A NOTE ON THE THEOREMS OF M. G. KREIN AND L. A. SAKHNOVICH ON CONTINUOUS ANALOGS OF ORTHOGONAL POLYNOMIALS ON THE CIRCLE.

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Abstract. Continuous analogs of orthogonal polynomials on the circle are solutions of a canonical system of differential equations, introduced and studied by M.G.Krein and recently generalized to matrix systems by L.A.Sakhnovich. In particular, $\int_{\mathbb{R}}\frac{|\log\det\tau'(\lambda)|}{1+\lambda^2}d\lambda<\infty$ if and only if $\int_0^\infty|P(r,\lambda)|^2dr<\infty$ for $\mathrm{Im}\lambda>0$, where τ' is the density of the absolutely continuous component of the spectral measure, and $P(r,\lambda)$ is the continuous analog of orthogonal polynomials. We point out that Krein's and Sakhnovich's papers contain an inaccuracy, which does not undermine known implications from these results, and prove the corrected statement: the convergence of the integrals above is equivalent not to the existence of the limit $\Pi(\lambda)=\lim\limits_{r\to\infty}P_*(r,\lambda)$ but to the convergence of a subsequence. Here $P_*(r,\lambda)$ is the continuous analog of the adjoint polynomials, and $\Pi(\lambda)$ is analytic for $\mathrm{Im}\lambda>0$. The limit as $r\to\infty$ does not necessarily converges even if τ is absolutely continuous. Also we show that $\Pi(\lambda)$ is unique if the coefficients are in L^2, but in general it can be defined only up to a constant multiple even if the coefficients are in L^p for any $p>2$.

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1. Introduction.

Orthogonal polynomials on the unit circle have interesting features that relate properties of their spectral measure to the properties of coefficients of generating recursive formulas (see Section 2 for more details). The present paper deals with continuous analogs of such polynomials.

The one dimensional analogs were introduced by M. G. Krein in [K]. They provide, in a sense, a generalization of the Fourier transform from $L^2(\mathbb{R})$ to $L^2(\mathbb{R},\tau)$. Here τ is an arbitrary, up to a mild constrain (3.2), Borel spectral measure on $\mathbb{R}$. In this generalization of the Fourier transform, the usual exponentials $e^{ir\lambda}$ are replaced with $p(r,\lambda)$, the continuous analog of orthogonal polynomials. We consider only "one sided" situation, that is, r is nonnegative and the Fourier transform is from a half-line to the whole line (see Section 3).

Note that the Fourier transform itself is a continuous analog of the expansion into the Fourier series, $\{e^{ir\lambda} \,|\, r \in \mathbb{R}_+, \lambda \in \mathbb{R}\}$ being analogous to $\{z^n \,|\, n \in \mathbb{Z}_+, |z| = 1\}$. Similarly, $\{p(r,\lambda) \,|\, r \in \mathbb{R}_+, \lambda \in \mathbb{R}\}$ is analogous to $\{\Phi_n(z) \,|\, n \in \mathbb{Z}_+, |z| = 1\}$, monic polynomials of degree n orthogonal on the unit circle with respect to an arbitrary probability Borel spectral measure τ. To add one more analogy, note that $\Phi_n(z) = z^n$ are the orthogonal polynomials with the normalized Lebesgue measure as the spectral measure.

The functions $p(r,\lambda)$, together with the continuous analog $p^*(r,\lambda)$ of the adjoint polynomials, are solutions of a canonical system of differential equations (3.1). The spectral measure τ is uniquely determined by these differential equations. In [S1–S5] L. A. Sakhnovich defined and studied matrix valued continuous analogs of orthogonal polynomials on the unit circle, and generalized Krein's results for this case (see Section 5).

Krein's differential equations are related to the study of the one dimensional continuous Schrödinger equation [D1,D5,DKu,K]. Also they can be used to solve an important factorization problem in the theory of analytic functions [A,G,Se].

