamma \in \Gamma$. We notice that $\text{u-lim}_{a\to\gamma} \lambda_a^* \mathbb{K}(\Gamma)\lambda_a = 0$, where u-lim means convergence in norm. Indeed for $T \in \mathbb{K}(\Gamma)$, the relations (3.9) and (3.10) imply that $||\lambda_a^* T \lambda_a|| = ||\mathbf{1}_{a\Gamma} T \mathbf{1}_{a\Gamma}||$, and then the Proposition 3.1 finishes the proof. Now, we compute the uniform limit of $\lambda_a^* T \lambda_a$ where $T \in \mathscr{C}(\widehat{\Gamma})_{\text{alg}}$ There is P be a non-commutative complex polynomial in $m+2$ variables and $\varphi_i \in C(\widehat{\Gamma})$ for $i = 1, \ldots, m$ such that

$$T = P(\varphi_1, \varphi_2, \ldots, \varphi_m, \partial, \partial^*)$$

and we set

$$T(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \partial^*).$$

Lemma 3.5 *There is $a_0 \in \Gamma$ such that*

$$\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma) \lambda_{a_0}.$$

Proof: The Proposition 3.4 and the relation (3.7) give us $\phi_k \in C(\widehat{\Gamma})$, $K \in \mathbb{K}(\Gamma)$ and integers $\alpha_k, \beta_k \geq 0$ such that

$$T = \sum_{k=1}^{n} \phi_k(Q) \partial^{*\alpha_k} \partial^{\beta_k} + K$$

and

$$T(\gamma) = \sum_{k=1}^{n} \phi_k(\gamma) \partial^{*\alpha_k} \partial^{\beta_k}.$$

Thus, it suffixes to compute $\text{u-lim}_{a \to \gamma} \lambda_a^* \varphi(Q) \partial^{*\alpha} \partial^{\beta} \lambda_a$ with $\varphi \in C(\widehat{\Gamma})$. We suppose $|a| \geq \alpha$ and take $f \in \ell^2(\Gamma)$. We will first show the result for $\varphi = 1$. Since

$$\begin{aligned} (\lambda_a^* \partial^{*\alpha} \partial^{\beta} \lambda_a f)(x) &= (\partial^{*\alpha} \partial^{\beta} \lambda_a f)(ax) = (\partial^{\beta} \lambda_a f)((ax)^{(\alpha)}) \\ &= \sum_{\{y \mid y^{(\beta)} = (ax)^{(\alpha)}\}} (\lambda_a f)(y) = \sum_{\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}} f(y), \end{aligned} \tag{3.11}$$

it suffices to show that the set $\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$ is independent of a if $|a| \geq \alpha$. This is the aim of our next lemma.

Lemma 3.6 *For $|a| \geq \alpha$ we have:*

$$\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} = \begin{cases} \emptyset & \text{for } |x| + \beta - \alpha < 0, \\ S^{|x|+\beta-\alpha} & \text{for } |x| < \alpha \text{ and } |x| + \beta - \alpha \geq 0, \\ x^{(\alpha)} S^{\beta} & \text{for } |x| \geq \alpha \text{ and } |x| + \beta - \alpha \geq 0. \end{cases}$$

Proof: Let $J_x = \{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$. Then

$$\begin{aligned} aJ_x &= \{ay \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} \\ &= \{y \mid y^{(\beta)} = (ax)^{(\alpha)}\} \cap a\Gamma \\ &= ((ax)^{(\alpha)} S^{\beta}(\Gamma)) \cap a\Gamma. \end{aligned}$$

We notice that $(ax)^{(\alpha)} S^{\beta} \subset S^{|a|+|x|-\alpha+\beta}$.

If $|x| - \alpha + \beta < 0$ then $((ax)^{(\alpha)} S^{\beta}) \cap a\Gamma = \emptyset$, so $aJ_x = \emptyset$. This clearly implies $J_x = \emptyset$.

If $|x| - \alpha + \beta \geq 0$ then $((ax)^{(\alpha)} S^{\beta}) \cap a\Gamma \neq \emptyset$. If we suppose that $|x| < \alpha$, i.e. $|(ax)^{(\alpha)}| < |a|$, we have $a \in (ax)^{(\alpha)}\Gamma$. Let b such that $a = (ax)^{(\alpha)} b$. Thus

$$\begin{aligned} ((ax)^{(\alpha)} S^{\beta}) \cap a\Gamma &= ((ax)^{(\alpha)} S^{\beta}) \cap (ax)^{(\alpha)} b\Gamma = (ax)^{(\alpha)} (S^{\beta} \cap b\Gamma) \\ &= (ax)^{(\alpha)} b S^{\beta - |b|} = a S^{\beta - |b|} = a S^{\beta + |x| - \alpha}. \end{aligned}$$

So we have $aJ_x = aS^{\beta+|x|-\alpha}$, hence $J_x = S^{\beta+|x|-\alpha}$.

