NONLINEAR TIME-DEPENDENT SCHRÖDINGER EQUATIONS: THE GROSS-PITAEVSKII EQUATION WITH DOUBLE-WELL POTENTIAL

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Abstract. We consider a class of Schrödinger equations with a symmetric double-well potential and an external, both repulsive and attractive, nonlinear perturbation. We show that, under certain conditions, the reduction of the time-dependent equation to a two-mode equation gives the dominant term of the solution with a precise estimate of the error.

1. Introduction

Following the recent observations of Bose-Einstein condensation, predicted more than 70 years ago, the theoretical analysis of the Gross-Pitaevskii equation

$$i\hbar\dot{\psi} = \left[-\frac{\hbar^2}{2m}\Delta + V\right]\psi + \epsilon|\psi|^2\psi, \quad \epsilon \in \mathrm{R},\ \dot{\psi} = \frac{\partial\psi}{\partial t},\ \Delta = \sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2},\ d \geq 1,$$

has attracted an increasing interest (see [6] for a review). The main rigorous results concern both the rigorous derivation of this equation [12] and the study of the existence of stationary solutions [1], [3]. On other hands, if the potential V is not zero then very few rigorous results there exist for the study of the solution of the time-dependent equation. In fact, these results concern the existence of solutions asymptotically given by solitary wave functions in the case that the discrete spectrum of the linear Schrödinger operator has only one non-degenerate eigenvalue [16], [18].

In this paper we would fill this gap for a class of nonlinear Schrödinger equations, in dimension $d = 1$ and $d = 3$, with double-well potential. Our main result consists in proving that the solution of the Gross-Pitaevskii equation is approximated, with a rigorous control of the error, by means of the solution of a two-dimensional dynamical system exactly solvable.

A similar investigation has been recently performed in [7] where the nonlinear perturbation is given by $\epsilon\langle\psi, g\psi\rangle g\psi$ and $g(x)$ is a given symmetric function.

Our paper is organized as follows.

In section 2 we introduce the main notations and we state the assumptions Hyp.1–5 on the potential. Moreover, we prove some results concerning the spectrum of the linear Schrödinger operator and we briefly compute the solution of the time-dependent Schrödinger equation in the absence of the nonlinear perturbation.

Key words and phrases. Nonlinear Schrödinger operator; Gross-Pitaevskii equation; norm estimate of evolution operators.

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In section 3 we prove the global existence of the solution of the Gross-Pitaevskii equation and the existence of the conservation laws (12) and (13). The global existence of the solution is proved for both repulsive and attractive nonlinear perturbation, where, in the second case, we have to assume that the strength of the nonlinear perturbation is small enough.

In section 4 we briefly introduce the two-level approximation which, roughly speaking, consists in projecting the Gross-Pitaevskii equation onto the two-dimensional space spanned by the eigenvectors of the linear Schrödinger operator associated to the two lowest eigenvalues. For practical purposes, it is more convenient to choose, as a basis of such a two-dimensional space, the two "single-well" states defined in (7). The dynamical system we obtain is exactly solvable as shown in appendix B.

In section 5 we give our main result (Theorem 4) proving the stability of the two-level approximation. Here, we make use of the comparison criterion between ordinary differential equations and of a "priori" estimate of the solution of the Gross-Pitaevskii equation. We underline that, in order to obtain such an estimate, the assumption Hyp.5 plays a crucial role.

In section 6 we consider the case where the strength of the nonlinear perturbation is of the same order of the splitting between the two lowest eigenvalues of the linear Schrödinger operator. In such a case we have that the stability result holds for times much larger than the unperturbed beating period and we give a criterion for the destruction of the beating motion between the two wells.

In section 7, following [8], we introduce a simple 1-dimensional model satisfying the conditions required in Sect. 2, in particular we prove that the uniform estimate (11) holds.

2. Assumptions and preliminary results

In this paper we consider the Cauchy problem

$$i\hbar\dot{\psi} = H_{\epsilon,\rho}\psi, \quad H_{\epsilon,\rho} = H_{0,\rho} + W \tag{1}$$

$$\psi(0,x) = \psi^0(x) \in L^2(\mathrm{R}^d), \quad \|\psi^0\| = 1, \quad d = 1, 3,$$

where

$$H_{0,\rho} = -\frac{\hbar^2}{2m}\Delta + V_\rho, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

V_ρ is a symmetric double-well potential depending on a real parameter ρ, $\rho \in [\rho^\star, +\infty)$ for some $\rho^\star$, and

$$W = \epsilon|\psi|^2, \quad |\epsilon| \leq \epsilon_0 \ \text{ for some } \ \epsilon_0 > 0,$$

is the nonlinear perturbation.

In the following, for the sake of definiteness, we set the units such that $\hbar = 1$ and $2m = 1$, we denote by C any positive constant independent of ρ, ϵ and t and we denote

$$\|\varphi\|_p = \|\varphi\|_{L^p} = \left\{ \int |\varphi(x)|^p dx \right\}^{1/p} \quad \text{and} \quad \|\varphi\| = \|\varphi\|_2.$$

Moreover, given $y = (y_1, \dots, y_m) \in \mathrm{R}^m$ for some $m \geq 1$, we denote

$$|y| = \max_{1 \leq j \leq m} |y_j|. \tag{2}$$

2.1. **Assumptions on the potential.** Here, we assume that the potential V_ρ is a regular symmetric function, bounded from below uniformly with respect to ρ and with a decreasing behavior at infinity:

Hypothesis 1. *The potential $V_\rho(x)$ is such that:*

i. *let $x' = (x_1, \dots, x_{d-1}) \in \mathrm{R}^{d-1}$ and $x = (x', x_d) \in \mathrm{R}^d$, then*
$$V_\rho(x', -x_d) = V_\rho(x', x_d);$$

ii. $V_\rho \in C(\mathrm{R}^d)$;

iii. $V_\rho(x)$ *is uniformly bounded from below, that is there exists a positive constant* $V_{min} > 0$ *independent of* ρ *such that*
$$V_\rho(x) \geq -V_{min}, \quad \forall x \in \mathrm{R}^d, \ \forall \rho \in [\rho^\star, +\infty);$$

iv. *there exist δ, $r > 0$ and a function $V_\infty(x)$ such that*
$$|V_\rho(x)| \leq V_\infty(x), \quad \forall x \in \mathrm{R}^d, \ |x| \geq r, \ \forall \rho \in [\rho^\star, +\infty),$$
and
$$\limsup_{|x|\to\infty} |x|^{1+\delta} V_\infty(x) = 0.$$

Lemma 1. *Let*
$$\sigma(H_{0,\rho}) = \sigma_{\mathrm{d}}(H_{0,\rho}) \cup \sigma_{\mathrm{ess}}(H_{0,\rho}) = \sigma_{\mathrm{pp}}(H_{0,\rho}) \cup \sigma_{\mathrm{c}}(H_{0,\rho})$$
be the spectrum of the self-adjoint realization of $H_{0,\rho}$ on the Hilbert space $L^2(\mathrm{R}^d)$, then the discrete spectrum is purely point and
$$\sigma_{\mathrm{d}}(H_{0,\rho}) = \sigma_{\mathrm{pp}}(H_{0,\rho}) \subset (-V_{min}, 0) \tag{3}$$
and
$$\sigma_{\mathrm{ess}}(H_{0,\rho}) = \sigma_{\mathrm{c}}(H_{0,\rho}) = [0, +\infty). \tag{4}$$

Proof. From the assumption Hyp.1-*iv.* it follows that V_ρ is a Kato-type potential, hence $\sigma_{\mathrm{ess}}(H_{0,\rho}) = [0, +\infty)$ (see Th. V-5.7 in [11]). In particular, since $V_\rho(x) = O(|x|^{-1-\delta})$, for some $\delta > 0$, then (4) follows from Th. X-3.9 in [11]. Finally, (3) immediately follows from (4) and Hyp.1-*iii.*. □