Another way to define $p(r,\lambda)$ and $p^*(r,\lambda)$ is by the formulas

$$p(r,\lambda) = e^{ir\lambda}\Big(1 - \int_0^r \Gamma_r(s,0)e^{-is\lambda}\Big)ds$$

$$p^*(r,\lambda) = 1 - \int_0^r \Gamma_r(s,0)e^{is\lambda}ds.$$

Here $\Gamma_r(s,t) = \overline{\Gamma_r(t,s)}$ is the resolvent of a positive integral operator S_r, that is

$$\Gamma_r(s,t) + \int_0^r H(s-u)\Gamma_r(u,t)du = H(s-t),$$

where $H(t) = -\overline{H(-t)}$ is locally integrable and

$$S_r f(x) = f(x) + \int_0^r H(x-t)f(t)dt.$$

The coefficient $a(r)$ of the equation (3.1) is $a(r) = \Gamma_r(0, r)$.

If τ' is the density of the absolutely continuous component of the spectral measure, then

$$\int_{\mathbb{R}} \frac{|\log \tau'(\lambda)|}{1+\lambda^2} d\lambda < \infty \tag{1.1}$$

if and only if

$$\int_0^\infty |p(r,\lambda)|^2 dr < \infty \tag{1.2}$$

for $\mathrm{Im}\lambda > 0$. In the center of our discussion is the existence of the limit

$$\Pi(\lambda) = \lim_{r\to\infty} p^*(r,\lambda), \tag{1.3}$$

where $\Pi(\lambda)$ is analytic for $\mathrm{Im}\lambda > 0$. An important relation, which follows from (3.1) and was noted by M. G. Krein in [K], is

$$|p^*(r,\lambda)|^2 - |p(r,\lambda)|^2 = 2\mathrm{Im}\lambda \int_0^r |p(s,\lambda)|^2 ds. \tag{1.4}$$

This a particular case of Lagrange identity, which is an analog of the Christofel-Darboux formula for orthogonal polynomials (see, for instance, [At]). Thus we must have $|\Pi(\lambda)|^2 = 2\mathrm{Im}\lambda \int_0^\infty |p(r,\lambda)|^2 dr$ if the integral converges.

In Section 6 we show that if the coefficients are square integrable, then the limit (1.3) converges, and so $\Pi(\lambda)$ is uniquely defined. The existence of this limit implies the convergence of the integrals (1.1) and (1.2), but the converse is not true in general. In Section 4 we prove that there are situations when (1.1) and (1.2) hold, but $\Pi(\lambda)$ has to be defined as a limit of a convergent subsequence. We show that this situation is not "pathological", but can occur even if the spectral measure τ is absolutely continuous with positive continuous density (Theorem 1). In another example (Theorem 2), this happens even though

$$|\Pi(\lambda)|^2 = \lim_{r\to\infty} |p^*(r,\lambda)|^2 = 2\mathrm{Im}\lambda \int_0^\infty |p(r,\lambda)|^2 dr < \infty,$$

and the coefficients are in L^p for any $p > 2$. Moreover, the function $\Pi(\lambda)$ can not be defined uniquely, but only up to a constant factor of absolute value one (up to left multiplication by a unitary matrix in the case of Sakhnovich's theorem). Section 6 contains several results related to the convergence of the limit (1.3) in the case of the Sakhnovich's differential equations.

Note that results of Section 6 apply to Krein's system, since it is a particular case of Sakhnovich's system. Two of the three results there are new even for Krein's system. At the same time results of Section 4 are stated for Krein's system, but are applicable for Sakhnovich's system as well.

The fundamental paper [K] presents a number of important results, though it does not contain proofs due to the type of the journal it was published in. Later proofs of Krein's results were given independently by the author in 1990 ([T1], partly published in [T2]) and L. A. Sakhnovich in 1998 ([S2–S4]). The main subject of [T1,T2] was to prove that the spectral measure τ is absolutely continuous with probability one if the coefficient $a(r)$ is a random function satisfying certain conditions.

In [T1,T2] the author noted and rectified an inaccuracy in the statement of Krein's theorem, and gave a proof of the corrected main theorem (see Section 3 for more details). Theorems 1 and 2 in Section 4 prove, in particular, that a part of the statement of Krein's theorem in [K] needs to be revised.