Finally, if $|x| \geq \alpha$, i.e. $|(ax)^{(\alpha)}| \geq |a|$, one has $(ax)^{(\alpha)} \in a\Gamma$. Thus, we obtain that $aJ_x = (ax)^{(\alpha)}S^\beta = ax^{(\alpha)}S^\beta$, hence $J_x = x^{(\alpha)}S^\beta$. $\square$

Now we treat the case of a general $\varphi \in C(\widehat{\Gamma})$. We have

$$(\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a f)(x) = \varphi(ax)(\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a f)(x).$$

Hence,

$$||\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a - \varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a|| \leq ||\varphi(aQ) - \varphi(\gamma)|| \cdot ||\partial^{*\alpha}\partial^\beta||$$

The right hand side tends to zero when a tends to γ. On the other hand, $\varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a$ is constant for $|a| \geq \alpha$. Thus, it suffices to choose $|a_0| \geq \max\{\alpha_k \mid k = 1, \ldots, n\}$ in the statement of the lemma to end the proof. $\square$

Remark: As seen in the proof, one may choose any a_0 such that $|a_0| \geq \deg(P)$. On the other hand, we stress that the limit is not a multiplicative function of T. For instance,

$$\operatorname*{u-lim}_{a\to\gamma} \lambda_a^*\partial^*\partial\lambda_a \neq (\operatorname*{u-lim}_{a\to\gamma} \lambda_a^*\partial^*\lambda_a) \cdot (\operatorname*{u-lim}_{a\to\gamma} \lambda_a^*\partial\lambda_a).$$

Therefore to describe the morphism of the algebra $\mathscr{C}(\widehat{\Gamma})$ onto its quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to improve our definition of the localizations at infinity.

4 Extensions to $\widetilde{\Gamma}$

4.1. Operators on $\ell^2(\widetilde{\Gamma})$. The space $\ell^2(\widetilde{\Gamma})$ is defined similarly to $\ell^2(\Gamma)$. Since $\Gamma \subset \widetilde{\Gamma}$, we have $\ell^2(\Gamma) \hookrightarrow l^2(\widetilde{\Gamma})$. As before, we embed $\widetilde{\Gamma}$ in $\ell^2(\widetilde{\Gamma})$ by sending x on $\chi_{\{x\}}$ and we notice that $\widetilde{\Gamma}$ is an orthonormal basis of $\ell^2(\widetilde{\Gamma})$. We now define $\widetilde{\partial} : l^2(\widetilde{\Gamma}) \to l^2(\widetilde{\Gamma})$ by

$$(\widetilde{\partial} f)(x) = f'(x) = \sum_{y'=x} f(y).$$

For $\alpha \in \mathbb{N}$, we set $f^{(\alpha)} = \widetilde{\partial}^\alpha f$. This notation is consistent with our old definition of $x^{(\alpha)}$, the restriction of x to $\mathbb{Z}_{|x|-\alpha}$.

We notice that $\widetilde{\partial}$ is a bounded operator on $\ell^2(\widetilde{\Gamma})$ and its adjoint $\widetilde{\partial}^*$ acts as follows:

$$(\widetilde{\partial}^* f)(x) = f(x').$$

Moreover, $\widetilde{\partial}^*/\sqrt{\nu}$ is an isometry on $\ell^2(\widetilde{\Gamma})$, in other terms:

$$\widetilde{\partial}\widetilde{\partial}^* = \nu \mathrm{Id}. \tag{4.12}$$

This implies $||\widetilde{\partial}|| = ||\widetilde{\partial}^*|| = \nu$.

We denote by $\widetilde{\mathscr{D}}$ the C^*-algebra generated by $\widetilde{\partial}$ and by $\widetilde{\mathscr{D}}_{\mathrm{alg}}$ the $*$-algebra generated by $\widetilde{\partial}$; both of them are unital.