2.2. **Assumptions on the discrete spectrum.** Here, we assume that the discrete spectrum of $H_{0,\rho}$ consists of only two non degenerate eigenvalues and that the splitting between these eigenvalues goes to zero as ρ goes to infinity. More precisely, we assume that:

Hypothesis 2. *Let $\sigma_{\mathrm{d}}(H_{0,\rho})$ be the discrete spectrum of $H_{0,\rho}$, we assume that*
$$\sigma_{\mathrm{d}}(H_{0,\rho}) = \{\lambda_{+,\rho}, \lambda_{-,\rho}\}$$
and that there exists $\tilde{\lambda} < 0$, independent of ρ, such that
$$\lambda_{+,\rho} < \lambda_{-,\rho} < \tilde{\lambda}, \ \forall \rho \in [\rho^\star, +\infty). \tag{5}$$

Lemma 2. *Let $\varphi_{\pm,\rho}$ be the normalized eigenvectors associated to $\lambda_{\pm,\rho}$, then:*

i. $\varphi_{\pm,\rho}$ *are real-valued functions such that* $\varphi_{\pm,\rho}(x', -x_d) = \pm\varphi_{\pm,\rho}(x', x_d)$;

ii. $\varphi_{\pm,\rho} \in H^1(\mathrm{R}^d)$;

iii. $\varphi_{\pm,\rho} \in L^\infty(\mathrm{R}^d)$, *in particular there exists a positive constant C independent of ρ such that*
$$\|\varphi_{\pm,\rho}\|_p \leq C, \ \forall \rho \in [\rho^\star, +\infty), \ p = 4 \ \text{ and } \ p = +\infty. \tag{6}$$

Proof. Property *i.* immediately follows from the assumption Hyp.1-*i.*. Properties *ii.* and *iii.*, for $p = +\infty$, follow from Lemma III.3.1 and Theorem III.3.2 in [4]. Finally, property *iii.*, for $p = 4$, follows from the normalization of the eigenvectors, from property *iii.* for $p = +\infty$ and from the Hölder inequality. □

Moreover, we assume that V_ρ is a family of "double-well" potentials with barrier between the wells that goes to infinity as ρ goes to infinity. In order to be more precise, let

$$B_{R,\rho} = \left\{ x = (x', x_d) \in \mathrm{R}^{d-1} \times \mathrm{R}^+ \ \text{ such that } \ V_\rho(x) \leq 0 \right\}$$

and

$$B_{L,\rho} = \left\{ x = (x', x_d) \in \mathrm{R}^{d-1} \times \mathrm{R}^- \ \text{ such that } \ V_\rho(x) \leq 0 \right\}.$$

From Hyp.1-*i.* it follows that $x = (x', x_d) \in B_{R,\rho}$ if, and only if, $(x', -x_d) \in B_{L,\rho}$. These sets are usually called "wells".

Hypothesis 3. *We assume that:*

i. $B_{R,\rho}$ *and* $B_{L,\rho}$ *are simply connected sets such that*

$$B_{R,\rho} \subset D_r \ \text{ and } \ B_{L,\rho} \subset D_r$$

for some $r > 0$ *independent of* ρ*, where* $D_r = \{x \in \mathrm{R}^d : |x| \leq r\}$*;*

ii. $\bar{B}_{R,\rho} \cap \bar{B}_{L,\rho} = \emptyset$ *where* $\bar{B}_{R,\rho}$ *and* $\bar{B}_{L,\rho}$ *respectively denote the closure of the sets* $B_{R,\rho}$ *and* $B_{L,\rho}$*;*

iii. let

$$\Gamma_\rho = \inf_\gamma \int_0^1 \sqrt{\max[V_\rho[\gamma(s)], 0]} |\dot{\gamma}(s)| ds,$$

be the Agmon distance between the two wells where γ *is any path connecting the two wells:*

$$\gamma \in AC([0,1], \mathrm{R}^d), \ \gamma(0) \in \partial B_{R,\rho} \ \text{ and } \ \gamma(1) \in \partial B_{L,\rho},$$

then, we assume that

$$\lim_{\rho \to +\infty} \Gamma_\rho = +\infty.$$

From these assumptions it follows that the splitting between the two eigenvalues $\lambda_{\pm,\rho}$ goes to zero as ρ goes to infinity. Indeed:

Lemma 3. *Let*

$$\omega_\rho = \frac{\lambda_{-,\rho} - \lambda_{+,\rho}}{2} \ \text{ and } \ \Omega_\rho = \frac{\lambda_{-,\rho} + \lambda_{+,\rho}}{2}$$

and

$$\varphi_{R,\rho} = \frac{1}{\sqrt{2}} \left[\varphi_{+,\rho} + \varphi_{-,\rho} \right] \ \text{ and } \ \varphi_{L,\rho} = \frac{1}{\sqrt{2}} \left[\varphi_{+,\rho} - \varphi_{-,\rho} \right]. \tag{7}$$

Then

$$\| \varphi_{R,\rho} \varphi_{L,\rho} \|_\infty \leq C \omega_\rho, \ \forall \rho \in [\rho^\star, +\infty), \tag{8}$$

for some positive constant C *independent of* ρ*,*

$$\lim_{\rho \to +\infty} \omega_\rho = 0 \tag{9}$$

and

$$\limsup_{\rho\to+\infty}\Omega_\rho=\Omega_\infty^{\sup}\quad \textit{and}\quad \liminf_{\rho\to+\infty}\Omega_\rho=\Omega_\infty^{\inf}$$

are such that $-V_{min}\le\Omega_\infty^{\inf}\le\Omega_\infty^{\sup}\le\tilde\lambda<0$.

Proof. From Lemma 2 it follows that $\varphi_{R,\rho}$ and $\varphi_{L,\rho}\in L^\infty(\mathrm{R}^d)$, (8) follows from standard WKB argument (see [9]). From the assumptions Hyp.1-3 it follows that (see [9]) $\omega_\rho\le Ce^{-\Gamma_\rho}$ for some $C>0$ independent of ρ; hence (9) follows. □

Remark 1. By definition and from the property *i.* of Lemma 2 it follows that $\varphi_{R,\rho}(x',-x_d)=\varphi_{L,\rho}(x',x_d)$; moreover, from (8), it follows that these functions are localized on only one of the "wells" $B_{R,\rho}$ and $B_{L,\rho}$, e.g.:

$$\int_{B_{R,\rho}}|\varphi_{R,\rho}(x)|^2dx=1+O(e^{-\Gamma_\rho}).$$

For such a reason we call them "single-well" (normalized) states.

2.3. **Solution of the unperturbed equation.** Let

$$P_c=1-\langle\varphi_{R,\rho},\cdot\rangle\varphi_{R,\rho}-\langle\varphi_{L,\rho},\cdot\rangle\varphi_{L,\rho}\tag{10}$$

be the projection operator onto the eigenspace associated to the continuous spectrum of $H_{0,\rho}$. We assume that:

Hypothesis 4. $P_c\psi^0=0$.

Under this assumption it follows that the solution of the unperturbed equation

$$i\dot\psi=H_{0,\rho}\psi,\quad \psi(0,x)=\psi^0(x),$$

is simply given by

$$\begin{aligned}\psi(t,x) &= e^{-i\Omega_\rho t}\left[\frac{c_++c_-}{\sqrt2}\cos(\omega_\rho t)+i\frac{c_--c_+}{\sqrt2}\sin(\omega_\rho t)\right]\varphi_{R,\rho}(x)+\\ &\quad+e^{-i\Omega_\rho t}\left[\frac{c_+-c_-}{\sqrt2}\cos(\omega_\rho t)-i\frac{c_++c_-}{\sqrt2}\sin(\omega_\rho t)\right]\varphi_{L,\rho}(x)\end{aligned}$$

where

$$c_\pm=\langle\varphi_{\pm,\rho},\psi^0\rangle.$$

Hence, $\psi(t,x)$ is, up to the phase factor $e^{-i(\Omega_\rho-\omega_\rho)t}$, a periodic function with period $T=\pi/\omega_\rho$.

