Extended energy levels for macroscopic Hall systems

Christian Ferrari and Nicolas Macris

Institute for Theoretical Physics

Ecole Polytechnique Fédérale
CH - 1015 Lausanne, Switzerland

Abstract

We study the spectrum of a random Schrödinger operator for an electron submitted to a magnetic field in a finite but macroscopic two dimensional system of linear dimensions equal to L. The y direction is periodic and in the x direction the electron is confined by two smooth increasing boundary potentials. We prove that, with large probability, in a subset of the first gap of the pure bulk Hamiltonian the spectrum of the full Hamiltonian consists only on eigenenergies whose eigenfuntions are extended, in the sense that their quantum mechanical currents are strictly positive/negative with respect to the size of the system.

1 Introduction

In this paper we investigate some spectral properties of the family of random Hamiltonians describing the dynamics of a spinless quantum particle lying on a cylinder of linear size L that are confined along the cylinder axes by two smooth boundaries whose support are separated by the distance L. The particle feel the effect of an external homogeneous magnetic field and a weak random potential supported in between the two boundaries. A precise statement of the model is given in section 2. The physical interest on the study of this kind of model is the integral quantum Hall effect. In fact in disordered and confined electronic systems subject to a uniform magnetic field, for example, in a two dimensional interface of an heterojunction we observe a quantization of the Hall conductivity [vKDP], [PG]. For the explanation of this effect Halperin [H] pointed out the importance of the boundary diamagnetic currents. Since many features of the integral quantum Hall effect can be described in the framework of one particle random magnetic Schrödinger operators, it is important to understand their spectral properties for finite but macroscopic samples with boundaries. A short review of the spectral properties of finite quantum Hall systems can be found in [FM2].

The study of these random magnetic Hamiltonians with boundaries is recent, before turn to this problem let us discuss previous results on magnetic random Schrödinger operators defined in $\mathbb{R}^2$. For the case without boundaries the pure bulk Hamiltonian

$$H_\omega^b = H_L + V_\omega$$

defined in the infinite plane $\mathbb{R}^2$ has spectrum contained in "Landau bands" $\bigcup_{n\geq 0}\left[(n+\frac{1}{2})B - \|V_\omega\|_\infty, (n+\frac{1}{2})B + \|V_\omega\|_\infty\right]$ (H_L being the Landau Hamiltonian, V_ω an Anderson–type random potential and B the magnetic field). It is proved that the spectrum near the band edges is pure point and the eigenfunctions are exponentially localized ([DMP1], [DMP2], [CH], [BCH], [W]). For a special model, where the impurities are point scatterers, it is also proved that, at the band center, the spectrum is pure point ([DMP3], [DMP4]).

For the system defined in $\mathbb{R}^2$, the case with only one boundary, given for example by a confining potential U or Dirichlet boundary condition at $x = 0$, is also satisfactory understood. In fact in this case it is proven that the spectrum of the Hamiltonian $H_\omega^e = H_L + U + V_\omega$ has absolutely continuous components inside the complement of Landau bands, this for $\frac{\|V_\omega\|_\infty}{B}$ sufficiently small ([MMP], [FGW], [F], [dBP]). The proof of this statements is essentially based on Mourre theory with conjugate operator y. Since $i[H, y] = p_y - Bx = v_y$, the positivity of $i[H, y]$ in spectral subspaces of H leads to the a.c. nature of the spectrum therein. This implies the positivity of v_y, that represents the velocity operator in the $y-$direction along the edge. Since the spectrum of $H_L + U$ is absolutely continuous, this result says that, if the perturbation is sufficiently small with respect to the strength of the magnetic field, part of the a.c. spectrum is preserved and thus states propagating along the edge survive.

For the case with two boundaries, say at distance L, few results are known [EJK], [CHS]. A first model consists in approximating the strip of size L by a parabolic channel where the confining potential is given by a parabolic wall. In this case it is shown [EJK] that if the perturbation V is periodic or if V is small enough and decay fast enough in the $y-$direction (that exclude Anderson-like random potentials), the a.c. spectrum survives in certain intervals.

If we consider an infinite strip with edges separated by a distance L, by a smart adaptation of the conjugate operator it can be proven [CHS] that, if $V = 0$ the existence of extended states is related to a.c. spectrum. For a general (random) potential we expect that these results turn out to be false, indeed the potential should create a tunnelling between the two boundaries and thus propagating edge states along the two boundaries should not exists for all times. In [CHS] the authors have shown that such states survive, but only for a finite time given by the quantum tunnelling time between the two edges.

In this work we address a similar question but for macroscopic finite systems with two confining walls (at distance L) along the $x-$direction and with the $y-$direction made L periodic. In this case, although the spectrum is discrete, extended edge states exist but we cannot derive them directly from a Mourre estimate as in the infinite case.

In Theorem 1 we show that, with large probability, the spectrum of the family of random Hamiltonians (U_ℓ and U_r being the two confining walls at distance L)

$$H_\omega = H_L + V_\omega + U_\ell + U_r$$

in an energy interval $\Delta \subset \left(\frac{1}{2}B + \|V_\omega\|_\infty, \frac{3}{2}B - \|V_\omega\|_\infty\right)$ consists in the union of two sets Σ_ℓ and Σ_r, and these sets are small perturbations of the spectra $\sigma(H_L + U_\ell)$ and $\sigma(H_L + U_r)$. As in [FM1], the eigenvalues are then classified by their quantum mechanical current along the periodic direction ($J_E = (\psi_E, v_y \psi_E)$). We show that the eigenfunctions corresponding the eigenvalues in Σ_ℓ and Σ_r have a uniform current with respect to L, this leads to the existence of extended states along the boundaries.

The physical importance of these propagating states is related to the diamagnetic boundary currents advocated in [H]. Indeed propagating states curry currents and thus contribute to the total Hall current in the sample, therefore they play an important role for the explanation of the integer quantum Hall effect in the framework of the so-called edge theory.

Finally, we remark that our result is consistent with the analysis of Combes et al. [CHS]. Indeed in our work we show the existence of propagating current carrying states for a finite system (size $L \times L$) while in [CHS] it is shown that, in an infinite strip (of width L), the current carrying edge states survive only for a finite time, and this time is much larger than the time spent for an electron, starting from the bottom of the sample, to reach its top (this time is indeed of order $\mathcal{O}(L)$ since the velocity is of order $\mathcal{O}(1)$).

2 The Model and Main Result

We study the spectral properties of the family of random Hamiltonians

$$H_\omega = H_L + U_\ell + U_r + V_\omega\,, \quad \omega \in \Omega_\Lambda \tag{2.1}$$

acting in the Hilbert space $L^2(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}])$ with periodic boundary conditions along y: $\psi(x, -\frac{L}{2}) = \psi(x, \frac{L}{2})$. We choose the Landau gauge in which the kinetic part has the form $H_L = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y - Bx)^2$ and the spectrum is given by the Landau levels: $\sigma(H_L) = \{(n + \frac{1}{2})B; n \in \mathbb{N}\}$. The potentials U_ℓ and U_r representing the confinement along the $x-$direction at $x = \pm\frac{L}{2}$ are supposed strictly monotonic, twice differentiable and satisfy

$$c_1|x + \tfrac{L}{2}|^{m_1} \le U_\ell(x) \le c_2|x + \tfrac{L}{2}|^{m_2} \qquad \text{for } x \le -\tfrac{L}{2} \tag{2.2}$$

$$c_1|x - \tfrac{L}{2}|^{m_1} \le U_r(x) \le c_2|x - \tfrac{L}{2}|^{m_2} \qquad \text{for } x \ge \tfrac{L}{2} \tag{2.3}$$

for some constants $0 < c_1 < c_2$, $2 \le m_1 < m_2 < \infty$ and $U_\ell(x) = 0$ for $x \ge -\frac{L}{2}$, $U_r(x) = 0$ for $x \le \frac{L}{2}$. The random potential V_ω is given by the sum of local perturbations located at the sites of a finite lattice $\Lambda = \{(n,m) \in \mathbb{Z}^2; n \in [-\frac{L}{2}, \frac{L}{2}], m \in [-\frac{L}{2}, \frac{L}{2}]\}$. Let $V \ge 0$, with $V \in C^2$, $\|V\|_\infty \le V_0$, $\operatorname{supp} V \subset \mathbb{B}(\mathbf{0}, \frac{1}{4})$ (the open ball centered at $(0,0)$ of radius $\frac{1}{4}$) and $X_{n,m}(\omega)$ the random strength of each local perturbation, then V_ω has the form

$$V_\omega = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x - n, y - m) \tag{2.4}$$

where $X_{n,m}$ are i.i.d. random variables with common bounded density $h \in C^2([-1,1])$. We denote by $\mathbb{P}_\Lambda$ the product measure defined on the fundamental set of all possible realizations $\Omega_\Lambda = [-1,1]^\Lambda$. Clearly for each realization

$\omega \in \Omega_\Lambda$ we have $\|V_\omega\| \leq V_0$ and we suppose $V_0 \ll B$.