4.2. Some technicalities. The next lemma will allow us to make the connection between $\mathscr{D}_{alg}$ and $\widetilde{\mathscr{D}}_{\mathrm{alg}}$.

Lemma 4.1 *We have for $|a| \geq \alpha$:*

$$\mathbf{1}_\Gamma \lambda_a^* \partial^{*\alpha} \partial^\beta \lambda_a \mathbf{1}_\Gamma = \mathbf{1}_\Gamma \widetilde{\partial^*}^\alpha \widetilde{\partial}^\beta \mathbf{1}_\Gamma.$$

Proof: For $f \in \ell^2(\widetilde{\Gamma})$, we have:

$$\begin{aligned}(\mathbf{1}_\Gamma \widetilde{\partial^*}^\alpha \widetilde{\partial}^\beta \mathbf{1}_\Gamma f)(x) &= \mathbf{1}_\Gamma(x) \sum_{y = x^{(\alpha)}} (\widetilde{\partial}^\beta \mathbf{1}_\Gamma f)(y) = \mathbf{1}_\Gamma(x) \sum_{y = x^{(\alpha)}} \sum_{z^{(\beta)} = y} (\mathbf{1}_\Gamma f)(z) \\ &= \mathbf{1}_\Gamma(x) \sum_{y^{(\beta)} = x^{(\alpha)}} \mathbf{1}_\Gamma(y) f(y).\end{aligned}$$

The proof of the lemma is finished by comparing this with (3.11) and by taking into account the next result, whose proof is identical to that of Lemma 3.6.

Lemma 4.2 *On $\widetilde{\Gamma}$, let $x \in \Gamma$ we have:*

$$\{y \in \Gamma \mid y^{(\beta)} = x^{(\alpha)}\} = \begin{cases} \emptyset & \textit{for } |x| + \beta - \alpha < 0, \\ S^{|x|+\beta-\alpha} & \textit{for } |x| < \alpha \textit{ and } |x| + \beta - \alpha \geq 0, \\ x^{(\alpha)} S^\beta & \textit{for } |x| \geq \alpha \textit{ and } |x| + \beta - \alpha \geq 0. \end{cases}$$

We will also need a result concerning the localization of the norm on $\widetilde{\mathscr{D}}_{alg}$.

Lemma 4.3 *Let $\widetilde{T} \in \widetilde{\mathscr{D}}_{alg}$. We have:*

$$||\widetilde{T}|| = ||\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma||.$$

Proof: Because of (4.12), we can suppose that $\widetilde{T} = \sum_{k=1}^{n} c_k \widetilde{\partial^*}^{\alpha_k} \widetilde{\partial}^{\beta_k}$. Let $\alpha = \max\{\alpha_k \mid k = 1, \ldots, n\}$. For each $\varepsilon > 0$, there is $g \in \ell^2(\widetilde{\Gamma})$ with compact support such that $||g|| = 1$ and

$$||\widetilde{T}g|| \geq ||\widetilde{T}|| - \varepsilon.$$

Note that if $y_1, y_2, \ldots, y_m$ are distinct points of Γ, and $a_1, a_2, \ldots, a_m$ are complex numbers and $x_1, x_2 \in \widetilde{\Gamma}$, we have

$$||\sum_{i=1}^{m} a_i x_1 y_i||^2 = \sum_{i=1}^{m} |a_i|^2 = ||\sum_{i=1}^{m} a_i x_2 y_i||^2. \tag{4.13}$$

Thus, since g has compact support, there are $x \in \widetilde{\Gamma}$, $(y_i)_{i=1,\ldots,m} \in \Gamma^m$ and $(a_i)_{i=1,\ldots,m} \in \mathbb{C}^m$ such that $|y_i| \geq \alpha$ and $g = \sum_{k=1}^{m} a_i x y_i$. We set $f = \sum_{k=1}^{m} a_i e y_i$. Then (4.13) gives us $||f|| = ||g|| = 1$. Since $|y_i| \geq \alpha$ for all $i = 1 \ldots m$, we obtain that $f \in \ell^2(\Gamma)$ and that $\widetilde{T}f \in \ell^2(\Gamma)$.