In particular, if ψ initially coincides with a single-well state, e.g. $\psi^0=\varphi_{R,\rho}$, then

$$\psi(t,x)=e^{-i(\Omega_\rho-\omega_\rho)t}\left[e^{-i\omega_\rho t}\cos(\omega_\rho t)\varphi_{R,\rho}(x)-ie^{-i\omega_\rho t}\sin(\omega_\rho t)\varphi_{L,\rho}(x)\right]$$

and the state $\psi(t,x)$ performs a beating motion. That is the state, initially localized on the well $B_{R,\rho}$, is localized on the other well $B_{L,\rho}$ after half a period and, after a whole period, it "returns" on the initial well, and so on. In particular, we have that the "center of mass" defined here as

$$\begin{aligned}\langle x_d\rangle^t &= \int_{\mathrm{R}^d}x_d|\psi(t,x)|^2p(x)dx\\ &= \left[\cos^2(\omega_\rho t)-\sin^2(\omega_\rho t)\right]\int_{\mathrm{R}^d}x_d|\varphi_{R,\rho}(x)|^2p(x)dx,\end{aligned}$$

for some real-valued function $p(x)$ such that $p(-x) = p(x)$ and $|p(x)| \leq C/[1+|x|]$ for some C, is a periodic function that periodically assumes positive and negative values.

2.4. **Assumption on the evolution operator.** Here, we assume that the restriction of the evolution operator $e^{-iH_{0,\rho}t}$ to the continuous eigenspace uniformly goes to zero as t goes to infinity:

Hypothesis 5. *We assume that for any p and p', $\frac{1}{p}+\frac{1}{p'}=1$ and $2 \leq p \leq \infty$, then there exists a positive constant C independent of ρ such that*

$$\left\|e^{-iH_{0,\rho}t}P_c\varphi\right\|_p \leq C < t >^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\varphi\|_{p'}, \quad < t >= \sqrt{1+t^2}, \ d=1,3. \tag{11}$$

Remark 2. We underline that, for fixed ρ, the above estimate has been proved by [10], in the case $d \geq 3$, and by [18], in the case $d = 1$, provided that 0 is neither an eigenvalue nor a resonance of $H_{0,\rho}$. Our assumption essentially concerns the validity of (11) **uniformly** with respect to ρ.

3. Global existence of the solution and conservation laws

Here, we prove that the Cauchy problem (1) admits a solution for all time provided that the assumptions Hyp.1–4 are satisfied and the strength of the nonlinear perturbation is small enough.

The following result holds.

Theorem 1. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ then Cauchy problem (1) admits a unique solution $\psi(t,x)$ for any $t \in \mathrm{R}$. Moreover the following conservation laws hold:*

$$\|\psi(t,\cdot)\| = \|\psi^0(\cdot)\| = 1 \tag{12}$$

and

$$E_\rho(\psi) = \langle H_{0,\rho}\psi, \psi\rangle + \frac{1}{2}\epsilon\|\psi^2\|^2 = E_\rho(\psi^0). \tag{13}$$

Proof. From Hyp.4 it follows that $\psi^0 = c_+\varphi_{+,\rho} + c_-\varphi_{-,\rho}$, $c_\pm = \langle\psi^0, \varphi_{\pm,\rho}\rangle$. From this fact and from Lemma 2 then $\psi^0 \in H^1$. Therefore, existence of the global solution $\psi \in C(\mathrm{R}, H^1)$ and the conservation laws (12) and (13) follow from known results (see, e.g., the papers quoted in [14] and [16]) for any $\epsilon > 0$ (repulsive nonlinear perturbation) and for any $\epsilon \in (-\epsilon_0, 0)$ for some $\epsilon_0 > 0$ (attractive nonlinear perturbation). □

Moreover:

Theorem 2. *There exists a positive constant C independent of ρ and ϵ such that $|E_\rho(\psi^0)| \leq C$ for any $\rho \in [\rho^\star, +\infty)$ and $\epsilon \in [-\epsilon_0, \epsilon_0]$.*

Proof. This estimate immediately follows from (13), from the assumption Hyp.4 and from Lemmas 2 and 3. □

Theorem 3. *The solution of equation (1) satisfies to the following uniform estimate*

$$\|\psi\|_4 \leq C, \quad \forall t \in \mathrm{R}, \ \forall \rho \in [\rho^\star, +\infty), \ \forall \epsilon \in [-\epsilon_0, \epsilon_0] \tag{14}$$

for some positive constant C independent of t, ρ and ϵ.

Proof. In order to prove (14) let

$$\tilde{V} = V + V_{min} \geq 0, \quad \Lambda = E + V_{min} \geq 0, \quad V_{min} = -\min_{x,\rho} V.$$

Then, the conservation law (13) can be written as

$$\|\nabla\psi\|^2 + \langle\tilde{V}\psi, \psi\rangle + \frac{1}{2}\epsilon\|\psi^2\|^2 = \Lambda$$

where, from Theorem 2, Λ is uniformly bounded with respect to ϵ and ρ. In particular, if we set

$$\chi = |\epsilon|^{1/2}\psi,$$

then the above equation takes the form

$$\|\nabla\chi\|^2 + \langle\tilde{V}\chi, \chi\rangle + \frac{1}{2}[\text{sign}(\epsilon)]\|\chi^2\|^2 = |\epsilon|\Lambda$$

from which it follows that

$$\|\nabla\chi\|^2 \leq |\epsilon|\Lambda + \frac{1}{2}\|\chi^2\|^2 = |\epsilon|\Lambda + \frac{1}{2}\|\chi\|_4^4 \tag{15}$$

From the Gagliardo-Niremberg inequality (see [14])

$$\|\chi\|_{2\sigma+2}^{2\sigma+2} \leq C\|\nabla\chi\|^{\sigma d}\|\chi\|^{2+\sigma(2-d)},$$

where we choose $\sigma = 1$ and $d = 1, 3$, it follows that

$$\|\chi\|_4^4 \leq C\|\nabla\chi\|^d\|\chi\|^{4-d} \leq C\|\nabla\chi\|^d|\epsilon|^{(4-d)/2} \tag{16}$$

since $\|\chi\| = |\epsilon|^{1/2}\|\psi\| = |\epsilon|^{1/2}$. By inserting this inequality in (15) it follows that $\|\nabla\chi\|$ satisfies to

$$\|\nabla\chi\|^2 \leq |\epsilon|\Lambda + C|\epsilon|^{(4-d)/2}\|\nabla\chi\|^d, \quad d = 1, 3 \tag{17}$$

for any $t \in \mathrm{R}$. From (17), from the fact that $\|\nabla\chi\| = O(|\epsilon|^{1/2})$ at $t = 0$ and by means of a continuity argument then estimate (14) follows.
Indeed, in the one-dimensional case $d = 1$ from (17) we immediately have that

$$\|\nabla\chi\| \leq C|\epsilon|^{1/2}.$$

From this fact and from (16) it follows that $\|\chi\|_4^4 \leq C\epsilon^2$ which finally implies that $\|\psi^2\| \leq C$.
In the 3-dimensional case we have that the inequality (17) takes the form $\|\nabla\chi\|^2 \leq \alpha|\epsilon| + \beta|\epsilon|^{1/2}\|\nabla\chi\|^3$, where $\alpha,\ \beta > 0$ are given constant, from which follows that $\|\nabla\chi\| \leq C|\epsilon|^{1/2}$ for any t since $\|\nabla\chi\| = O(|\epsilon|^{1/2})$ at $t = 0$. Hence, as above, it follows that $\chi\|_4^4 \leq C\epsilon^2$ and, finally, $\|\psi^2\| \leq C$. □