We explain some properties of three simpler random Hamiltonians. First the pure wall Hamiltonians

$$H_\alpha^0 = H_L + U_\alpha \qquad\qquad \alpha = \ell, r \ . \tag{2.5}$$

Their properties are discussed in [FM1], the spectrum is given by $\sigma(H_\alpha^0) = \{E_{nk}^\alpha : n \in \mathbb{N}, k \in \frac{2\pi}{L}\mathbb{Z}\}$, here we only need the property ($k = \frac{2\pi}{L}m$, $m \in \mathbb{Z}$)

$$\left| E^\alpha_{0\frac{2\pi}{L}(m+1)} - E^\alpha_{0\frac{2\pi}{L}m} \right| \geq \frac{C_0}{L} \qquad \alpha = \ell, r \tag{2.6}$$

for each k such that $E_{0k}^\alpha \in \Delta_\varepsilon = \left(\frac{1}{2}B + V_0 + \varepsilon, \frac{3}{2}B - V_0 - \varepsilon\right)$ and $C_0 > 0$ (uniformly in k). We will suppose that the following hypothesis is fulfilled (see Appendix C)

Hypothesis 1. *There exists L_0 and $d_0 > 0$ such that for all $L > L_0$*

$$\mathrm{dist}\left(\sigma(H_\ell^0) \cap \Delta_\varepsilon, \sigma(H_r^0) \cap \Delta_\varepsilon\right) \geq \frac{d_0}{L} \ . \tag{2.7}$$

Second the edge Hamiltonians

$$H_\alpha = H_L + U_\alpha + V_\omega^\alpha \tag{2.8}$$

where $V_\omega^\alpha = V_\omega|_{\Lambda_\alpha}$ with $\Lambda_r = \left\{(n,m) \in \mathbb{Z}^2; n \in [\frac{L}{2} - \frac{3D}{4} - 1, \frac{L}{2}], m \in [-\frac{L}{2}, \frac{L}{2}]\right\}$ and $\Lambda_\ell = \left\{(n,m) \in \mathbb{Z}^2; n \in [-\frac{L}{2}, -\frac{L}{2} + \frac{3D}{4} + 1], m \in [-\frac{L}{2}, \frac{L}{2}]\right\}$, where $D = \sqrt{L}$. Since the perturbation has compact support and the essential spectrum of H_α^0 is given by the Landau levels, the spectrum of H_α is given by

$$\sigma(H_\alpha) = \{E_\kappa^\alpha : \kappa \in \mathbb{Z}\} \ . \tag{2.9}$$

One can prove ([M], Theorem 1) that, for each $\omega \in \Omega_{\Lambda_\alpha} = [-1,1]^{\Lambda_\alpha}$ (the restriction of the configurations ω to the sublattice Λ_α) and for each κ such that $E_\kappa^\alpha \in \Delta = (B - \delta, B + \delta) \subset \Delta_\varepsilon$ the distance between two consecutive eigenvalues satisfies

$$\left|E_{\kappa+1}^\alpha - E_\kappa^\alpha\right| \geq \frac{C}{L} \qquad\qquad \alpha = \ell, r \tag{2.10}$$

where $C > 0$ (uniformly in κ). Moreover for each $E_\kappa^\alpha \in \Delta$ the quantum mechanical current associated to the corresponding eigenfunctions is strictly positive/negative uniformly in L (see Appendix B)

$$\left| J_{E_\kappa^\alpha} \right| \geq C' > 0 \qquad\qquad \alpha = \ell, r \; . \tag{2.11}$$

Thus, following the classification of the eigenvalues via the quantum mechanical current, the eigenvalues E_κ^α are extended.

Finally we remark that the Hamiltonian $H_L + V_\omega|_{\tilde\Lambda}$ ($\tilde\Lambda \subset \Lambda$) has a point spectrum contained in Landau bands

$$\sigma(H_L + V_\omega|_{\tilde\Lambda}) \subset \bigcup_{n \geq 0} \left[(n + \tfrac{1}{2})B - V_0, (n + \tfrac{1}{2})B + V_0 \right] \; . \tag{2.12}$$

When $\tilde\Lambda$ is given by

$$\Lambda_b \equiv \tilde\Lambda = \left\{ (n,m) \in \mathbb{Z}^2; n \in [-\tfrac{L}{2} + (\tfrac{D}{4} - 1), \tfrac{L}{2} - (\tfrac{D}{4} - 1)], m \in [-\tfrac{L}{2}, \tfrac{L}{2}] \right\}$$

we call the Hamiltonian $H_L + V_\omega|_{\Lambda_b}$ the bulk Hamiltonian and we denote it by H_b.

All these Hamiltonians and (2.1) are densely defined self-adjoint operators.

We now state the main result of this paper.

Theorem 1. *Let V_0 small enough, fix $\varepsilon > 0$ and let $0 < \delta \equiv \delta(V_0) < \frac{B}{2} - V_0 - \varepsilon$. Suppose that $(H1)$ hold. Then there exists $\mu > 0$, $\bar L$ such that if $L > \bar L$ one can find a set $\hat\Omega \subset \Omega_\Lambda$ of realizations of the random potential V_ω with $\mathbb{P}_\Lambda(\hat\Omega) \geq 1 - L^{-\nu}$ ($\nu \gg 1$) such that for all $\omega \in \hat\Omega$ the spectrum of H_ω in $\Delta = (B - \delta, B + \delta)$ is the unions of two sets Σ_ℓ and Σ_r with the following properties:*

a) $\mathcal{E}_\kappa^\alpha \in \Sigma_\alpha$ ($\alpha = \ell, r$) are a small perturbation of $E_\kappa^\alpha \in \sigma(H_\alpha) \cap \Delta$ with

$$|\mathcal{E}_\kappa^\alpha - E_\kappa^\alpha| \leq e^{-\mu\sqrt{B}\sqrt{L}} \; . \tag{2.13}$$

b) For $\mathcal{E}^\alpha_\kappa \in \Sigma_\alpha$ the current $J_{\mathcal{E}^\alpha_\kappa}$ of the associated eigenstate satisfies

$$|J_{\mathcal{E}^\alpha_\kappa} - J_{E^\alpha_\kappa}| \leq e^{-\mu\sqrt{B}\sqrt{L}} \, . \tag{2.14}$$

That is the eigenfunctions associated to the eigenvalues (of H_ω) in Δ are extended.

The main tools for the proof of Theorem 1 are given in section 3. Basically they consist in a Wegner estimate for the random Hamiltonians H_α ($\alpha = \ell, r$) and a decoupling scheme that links the resolvent of the full Hamiltonian H_ω to those of H_ℓ, H_r and H_b. In section 4 we prove two propositions that give part *a*) and *b*) of Theorem 1. Finally in appendix A we prove some technical results, in appendix B we prove (2.11) and in appendix C we discuss the Hypothesis 1.

We introduce a useful notation. Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right]$, then one can check that

$$|\boldsymbol{x} - \boldsymbol{x}'|_\star \equiv \inf_{n \in \mathbb{Z}} \sqrt{(x-x')^2 + (y - y' - nL)^2} \tag{2.15}$$

has the properties of a distance on $\mathbb{R} \times \mathbb{S}_L$ and that it is related to the Euclidian distance $|\boldsymbol{x} - \boldsymbol{x}'| \equiv \sqrt{(x-x')^2 + (y-y')^2}$ by

$$|\boldsymbol{x} - \boldsymbol{x}'|_\star \leq |\boldsymbol{x} - \boldsymbol{x}'| \, . \tag{2.16}$$

The interest of $|\cdot|_\star$ is that, since we are working with a cylindrical geometry all decay estimates are naturally expressed in terms of this distance.

3 Wegner Estimates and Decoupling Scheme

We first give a Wegner estimate for the Hamiltonians H_α ($\alpha = \ell, r$). We introduce some notations. Denote by P^α_{0k} the projector of H^0_α onto the eigenvalue E^α_{0k} and let $\Delta_\alpha = \bigcup_{\tilde{k}_0 \leq k \leq \tilde{k}_1} \left(E^\alpha_{0k-1} + \delta_0, E^\alpha_{0k} - \delta_0\right) \equiv \bigcup_{\tilde{k}_0 \leq k \leq \tilde{k}_1} I_k$, for some $-\infty \ll \tilde{k}_0, \tilde{k}_1 \ll \infty$ and $\delta_0 \ll \frac{C_0}{L}$.

Proposition 1. *Let* V_0 *sufficiently small with respect to* B, $E \in \Delta_\alpha \cap \Delta_\varepsilon$ *and* $I = [E-\bar\delta, E+\bar\delta] \subset I_k$. *Then*

$$\mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{dist}(\sigma(H_\alpha),E) < \bar\delta\right\} \leq \|h\|_\infty \bar\delta \operatorname{dist}(I, E^\alpha_{0\bar k})^{-2} V_0^2 L^4 \tag{3.1}$$

where $E^\alpha_{0\bar k}$ *is the closest eigenvalue of* $\sigma(H^0_\alpha)$ *to the interval* I.

Proof. We first observe that $V_{\boldsymbol{i}}^{1/2} P^\alpha_{0k} V_{\boldsymbol{j}}^{1/2}$ is trace class. Indeed, using $\|AB\|_i \leq \|A\|\|B\|_i$ $(i = 1,2)$ and $\|AB\|_1 \leq \|A\|_2\|B\|_2$ we get $\|V_{\boldsymbol{i}}^{1/2} P^\alpha_{0k} V_{\boldsymbol{j}}^{1/2}\|_1 \leq \|V_{\boldsymbol{i}}^{1/2} P^\alpha_{0k}\|_2 \|P^\alpha_{0k} V_{\boldsymbol{j}}^{1/2}\|_2 \leq V_0 \|P^\alpha_{0k}\|_1^2 \leq V_0$.