Moreover, using once more the fact that $|y_i| \geq \alpha$ and (4.13), we have, with $z \in \Gamma$,

$$\begin{aligned}
||\widetilde{T}g|| &= ||\sum_{k=1}^{n}\sum_{i=1}^{m} c_k a_i \widetilde{\partial^*}^{\alpha_k} \widetilde{\partial}^{\beta_k} x y_i|| = ||\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{|z|=\beta_k} c_k a_i (x y_i)^{(\alpha_k)} z|| \\
&= ||\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{|z|=\beta_k} c_k a_i x (y_i)^{(\alpha_k)} z|| = ||\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{|z|=\beta_k} c_k a_i e (y_i)^{(\alpha_k)} z|| \\
&= ||\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{|z|=\beta_k} c_k a_i (e y_i)^{(\alpha_k)} z|| = ||\sum_{k=1}^{n}\sum_{i=1}^{m} c_k a_i \widetilde{\partial^*}^{\alpha_k} \widetilde{\partial}^{\beta_k} e y_i|| = ||\widetilde{T}f||.
\end{aligned}$$

Hence, there is $f \in \ell^2(\widetilde{\Gamma})$ such that

$$||\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma f|| = ||\mathbf{1}_\Gamma \widetilde{T} f|| = ||\widetilde{T}f|| = ||\widetilde{T}g|| \geq ||\widetilde{T}|| - \varepsilon.$$

This shows us that

$$||\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma|| \geq ||\widetilde{T}||.$$

The reverse inequality is obvious. $\square$

5 Main results

5.1. Morphism. In the following, a morphism will be a morphism of C^*-algebras. To describe to quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$, we need to find an adapted morphism. We have the following result.

Theorem 5.1 *For each $\gamma \in \partial\Gamma$ there is a unique morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ such that $\Phi_\gamma(\partial) = \widetilde{\partial}$ and $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$, for all $\varphi \in C(\widehat{\Gamma})$. One has $\Phi_\gamma(\mathbb{K}(\Gamma)) = 0$.*

Proof: Let $T \in \mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$. We keep the notations introduced before Lemma 3.5. Then by Lemma 3.5 we have $\text{u-lim}_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma)\lambda_{a_0}$. We recall that

$$T(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \partial^*).$$

Now let

$$\widetilde{T}(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \widetilde{\partial}, \widetilde{\partial}^*).$$

From Lemma 4.1 and (4.12) it follows that one can choose a_0 such that

$$\lambda_{a_0}^* T(\gamma)\lambda_{a_0} = \mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma.$$

Using Lemma 4.3 we obtain:

$$\begin{aligned} ||\widetilde{T}(\gamma)|| &= ||\mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma|| = ||\lambda_{a_0}^* T(\gamma)\lambda_{a_0}|| \\ &= ||\operatorname*{u-lim}_{a\to\gamma} \lambda_a^* T \lambda_a|| \le ||T||. \end{aligned}$$

This shows that there is a linear mapping $\Phi_\gamma^0 : \mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}} \to \widetilde{\mathscr{D}}$ such that $\Phi_\gamma^0(T) = T(\gamma)$ and that this map is a contraction. It is clear that it is also multiplicative. The density of $\mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$ in $\mathscr{C}(\widehat{\Gamma})$ allows us to extend Φ_γ^0 to a morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ which clearly satisfies the conditions of the theorem. The uniqueness of Φ_γ is obvious and the last assertion of the theorem follows from the Proposition 3.2. □

5.2. The case $\nu > 1$. We shall improve the theorem under the hypothesis $\nu > 1$, which will be assumed in all this subsection.

We recall that an isometry is said to be *proper* if it is not unitary. The operators ∂^* and $\widetilde{\partial}^*$ are proper isometries and the dimensions of the kernels of ∂ and $\widetilde{\partial}$ are infinite. For instance in the case of ∂, let a, b be different letters of $\mathscr{A}$. Choose $g \in \ell^2(\Gamma a)$ and $h \in \ell^2(\Gamma b)$ such that $h(xb) = g(xa)$ for all $x \in \Gamma$, then $g - h$ is in $\mathrm{Ker}(\partial)$.

Let $\mathbb{T}$ be the unit circle of $\mathbb{R}^2$ and H^2 the closure of the subspace spanned by $\{e^{inQ}, n \in \mathbb{N}\}$ in $\ell^2(\mathbb{T})$. For $g \in L^\infty(\mathbb{T})$, we define the *Toeplitz operator* T_g on H^2 by $T_g h = P_{H^2} g h$, where P_{H^2} is the projection on H^2. We denote by $\mathscr{T}$ the C^*-algebra generated by T_z. The next theorem is due to Coburn (see [Da] for a proof).