4. Two-level approximation

Let us write the solution of equation (1) in the form

$$\psi(t, x) = a_R(t)\varphi_{R,\rho}(x) + a_L(t)\varphi_{L,\rho}(x) + \psi_c(t, x), \tag{18}$$

where $a_R(t)$ and $a_L(t)$ are unknown functions depending on time and $\psi_c = P_c\psi$, P_c, defined in (10), is the projection onto the continuous eigenspace, that is the space orthogonal to the two-dimensional space spanned by the two "single-well" state $\varphi_{R,\rho}$ and $\varphi_{L,\rho}$, i.e.:

$$\langle\psi_c, \varphi_{R,\rho}\rangle = \langle\psi_c, \varphi_{L,\rho}\rangle = 0, \quad \forall t.$$

From the conservation law (12) it follows that

$$|a_R(t)|^2 + |a_L(t)|^2 + \|\psi_c(t,\cdot)\|^2 = 1. \tag{19}$$

By substituting ψ by (18) in equation (1) we obtain that a_R, a_L and ψ_c must satisfy to the system of differential equations

$$\left\{ \begin{array}{rcl} i\dot{a}_R &=& \Omega a_R - \omega a_L + \langle \varphi_{R,\rho}, W\psi\rangle \\ i\dot{a}_L &=& \Omega a_L - \omega a_R + \langle \varphi_{L,\rho}, W\psi\rangle \\ i\dot{\psi}_c &=& H_0\psi_c + P_c W\psi \end{array} \right. \tag{20}$$

By substituting again ψ by (18) in the first two equations of the above system then we obtain that these equations take the form

$$\left\{ \begin{array}{rcl} i\dot{a}_R &=& \Omega a_R - \omega a_L + \epsilon c|a_R|^2 a_R + \epsilon r_R \\ i\dot{a}_L &=& \Omega a_L - \omega a_R + \epsilon c|a_L|^2 a_L + \epsilon r_L \end{array} \right. \tag{21}$$

where

$$c = \|\varphi_{R,\rho}^2\|^2 = \|\varphi_{L,\rho}^2\|^2$$

and where r_R and r_L are given by

$$\begin{array}{rcl} r_R &=& \left[\langle \varphi_{R,\rho}, |\psi|^2\varphi_{L,\rho}\rangle a_L + \langle \varphi_{R,\rho}, |\psi|^2\psi_c\rangle\right] + 2a_R\Re\left[a_R\langle \varphi_{R,\rho}^2, (a_L\varphi_{L,\rho}\psi_c)\varphi_{R,\rho}\rangle\right] \\ r_L &=& \left[\langle \varphi_{L,\rho}, |\psi|^2\varphi_{R,\rho}\rangle a_R + \langle \varphi_{L,\rho}, |\psi|^2\psi_c\rangle\right] + 2a_L\Re\left[a_L\langle \varphi_{L,\rho}^2, (a_R\varphi_{R,\rho}\psi_c)\varphi_{L,\rho}\rangle\right] \end{array}$$

We denote **two-level approximation** the solutions b_R and b_L of the system of ordinary differential equations

$$\left\{ \begin{array}{rcl} i\dot{b}_R &=& \Omega b_R - \omega b_L + \epsilon c|b_R|^2 b_R \\ i\dot{b}_L &=& \Omega b_L - \omega b_R + \epsilon c|b_L|^2 b_L \end{array} \right. , \quad b_{R,L}(0) = a_{R,L}(0). \tag{22}$$

obtained by neglecting the remainder terms r_R and r_L in (21). It is easy to see that the solution of this system satisfies to the following conservation law

$$|b_R(t)|^2 + |b_L(t)|^2 = |b_R(0)|^2 + |b_L(0)|^2 = |a_R(0)|^2 + |a_L(0)|^2 = 1, \tag{23}$$

and, moreover, it is also possible to explicitely compute (see appendix B) the solution of (22) by means of elliptic functions.

5. Stability of the two-level approximation

Our main result consists in proving the stability of the two-level approximation when we restore the remainder terms r_R and r_L.

We prove that:

Theorem 4. *Let $\psi_c = P_c\psi$, $a_R(t) = \langle \psi, \varphi_{R,\rho}\rangle$ and $a_L(t) = \langle \psi, \varphi_{L,\rho}\rangle$, where ψ is the solution of equation (1), let $b_R(t)$ and $b_L(t)$ be the solution of the system of ordinary differential equations (22) and let $\beta = \max[\omega, \epsilon]$. Then there exists a positive constant C independent of ρ, ϵ and t such that:*

1-dimensional case: $d = 1$. *Let $q_1(t) = C\epsilon\left[e^{C\beta t} - 1 + t^{1/2}\right]$, then for any $t \geq 0$*

$$|b_{R,L}(t) - a_{R,L}(t)| \leq q_1(t) \tag{24}$$

and

$$\|\psi_c\|_\infty \leq C\epsilon t^{1/2} \quad \textit{and} \quad \|\psi_c\| \leq q_1(t); \tag{25}$$

3-dimensional case $d = 3$. *Let* $q_3(t) = C\epsilon\left[e^{C\beta t} - 1\right]$, *then for any* $t \geq 0$

$$|b_{R,L}(t) - a_{R,L}(t)| \leq q_3(t) \tag{26}$$

and

$$\|\psi_c\|_\infty \leq C\epsilon \quad \textit{and} \quad \|\psi_c\| \leq q_3(t). \tag{27}$$

Proof. For the sake of simplicity, hereafter, let us drop out ρ where this does not cause misunderstanding. In order to prove the theorem we introduce the "slow time" $\tau = \omega t$ and let

$$A_{R,L}(\tau) = e^{-i\Omega t} a_{R,L}(t), \quad B_{R,L}(\tau) = e^{-i\Omega t} b_{R,L}(t) \tag{28}$$

and

$$R_{R,L}(\tau) = \frac{\epsilon}{\omega} e^{-i\Omega t} r_{R,L}(t) \text{ and } \eta = \frac{\epsilon c}{\omega} \in \mathrm{R}.$$

Then (21) and (22) respectively take the form (here $'$ denotes the derivative with respect to τ)

$$\left\{ \begin{array}{rcl} A'_R &=& iA_L - i\eta|A_R|^2 A_R + R_R \\ A'_L &=& iA_R - i\eta|A_L|^2 A_L + R_L \end{array} \right. \tag{29}$$

and

$$\left\{ \begin{array}{rcl} B'_R &=& iB_L - i\eta|B_R|^2 B_R \\ B'_L &=& iB_R - i\eta|B_L|^2 B_L \end{array} \right. \tag{30}$$

satisfying to the same initial condition

$$B_{R,L}(0) = A_{R,L}(0) = a_{R,L}(0)$$

and, since (19) and (23), such that

$$|B_R(\tau)|^2 + |B_L(\tau)|^2 = 1, \quad |A_R(\tau)|^2 + |A_L(\tau)|^2 \leq 1. \tag{31}$$

In a more concise way, with an obvious meaning of the notation, we can write (29) and (30) as

$$A' = f(A) + R \quad \text{and} \quad B' = f(B), \quad A(0) = B(0) = a(0), \tag{32}$$

where $A, B \in S^2$ since (31), $S^2 = \{(z_1, z_2) \in \mathrm{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$.