In [S1–S5] L. A. Sakhnovich defined and studied matrix valued continuous analogs of orthogonal polynomials on the unit circle, and proved matrix generalizations of Krein's results. Unfortunately, these works contain the same kind of inaccuracy as [K]. In Section 5 we present the corrected statement, and the corrected part of the proof.

We emphasize that the inaccuracy in the statement of Krein's and Sakhnovich's theorems is not significant, and does not undermine known implications from these important results. For instance, if (1.1) and (1.2) hold, then there is the function $\Pi(\lambda)$ which is analytic and has no zeros for $\mathrm{Im}\lambda > 0$, and

$$\tau'(\lambda) = \frac{1}{2\pi|\Pi(\lambda)|^2}$$

for Lebesgue almost all $\lambda \in \mathbb{R}$ (there is an analogous matrix version proved by L. A. Sakhnovich in [S4]). This result remains unchanged even if the limit $\lim\limits_{r\to\infty} p^*(r,\lambda)$ diverges, and the nonuniqueness of $\Pi(\lambda)$ mentioned above takes place.

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2. ORTHOGONAL POLYNOMIALS ON THE CIRCLE.

If $\{\Phi_n(z)\}_{n=0}^\infty$ are monic polynomials of degree n orthogonal on the unit circle with respect to a probability Borel measure τ then there exists a sequence of complex numbers $\{a_n\}_{n=0}^\infty$ such that

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) - \bar{a}_n\Phi_n^*(z) \\ \Phi_{n+1}^*(z) &= \Phi_n^*(z) - a_n z\Phi_n(z) \end{aligned} \tag{2.1}$$

where $\{\Phi_n^*(z)\}_{n=0}^\infty$ is an auxiliary sequence of polynomials and

$$\Phi_0(z) = \Phi_0^*(z) = 1.$$

The polynomials $\Phi_n^*(z)$ are adjoint to the orthogonal polynomials $\Phi_n(z)$ in the sense that $\Phi_n^*(z) = 1 + \cdots + \bar{c}_j z^{n-j} + \cdots$ if $\Phi_n^*(z) = z + \cdots + c_j z^j + \cdots$.

So called circular (reflection, Shur's) parameters $\{a_n\}_{n=0}^{\infty}$ satisfy $|a_n| \leqslant 1$. Moreover,

$$|a_n| < 1 \tag{2.2}$$

for all n if and only if the measure is not concentrated in a finite number of atoms.

Conversely, if conditions (2.2) are satisfied, then there exists a unique Borel probability measure τ on the unit circle such that polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$, defined by (2.1), are orthogonal with respect to τ.

The theory of orthogonal polynomials on the circle was developed by G. Szegö, N. I. Akhiezer, L. Ya. Geronimus *et al.* ([A,G,Se]). The following theorem is a combination of results of G. Szegö, A. N. Kolmogorov, M. G. Krein and L. Ya. Geronimus (see [G]).

Theorem. *The linear span of $\{\Phi_n(z)\}_{n=0}^{\infty}$ is not dense in L^2_τ if and only if any of the following five equivalent statements hold*

(I)

$$\int_0^{2\pi} \log \tau'(e^{i\,\theta}) d\theta > -\infty \tag{2.3}$$

where τ' is the density of the absolutely continuous component of τ with respect to the Lebesgue measure on the unit circle.

(II) *There exists at least one z in the unit disk $D = \{z : |z| < 1\}$ such that*

$$\sum_{n=0}^{\infty} |\Phi_n(z)|^2 < \infty. \tag{2.4}$$

(III) *There exists at least one $z \in D$ such that*

$$\liminf_{n\to\infty} |\Phi_n^*(z)| < \infty.$$

(IV) *The series (2.4) converges uniformly on compact subsets of D.*

(V) *There exists a function $\Pi(z)$, analytic in D, such that the limit*

$$\Pi(z) = \lim_{n\to\infty} \Phi_n^*(z) \tag{2.5}$$

is uniformly convergent on compact subsets of D.