We have $E \in \Delta_\alpha \cap \Delta_\varepsilon$, and $I = [E-\bar\delta, E+\bar\delta]$ for $\bar\delta$ small enough (we require that $I \subset \Delta_\alpha \cap \Delta_\varepsilon$). By the Chebyshev inequality we have

$$\mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{dist}(\sigma(H_\alpha),E) < \bar\delta\right\} = \mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{Tr} P_\alpha(I) \geq 1\right\} \leq \mathbb{E}_{\Lambda_\alpha}\left\{\operatorname{Tr} P_\alpha(I)\right\} \tag{3.2}$$

where $\mathbb{E}_{\Lambda_\alpha}$ is the expectation with respect to the random variables in Λ_α. We first give an estimate on $\operatorname{Tr} P_\alpha(I)$. Let $E^\alpha_{0\bar k}$ the closest eigenvalue of $\sigma(H^0_\alpha)$ to I and k_i $(i = 0,1)$ s.t. $\operatorname{dist}(E^\alpha_{0\bar k}, E^\alpha_{0k_i}) = \mathcal{O}(1)$ (we have $\mathcal{O}(L)$ $k's$ in between $k_0 < k_1$). Let also $P^\alpha_> = \sum_{k>k_1} P^\alpha_{0k}$ and $P^\alpha_< = \sum_{k<k_0} P^\alpha_{0k}$.
Using $P^\alpha_>(H^0_\alpha - E)P^\alpha_> \geq 0$ and $P^\alpha_> R^0_\alpha(E) P^\alpha_> \leq \operatorname{dist}(E^\alpha_{0k_1+1}, E)^{-1} P^\alpha_>$ we can write

$$\begin{aligned} P_\alpha(I)P^\alpha_>P_\alpha(I) &= P_\alpha(I)P^\alpha_>(H_\alpha - E)^{1/2} R^0_\alpha(E)(H_\alpha - E)^{1/2} P^\alpha_> P_\alpha(I) \\ &\leq \operatorname{dist}(E^\alpha_{0k_1+1}, E)^{-1}\left[P_\alpha(I)(H_\alpha - E)P^\alpha_> P_\alpha(I) - P_\alpha(I)V^\alpha_\omega P^\alpha_> P_\alpha(I)\right] \end{aligned} \tag{3.3}$$

and thus

$$\|P_\alpha(I)P^\alpha_>P_\alpha(I)\| \leq \operatorname{dist}(E^\alpha_{0k_1+1}, E)^{-1}\left(\tfrac{|I|}{2} + V_0\right) \leq \tfrac{1}{4} \tag{3.4}$$

if, as we can suppose, V_0 is sufficiently small ($\operatorname{dist}(E^\alpha_{0k_1+1}, E)^{-1} V_0 = \mathcal{O}\left(\frac{V_0}{B}\right)$). In a similar way we get

$$\|P_\alpha(I)P^\alpha_<P_\alpha(I)\| \leq \operatorname{dist}(E^\alpha_{0k_0-1}, E)^{-1}\left(\tfrac{|I|}{2} + V_0\right) \leq \tfrac{1}{4}\ . \tag{3.5}$$

Using $1 = P^\alpha_< + P^\alpha_> + \sum_{k_0 \leq k \leq k_1} P^\alpha_{0k}$, (3.4) and (3.5) we obtain

$$\operatorname{Tr} P_\alpha(I) \leq 2 \sum_{k_0 \leq k \leq k_1} \operatorname{Tr} P_\alpha(I) P^\alpha_{0k} P_\alpha(I)\ \ . \tag{3.6}$$

Since

$$\operatorname{dist}(I,E^\alpha_{0k})^2 P_\alpha(I)^2 \le (P_\alpha(I)(H_\alpha - E^\alpha_{0k})P_\alpha(I))^2 \tag{3.7}$$

and $\operatorname{dist}(I,E^\alpha_{0k})^{-1} \le \operatorname{dist}(I,E^\alpha_{0\bar k})^{-1}$ ($\forall\, k_0 \le k \le k_1$) it follows that

$$\begin{aligned} \operatorname{Tr} P^\alpha_{0k} P_\alpha(I) P^\alpha_{0k} &\le \operatorname{dist}(I,E^\alpha_{0\bar k})^{-2} \times \\ &\times \operatorname{Tr}(P^\alpha_{0k}P_\alpha(I)(H_\alpha - E^\alpha_{0k})P_\alpha(I)(H_\alpha - E^\alpha_{0k})P_\alpha(I)P^\alpha_{0k}) \\ &\le \operatorname{dist}(I,E^\alpha_{0\bar k})^{-2} \operatorname{Tr}(P^\alpha_{0k} V^\alpha_\omega P_\alpha(I) V^\alpha_\omega P^\alpha_{0k}) \ . \end{aligned} \tag{3.8}$$

Thus

$$\operatorname{Tr} P_\alpha(I) \le 2\cdot\mathcal{O}(L)\cdot\operatorname{dist}(I,E^\alpha_{0\bar k})^{-2} \sup_{k_0\le k\le k_1} \operatorname{Tr}(P^\alpha_{0k}V^\alpha_\omega P_\alpha(I)V^\alpha_\omega P^\alpha_{0k}) \tag{3.9}$$

and then

$$\mathbb{E}_{\Lambda_\alpha}\{\operatorname{Tr} P_\alpha(I)\} \le 2\cdot\mathcal{O}(L)\cdot\operatorname{dist}(I,E^\alpha_{0\bar k})^{-2} \sup_{k_0\le k\le k_1} \mathbb{E}_{\Lambda_\alpha}\{\operatorname{Tr}(P^\alpha_{0k}V^\alpha_\omega P_\alpha(I)V^\alpha_\omega P^\alpha_{0k})\} \ . \tag{3.10}$$

Writing $V^\alpha_\omega = \sum_{\boldsymbol{i}\in\Lambda_\alpha} X_{\boldsymbol{i}}(\omega)V_{\boldsymbol{i}}$

$$\begin{aligned} \operatorname{Tr} P^\alpha_{0k}V^\alpha_\omega P_\alpha(I)V^\alpha_\omega P^\alpha_{0k} &= \sum_{\boldsymbol{i},\boldsymbol{j}\in\Lambda^2_\alpha} X_{\boldsymbol{i}}(\omega)X_{\boldsymbol{j}}(\omega)\operatorname{Tr} P^\alpha_{0k}V_{\boldsymbol{i}}P_\alpha(I)V_{\boldsymbol{j}}P^\alpha_{0k} \qquad (3.11) \\ &= \sum_{\boldsymbol{i},\boldsymbol{j}\in\Lambda^2_\alpha} X_{\boldsymbol{i}}(\omega)X_{\boldsymbol{j}}(\omega)\operatorname{Tr} V^{1/2}_{\boldsymbol{j}}P^\alpha_{0k}V^{1/2}_{\boldsymbol{i}}V^{1/2}_{\boldsymbol{i}}P_\alpha(I)V^{1/2}_{\boldsymbol{j}} \ . \end{aligned}$$

Since $V^{1/2}_{\boldsymbol{j}}P^\alpha_{0k}V^{1/2}_{\boldsymbol{i}}$ is trace class we can introduce the singular value decomposition

$$V^{1/2}_{\boldsymbol{j}}P^\alpha_{0k}V^{1/2}_{\boldsymbol{i}} = \sum_{n=0}^\infty \mu_n(u_n,.)v_n \tag{3.12}$$

where $\sum_{n=0}^\infty \mu_n = \|V^{1/2}_{\boldsymbol{j}}P^\alpha_{0k}V^{1/2}_{\boldsymbol{i}}\|_1$. Then

$$\begin{aligned} \operatorname{Tr} V^{1/2}_{\boldsymbol{j}}P^\alpha_{0k}V^{1/2}_{\boldsymbol{i}}V^{1/2}_{\boldsymbol{i}}P_\alpha(I)V^{1/2}_{\boldsymbol{j}} &= \sum_{n=0}^\infty \mu_n(u_n, V^{1/2}_{\boldsymbol{i}}P_\alpha(I)V^{1/2}_{\boldsymbol{j}}v_n) \\ \le \sum_{n=0}^\infty \mu_n(v_n, V^{1/2}_{\boldsymbol{j}}P_\alpha(I)V^{1/2}_{\boldsymbol{j}}v_n)^{1/2}&(u_n, V^{1/2}_{\boldsymbol{i}}P_\alpha(I)V^{1/2}_{\boldsymbol{i}}u_n)^{1/2} \\ \le \tfrac{1}{2}\sum_{n=0}^\infty \mu_n\left\{(v_n, V^{1/2}_{\boldsymbol{j}}P_\alpha(I)V^{1/2}_{\boldsymbol{j}}v_n)\right.&\left. + (u_n, V^{1/2}_{\boldsymbol{i}}P_\alpha(I)V^{1/2}_{\boldsymbol{i}}u_n)\right\} \ . \end{aligned} \tag{3.13}$$

An application of the spectral averaging theorem (see [CH]) shows that

$$\mathbb{E}_{\Lambda_\alpha}\{(v_n, V_{\boldsymbol{j}}^{1/2} P_\alpha(I) V_{\boldsymbol{j}}^{1/2} v_n)\} \leq \|h\|_\infty 2\bar{\delta} \tag{3.14}$$

as well as for the term with $\boldsymbol{j}$ replacing $\boldsymbol{i}$ and v_n replacing u_n. Combining (3.10), (3.13), (3.14) and (3.11) we get

$$\begin{aligned}
\mathbb{E}_{\Lambda_\alpha}\{\operatorname{Tr} P_\alpha(I)\} &\leq 4 \cdot \mathcal{O}(L) \cdot \|h\|_\infty \bar{\delta}\, \operatorname{dist}(I, E_{0\bar{k}}^\alpha)^{-2} V_0^2 \sum_{\boldsymbol{i},\boldsymbol{j} \in \Lambda_\alpha^2} \|V_{\boldsymbol{j}}^{1/2} P_{0k}^\alpha V_{\boldsymbol{i}}^{1/2}\|_1 \\
&\leq 4 \cdot \mathcal{O}(L) \cdot \|h\|_\infty \bar{\delta}\, \operatorname{dist}(I, E_{0\bar{k}}^\alpha)^{-2} V_0^2 |\Lambda_\alpha|^2 \,. \qquad (3.15)
\end{aligned}$$