Lemma 4. *The function* $f : S^2 \to C^2$ *satisfies to the Lipschitz condition:*

$$|f(A) - f(B)| \leq L|A - B|, \ L = 1 + 3\eta. \tag{33}$$

Proof. According with the notation (2) we have that

$$|f(A) - f(B)| = \max\left[|f_R|, |f_L|\right]$$

where $|A| \leq 1$ and $|B| \leq 1$ since $A,\ B \in S^2$ and where

$$\begin{array}{rcl} f_R &=& (A_L - B_L) - \eta(|A_R|^2 A_R - |B_R|^2 B_R) \\ f_L &=& (A_R - B_R) - \eta(|A_L|^2 A_L - |B_L|^2 B_L) \end{array}$$

Then (33) immediately follows since

$$f_R = (A_L - B_L) - \eta\left[|B_R|^2(A_R - B_R) + A_R(|A_R| + |B_R|)(|A_R| - |B_R|)\right]$$

where $||A_R| - |B_R|| \leq |A_R - B_R|$ and where the other term f_L will be treated in the same way. □

We discuss the 3-dimensional case at first and then we treat the 1-dimensional case.

5.1. **3-dimensional case.** Let $d = 3$. We prove that:

Lemma 5. *The solution ψ of equation (1) satisfies to the following uniform estimate*

$$\|\psi_c\|_\infty \leq C\epsilon, \quad \forall t \geq 0, \tag{34}$$

for some positive constant C independent of ρ, ϵ and t.

Proof. In order to prove the above estimate we make use of the third equation of (20) from which it follows that

$$\psi_c(t,x) = -i\epsilon \int_0^t e^{-iH_0(t-s)} P_c |\psi(s,\cdot)|^2 \psi(s,\cdot) ds \tag{35}$$

since $\psi_c^0 = P_c\psi^0 = 0$ from assumption Hyp.4. Now, let $p = \infty$ and $p' = 1$, then from (11) and (35) it follows that

$$\begin{aligned} \|\psi_c\|_p &\leq \epsilon \int_0^t \left\| e^{-iH_0(t-s)} P_c |\psi|^2 \psi \right\|_p ds \\ &\leq C\epsilon \int_0^t < t-s >^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\psi^3\|_{p'} ds \\ &\leq C\epsilon \int_0^t < t-s >^{-d/2} \|\psi^3\|_1 ds \end{aligned} \tag{36}$$

from which (34) follows since $\int_0^t < t-s >^{-d/2} ds \leq C$ for any $t \geq 0$, where $d = 3$, and since

$$\|\psi^3\|_1 = \|\psi\psi^2\|_1 \leq \|\psi\|_2 \|\psi^2\|_2 \leq C,$$

from (12), the Hölder inequality and (14). □

Lemma 6. *The remainder terms r_R and r_L satisfy to the following uniform estimate:*

$$\max\left[|r_R|, |r_L|\right] \leq C\beta, \quad \forall t > 0, \tag{37}$$

for some positive constant C independent of ρ, ϵ and t.

Proof. Let us only consider the term $|r_R|$, the other term $|r_L|$ could be treated in the same way. By definition and since $\max[|a_R|, |a_L|] \leq 1$ then it follows that

$$\begin{aligned} |r_R| \leq & + \left|\langle \varphi_{R,\rho}\bar{\varphi}_{L,\rho}, |\psi|^2 \rangle\right| && (38) \\ & + \left|\langle \varphi_{R,\rho}|\psi|^2, \psi_c \rangle\right| && (39) \\ & + 2\left|\langle \varphi_{R,\rho}^2 \bar{\varphi}_{L,\rho}\bar{\varphi}_{R,\rho}, \psi_c \rangle\right| && (40) \end{aligned}$$

and we estimate separately each term.

From Lemma 3, equation (12) and the Hölder inequality it follows that the term (38) satisfies to the following estimate:

$$\left|\langle \varphi_{R,\rho}\varphi_{L,\rho}, |\psi|^2 \rangle\right| \leq \|\varphi_{R,\rho}\bar{\varphi}_{L,\rho}\|_\infty \cdot \|\psi^2\|_1 \leq C\omega.$$

From (12), Lemma 5 and the Hölder inequality it follows that the terms (39) and (40) satisfy to the following estimates:

$$\left|\langle \varphi_{R,\rho}|\psi|^2, \psi_c \rangle\right| \leq \|\varphi_{R,\rho}\|_\infty \cdot \|\psi^2\|_1 \cdot \|\psi_c\|_\infty \leq C\epsilon$$

and

$$\left|\langle \varphi_{R,\rho}^2 \bar{\varphi}_{L,\rho}\bar{\varphi}_{R,\rho}, \psi_c \rangle\right| \leq \|\psi_c\|_\infty \cdot \|\varphi_{R,\rho}^2\|_1 \cdot \|\varphi_{R,\rho}\varphi_{L,\rho}\|_\infty \leq C\epsilon\omega.$$

Collecting all these estimates we obtain the proof of the Lemma. □

The proof of the Theorem, in the case $d = 3$, is almost done. Indeed, equations (32) can be rewritten in the integral form:

$$A(\tau) = A(0) + \int_0^\tau f[A(s)]ds + \int_0^\tau Rds \tag{41}$$

and

$$B(\tau) = B(0) + \int_0^\tau f[B(s)]ds \tag{42}$$

from which it follows that

$$\begin{aligned} |A(\tau) - B(\tau)| &\leq \int_0^\tau |f[A(s)] - f[B(s)]|\, ds + \int_0^\tau |R|ds \\ &\leq L\int_0^\tau |A(s) - B(s)|\, ds + C\frac{\epsilon\beta}{\omega}\tau \end{aligned}$$

since Lemmas 4 and 6. From this inequality and by means of the Gronwall's Lemma [15] we finally obtain that

$$|A(\tau) - B(\tau)| \leq \frac{C}{L}\frac{\epsilon\beta}{\omega}\left[e^{L\tau} - 1\right] \tag{43}$$

proving so (26) since

$$\frac{\omega + \epsilon}{C'\omega} \leq L = 1 + 3\eta \leq C'\frac{\omega + \epsilon}{\omega},$$

for some $C' > 0$, which implies that $\frac{\beta}{L\omega} \leq C$ for some $C > 0$. Finally, the second equation in (27) follows from (19), (23) and (26).

5.2. **1-dimensional case.** Let $d = 1$. In such a case the proof is very similar to the one given in the 3-dimensional case. We just discuss the steps of the proof which are different with respect to the other case.

Lemma 7. *The solution ψ of equation (1) satisfies to the following uniform estimate*

$$\|\psi_c\|_\infty \leq C\epsilon t^{1/2},\ \forall t \geq 0, \tag{44}$$

for some positive constant C independent of ρ, ϵ and t.

Proof. This estimate directly comes from (11), (14) and (36). □

Lemma 8. *The remainder terms r_R and r_L satisfy to the following uniform estimate:*

$$\max\left[|r_R|, |r_L|\right] \leq C[\epsilon t^{1/2} + \omega], \quad \forall t \geq 0, \tag{45}$$

for some positive constant C independent of ρ, ϵ and t.