Theorem 5.2 *If S is proper isometry, then there is a unique isomorphism φ of $\mathscr{T}$ onto $\mathscr{S}$, the C^*-algebra generated by S, such that $\varphi(T_z) = S$.*

Thus, we see that there is a unique isomorphism φ of $\mathscr{D}$ onto $\widetilde{\mathscr{D}}$ such that $\varphi(\partial) = \varphi(\widetilde{\partial})$. We can rewrite our theorem as follows.

Theorem 5.3 *Let $\nu > 1$ and let $\gamma \in \partial\Gamma$. There is a unique morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D}$ such that $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$, $\forall \varphi \in C(\widehat{\Gamma})$ and $\Phi_\gamma(D) = D$, $\forall D \in \mathscr{D}$.*

On the other hand, when $\nu = 1$, there is no isomorphism $\mathscr{J} : \mathscr{D} \to \widetilde{\mathscr{D}}$ such that $\mathscr{J}(\partial) = \widetilde{\partial}$ because $\widetilde{\mathscr{D}}$ is commutative. Thus, one cannot hope such a theorem in that case.

Another way of proving Theorem 5.3 is through the use of the next proposition.

Proposition 5.4 *If $\nu \geq 1$ then $\{\partial^{*\alpha}\partial^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of the vector space $\mathscr{D}_{\mathrm{alg}}$. One has $\nu > 1$ if and only if $\{\widetilde{\partial^*}^{\alpha}\widetilde{\partial}^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of space $\widetilde{\mathscr{D}}_{\mathrm{alg}}$.*

Proof: Assume that $\sum_{i=1}^n \lambda_i \partial^{*\alpha_i}\partial^{\beta_i} = 0$, where (α_i, β_i) are distinct couples and $\lambda_i \neq 0$. We set $\alpha = \min(\alpha_i, i = 1, \ldots, n)$ and $I = \{i \mid \alpha_i = \alpha\}$. We take $x \in \Gamma$ such that $|x| = \alpha$ and we obtain $\sum_{i\in I} \lambda_i (\partial^{\beta_i} f)(e) = 0$. Notice that $\{\beta_i\}_{i\in I}$ are pairwise distinct by hypotheses. Now, by taking $i_0 \in I$ and f the characteristic function of $S_{\beta_{i_0}}$, we get that $\lambda_{i_0} = 0$ which is a contradiction.

Now assume $\nu > 1$ and let $\sum_{i=1}^n \lambda_i \widetilde{\partial^*}^{\alpha_i}\widetilde{\partial}^{\beta_i} = 0$, with (α_i, β_i) pairwise distinct and $\lambda_i \neq 0$. We fix $x \in \widetilde{\Gamma}$ on and set $\alpha = \max(\alpha_i, i = 1, \ldots, n)$. We have:

$$\sum_{i=1}^n \lambda_i \sum_{y \in x^{(\alpha_i)} S^{\beta_i}} f(y) = 0.$$

Noticing that $x^{(\alpha)}S^{\beta} \cap x^{(\alpha')}S^{\beta'} = \emptyset$ if and only if $\alpha' - \alpha \neq \beta' - \beta$, we can suppose that there is k such that $\alpha_i - \beta_i = k$ for all $i = 1, \dots, n$. Moreover, since for all $l = 1, \dots, (\alpha - k)$,

$$x^{(\alpha-l)}S^{\alpha-k-l} \subset x^{(\alpha-1)}S^{\alpha-k-1} \nsubseteq x^{(\alpha)}S^{\alpha-k},$$

by the definition of α we obtain:

$$\cup_{\alpha_i \neq \alpha} x^{(\alpha_i)}S^{\beta_i} \nsubseteq x^{(\alpha)}S^{\beta}.$$

We now choose y_0 strictly in the $x^{(\alpha)}S^{\beta}$. Hence, by taking $f = \chi_{\{y_0\}}$, we obtain $\lambda_{i_0} = 0$ which is a contradiction. Finally, $\{\widetilde{\partial^*}^{\alpha}\widetilde{\partial}^{\beta}\}_{\alpha,\beta\in\mathbb{N}}$ is not a basis if $\nu = 1$ because $\widetilde{\partial}\widetilde{\partial}^* = \widetilde{\partial}^*\widetilde{\partial} = 1$. $\square$

5.3. Description of $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$. We now compute the quotient of $\mathscr{C}(\widehat{\Gamma})$ by $\mathbb{K}(\Gamma)$.