□

We now turn to the decoupling scheme. By a decoupling formula [BG], [BCD] the resolvent $R(z) = (z - H_\omega)^{-1}$ can be expressed, up to a small term, as the sum of $R_\alpha(z) = (z - H_\alpha)^{-1}$ ($\alpha = \ell, r$) and $R_b(z) = (z - H_b)^{-1}$. We set $D = \sqrt{L}$ and introduce the characteristic functions

$$\begin{aligned}
\tilde{J}_\ell(x) &= \chi_{]-\infty, -\frac{L}{2} + \frac{D}{2}]}(x) \qquad \tilde{J}_b(x) = \chi_{[-\frac{L}{2} + \frac{D}{2}, \frac{L}{2} - \frac{D}{2}]}(x) \\
\tilde{J}_r(x) &= \chi_{[\frac{L}{2} - \frac{D}{2}, +\infty[}(x) \,. \qquad (3.16)
\end{aligned}$$

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leq 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $\sup_x |\partial_x^n J_i(x)| \leq 2$, $n = 1, 2$, and such that

$$\begin{aligned}
J_\ell(x) &= \begin{cases} 1 & \text{if } x \leq -\frac{L}{2} + \frac{3D}{4} \\ 0 & \text{if } x \geq -\frac{L}{2} + \frac{3D}{4} + 1 \end{cases} \qquad J_b(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{L}{2} - \frac{D}{4} \\ 0 & \text{if } |x| \geq \frac{L}{2} - \frac{D}{4} + 1 \end{cases} \\
J_r(x) &= \begin{cases} 1 & \text{if } x \geq \frac{L}{2} - \frac{3D}{4} \\ 0 & \text{if } x \leq \frac{L}{2} - \frac{3D}{4} - 1 \end{cases} \,. \qquad (3.17)
\end{aligned}$$

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ and the decoupling formula is [BG]

$$R(z) = \left(\sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) (1 - \mathcal{K}(z))^{-1} \tag{3.18}$$

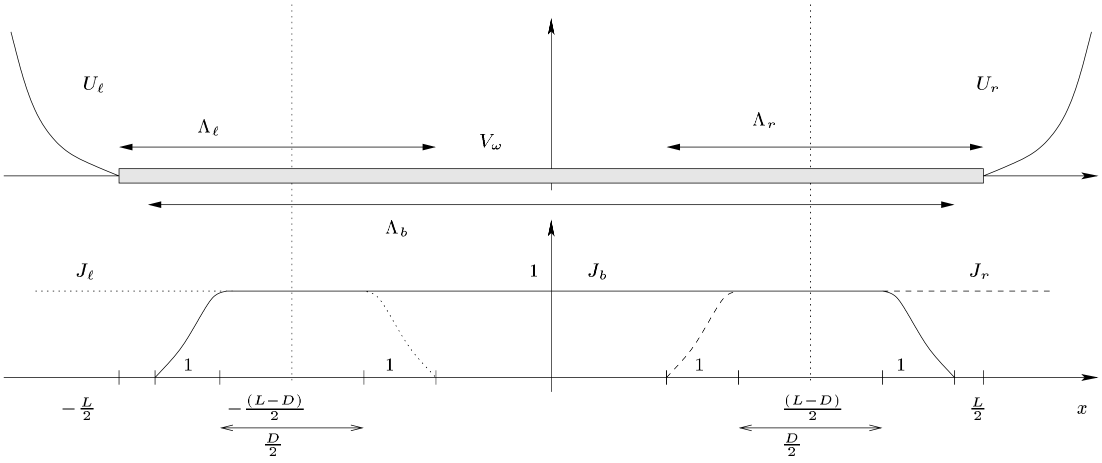

Figure 1: *The system of decoupling functions J_i ($i \in \mathcal{I}$).*

where

$$\mathcal{K}(z) = \sum_{i \in \mathcal{I}} K_i(z) = \sum_{i \in \mathcal{I}} \tfrac{1}{2}[p_x^2, J_i] R_i(z) \tilde{J}_i \; . \tag{3.19}$$

The main result of this part is a lemma about $\|\mathcal{K}(z)\|$ for z such that $\mathrm{dist}(z, \sigma(H_\alpha)) \geq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}$, for $0 < \bar{\mu} < \frac{\bar{\gamma}}{12} - 1$, and $\mathrm{dist}(z, \sigma(H_b)) \geq \varepsilon$.

Proposition 2. *Let $\varepsilon > 0$, and $z \in \Delta_\varepsilon$ such that* $\mathrm{dist}(z, \sigma(H_\ell) \cup \sigma(H_r)) \geq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}$. *Then for L large enough there exists $C(B, V_0, \varepsilon) > 0$ and $\tilde{\gamma} > 0$ independent of L such that*

$$\|\mathcal{K}(z)\| \leq C(B, V_0, \varepsilon) e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} \; . \tag{3.20}$$

Proof. Computing the commutator in the definition of $K_i(z)$ we have

$$K_i(z) \;\; = \;\; -\tfrac{1}{2}(\partial_x^2 J_i) R_i(z) \tilde{J}_i - (\partial_x J_i) \partial_x R_i(z) \tilde{J}_i \tag{3.21}$$

and then

$$\|K_b(z)\| \;\; \leq \;\; \tfrac{1}{2}\|(\partial_x^2 J_b) R_b(z) \tilde{J}_b\| + \|(\partial_x J_b) \partial_x R_b(z) \tilde{J}_b\| \tag{3.22}$$

$$\begin{aligned} \|K_\alpha(z)\| \;\; &\leq \;\; \tfrac{1}{2}\|(\partial_x^2 J_\alpha) R_\alpha^b(z) \tilde{J}_\alpha\| + \tfrac{1}{2}\|(\partial_x^2 J_\alpha) R_\alpha^b(z) U_\alpha\| \; \mathrm{dist}(z, \sigma(H_\alpha))^{-1} \\ &+ \;\; \|(\partial_x J_\alpha) \partial_x R_\alpha^b(z) \tilde{J}_\alpha\| + \|(\partial_x J_\alpha) \partial_x R_\alpha^b(z) U_\alpha\| \; \mathrm{dist}(z, \sigma(H_\alpha))^{-1} \end{aligned} \tag{3.23}$$

where $R^b_\alpha(z) = (z - [H_L + V^\alpha_\omega])^{-1}$.

we have to estimate norms of the type $\|f \partial^\alpha_x R(z) g\|$ $(\alpha = 0, 1)$ where here $\tilde{R}(z)$ is $R_b(z)$ or $R^b_\alpha(z)$, $f = \partial^m_x J_i$ and $g = \tilde{J}_i$ or $g = U_\alpha$.

Using the second resolvent formula we develop $\tilde{R}(z)$ in its series, denote $V_\omega|_{\tilde{\Lambda}} \equiv W$ $(\tilde{\Lambda} = \Lambda_b$ or $\Lambda_\alpha)$

$$\tilde{R}(z) = \sum_{n=0}^{\infty} R_0(z)[W R_0(z)]^n \tag{3.24}$$

where $R_0(z) = (z - H_L)^{-1}$. The norm convergence is insured since we are in a spectral gap, indeed

$$\|W R_0(z)\| \le V_0 \operatorname{dist}(z, \sigma(H_L))^{-1} \le \frac{V_0}{V_0 + \varepsilon} < 1 \, . \tag{3.25}$$

Therefore

$$\|f \partial^\alpha_x \tilde{R}(z) g\| \le \sum_{n=1}^{\infty} \|f \partial^\alpha_x R_0(z) \, [W R_0(z)]^n \, g\| \tag{3.26}$$

and we have to control the operator norms $\|f \partial^\alpha_x R_0(z) \, [W R_0(z)]^n \, g\|$.

We now estimate the norm $(\|\varphi\| = 1)$

$$\|f \partial^\alpha_x R_0(z) \, [W R_0(z)]^n \, g\varphi\|^2 = \int_{\operatorname{supp} f} |f(\boldsymbol{x})|^2 \left[|(\partial^\alpha_x R_0(z) \, [W R_0(z)]^n \, g\varphi)(\boldsymbol{x})|\right]^2 \, \mathrm{d}\boldsymbol{x} \, , \tag{3.27}$$

and

$$\begin{aligned} \mathcal{J} \quad &\equiv \quad |(\partial^\alpha_x R_0(z) \, [W R_0(z)]^n \, g\varphi)(\boldsymbol{x})| \le \int_{\operatorname{supp} g} \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_n \, \times \qquad (3.28) \\ &\times \quad |\partial^\alpha_x R_0(\boldsymbol{x}, \boldsymbol{x}_1; z)||W(\boldsymbol{x_1})||R_0(\boldsymbol{x}_1, \boldsymbol{x}_2; z)| \ldots |W(\boldsymbol{x_n})||R_0(\boldsymbol{x}_n, \boldsymbol{x}'; z)||g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')| \, . \end{aligned}$$