Proof. As done in the 3-dimensional case we have to estimate the terms (38), (39) and (40). In particular, by means of the same arguments and from Lemma 7 we have that:

$$\left|\langle\varphi_{R,\rho}\bar{\varphi}_{L,\rho}, |\psi|^2\rangle\right| \leq \|\varphi_{R,\rho}\bar{\varphi}_{L,\rho}\|_\infty \cdot \|\psi^2\|_1 \leq C\omega,$$

$$\left|\langle\varphi_{R,\rho}|\psi|^2, \psi_c\rangle\right| \leq \|\varphi_{R,\rho}\|_\infty \cdot \|\psi^2\|_1 \cdot \|\psi_c\|_\infty \leq C\epsilon t^{1/2}$$

and

$$\left|\langle\varphi^2_{R,\rho}\bar{\varphi}_{L,\rho}\bar{\varphi}_{R,\rho},\psi_c\rangle\right| \leq \|\psi_c\|_\infty \cdot \|\varphi^2_{R,\rho}\|_1 \cdot \|\varphi_{R,\rho}\varphi_{L,\rho}\|_\infty \leq C\epsilon\omega t^{1/2}.$$

The Lemma has been so proved. □

Hence, we have that

$$|A(\tau) - B(\tau)| \leq L\int_0^\tau |A(s) - B(s)|\, ds + C\epsilon\tau + C\frac{\epsilon^2}{\omega^{3/2}}\tau^{3/2}$$

from which it follows that

$$|a(t) - b(t)| \leq L\omega\int_0^t |a(t') - b(t')|\, dt' + C\epsilon\omega t + C\epsilon^2 t^{3/2}.$$

Let

$$h(t) = |a(t) - b(t)| + \frac{3C\epsilon^2}{2L\omega}t^{1/2},$$

then $h(t)$ is a real and positive valued function satisfying to the inequality

$$h(t) \leq L\omega\int_0^t h(t')dt' + C\epsilon\beta t.$$

From this inequality and from the Gronwall's lemma it follows that

$$h(t) \leq \frac{C\epsilon\beta}{L\omega}\left[e^{L\omega t} - 1\right]$$

from which we have the estimate (24) since $\epsilon/L\omega \leq C$ for some $C > 0$. The second estimate in (25) follows from (19), (23) and (24). □

6. Criterion for the destruction of the beating motion

Now, we apply our result to the case of a simultaneous limit of small perturbation and large barrier. In particular, let

$$\lim_{\epsilon\to 0,\ \rho\to+\infty} \eta = \tilde{\eta} \in \mathrm{R}, \quad \eta = \frac{c\epsilon}{\omega_\rho},\ c = \|\varphi^2_{R,\rho}\|^2, \tag{46}$$

that is the strength of the nonlinear perturbation is of the same order of the splitting, i.e. $\omega \sim \epsilon$. Here, $\tilde{\eta}$ plays here the role of the nonlinearity parameter.

In such a case we have that for any $\alpha' \in (0,1)$ ($\alpha' \in (0,1/2)$ in the 1-dimensional case) there exists $C' > 0$ such that

$$|a(t) - b(t)| \leq C'\omega^{\alpha'}, \quad \forall t \in [0, C'\omega^{-1}\ln(1/\omega)]. \tag{47}$$

Hence, the stability result holds for times much larger than the beating period $T = \pi/\omega_\rho$. From this fact and from the results given in appendix B it follows that the motion of the "center of mass" is asymptotically (in the limit (46)) given by

$$\langle x_d\rangle^t = \int_{\mathrm{R}} x_d|\psi(t,x)|^2 p(x)dx \sim z(\omega_\rho t)\left[\int_{\mathrm{R}} x_d|\varphi_{R,\rho}(t,x)|^2 p(x)dx\right] \tag{48}$$

where $z(\tau)$ is the fractional imbalance function introduced in appendix B given by (64) and (65). In particular, we still have the beating motion, that is $\langle x_d\rangle^t$ is, up to a small error, a periodic function that assumes both positive and negative values if the modulus k of the elliptic function defined in (66) is smaller than 1, the period can be computed as function of k. In contrast, if k is larger than 1 then $\langle x_d\rangle^t$ is definetely positive or negative and we have the destruction of the beating motion.

In particular, we have the following criterion:

Theorem 5. *Let (see appendix B for the definition of z and θ)*

$$\tilde{I} = \sqrt{1-[z(0)]^2}\cos[\theta(0)] - \frac{1}{4}\tilde{\eta}[z(0)].$$

If $|\tilde{I}| < 1$ then we still have a beating motion between the two wells for the solution of (1) for ρ large enough and ϵ small enough. In contrast, if $|\tilde{I}| > 1$ then we have the destruction of the beating motion for ρ large enough and ϵ small enough.

Proof. In order to prove this theorem we simply observe that the condition $|\tilde{I}| < 1$ (respectively $|\tilde{I}| > 1$) implies that the law of conservation is such that $|I| < 1$ (respectively $|I| > 1$), for ρ large enough and ϵ small enough, and that the modulus k defined in (66) is less (respectively larger) than 1. From this fact, from the explicit expression of the function $z(\tau)$, from equation (48) and from the stability result proved in the previous section the theorem follows. □

Remark 3. We underline that a similar criterion has been recently given by [7] for a class of double-well Schrödinger equations under the effect of a nonlinear perturbation of the form $\epsilon\langle\psi, g\psi\rangle g\psi$ where g is a given symmetric function. Here, we extend the validity of such a criterion to the Gross-Pitaevskii equation.

Remark 4. In particular, if the state is initially prepared on only one of the two wells, for instance $\psi^0 = \varphi_{R,\rho}$, then it follows that $z(0) = 1$ and $\tilde{I} = -\frac{1}{4}\tilde{\eta}$. In such a case we still have the beating motion in the simultaneous limit of small perturbation and large barrier if $|\tilde{\eta}|$ is less than 4. In contrast, if $|\tilde{\eta}| > 4$ then we have the destruction of the beating motion. As a result, it follows that the results obtained by [13] and [17] on coherent oscillations between couple Bose-Einstein condensates acquire full validity on a rigorous mathematical basis.

7. The "toy model"

In this section we consider the explicit one-dimensional model introduced in [8], where the double well potential is given by means of two attractive delta functions at $x = \pm a$, with negative strength $-b$, and one repulsive delta function at $x = 0$, with positive strength ρ:

$$V_\rho = -b\delta(x-a) - b\delta(x+a) + \rho\delta(x)$$

where $a > 0$ and $b > 0$ are fixed and $\rho > 0$ is large enough. We prove now that in such a case the results of Lemmas 1, 2, 3 and the bound (11) are satisfied provided that $ab > 1$ and $\rho \in [\rho^\star, +\infty)$ for some $\rho^\star > 0$ large enough.

The operator $H_{0,\rho}$ is a self-adjoint operator on the domain $H^2(\mathrm{R} - \{0, \pm a\})$ with the boundary conditions

$$\varphi(x+0) = \varphi(x-0),\ x = 0, \pm a, \tag{49}$$

and

$$\varphi'(x+0) - \varphi'(x-0) = -b\varphi(x),\ x = \pm a, \tag{50}$$

$$\varphi'(x+0) - \varphi'(x-0) = \rho\varphi(x),\ x = 0. \tag{51}$$

The essential spectrum of $H_{0,\rho}$ is purely absolutely continuous [2] and it coincides with the positive real axis. The eigenvectors of $H_{0,\rho}$ are the solution of the equation $[H_{0,\rho} - \lambda]\varphi = 0$, $\lambda = k^2 < 0$, $\Im k \geq 0$, given by

$$\varphi(x) = A_j e^{|k|x} + B_j e^{-|k|x}, \quad \text{if} \quad y_{j-1} < x < y_j,\ j = 1, 2, 3, 4,$$

where

$$y_0 = -\infty,\ y_1 = -a,\ y_2 = 0,\ y_3 = a,\ y_4 = +\infty$$

$A_4 = B_1 = 0$ and A_j and B_j, $j = 1, 2, 3, 4$, satisfy to the matching conditions (49), (50) and (51). It is not hard to verify that for $B_1 = 0$ and any $A_1 \neq 0$ then the matching conditions imply that

$$A_4 = A_1 \frac{a^2 b^2 \rho \mathsf{h}(w) \mathsf{g}_\rho(w)}{2|k|}$$

and

$$B_2 = A_1 \frac{b}{2|k| e^{2|k|a}},\ A_2 = A_1 \frac{-b + 2|k| ab \mathsf{h}(w) e^{2|k|a}}{2|k| e^{2|k|a}},$$

$$B_3 = A_1 \frac{b - ab\rho \mathsf{h}(w) e^{2|k|a}}{2|k| e^{2|k|a}},\ A_3 = A_1 \frac{b + ab\rho \mathsf{g}_\rho(w) e^{2|k|a}}{2|k| e^{2|k|a}}$$

where

$$w = 2iak$$

and where we have introduced the functions

$$\mathsf{f}(w) = \begin{cases} 1 & ,\ w = 0 \\ \frac{e^w - 1}{w} & ,\ w \neq 0 \end{cases}$$

$$\mathsf{h}(w) = \frac{1}{ab} - \mathsf{f}(w)$$

and

$$\mathsf{g}_\rho(w) = \mathsf{h}(w) - \frac{1}{a\rho} \left[w \mathsf{f}(w) + 2 + \frac{w}{ab} \right].$$