Moreover, the statements (I–V) are equivalent to the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Note that in (I) the integral is always less than $+\infty$, and that there is no restrictions on the singular part of τ.

3. KREIN'S THEOREM.

In [K] M. G. Krein studied the following canonical system of ordinary differential equations

$$\begin{aligned} \frac{d}{dr}\, p\,(r,\lambda) &= \boldsymbol{i}\lambda\, p(r,\lambda) - \overline{a(r)}\, p^*(r,\lambda) \\ \frac{d}{dr}\, p^*(r,\lambda) &= -\, a(r)\, p(r,\lambda) \end{aligned} \tag{3.1}$$

where $a(\cdot)$ is a measurable locally integrable function on $[0,\infty)$, and the initial conditions are

$$p(0,\lambda) = p^*(0,\lambda) = 1.$$

There is a unique Borel measure τ on $\mathbb{R}$, which called the spectral measure, such that

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2} d\tau(\lambda) < \infty \tag{3.2}$$

and the map $\mathfrak{U} : L^2_{[0,\infty)} \to L^2_\tau$ defined by

$$\mathfrak{U}f(\lambda) = \int_0^\infty f(r) p(r,\lambda) dr \tag{3.3}$$

is an isometry. Conversely, if the condition (3.2) is satisfied for a Borel measure τ on $\mathbb{R}$, then there exists a unique measurable locally integrable function $a(\cdot)$ on $\mathbb{R}_+$ such that $\mathfrak{U}$ defined by (3.1) and (3.3) is an isometry $\mathfrak{U} : L^2_{[0,\infty)} \to L^2_\tau$.

A simple example is the situation when $a(r) \equiv 0$ and $\mathfrak{U}$ is the usual Fourier transform. In this case τ is the Lebesgue measure normalized by 2π. For a more detailed study see [AR,R,D2–5,DKu].

The following important result was stated by M. G. Krein in [K] without a proof.

Theorem. *The isometry $\mathfrak{U}$ is not onto if and only if any of the following five equivalent statements hold*

(I)

$$\int_{\mathbb{R}} \frac{\log \tau'(\lambda)}{1+\lambda^2} d\lambda > -\infty \tag{3.4}$$

where τ' is the density of the absolutely continuous component of τ with respect to the Lebesgue measure on $\mathbb{R}$.

(II) *There exists at least one λ in the domain $H = \{\lambda : Im\lambda > 0\}$ such that*

$$\int_0^\infty |p(r,\lambda)|^2 dr < \infty. \tag{3.5}$$

(III) *There exists at least one $\lambda \in H$ such that*

$$\liminf_{r\to\infty} |p^*(r,\lambda)| < \infty. \tag{3.6}$$

(IV) *The integral* (3.5) *converges uniformly on compact subsets of H.*

(V) *There exists a function* $\Pi(\lambda)$*, analytic in* H*, such that on any compact subset of* H *there is a uniform limit*

$$\Pi(\lambda) = \lim_{n\to\infty} p^*(r_n, \lambda) \tag{3.7}$$

for a sequence $r_n \to \infty$*.*

Parts (III,V) of this theorem were not stated in [K] correctly. Namely, it was written as if (I,II,IV) were equivalent to:

(III') There exists at least one $\lambda \in H$ such that $\sup_{r\geqslant 0} |p^*(r,\lambda)| < \infty$.

(V') The limit $\Pi(\lambda) = \lim_{r\to\infty} p^*(r,\lambda)$ converges uniformly on compact subsets of H.

In Section 4 we present two counterexamples. The corrected statement appeared first in [T1,T2].

In Section 4 we prove that, in general, $\Pi(\lambda)$ can not be defined uniquely, but only up to a factor of absolute value one. In Section 6 we show that if $a(r) \in L^2[0,\infty)$ then (I–V) as well as (III',V') hold, but the result is sharp in the sense of Theorem 2 and Remark 6.1.

Note that in (I) the integral is always less than $+\infty$, and that there is no restrictions on the singular part of τ.