Now from the estimate of Lemma 1, Appendix A and taking out $W(\cdot)$ we get

$$\begin{aligned} \mathcal{J} \quad &\le \quad \underbrace{\left(cB^2 \tfrac{V_0}{V_0+\varepsilon}\right)^n}_{\equiv A^n} \int_{\operatorname{supp} g} \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\{\boldsymbol{x}_i\} e^{-\bar{\gamma}\sqrt{B} \sum_{i=0}^n |\boldsymbol{x}_i - \boldsymbol{x}_{i+1}|_\star} \times \\ &\times \quad |\Phi^1(|\boldsymbol{x} - \boldsymbol{x}_1|_\star)| \ldots |\Phi^0(|\boldsymbol{x}_n - \boldsymbol{x}'|_\star)||g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')| \end{aligned} \tag{3.29}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}$ and $\boldsymbol{x}_{n+1} = \boldsymbol{x}'$. Splitting the the exponential and making the change of variables $\boldsymbol{x} - \boldsymbol{x}_1 = -\boldsymbol{z}_1$, …, $\boldsymbol{x}_{n-1} - \boldsymbol{x}_n = -\boldsymbol{z}_n$ we get (with $\boldsymbol{x}_n = \boldsymbol{x}_n(\{\boldsymbol{z}_i\}, \boldsymbol{x})$)

$$\begin{aligned} \mathcal{J} \quad &\le \quad A^n \sup_{\boldsymbol{z}_1 \ldots \boldsymbol{z}_n} \left\{ \int_{\operatorname{supp} g} e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x} - \boldsymbol{x}'|_\star} |g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')||\Phi^0(|\boldsymbol{x}_n - \boldsymbol{x}'|_\star)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x}_n - \boldsymbol{x}'|_\star} \, \mathrm{d}\boldsymbol{x}' \right\} \times \\ &\times \quad \left[\int_{\mathbb{R}^2} |\Phi^1(|\boldsymbol{z}|)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{z}|} \, \mathrm{d}\boldsymbol{z}\right] \left[\int_{\mathbb{R}^2} |\Phi^0(|\boldsymbol{z}|)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{z}|} \, \mathrm{d}\boldsymbol{z}\right]^{n-1} \, . \end{aligned} \tag{3.30}$$

Using Schwartz inequality and splitting the exponential we get

$$\begin{aligned} \mathcal{J} &\leq A^n[\ldots][\ldots]^{n-1} \sup_{\boldsymbol{x}'\in \operatorname{supp} g} e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x}-\boldsymbol{x}'|_\star} \left\{\int_{\mathbb{R}^2} |\Phi^0(|\boldsymbol{w}|)|^2 e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{w}|}\,\mathrm{d}\boldsymbol{w}\right\}^{1/2} \times \\ &\times \left(\sup_{\boldsymbol{x}'\in \operatorname{supp} g} e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|x-x'|}|g(x')|^2\right)^{1/2} \|\varphi\| , \end{aligned} \tag{3.31}$$

and since U_α do not grow to fast (see (2.2), (2.3)) $(\sup\ldots)^{1/2}$ is bounded by a numerical constant. Moreover the terms $[\ldots]$, $[\ldots]^{n-1}$ and $\{\ldots\}^{1/2}$ are also bounded by a constant $\tilde{C}(B)$ depending only on B and not on L. This leads to

$$\|f\partial_x^\alpha \left[R_0(z)\right]^n g\varphi\| \leq \|f\|_\infty \hat{C}(B)(\tilde{C}A)^n e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}D}\|\varphi\| . \tag{3.32}$$

Therefore, if V_0 (i.e. A) is small enough the series (3.26) converges and

$$\|f\partial_x^\alpha \tilde{R}(z)g\| \leq \tilde{C}(B,V_0)\sqrt{L}e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}D} , \tag{3.33}$$

this imply

$$\|K_b(z)\| \leq \varepsilon^{-1}\sqrt{L}C(B,V_0)e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}\sqrt{L}} \tag{3.34}$$

$$\|K_\alpha(z)\| \leq \sqrt{L}e^{\bar{\mu}\sqrt{B}\sqrt{L}}C(B,V_0)e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}\sqrt{L}} \qquad \alpha = \ell, r \tag{3.35}$$

thus $\|\mathcal{K}(z)\| \leq C(B,V_0,\varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}}$ where $2\tilde{\gamma} = \frac{\bar{\gamma}}{12} - \bar{\mu}$. ☐

By the way we remark that in the proof above we have proved the following statement (see (3.33)) that will be useful in the next section

$$\|(1-\tilde{J}_\alpha)\tilde{R}_b(z)g\| \leq \bar{C}(B,V_0,\varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} . \tag{3.36}$$

where $g = U_\alpha$ or $g = \chi_B$ ($B \subset \mathbb{R}\times[-\frac{L}{2},\frac{L}{2}]$) with $\operatorname{dist}(\operatorname{supp} g, \operatorname{supp}(1-\tilde{J}_\alpha)) = \mathcal{O}(D)$ and $\tilde{R}_b(z)$ a resolvent associated to a generic bulk Hamiltonian $(H_L + V_\omega|_{\tilde{\Lambda}})$.

4 Projector estimates and the proof of Theorem 1

In this section we prove two propositions that lead to Theorem 1. Let $\mathcal{D}' = \{\kappa : E_\kappa^\alpha \in \Delta, \alpha = \ell, r\}$, $\operatorname{card}(\mathcal{D}') = \mathcal{O}(L)$. Where $\Delta \subset \Delta_\varepsilon$ is given in section 2.

Proposition 3. *For L large enough, with probability greater then $1 - L^{-\nu}$, we have for all $\kappa \in \mathcal{D}'$*

$$\|P - P_\alpha(E^\alpha_\kappa)\| \le e^{-\gamma\sqrt{B}\sqrt{L}} \tag{4.1}$$

where $P_\alpha(E^\alpha_\kappa)$ is the projector associated to H_α onto E^α_κ and P is the projector associated to H_ω onto $\{z \in \mathbb{C} : |z - E^\alpha_\kappa| \le e^{-\tilde{\mu}\sqrt{B}\sqrt{L}}\}$.

Proof. (1): Let $\mathcal{E} = \{k : E^\alpha_{0k} \in \Delta\}$ $(\mathrm{card}(\mathcal{E}) = \mathcal{O}(L))$ and let

$$\hat{\Omega}_\ell = \{\omega \in \Omega_{\Lambda_\ell} : \mathrm{dist}(E^r_{0k}, \sigma(H_\ell)) \ge L^{-\sigma}, \forall k \in \mathcal{E}\} \ , \tag{4.2}$$

with $\sigma > 10$, this set has probability

$$\mathbb{P}_{\Lambda_\ell}(\hat{\Omega}_\ell) \ge 1 - L^{-(\sigma-8)} \ . \tag{4.3}$$

Indeed for a fixed $k \in \mathcal{E}$, using Proposition 1 and $(H1)$ one gets

$$\begin{aligned} &\mathbb{P}_{\Lambda_\ell}\left\{\omega \in \Omega_{\Lambda_\ell} : \mathrm{dist}(E^r_{0k}, \sigma(H_\ell)) \ge L^{-\sigma}, \text{ for one } k \in \mathcal{E}\right\} \\ \ge\ & 1 - C'(h, V_0) L^{-\sigma} L^4 \left(\tfrac{d_0}{L} - L^{-\sigma}\right)^{-2} \ge 1 - C(h, V_0) L^{6-\sigma} \ . \end{aligned} \tag{4.4}$$

Let also

$$\hat{\Omega}_r = \{\omega \in \Omega_{\Lambda_r} : \mathrm{dist}(E^\ell_\kappa, \sigma(H_r)) \ge L^{-3\sigma}, \forall \kappa \in \mathcal{D}'\} \ , \tag{4.5}$$

this set has probability

$$\mathbb{P}_{\Lambda_r}(\hat{\Omega}_r) \ge 1 - L^{-(\sigma-6)} \ . \tag{4.6}$$

Indeed

$$\begin{aligned} &\mathbb{P}_{\Lambda_r}\left\{\omega \in \Omega_{\Lambda_r} : \mathrm{dist}(E^\ell_\kappa, \sigma(H_r)) \ge L^{-3\sigma}, \text{ for one } \kappa \in \mathcal{D}'\right\} \\ \ge\ & 1 - C'(h, V_0) L^{-3\sigma} L^4 \left(L^{-\sigma} - L^{-3\sigma}\right)^{-2} \ge 1 - C(h, V_0) L^{4-\sigma} \ . \end{aligned} \tag{4.7}$$

It follows that the set $\hat{\Omega} = \hat{\Omega}_\ell \times \Omega_b \times \hat{\Omega}_r$ $(\Omega_b = \Omega|_{\Lambda_b \setminus (\Lambda_\ell \cup \Lambda_r)})$ has probability

$$\mathbb{P}_\Lambda(\hat{\Omega}) \ge 1 - L^{-\nu} \tag{4.8}$$

with $\nu = \sigma - 9 > 1$.

(2): We now work with a given $\omega \in \hat{\Omega}$. Take $\bar{\mu} > 0$ as in Proposition 2 and L large enough such that for all $\kappa \in \mathcal{D}'$ $\Gamma_\kappa = \{z \in \mathbb{C} : |z - E^\ell_\kappa| \leq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\} \cap \sigma(H_r) = \emptyset$, and remark that $\operatorname{Tr} P_b(\Delta) = 0$ (P_b the projector associated to H_b).

We need to introduce two auxiliary Hamiltonians H_1 and H_2 defined as follows:

$$H_1 = H_L + V^\ell_\omega|_{\Lambda_1} \tag{4.9}$$

$$H_2 = H_L + V^\ell_\omega|_{\Lambda_2} + U_\ell \tag{4.10}$$

where $\Lambda_2 = \left\{(n,m) \in \mathbb{Z}^2; n \in [-\frac{L}{2}, -\frac{L}{2} + (\frac{D}{4} - 1)], m \in [-\frac{L}{2}, \frac{L}{2}]\right\}$, and $\Lambda_1 = \Lambda_\ell \backslash \Lambda_2$, of course $H_\alpha = H_2 + V^\ell_\omega|_{\Lambda_1}$.