Then $\lambda = k^2 < 0$, $\Im k \geq 0$, is an eigenvalue of $H_{0,\rho}$ if $k = i|k|$ is such that $A_4 = 0$, that is $|k|$ is a solution of the equation

$$h(w) = 0 \ \Rightarrow\ 2|k| - b = -\frac{b}{e^{2|k|a}} \tag{52}$$

or

$$g_\rho(w) = 0 \ \Rightarrow\ 2|k| - b = -\frac{\rho - 2|k|}{\rho + 2|k|} \frac{b}{e^{2|k|a}}. \tag{53}$$

It is not hard to see that if $ab > 1$ and ρ is large enough then equation (52) has only one real solution $|k_-|$ independent of ρ and equation (53) has only one real solution $|k_+^\rho|$ such that $|k_-| - |k_+^\rho| = O(\rho^{-1})$, hence

$$\omega^\rho = \lambda_- - \lambda_+^\rho = O(\rho^{-1}), \quad \text{as} \ \ \rho \to \infty.$$

The associated eigenvectors $\varphi_\pm$ (let me drop out the dependence of ρ for the sake of simplicity) are given by

$$\varphi_-(x) = A_{1,-} \begin{cases} -e^{|k_-|x}, & x < -a \\ \frac{b}{2|k_-|e^{2|k_-|a}} \left(e^{|k_-|x} - e^{-|k_-|x} \right), & -a < x < 0 \\ \frac{b}{2|k_-|e^{2|k_-|a}} \left(e^{|k_-|x} - e^{-|k_-|x} \right), & 0 < x < a \\ e^{|k_-|x}, & a < x \end{cases}$$

and

$$\varphi_+(x) = A_{1,+} \begin{cases} e^{|k_+|x}, & x < -a \\ \frac{b}{2|k_+|e^{2|k_+|a}} \left(-\frac{\rho - 2|k_+|}{\rho + 2|k_+|} e^{|k_+|x} + e^{-|k_+|x} \right), & -a < x < 0 \\ \frac{b}{2|k_+|e^{2|k_+|a}} \left(-\frac{\rho - 2|k_+|}{\rho + 2|k_+|} e^{-|k_+|x} + e^{|k_+|x} \right), & 0 < x < a \\ e^{|k_+|x}, & a < x \end{cases}$$

where $A_{1,\pm}$ are chosen such that $\|\varphi_\pm\| = 1$. In particular, we have that

$$\begin{aligned} &\varphi_+(-x) = \varphi_+(x),\ \varphi_-(-x) = -\varphi_-(x),\ \forall x \in \mathrm{R} \\ &\varphi_+(x) = \varphi_-(x) \left[1 + O(\rho^{-1})\right],\ \forall x \in \mathrm{R}^+,\ \text{ as } \ \rho \to \infty \\ &|A_{1,+}| = |A_{1,-}| \left[1 + O(\rho^{-1})\right],\ \text{ as } \ \rho \to \infty \\ &|A_{1,-}| = \left\{ \frac{1}{|k|e^{2|k|a}} + \frac{b^2}{2|k|^2 e^{4|k|a}} \left[2a + \frac{1}{|k|} \cosh(2|k|a) \right] \right\}^{-1/2} \end{aligned}$$

Hence, Lemma 1, equation (5), Lemma 2 and Lemma 3 follow.

In order to prove the bound (11) let

$$\Gamma_{\rho,k} = \begin{pmatrix} -\left(-\frac{1}{b} + \frac{i}{2k}\right) & -\frac{i}{2k} e^{ika} & -\frac{i}{2k} e^{i2ka} \\ -\frac{i}{2k} e^{ika} & -\left(\frac{1}{\rho} + \frac{i}{2k}\right) & -\frac{i}{2k} e^{ika} \\ -\frac{i}{2k} e^{i2ka} & -\frac{i}{2k} e^{ika} & -\left(-\frac{1}{b} + \frac{i}{2k}\right) \end{pmatrix},$$

$$K_\rho(x, y; k) = \frac{i}{2k} e^{ik|x-y|} - \frac{1}{4k^2} \sum_{r,s=1}^{3} \left(\Gamma^{-1}_{\rho,k}\right)_{rs} e^{ik|x-y_r|} e^{ik|y-y_s|}; \tag{54}$$

we recall (see [2]) that the resolvent operator $[H_{0,\rho} - \lambda]^{-1}$ is an integral operator with kernel K_ρ:

$$\left([H_{0,\rho} - \lambda]^{-1} \phi\right)(x) = \int_{\mathrm{R}} K_\rho(x, y; k) \phi(y) dy, \quad \lambda = k^2. \tag{55}$$

By means of a simple computation it follows that

$$\det \Gamma_{\rho,k} = \frac{a^3}{w} \mathsf{h}(w) \mathsf{g}_\rho(w), \quad \text{where} \quad w = i2ka,$$

and that the elements of the inverse matrix $\Gamma^{-1}_{\rho,k}$ are given by

$$\begin{aligned} &\left(\Gamma^{-1}_{\rho,k}\right)_{1,1} = \left(\Gamma^{-1}_{\rho,k}\right)_{3,3} = \frac{-\mathsf{f}(w) + \frac{1}{a}\left(\frac{1}{b} - \frac{1}{\rho}\right) - \frac{w}{a^2 \rho b}}{a \mathsf{g}_\rho(w) \mathsf{h}(w)}, \\ &\left(\Gamma^{-1}_{\rho,k}\right)_{1,2} = \left(\Gamma^{-1}_{\rho,k}\right)_{2,1} = \left(\Gamma^{-1}_{\rho,k}\right)_{2,3} = \left(\Gamma^{-1}_{\rho,k}\right)_{3,2} = -\frac{e^{w/2}}{a \mathsf{g}_\rho(w)}, \\ &\left(\Gamma^{-1}_{\rho,k}\right)_{2,2} = \frac{2 + \frac{w}{ab} + w\mathsf{f}(w)}{a \mathsf{g}_\rho(w)}, \\ &\left(\Gamma^{-1}_{\rho,k}\right)_{1,3} = \left(\Gamma^{-1}_{\rho,k}\right)_{3,1} = \frac{e^w}{a^2 \rho \mathsf{g}_\rho(w) \mathsf{h}(w)}. \end{aligned} \tag{56}$$

We underline that when $ab > 1$ then there exist a positive constant C and $\rho^\star > 0$ such that

$$|\mathsf{h}(w)| > C \quad \text{and} \quad |\mathsf{g}_\rho(w)| > C,\ w = 2iak,\ \forall k \in \mathrm{R}^+,\ \forall \rho \in [\rho^\star, +\infty), \tag{57}$$

in particular $\lambda = 0$ is neither an eigenvalue nor a resonance.