4. TWO RESULTS ON NONCONVERGENCE.

Theorem 1. *There exists a real-valued* C^∞ *function* $a(r)$ *such that the spectral measure* τ *is absolutely continuous with positive continuous density, statements (I–V) of Krein's theorem hold, but*

$$\liminf_{r\to\infty} |p^*(r,\lambda)| < \limsup_{r\to\infty} |p^*(r,\lambda)| \tag{4.1}$$

for any $\lambda \in H$*. In addition, the* lim sup *in (4.1) can be either finite or identically* $+\infty$ *on* H*.*

First, we need the following lemma.

Lemma 4.1. *Suppose* $b(r)$ *be a real nonzero continuous function with support in* $[0,2]$ *such that* $b(r)$ *is nonnegative for* $r \in [0,1]$*, and* $b(r) = -b(r-2)$ *for* $r \in [1,2]$*. Let* $p_\varepsilon(r,\lambda)$*,* $p^*_\varepsilon(r,\lambda)$ *be the solutions of (3.1) with*

$$a(r) = a_\varepsilon(r) = -\tfrac{\log|\log\varepsilon|}{\varepsilon} b(\tfrac{r}{\varepsilon}),$$

$0 < \varepsilon < 1$*, and initial conditions* $p_\varepsilon(0,\lambda) = c$*,* $p^*_\varepsilon(0,\lambda) = c^*$*.*

If λ *is any complex number, then*

$$p_\varepsilon(2\varepsilon,\lambda) = c + o(\sqrt{\varepsilon})_{\varepsilon\to 0} \qquad \text{and} \qquad p^*_\varepsilon(2\varepsilon,\lambda) = c^* + o(\sqrt{\varepsilon})_{\varepsilon\to 0}, \tag{4.2}$$

where the limits are uniform for λ*,* c*,* c^* *in any compact subset of* $\mathbb{C}$*. If in addition* $c \neq -c^*$*, then*

$$\lim_{\varepsilon\to 0} |p_\varepsilon(\varepsilon,\lambda)| = \lim_{\varepsilon\to 0} |p^*_\varepsilon(\varepsilon,\lambda)| = \infty. \tag{4.3}$$

Proof. First, we consider differential equations

$$\begin{aligned} \frac{d}{dr}\, q_\varepsilon(r) &= -\, a_\varepsilon(r)\, q^*_\varepsilon(r) \\ \frac{d}{dr}\, q^*_\varepsilon(r) &= -\, a_\varepsilon(r)\, q_\varepsilon(r) \end{aligned} \tag{4.4}$$

with initial conditions $q_\varepsilon(0) = c$, $q^*_\varepsilon(0) = c^*$. Then we have

$$\begin{aligned} q_\varepsilon(r) - q^*_\varepsilon(r) &= (c - c^*) \exp\Big\{ \int_0^r a_\varepsilon(r) dr \Big\}, \\ q_\varepsilon(r) + q^*_\varepsilon(r) &= (c + c^*) \exp\Big\{ -\int_0^r a_\varepsilon(r) dr \Big\}. \end{aligned} \tag{4.5}$$

Hence $q_\varepsilon(2\varepsilon) = c$ and $q^*_\varepsilon(2\varepsilon) = c^*$. Moreover,

$$q_\varepsilon(\varepsilon) = q^*_\varepsilon(\varepsilon) + o(1)_{\varepsilon\to 0} = \frac{1}{2}(c + c^*) \exp\Big\{ \log|\log\varepsilon| \cdot \int_0^1 b(r) dr \Big\} + o(1)_{\varepsilon\to 0}.$$

Thus our aim is to show that for $0 \leqslant r \leqslant 2\varepsilon$ we have

$$|p_\varepsilon(r,\lambda) - q_\varepsilon(r)| = o(\sqrt{\varepsilon})_{\varepsilon\to 0} \qquad \text{and} \qquad |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)| = o(\sqrt{\varepsilon})_{\varepsilon\to 0}.$$