From the decoupling formula (3.18) we have

$$\begin{aligned} R(z) - R_\ell(z) &= \left(\sum_{i\in\mathcal{I}} J_i R_i(z) \tilde{J}_i\right)\left(\sum_{n=1}^{\infty} \mathcal{K}(z)^n\right) - (1 - J_\ell) R_\ell(z) \\ &- J_\ell R_\ell(z)(1 - \tilde{J}_\ell) + J_b R_b(z) \tilde{J}_b + J_r R_r(z) \tilde{J}_r \ . \end{aligned} \tag{4.11}$$

integrating over $\partial\Gamma_\kappa$ and taking the operator norm we get

$$\begin{aligned} \|P - P_\ell(E^\ell_\kappa)\| &\leq e^{-\bar{\mu}\sqrt{B}\sqrt{L}} \left(\sum_{i\in\mathcal{I}} \sup_{z\in\partial\Gamma_\kappa} \|R_i(z)\|\right) \frac{\sup_{z\in\partial\Gamma_\kappa} \|\mathcal{K}(z)\|}{1 - \sup_{z\in\partial\Gamma_\kappa} \|\mathcal{K}(z)\|} \\ &+ \|(1 - J_\ell) P_\ell(E^\ell_\kappa)\| + \|J_\ell P_\ell(E^\ell_\kappa)(1 - \tilde{J}_\ell)\| \\ &= a + b + c \ . \end{aligned} \tag{4.12}$$

For the first term we note that for L large enough $e^{-\bar{\mu}\sqrt{B}\sqrt{L}} \sup_{z\in\partial\Gamma_\kappa} \|R_i(z)\| \leq 1$ ($i \in \mathcal{I}$). Indeed, for $i = \ell$ we have $\sup_{z\in\partial\Gamma_\kappa} \|R_\ell(z)\| = e^{\bar{\mu}\sqrt{B}\sqrt{L}}$ by construction, for $i = b$ we have $\sup_{z\in\partial\Gamma_\kappa} \|R_b(z)\| = \varepsilon^{-1}$ and for $i = r$ $\sup_{z\in\partial\Gamma_\kappa} \|R_r(z)\| = \left(L^{-3\sigma} - e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\right)^{-1}$. Then, applying Proposition 2 we get

$$a \leq 2C(B, V_0, \varepsilon) e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} \ . \tag{4.13}$$

For the second and third term we first observe that by the second resolvent formula

$$\frac{P_\ell(E^\ell_\kappa)}{(z - E^\ell_\kappa)} = (z - H_1)^{-1} P_\ell(E^\ell_\kappa) + (z - H_1)^{-1} [V^\ell_\omega|_{\Lambda_2} + U_\ell] \frac{P_\ell(E^\ell_\kappa)}{(z - E^\ell_\kappa)} \ . \tag{4.14}$$

and integrating (4.14) along $\partial\Gamma\kappa$ we obtain (using $\sigma(H_1) \cap \Delta_\varepsilon = \emptyset$)

$$\begin{aligned} P_\ell(E^\ell_\kappa) &= R_1(E^\ell_\kappa)[V^\ell_\omega|_{\Lambda_2} + U_\ell] P_\ell(E^\ell_\kappa) & (4.15) \\ &= P_\ell(E^\ell_\kappa)[V^\ell_\omega|_{\Lambda_2} + U_\ell] R_1(E^\ell_\kappa) \ . & (4.16) \end{aligned}$$

Therefore, using (4.15) for b and (4.16) for c we get

$$b \leq \|(1-J_\ell)R_1(E^\ell_\kappa)[V^\ell_\omega|_{\Lambda_2} + U_\ell]\| \leq \|(1-\tilde{J}_\ell)R_1(E^\ell_\kappa)[V^\ell_\omega|_{\Lambda_2} + U_\ell]\| \tag{4.17}$$

$$c \leq \|(1-\tilde{J}_\ell)R_1(E^\ell_\kappa)[V^\ell_\omega|_{\Lambda_2} + U_\ell]\| \ . \tag{4.18}$$

Using (3.36) we get

$$\begin{aligned} b + c &\leq 2\left(V_0 L^2 \|(1-\tilde{J}_\ell)R_1(E^\ell_\kappa)\chi_{\Lambda_2}\| + \|(1-\tilde{J}_\ell)R_1(E^\ell_\kappa)U_\ell\|\right) \\ &\leq 2\bar{C}(B, V_0, \varepsilon)L^2 e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} \ . \end{aligned} \tag{4.19}$$

Thus

$$\|P - P_\ell(E^\ell_\kappa)\| \leq e^{-\gamma\sqrt{B}\sqrt{L}} \ . \tag{4.20}$$

Now take $\bar{\Gamma}_\kappa = \{z \in \mathbb{C} : |z - E^r_\kappa| \leq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\}$, then $\bar{\Gamma}_\kappa \cap \sigma(H_\ell) = \emptyset$ and exactly as for (4.20) we get

$$\|P - P_r(E^r_\kappa)\| \leq e^{-\gamma\sqrt{B}\sqrt{L}} \ . \tag{4.21}$$

□

Since the norm difference of the projectors is strictly less then 1 (for L large enough) their dimensions are the same, thus each $\mathcal{E}^\alpha_\kappa \in \sigma(H_\omega)$ is a small perturbation of E^α_κ: this give part *a*) of Theorem 1.

Proposition 4. *Let* $\omega \in \hat{\Omega}$. *Then there exists* $\hat{\mu} > 0$ *such that the current associated to each eigenvalue* $\mathcal{E}^\alpha_\kappa$ *of* H_ω *in* Δ *satisfies*

$$\left|J_{\mathcal{E}^\alpha_\kappa} - J_{E^\alpha_\kappa}\right| \leq e^{-\hat{\mu}\sqrt{B}\sqrt{L}} \ . \tag{4.22}$$

Proof. Let $J_{\mathcal{E}^\alpha_\kappa} = \operatorname{Tr} v_y P(\mathcal{E}^\alpha_\kappa)$ the current associated to the eigenvalue $\mathcal{E}^\alpha_\kappa \in \sigma(H_\omega)$ and $J_{E^\alpha_\kappa} = \operatorname{Tr} v_y P_\alpha(E^\alpha_\kappa)$ that associated to the eigenvalue E^α_κ of H_α. First we observe that $v_y P(\mathcal{E}^\alpha_\kappa)$ is trace class. Indeed, $v_y P(\mathcal{E}^\alpha_\kappa) = v_y P(\mathcal{E}^\alpha_\kappa) P(\mathcal{E}^\alpha_\kappa)$ with $v_y P(\mathcal{E}^\alpha_\kappa)$ bounded and $\|P(\mathcal{E}^\alpha_\kappa)\|_1 = \operatorname{Tr} P(\mathcal{E}^\alpha_\kappa) = \operatorname{Tr} P_\alpha(E^\alpha_\kappa) = 1$.

$$\begin{aligned} \|v_y P(\mathcal{E}^\alpha_\kappa)\|_1^2 &\leq \|v_y P(\mathcal{E}^\alpha_\kappa)\|^2 \leq \|P(\mathcal{E}^\alpha_\kappa) v_y^2 P(\mathcal{E}^\alpha_\kappa)\| \\ &\leq 2\|P(\mathcal{E}^\alpha_\kappa)(H_\omega - V_\omega)P(\mathcal{E}^\alpha_\kappa)\| \leq (3B + 2V_0) \end{aligned} \tag{4.23}$$

to get the second inequality one has simply added positive terms to v_y^2. Similarly

$$\|v_y P_\alpha(E_\kappa^\alpha)\|_1^2 \le (3B + 2V_0) \ . \tag{4.24}$$

With the help of the identity

$$\begin{aligned} P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha) &= [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]^2 + [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]P_\alpha(E_\kappa^\alpha) \\ &+ P_\alpha(E_\kappa^\alpha)[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)] \end{aligned} \tag{4.25}$$

we get

$$\begin{aligned} |J_{\mathcal{E}_\kappa^\alpha} - J_{E_\kappa^\alpha}| &= |\mathrm{Tr}\, v_y[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]| \le \left|\mathrm{Tr}\, v_y[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]^2\right| \\ &+ |\mathrm{Tr}\, v_y[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]P_\alpha(E_\kappa^\alpha)| \\ &+ |\mathrm{Tr}\, v_y P_\alpha(E_\kappa^\alpha)[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]| \ . \end{aligned} \tag{4.26}$$

and then, from (4.23) and (4.24), we get

$$\begin{aligned} |J_{\mathcal{E}_\kappa^\alpha} - J_{E_\kappa^\alpha}| &\le 2\left(\|v_y P(\mathcal{E}_\kappa^\alpha)\|_1 + \|v_y P_\alpha(E_\kappa^\alpha)\|_1\right)\|P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)\| \qquad (4.27) \\ &\le 4\sqrt{2} V_0 L^{3/2}(3B + 2V_0)^{1/2}\|P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)\| \ . \end{aligned}$$

Combining this last inequality with Proposition 3 we get the result. ☐

From Proposition 4 and the result of Appendix B given in (2.11) we see that the eigenvalues $\mathcal{E}_\kappa^\alpha$ are extended: this leads to part *b)* of Theorem 1.