Theorem 5.5 *There is a unique morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$ such that $\Phi(\partial) = \widetilde{\partial} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. This morphism is surjective and its kernel is $\mathbb{K}(\Gamma)$.*

Proof: According to the Theorem 5.1 there is a morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}^{\partial\Gamma}$ such that $(\Phi(\partial))(\gamma) = \widetilde{\partial}$ and $(\Phi(\varphi(Q)))(\gamma) = \varphi(\gamma)$, for all $\gamma \in \partial\Gamma$ and all $\varphi \in C(\widehat{\Gamma})$. Since the images of ∂ and $\varphi(Q)$ belong to the C^*-subalgebra $C(\partial\Gamma, \widetilde{\mathscr{D}})$, and since $\mathscr{C}(\widehat{\Gamma})$ is generated by ∂ and such $\varphi(Q)$, it follows that the range of Φ is included in $C(\partial\Gamma, \widetilde{\mathscr{D}})$. We have $C(\partial\Gamma, \widetilde{\mathscr{D}}) \cong \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$, so we get the required morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$. Now since $\Phi(\partial) = \widetilde{\partial} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$, and since any function in $C(\partial\Gamma)$ is the restriction of some function from $C(\widehat{\Gamma})$, it follows that Φ is surjective. Its uniqueness is clear. We now compute the kernel.

As seen in the Theorem 5.1 $\mathbb{K}(\Gamma) \subset \mathrm{Ker}(\Phi)$. In the remainder of this section we shall prove the reverse inclusion. For this we need some preliminary lemmas.

Lemma 5.6 *Let $R = \varphi(Q)\partial^{*\alpha}\partial^{\beta}$ and $\mathscr{U} = \{a_i\Gamma\}_{i=1,\dots,n}$ be a disjoint covering of $\partial\Gamma$. For each $\varepsilon > 0$ there are $c_1, c_2, \dots, c_m \in \mathrm{Ran}(\varphi)$ and there is a disjoint covering $\mathscr{U}' = \{b_j\Gamma\}_{j=1,\dots,m}$ of $\partial\Gamma$ finer than $\mathscr{U}$ such that $\|\mathbf{1}_{U'}R - R'\| \leq \varepsilon$, where $R' = \sum_{j=1}^{m} \mathbf{1}_{b_j\Gamma}c_j\partial^{*\alpha}\partial^{\beta}$ and $U' = \cup_{j=1}^{m} b_j\Gamma$.*

Proof: Let $\varepsilon > 0$, we denote $\varepsilon/\|\partial^{*\alpha}\partial^{\beta}\|$ by ε'. Since $\varphi(\partial\Gamma)$ is compact, there are $\gamma_1, \gamma_2, \dots, \gamma_N \subset \partial\Gamma$ such that $\varphi(\partial\Gamma) \subset \cup_{k=1}^{N} D(\varphi(\gamma_k), \varepsilon')$, where $D(z, r)$ is the complex open disk of center z and ray r. The open sets

$$\mathscr{O}_{i,k} = a_i\widehat{\Gamma} \cap \varphi^{-1}(D(\varphi(\gamma_k), \varepsilon'))$$

cover $\partial\Gamma$. The Proposition 2.4 gives us a disjoint covering $\{b_j\Gamma\}_{j=1,\dots,m}$ of $\partial\Gamma$ such that for each $j \in \{1, \dots, m\}$ there are i and k such that $b_j\widehat{\Gamma} \subset \mathscr{O}_{i,k}$. To simplify the notations, we will denote by γ_j the γ_k associated to $b_j\Gamma$. We set $\mathscr{U}' = \{b_j\Gamma\}_{j=1,\dots,m}$ and

$$R' = \sum_{j=1}^{n} \mathbf{1}_{b_j\Gamma}\varphi(\gamma_j)\partial^{*\alpha}\partial^{\beta}.$$

Remembering that $\sup_{x \in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)| \leq \varepsilon'$, we obtain:

$$\begin{aligned}
\|(R' - \mathbf{1}_{U'}R)f\|^2 &= \sum_{x\in\Gamma} |\sum_{j=1}^{m} \mathbf{1}_{b_j\Gamma}(x)(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^{\beta}f)(x)|^2 \\
&= \sum_{j=1}^{m} \sum_{x\in b_j\Gamma} |(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^{\beta}f)(x)|^2 \\
&\leq \sum_{j=1}^{m} \sup_{x\in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)|^2 \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^{\beta}f)(x)|^2 \\
&\leq \varepsilon'^2 \sum_{j=1}^{m} \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^{\beta}f)(x)|^2 \\
&\leq \varepsilon^2\|\partial^{*\alpha}\partial^{\beta}\|^{-2} \cdot \|\partial^{*\alpha}\partial^{\beta}\|^2 \cdot \|f\|^2 = \varepsilon^2\|f\|^2.
\end{aligned}$$