To show this, we use Gronwall's lemma: *if $\alpha(r)$ is a nonnegative integrable function such that for some constants $c_1, c_2 \geqslant 0$ we have $\alpha(r) \leqslant c_1 \int_0^r \alpha(s)ds + c_2$, then $\alpha(r) \leqslant c_2 e^{c_1 r}$.*

First, we can use this lemma to estimate $\alpha(r) = |p_\varepsilon(r,\lambda)| + |p^*_\varepsilon(r,\lambda)|$. By (3.1) and the definition of $p_\varepsilon(r,\lambda)$ and $p^*_\varepsilon(r,\lambda)$ we have

$$|p_\varepsilon(r,\lambda)| + |p^*_\varepsilon(r,\lambda)| \leqslant (|c| + |c^*|) e^{M_\varepsilon r} \tag{4.6}$$

where

$$M_\varepsilon = |\lambda| + \tfrac{\log|\log\varepsilon|}{\varepsilon} \max_{0\leqslant s\leqslant 1} b(s).$$

Then we use Gronwall's lemma once more to estimate $\alpha(r) = |p_\varepsilon(r,\lambda) - q_\varepsilon(r)| + |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)|$. Using (3.1) and (4.5) we obtain $\alpha(r) \leqslant c_1 \int_0^r \alpha(s)ds + c_2$ with $c_1 = M_\varepsilon$ and $c_2 \geqslant r|\lambda p_\varepsilon(s,\lambda)|$ for any $0 \leqslant s \leqslant r$. By estimate (4.6) we can use $c_2 = 2\varepsilon|\lambda|(|c| + |c^*|)e^{2M_\varepsilon \varepsilon}$ when $0 \leqslant r \leqslant 2\varepsilon$. Hence we have

$$|p_\varepsilon(r,\lambda) - q_\varepsilon(r)| + |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)| \leqslant 2\varepsilon|\lambda|(|c| + |c^*|)e^{4M_\varepsilon\varepsilon} = o(\sqrt{\varepsilon})_{\varepsilon\to 0}$$

for $0 \leqslant r \leqslant 2\varepsilon$, which completes the proof. □

Proof of Theorem 1. In this proof $n \to \infty$ means that the limit is taken over positive integers, and $r \to \infty$ means that the limit is taken over positive reals.

We fix a function $b(r)$ which satisfies the conditions of Lemma 4.1. Let $a(r)$ be defined by

$$a(r) = \sum_{n=1}^{\infty} (2^n \log n) b(2^n r - n2^n) = \sum_{n=1}^{\infty} a_{\varepsilon_n}(r - n),$$

where $a_\varepsilon(\cdot)$ is defined as in Lemma 4.1, and $\varepsilon_n = 2^{-n}$. This sum is a C^∞ function since for any r the sum contains at most one nonzero term. Then by Lemma 4.1 we have

$$|p^*(n,\lambda) - p^*(n+2^{1-n},\lambda)| = o(2^{-n/2})_{n\to\infty}.$$

Note that $p^*(r,\lambda)$ does not change when r is in an interval $[n+2^{1-n}, n+1]$ since $a(r)=0$ on such intervals, and so

$$|p^*(n,\lambda) - p^*(n+1,\lambda)| = o(2^{-n/2})_{n\to\infty}. \tag{4.7}$$

Hence the limit $\lim_{n\to\infty} p^*(n,\lambda)$ exists and is finite for any $\lambda \in \mathbb{C}$. Note that $\lim_{n\to\infty} p^*(n,\lambda) \neq 0$ for $\mathrm{Im}\lambda \geqslant 0$ since, by (3.1),

$$\frac{d}{dr}\left(|p^*(r,\lambda)|^2 - |p(r,\lambda)|^2\right) = 2\mathrm{Im}\lambda |p(r,\lambda)|^2 \geqslant 0. \tag{4.8}$$

By the same argument, for any $\mathrm{Im}\lambda > 0$ we have $\lim_{n\to\infty} p(n,\lambda) = 0$ since $\frac{d}{dr}p(r,\lambda) = \boldsymbol{i}\lambda$ on any interval $[n+2^{1-n}, n+1]$. Hence $p(r,\lambda) \neq p^*(r,\lambda)$ for any $r>0$ and $\mathrm{Im}\lambda > 0$. Lemma 4.1 implies that

$$\lim_{n\to\infty} |p(n+2^{-n},\lambda)| = \lim_{n\to\infty} |p^*(n+2^{-n},\lambda)| = \infty.$$