A Estimate of the Green function $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$

In this appendix we give the necessary decay property of the kernel $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$ with periodic boundary conditions along y. The exact formula for $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$ can be found in [FM1]. We introduce the following notation

$$\begin{aligned} &\Phi^\alpha(|\boldsymbol{x} - \boldsymbol{x}'|_\star) \\ = &\begin{cases} 1 + \left|\ln\left(\frac{B}{2}|\boldsymbol{x} - \boldsymbol{x}'|_\star^2\right)\right| & , \ \alpha = 0 \\ 1 + \left[\left|\ln\left(\frac{B}{2}|\boldsymbol{x} - \boldsymbol{x}'|_\star^2\right)\right| + \left(1 + \left|\ln\left(\frac{B}{2}|\boldsymbol{x} - \boldsymbol{x}'|_\star^2\right)\right|\right)|\boldsymbol{x} - \boldsymbol{x}'|_\star^{-1}\right] & , \ \alpha = 1 \ . \end{cases} \end{aligned} \tag{A.1}$$

Lemma 1. *If* $|\mathcal{I}m\, z| \leq 1$, $\mathcal{R}e\, z \in \left]\frac{1}{2}B, \frac{3}{2}B\right[$ *then, for* L *large enough, there exists* $C(z,B)$ *positive constant independent of* L *such that* $(\alpha = 0,1)$

$$\begin{aligned} |\partial_x^\alpha R_0(\boldsymbol{x}, \boldsymbol{x}'; z)| &\leq C'(z,B) e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|_\star^2} \Phi^\alpha(|\boldsymbol{x}-\boldsymbol{x}'|_\star) \\ &\leq C(z,B) e^{-\bar{\gamma}\sqrt{B}|\boldsymbol{x}-\boldsymbol{x}'|_\star} \Phi^\alpha(|\boldsymbol{x}-\boldsymbol{x}'|_\star) \end{aligned} \tag{A.2}$$

where $C(z,B) = cB^2 \operatorname{dist}(z, \sigma(H_L))^{-1}$ *with* c *a numerical positive constant and* $\bar{\gamma} = \frac{1}{16}$.

Proof. As in [FM1] we can prove that (for L large enough the logarithmic divergences appear only for $|m| \leq 1$ and the sum over $|m| > 1$ converge)

$$|\partial_x^\alpha R_0(\boldsymbol{x}, \boldsymbol{x}'; z)| \leq \tfrac{C'(z,B)}{3} e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2} + \sum_{|m|\leq 1} |\partial_x^\alpha R_0^\infty(x\, y - mL, \boldsymbol{x}'; z)| \tag{A.3}$$

with

$$\begin{aligned} &|\partial_x^\alpha R_0^\infty(\boldsymbol{x}, \boldsymbol{x}'; z)| \qquad\qquad (\text{A.4}) \\ \leq\ &\begin{cases} \tfrac{C'(z,B)}{3} e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2} \left\{1 + \mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}(|\boldsymbol{x}-\boldsymbol{x}'|) \left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right|\right\}, & \alpha = 0 \\ \tfrac{C'(z,B)}{3} e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2} \Big\{1 + \mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}(|\boldsymbol{x}-\boldsymbol{x}'|) \Big[\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right| & \\ + \left(1 + \left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right|\right) |\boldsymbol{x}-\boldsymbol{x}'|^{-1}\Big]\Big\}, & \alpha = 1\,. \end{cases} \end{aligned}$$

Now, using $|\boldsymbol{x}-\boldsymbol{x}'|_\star \leq |\boldsymbol{x}-\boldsymbol{x}'|$, we can replace the Euclidean distance with the distance $|\cdot|_\star$ in all the terms in the RHS of (A.3), since all these functions are decreasing. To obtain the same bound for the terms $|m| \leq 1$ in the sum we just drop the characteristic functions $\mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}$. ☐

B Extended nature of E_κ^α

In this appendix we prove following [F] that the eigenstates corresponding to the eigenvalues of H_α $(\alpha = \ell, r)$ in a energy interval $\Delta = (B-\delta, B+\delta) \subset \Delta_\varepsilon$ are extended in the sense that its current is strictly positive/negative uniformly in L, that is, if we have $H_\alpha \psi_\kappa^\alpha = E_\kappa^\alpha \psi_\kappa^\alpha$ then

$$|(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| \geq C' > 0\,. \tag{B.1}$$

From the eigenvalue equation we have

$$\|(H_\alpha^0 - E_\kappa^\alpha)\psi_\kappa^\alpha\|^2 = \|V_\omega^\alpha \psi_\kappa^\alpha\|^2 \leq V_0^2 \ . \tag{B.2}$$

We now develop ψ_κ^α of the eigenfunctions of H_α^0 denoted $\left\{\phi_{nk}(x,y) = \frac{e^{iky}}{\sqrt{L}}\varphi_{nk}(x)\right\}_{n\in\mathbb{N}, k\in\frac{2\pi}{L}\mathbb{Z}}$ where φ_{nk} is the solution on the eigenvalue problem $[\frac{1}{2}p_x^2 + \frac{1}{2}(k-Bx)^2 + U_\alpha]\varphi_{nk} = E_{nk}^\alpha \varphi_{nk}$.

$$\psi_\kappa^\alpha(x,y) = \sum_{n=0}^{\infty} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} \psi_n(k)\phi_{nk}(x,y) \ , \tag{B.3}$$

and of course

$$\|\psi_\kappa^\alpha\|^2 = \sum_{n=0}^{\infty} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} |\psi_n(k)|^2 = 1 \ . \tag{B.4}$$

From (B.3) the equation (B.2) becomes

$$\sum_{n=0}^{\infty} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} |\psi_n(k)|^2 \left(E_{nk}^\alpha - E_\kappa^\alpha\right)^2 \leq V_0^2 \tag{B.5}$$

thus since each term in the sum is positive we have

$$\sum_{k\in\frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 \left(E_{0k}^\alpha - E_\kappa^\alpha\right)^2 \leq V_0^2 \tag{B.6}$$

We remark that for $n \geq 1$ one has $|E_{nk}^\alpha - E_\kappa^\alpha| \geq \frac{B}{2} - \delta$, this leads to

$$\|\psi_\star\|^2 \equiv \sum_{n=1}^{\infty} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} |\psi_n(k)|^2 \leq \frac{V_0^2}{(\frac{B}{2} - \delta)^2} \ . \tag{B.7}$$

Let $k^\star$ such that $|E_{0k^\star}^\alpha - E_\kappa^\alpha|$ is minimal, and for a fixed a independent of L let $\mathcal{A} = [k^\star - a, k^\star + a]$. Then from (B.5)

$$\begin{aligned} V_0^2 &\geq \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 \left(E_{0k}^\alpha - E_\kappa^\alpha\right)^2 \geq \sum_{k\in\mathcal{A}^c} |\psi_0(k)|^2 \left(E_{0k}^\alpha - E_\kappa^\alpha\right)^2 \\ &\geq \inf_{k\in\mathcal{A}^c} \left(E_{0k}^\alpha - E_\kappa^\alpha\right)^2 \sum_{k\in\mathcal{A}^c} |\psi_0(k)|^2 \end{aligned} \tag{B.8}$$

thus

$$\sum_{k\in\mathcal{A}^c} |\psi_0(k)|^2 \leq V_0^2 \sup_{k\in\mathcal{A}^c} \left(E_{0k}^\alpha - E_\kappa^\alpha\right)^{-2} \ . \tag{B.9}$$

From (B.4) and (B.7) we get

$$1 \geq \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 \geq 1 - \tfrac{V_0^2}{(\frac{B}{2}-\delta)^2} \ . \tag{B.10}$$

Combining the last equation and (B.9) we get

$$\sum_{k \in \mathcal{A}} |\psi_0(k)|^2 \geq 1 - V_0^2 \left[\tfrac{1}{(\frac{B}{2}-\delta)^2} + \sup_{k \in \mathcal{A}^c} \left(E^\alpha_{0k} - E^\alpha_\kappa\right)^{-2} \right] \ . \tag{B.11}$$

Decompose now ψ^α_κ as $\psi^\alpha_\kappa = \psi_0 + \psi_\star$, then

$$|(\psi^\alpha_\kappa, v_y \psi^\alpha_\kappa)| \geq |(\psi_0, v_y \psi_0)| - |(\psi_\star, v_y \psi_\star)| - 2|(\psi_\star, v_y \psi_0)| \tag{B.12}$$

the first term can be written as

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}x \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y \left\{ \sum_{k' \in \frac{2\pi}{L}\mathbb{Z}} \psi_0^*(k') \frac{e^{-ik'y}}{\sqrt{L}} \varphi^*_{0k'}(x) \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \psi_0(k) v_y \frac{e^{iky}}{\sqrt{L}} \varphi_{0k}(x) \right\} \\
= & \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 \int_{\mathbb{R}} \mathrm{d}x \, (k - Bx) \, |\varphi_{0k}(x)|^2 \\
= & \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 \partial_{\hat{k}} E^\alpha_0(\hat{k}) \Big|_{\hat{k}=k}
\end{aligned} \tag{B.13}$$

The partial derivative of E^α_0 is the current $\partial_{\hat{k}} E^\alpha_0(\hat{k}) \Big|_{\hat{k}=k} = J_{E^\alpha_{0k}}$, thus

$$\begin{aligned}
|(\psi_0, v_y \psi_0)| &\geq \left| \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} |\psi_0(k)|^2 J_{E^\alpha_{0k}} \right| \\
&\geq |J_{E^\alpha_{0\bar{k}}}| \left\{ 1 - V_0^2 \left[\tfrac{1}{(\frac{B}{2}-\delta)^2} + \sup_{k \in \mathcal{A}^c} \left(E^\alpha_{0k} - E^\alpha_\kappa\right)^{-2} \right] \right\}
\end{aligned} \tag{B.14}$$

for a suitable $\bar{k} \in \mathcal{A}$, and we have $|J_{E^\alpha_{0\bar{k}}}| > 0$. The second term can be bounded as follows $|(\psi_\star, v_y \psi_\star)| \leq \|\psi_\star\| \|v_y \psi_\star\| \leq \frac{V_0}{\frac{B}{2}-\delta} \|v_y \psi_\star\|$ and

$$\begin{aligned}
\|v_y \psi_\star\|^2 &= 2 \left(\psi_\star, \tfrac{1}{2} (p_y - Bx)^2 \psi_\star \right) \\
&\leq 2 \left(\psi_\star, \left[\tfrac{1}{2} p_x^2 + \tfrac{1}{2} (p_y - Bx)^2 + U_\alpha \right] \psi_\star \right) \\
&+ 2 \left(\psi_0, \left[\tfrac{1}{2} p_x^2 + \tfrac{1}{2} (p_y - Bx)^2 + U_\alpha \right] \psi_0 \right) = 2 \left(\psi^\alpha_\kappa, H^0_\alpha \psi^\alpha_\kappa \right) \\
&= 2(\psi^\alpha_\kappa, H_\alpha \psi^\alpha_\kappa) - 2(\psi^\alpha_\kappa, V^\alpha_\omega \psi^\alpha_\kappa) \leq 2(E^\alpha_\kappa + V_0) \ .
\end{aligned} \tag{B.15}$$

This leads to the bound

$$|(\psi_\star, v_y \psi_\star)| \leq \tfrac{V_0}{\frac{B}{2}-\delta} \sqrt{2(E^\alpha_\kappa + V_0)} \tag{B.16}$$

A similar argument gives the same bound for the third term.