By denoting $\varphi(\lambda_j)$ by c_j we obtain the result. $\square$

Lemma 5.7 *Let* $T = \sum_{k=1}^{n} \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$ *with* $\varphi_k \in C(\widehat{\Gamma})$ *and let* $\varepsilon > 0$. *There are a compact operator* K, *a disjoint covering* $\{a_j\Gamma\}_{j=1,\dots,m}$ *of* $\partial\Gamma$ *and an operator* S *of the form*

$$S = \sum_{k=1}^{n}\sum_{j=1}^{m} \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}, \tag{5.14}$$

such that $\min\{|a_j| \mid j = 1, \dots, m\} \geq \max\{\alpha_k \mid k = 1, \dots, n\}$, $\gamma_{j,k} \in \partial\Gamma$ *and*

$$\|T - S - K\| \leq \varepsilon.$$

Proof: We denote by $\alpha = \max\{\alpha_k \mid k = 1, \dots, n\}$. We set $T_k = \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$. Setting $\mathscr{U}_0 = \cup_{\{a||a|=\alpha\}}\{a\Gamma\}$, we now apply the Lemma 5.6 inductively for $k = 1, \dots, n$ with ε/n instead of ε, $\mathscr{U} = \mathscr{U}_{k-1}$ and $R = T_k$, denoting $\mathscr{U}'$ by $\mathscr{U}_k$ and R' by S_k. Then, we obtain for $k \in \{1, \dots, n\}$:

$$\|\mathbf{1}_{U_k} T_k - S_k\| \leq \frac{\varepsilon}{k}$$

Since $\mathscr{U}_{k+1}$ is finer than $\mathscr{U}_k$ for $k = 1, \dots, n-1$, we obtain:

$$\|\mathbf{1}_{U_n} \sum_{k=1}^{n} (T_k - S_k)\| \leq \varepsilon.$$

Hence,

$$\|T - \mathbf{1}_{U_n^c} T - \mathbf{1}_{U_n} \sum_{k=1}^{n} S_k\| \leq \varepsilon.$$

Finally, we denote the compact operator $\mathbf{1}_{U_n^c} T$ by K, $(\mathbf{1}_{U_n} \sum_{k=1}^{n} S_k)$ by S and $\mathscr{U}_n$ by $\{a_j\Gamma\}_{j=1,\dots,m}$ to finish the proof. □

We now go back to the proof of Theorem 5.5. Let $T \in \operatorname{Ker}(\Phi)$. For each $\varepsilon > 0$ there is $T' \in \mathscr{C}(\widehat{\Gamma})_{\text{alg}}$ such that $\|T - T'\| \leq \varepsilon$. By relation (3.7) and Proposition 3.4, we can write:

$$T' = \sum_{k=1}^{n} \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k} + K,$$

where $K \in \mathbb{K}(\Gamma)$ and $\varphi_k \in C(\widehat{\Gamma})$. Thus $\|\Phi(T')\| \leq \varepsilon$. Using Lemma 5.7, we get an operator S of the form (5.14) and a compact operator K_1 such that $\|T' - S - K_1\| \leq \varepsilon$. This implies that $\|\Phi(S)\| \leq 2\varepsilon$.

Lemma 5.8 *There is $K_2 \in \mathbb{K}(\Gamma)$ such that:*

$$\|S - K_2\| \leq \|\Phi(S)\| \leq 2\varepsilon.$$

The proof will be given below. By using this we get

$$\begin{aligned}
\|T - K_1 - K_2\| &= \|T - T' + T' - S - K_1 + S - K_2\| \\
&\leq \|T - T'\| + \|T' - S - K_1\| + \|S - K_2\| \\
&\leq 4\varepsilon.
\end{aligned}$$

Finally, the space $\mathbb{K}(\Gamma)$ being closed, we conclude that $T \in \mathbb{K}(\Gamma)$. This finishes the proof of the Theorem.