In order to prove the bound (11) uniformly with respect to ρ we observe that a direct computation gives that

$$\sum_{r=1}^{3} \left(\Gamma_{\rho,k}^{-1}\right)_{s,r} = wF_s^\rho(w), \quad s = 1,2,3 \tag{58}$$

where we set

$$F_1^\rho(w) = F_3^\rho(w) = \frac{a\rho \mathsf{f}(w)\mathsf{f}(w/2) - \frac{2}{ab} - \frac{\rho}{b}\mathsf{f}(w/2) + 2\mathsf{f}(w)}{2a^2\rho \mathsf{g}_\rho(w)\mathsf{h}(w)} \tag{59}$$

and

$$F_2^\rho(w) = \frac{-\mathsf{f}(w/2) + \frac{1}{ab} + \mathsf{f}(w)}{a\mathsf{g}_\rho(w)}. \tag{60}$$

Now, we are ready to prove that

Theorem 6. *There exists a positive constant C, independent of ρ such that*

$$\|e^{-itH_{0,\rho}} P_c \varphi\|_\infty \leq C < t >^{-1/2} \|\varphi\|_1. \tag{61}$$

Proof. In order to prove the above estimate let γ be an anti-clockwise curve, surrounding the absolute continuous spectrum $\sigma_{\rm ac}(H_{0,\rho}) = [0, +\infty)$, with endpoints $+\infty + i0$ and $+\infty - i0$; the spectral theorem gives that

$$\begin{aligned} \left(e^{-itH_{0,\rho}} P_c \varphi\right)(x) &= \int_\gamma e^{-i\lambda t} \left([H_{0,\rho} - \lambda]^{-1}\varphi\right)(x) d\lambda \\ &= i \int_{\mathrm{R}+i0} 2k e^{-ik^2 t} dk \int_{\mathrm{R}} K_\rho(x,y;k)\varphi(y) dy \\ &= \int_{\mathrm{R}} U_\rho^t(x,y)\varphi(y) dy \end{aligned}$$

where

$$U_\rho^t(x,y) = i \int_{\mathrm{R}+i0} 2k e^{-ik^2 t} K_\rho(x,y;k) dk.$$

We prove that if $a, b > 0$ are such that $ab > 1$ and if $\rho^\star > 0$ is large enough in order to have (57) then, the following asymptotic behavior uniformly holds for any $\rho \geq \rho^\star$:

$$\max_{x,y\in\mathrm{R}} |\, U_\rho^t(x,y)| \leq C < t >^{-1/2}. \tag{62}$$

Now, we have that

$$\begin{aligned} U_\rho^t(x,y) &= i \int_{\mathrm{R}+i0} e^{-ik^2 t} e^{ik|x-y|} dk + V_\rho^t(x,y) \\ &= \frac{i\sqrt{\pi}}{\sqrt{it}} e^{i|x-y|^2/4t} + V_\rho^t(x,y) \end{aligned}$$

where

$$V_\rho^t(x,y) = \frac{1}{2} \sum_{r,s=1}^{3} \int_{\mathrm{R}+i0} e^{-ik^2 t} e^{iku_{r,s}} \left(\Gamma_{\rho,k}^{-1}\right)_{r,s} \frac{dk}{k}$$

and where we set $u_{r,s} = |x - y_r| + |y - y_s|$ and we recall that the Fourier transform of e^{-ik^2t} is $\sqrt{\pi}e^{iu^2/4t}/\sqrt{it}$. Now, let

$$u_{r,s} = u_{2,s} + u_r,\ u_r(x) = |x - y_r| - |x - y_2|,$$

then we can write

$$V^t_\rho(x,y) = V^t_{\rho,1}(x,y) + V^t_{\rho,2}(x,y)$$

where

$$\begin{aligned} V^t_{\rho,1}(x,y) &= \frac{1}{2}\sum_{r,s=1}^{3}\int_{\mathrm{R}+i0} e^{-ik^2t}e^{iku_{2,s}}\left(\Gamma^{-1}_{\rho,k}\right)_{r,s}\frac{dk}{k} \\ &= \frac{1}{2}\sum_{s=1}^{3}\int_{\mathrm{R}+i0} e^{-ik^2t}e^{iku_{2,s}}F^\rho_s(w)\frac{w}{k}dk \\ &= ia\sum_{s=1}^{3}\int_{\mathrm{R}+i0} e^{-ik^2t}e^{iku_{2,s}}F^\rho_s(w)dk \\ &= O(t^{-1/2}) \end{aligned}$$

from (58), uniformly with respect to ρ, since

$$\sup_{\rho\in[\rho^\star,+\infty),\ k\in[0,+\infty)} |F^\rho_s(2iak)| \le C,\ s = 1,2,3,$$

and

$$V^t_{\rho,2}(x,y) = \frac{1}{2}\sum_{r,s=1}^{3}\int_{\mathrm{R}+i0} e^{-ik^2t}e^{iku_{2,s}}\left(\Gamma^{-1}_{\rho,k}\right)_{r,s}\frac{e^{iku_r}-1}{k}dk = O(t^{-1/2})$$

by means of the same arguments since

$$\left|\frac{e^{iku_r}-1}{k}\right| \le C,\ \forall k \in (0,+\infty),$$

where $|u_r| = ||x - y_r| - |x - y_2|| \le a$, and

$$\sup_{\rho\in[\rho^\star,+\infty),\ k\in[0,+\infty)} \left|\left(\Gamma^{-1}_{\rho,k}\right)_{r,s}\right| \le C;\ r,s = 1,2,3.$$

Collecting all these results, it follows the estimate (uniform with respect to ρ)

$$U^t_\rho(x,y) = O(t^{-1/2}).$$

□

Remark 5. We underline that from the estimate (61), from the obvious estimate $\|e^{-itH_{0,\rho}}P_c\phi\|_2 \le \|\phi\|_2$ and from the Riesz-Thorin interpolation theorem then (11) follows.

Appendix A. Explicit solution of the two-level approximation

Here, following [13] we give the solution of the two-level approximation (22) where $b_R(t)$ and $b_L(t)$ satisfies to the condition (23). By means of the change of variables (28) then the system of ordinary differential equations (22) takes the form (30). Moreover, if we introduce the relative phase

$$\theta(\tau) = \arg[B_R(\tau)] - \arg[B_L(\tau)]$$

and the fractional imbalance function

$$z(\tau) = |B_R(\tau)|^2 - |B_L(\tau)|^2$$

then it follows that these two functions have to satisfy to the following system of ordinary differential equations

$$\left\{ \begin{array}{rcl} z' & = & 2\sqrt{1-z^2}\sin\theta \\ \theta' & = & -\frac{2z}{\sqrt{1-z^2}}\cos\theta - \eta z \end{array} \right.$$

which admits the following conservation laws directly coming from (13)

$$I = \sqrt{1-z^2}\cos\theta - \frac{1}{4}\eta z^2. \tag{63}$$

Hence, $z(\tau)$ satisfies to the following differential equation of the first order

$$(z')^2 = 4(1-z^2) - 4\left(I + \frac{1}{4}\eta z^2\right)^2$$

which solution is given by means of the elliptic Jacobian functions (see §7.10 [5]):

$$z(\tau) = A\mathrm{cn}\left[A\eta(\tau-\tau_0)/2k, k\right], \quad \text{if } k<1, \tag{64}$$

or

$$z(\tau) = A\mathrm{dn}\left[A\eta(\tau-\tau_0)/2, 1/k\right], \quad \text{if } k>1, \tag{65}$$

where τ_0 depends on the initial condition,

$$A = \frac{2\sqrt{2}}{\eta}\left[\sqrt{\frac{1}{4}\eta^2 + 1 + I\eta} - J\right]^{1/2}, \; J = 1 + \frac{1}{2}I\eta,$$

and

$$k^2 = \frac{1}{2}\left[1 - \frac{J}{\sqrt{\frac{1}{4}\eta^2 + 1 + I\eta}}\right] = \frac{1}{2}\left[1 - \frac{J}{\sqrt{J^2 + \frac{1}{4}\eta^2(1-I^2)}}\right]. \tag{66}$$

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