Finally

$$\begin{aligned} |(\psi^{\alpha}_{\kappa}, v_y \psi^{\alpha}_{\kappa})| \;&\geq\; |J_{E^{\alpha}_{0\bar{k}}}| \left\{ 1 - V_0^2 \left[\frac{1}{(\frac{B}{2}-\delta)^2} + \sup_{k \in \mathcal{A}^c} \left(E^{\alpha}_{0k} - E^{\alpha}_{\kappa}\right)^{-2} \right] \right\} \\ &-\; 3 \frac{V_0}{\frac{B}{2}-\delta} \sqrt{2(E^{\alpha}_{\kappa} + V_0)} \end{aligned} \tag{B.17}$$

that is strictly positive for a sufficiently small $V_0 > 0$ (we can remark that the important condition is $V_0 \ll B$).

C About Hypothesis 1

In this section we give some details about Hypothesis 1, in particular we shows a possibility to realize it. This is done by the addition of a magnetic flux $0 \leq \Phi \leq 2\pi$ at the origin (i.e. along the cylinder axes) and taking two symmetric confining walls $U_\ell(-x) = U_r(x) \equiv U(x)$.

In this case the pure walls Hamiltonians are

$$H^0_\ell[\Phi] \;=\; \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx + \tfrac{\Phi}{L}\right)^2 + U(-x) \tag{C.1}$$

$$H^0_r[\Phi] \;=\; \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx + \tfrac{\Phi}{L}\right)^2 + U(x)\ . \tag{C.2}$$

The spectra of these Hamiltonians are

$$\sigma(H^0_\alpha[\Phi]) = \{E^{\alpha}_{nk}(\Phi) : n \in \mathbb{N}, k \in \tfrac{2\pi}{L}\mathbb{Z}\}\ . \tag{C.3}$$

Here we consider only the first spectral branches, and we write $E^{\alpha}_{0k}(\Phi) \equiv E^{\alpha}\,(k;\Phi)$, moreover we have $(k = \frac{2\pi}{L}m)$

$$E^{\alpha}\left(\tfrac{2\pi}{L}m;\Phi\right) = E^{\alpha}\left(\tfrac{2\pi}{L}m + \tfrac{\Phi}{L}\right)\ . \tag{C.4}$$

From the symmetry of the walls, for $\Phi = 0$

$$E^{\ell}\left(\tfrac{2\pi}{L}m\right) = E^{r}\left(\tfrac{2\pi}{L}(-m)\right) \quad \forall\, m \in \mathbb{Z} \tag{C.5}$$

therefore

$$\begin{aligned} E^{\ell}\left(\tfrac{2\pi m}{L}+\tfrac{\Phi}{L}\right) &= E^{\ell}\left(\tfrac{2\pi m}{L}\right)+\partial_{\hat{k}}E^{\ell}(\tilde{k}_{\ell})\frac{\Phi}{L} && \text{(C.6)}\\ E^{r}\left(\tfrac{2\pi(-m)}{L}+\tfrac{\Phi}{L}\right) &= E^{r}\left(\tfrac{2\pi(-m)}{L}\right)+\partial_{\hat{k}}E^{r}(\tilde{k}_{r})\frac{\Phi}{L} && \text{(C.7)} \end{aligned}$$

for a suitable $\frac{2\pi}{L}(-m) \leq \tilde{k}_{\ell} \leq \frac{2\pi}{L}(-m)+\frac{\Phi}{L}$ and $\frac{2\pi}{L}m \leq \tilde{k}_{r} \leq \frac{2\pi}{L}m+\frac{\Phi}{L}$. Thus

$$\begin{aligned} \left|E^{\ell}\left(\tfrac{2\pi m}{L}+\tfrac{\Phi}{L}\right)-E^{r}\left(\tfrac{2\pi(-m)}{L}+\tfrac{\Phi}{L}\right)\right| &= \frac{\Phi}{L}\left|\partial_{\hat{k}}E^{r}(\hat{k}_{r})-\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})\right| \\ &\geq 2\frac{\Phi}{L}|\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})| \geq 2\mathcal{C}\frac{\Phi}{L} && \text{(C.8)} \end{aligned}$$

where $\mathcal{C}>0$. A similar argument shows that

$$\begin{aligned} &\left|E^{\ell}\left(\tfrac{2\pi(m-1)}{L}+\tfrac{\Phi}{L}\right)-E^{r}\left(\tfrac{2\pi(-m)}{L}+\tfrac{\Phi}{L}\right)\right| \\ = &\left|\frac{\Phi}{L}\left[\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})-\partial_{\hat{k}}E^{r}(\hat{k}_{r})\right]-\tfrac{2\pi}{L}\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})\right| \geq \left|2\frac{\Phi}{L}|\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})|-\frac{2\pi}{L}|\partial_{\hat{k}}E^{\ell}(\hat{k}_{\ell})|\right| \\ \geq &\ 2\mathcal{C}\frac{|\Phi-\pi|}{L} && \text{(C.9)} \end{aligned}$$

Then, for a fixed $0<\Phi^{\star}<\pi$ or $\pi<\Phi^{\star}<2\pi$

$$\operatorname{dist}\left(\sigma(H^{0}_{\ell}[\Phi^{\star}])\cap\Delta_{\varepsilon},\sigma(H^{0}_{r}[\Phi^{\star}])\cap\Delta_{\varepsilon}\right) \geq \frac{d_0}{L}\ . \qquad \text{(C.10)}$$

Acknowledgements

We wish to thank J.M. Combes, P. Exner, J. Fröhlich and P.A. Martin for helpful discussions. The work of C.F. was supported by a grant from the Fonds National Suisse de la Recherche Scientifique No. 20 - 55654.98.

References

[BCD] P. Briet, J.M. Combes, P. Duclos: Spectral stability under thunneling. Commun. Math. Phys. **126**, 133 (1989)

[BG] F. Bentosela, V. Grecchi: Stark Wannier Ladders. Commun. Math. Phys. **142**, 169 (1991)

[BCH] J.M. Barbaroux, J.M. Combes, P.D. Hislop: Localization near band edges for random Schrödinger operators. Helv. Phys. Acta **70**, 16 (1997)

[CH] J.M. Combes, P.D. Hislop: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. **177**, 603 (1996)

[CHS] J.M. Combes, P.D. Hislop, E. Soccorsi: Edge states for quantum Hall hamiltonians. mp-arc/02-172

[dBP] S. de Bièvre, J.V. Pulé: Propagating edge states for magnetic Hamiltonian. Math. Phys. Electr. J. **5**, no. 3 (1999)

[DMP1] T.C. Dorlas, N. Macris, J.V. Pulé: Localisation in a single-band approximation to random Schrödinger operators in a magnetic field. Helv. Phys. Acta **68**, 330 (1995)

[DMP2] T.C. Dorlas, N. Macris, J.V. Pulé: Localization in single Landau bands. J. Math. Phys. **37**, 1574 (1996)

[DMP3] T.C. Dorlas, N. Macris, J.V. Pulé: The nature of the spectrum for a Landau Hamiltonian whit delta impurities. J. Stat. Phys. **87**, 847 (1997)

[DMP4] T.C. Dorlas, N. Macris, J.V. Pulé: Characterization of the spectrum of the Landau Hamiltonian with delta impurities. Commun. Math. Phys. **204**, 367 (1999)

[EJK] P. Exner, A. Joye, H. Kovarik: Magnetic transport in a straight parabolic channel. J. Phys. A: Math. Gen. **34**, 9733 (2001)

[F] C. Ferrari: Dynamique d'une particule quantique dans un champ magnétique inhomogène. Diploma work, EPFL (1999).

[FGW] J. Fröhlich, G.M. Graf, J. Walcher: On the extended nature of edge states of quantum Hall Hamiltonians. Ann. Henri Poincaré **1**, 405 (2000)

[FM1] C. Ferrari, N. Macris: Intermixture of extended edge and localized bulk energy levels in macroscopic Hall systems. *math-ph/0011013*

[FM2] C. Ferrari, N. Macris: Spectral properties of finite quantum Hall systems. To appear in J. Oper. Theor. (mp-arc/02-121)

[H] B.I. Halperin: Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. Phys. Rev. B **25**, 2185 (1982)

[M] N. Macris: Spectral flow and level spacing of edge states for quantum Hall Hamiltonians. Preprint

[MMP] N. Macris, P.A. Martin and J.V. Pulé: On Edge States In Semi-Infinite Quantum Hall Systems. J. Phys. A: Math. Gen. **32**, 1985 (1999)

[PG] R.E. Prange and S.M. Girvin: *The Quantum Hall Effect.* New York: Graduate Texts in Contemporary Physics, Springer, 1987

[vKDP] K. v. Klitzing, G. Dorda, M. Pepper: New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance. Phys. Rev. Lett. **45**, 494 (1980)

[W] W.M. Wang: Microlocalization, percolation and Anderson localization for the magnetic Schrödinger operator with a random potential. J. of Funct. Anal. **146**, 1 (1997)