Proof of Lemma 5.8: First, we should remark that for each $a \in \Gamma$ and α, $\beta \geq 0$, the Proposition 3.4 gives us that $\mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta} - \mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta}\mathbf{1}_{a\Gamma}$ is a compact operator. We introduce

$$S' = \sum_{k=1}^{n}\sum_{j=1}^{m} \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}$$

and we set $K_2 = S - S'$ which is a compact operator. Let $f \in \ell^2(\Gamma)$. Since $\{a_j\Gamma\}_{j=1,\dots,m}$ is a disjoint covering of $\partial\Gamma$, we have:

$$\begin{aligned}
\|S'f\|^2 &= \sum_{x\in\Gamma} |\sum_{k=1}^{n}\sum_{j=1}^{m}(\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}f)(x)|^2 \\
&= \sum_{j=1}^{m}\sum_{x\in\Gamma} |\sum_{k=1}^{n}(\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}f)(x)|^2 \\
&\leq \sum_{j=1}^{m} \|\sum_{k=1}^{n}\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}\|^2 \cdot \|\mathbf{1}_{a_j\Gamma}f\|^2.
\end{aligned}$$

Now we use (3.9) and (3.10) and get:

$$\|\mathbf{1}_{a_j\Gamma}(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k})\mathbf{1}_{a_j\Gamma}\| = \|\lambda_{a_j}^{*}(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k})\lambda_{a_j}\|.$$

Since $|a_j| \geq \max\{\alpha_k \mid k = 1,\dots,n\}$, the Lemmas 4.1 and 4.3 give us:

$$\begin{aligned}
\|\lambda_{a_j}^{*}(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k})\lambda_{a_j}\| &= \|\mathbf{1}_{\Gamma}(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial^*}^{\alpha_k}\widetilde{\partial}^{\beta_k})\mathbf{1}_{\Gamma}\| \\
&= \|\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial^*}^{\alpha_k}\widetilde{\partial}^{\beta_k}\|.
\end{aligned}$$

For each j we choose $\gamma_j \in a_j\partial\Gamma$. The family $\{a_j\Gamma\}_{j=1,\dots,m}$ is a disjoint covering of $\partial\Gamma$, so we have $\lim_{x\to\gamma_j}\chi_{a_j\Gamma}(x) = 1$ and $\lim_{x\to\gamma_j}\chi_{a_i\Gamma}(x) = 0$ for $i \neq j$. Hence $\Phi_{\gamma_j}(S') = \sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial^*}^{\alpha_k}\widetilde{\partial}^{\beta_k}$. We obtain

$$\|S'f\|^2 \leq \sum_{j=1}^{m}\|\Phi_{\gamma_j}(S')\|^2 \cdot \|\mathbf{1}_{a_j\Gamma}f\|^2 \leq \sup_{\gamma\in\partial\Gamma}\|\Phi_{\gamma}(S')\|^2 \cdot \|f\|^2.$$

Finally, since $\mathbb{K}(\Gamma) \subset \mathrm{Ker}(\Phi)$, $\|\Phi(S)\| = \|\Phi(S')\| = \sup_{\gamma\in\partial\Gamma} \|\Phi_\gamma(S')\|$. This finishes the proof. $\square$

In the case $\nu > 1$, we can reformulate the theorem as follows:

Theorem 5.9 *Let* $\nu > 1$*. There is a unique morphism* $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ *such that* $\Phi(D) = D \otimes 1$ *for all* $D \in \mathscr{D}$ *and* $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$*. This morphism is surjective and its kernel is* $\mathbb{K}(\Gamma)$*.*

Once again this Theorem is false if $\nu = 1$. As a corollary we obtain the following result.

Proposition 5.10 *If* $\nu > 1$ *then* $\mathscr{D} \cap \mathbb{K}(\Gamma) = \{0\}$ *and if* $\nu = 1$ *one has* $\mathbb{K}(\Gamma) \subset \mathscr{D}$*.*

Proof: Let $\nu > 1$ and $T \in \mathscr{D} \cap \mathbb{K}(\Gamma)$. Theorem 5.9 gives us that $\Phi(T) = T \otimes 1$ and that $\Phi(T) = 0$ since T is compact. For $\nu = 1$, as seen in the proof of Proposition 3.2, it suffices to prove that $\delta_{x,x}$ is in $\mathscr{D}$. But this is clear since $\delta_{x,x} = \partial^{*|x+1|}\partial^{|x+1|} - \partial^{*|x|}\partial^{|x|}$. $\square$

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