Note that if in Lemma 4.1 we define $a_\varepsilon(r) = -\frac{M}{\varepsilon} b(\frac{r}{\varepsilon})$, then

$$\liminf_{r\to\infty} |p^*(r,\lambda)| < \limsup_{r\to\infty} |p^*(r,\lambda)| < \infty$$

for any large enough M.

In order to complete the proof we need to show that the spectral measure τ is absolutely continuous with positive continuous density. The estimates (4.7) and Lemma 4.1 shows that the limit $\Pi(\lambda) = \lim_{n\to\infty} p^*(n,\lambda)$ converges uniformly on compact sets of $\lambda \in \mathbb{C}$. As a byproduct we have proved that $\Pi(\lambda)$ is continuous for $\lambda \in \mathbb{C}$ and has no zeros in the closed half-plane $\mathrm{Im}\lambda \geqslant 0$. In particular, this is so for real λ.

For the rest of the proof we assume $\lambda \in \mathbb{R}$. Let τ_r be the measure absolutely continuous with respect to the Lebesgue measure with the density

$$\frac{d\tau_r(\lambda)}{d\lambda} = \frac{1}{2\pi |p^*(r,\lambda)|^2}.$$

Then τ_r converges weakly to τ as $r \to \infty$ (see, for instance, [T2]). By the previous paragraph,

$$\frac{d\tau(\lambda)}{d\lambda} = \lim_{n\to\infty} \frac{1}{2\pi |p^*(n,\lambda)|^2} = \frac{1}{2\pi |\Pi(\lambda)|^2}$$

is a positive continuous function on $\mathbb{R}$, which completes the proof. ☐

Theorem 2. *There exists a C^∞ function $a(r)$ such that (I–V) of Krein's theorem hold, $a(r) \in L^p[0,\infty)$ for any $p > 2$, and $\lim_{r\to\infty} a(r) = 0$. In addition, for a function $\Pi(\lambda)$, which is analytic in $H = \{\lambda : Im\lambda > 0\}$, we have that*

$$\lim_{r\to\infty} p(r,\lambda) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} |p^*(r,\lambda)| = |\Pi(\lambda)| \tag{4.9}$$

for any $\lambda \in H$.

However $\Pi(\lambda)$ is not unique in the following sense: for any complex θ of absolute value one there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} p^*(t_n,\lambda) = \theta\Pi(\lambda). \tag{4.10}$$

Proof. We will construct a function $a(r)$ which is piecewise constant, and then can be approximated by a C^∞ function that still has the desired properties.

First, note that the system of differential equations

$$\begin{aligned} \frac{d}{dr}\, q(r) \quad &= -\,\overline{a(r)}\, q^*(r) \\ \frac{d}{dr}\, q^*(r) \quad &= -\,a(r)\, q(r) \end{aligned} \tag{4.11}$$

with constant coefficient $a(r) = -b$ has a matrix solution

$$Q(r) = \begin{pmatrix} \cosh |b|r & \frac{\overline{b}}{|b|}\sinh |b|r \\ \frac{b}{|b|}\sinh |b|r & \cosh |b|r \end{pmatrix}. \tag{4.12}$$

Now let b be positive real and

$$a_{b,\xi,\varepsilon}(r) = \begin{cases} -b, & \text{for } 0 \leqslant r \leqslant \varepsilon, \\ \overline{\xi} b, & \text{for } \varepsilon \leqslant r \leqslant 2\varepsilon, \\ 0, & \text{for } r \geqslant 2\varepsilon, \end{cases} \tag{4.13}$$

where the constant $\xi \in \mathbb{C}$ is such that $|\xi| = 1$. Let $q(r) = q_{b,\xi,\varepsilon}(r)$ and $q^*(r) = q^*_{b,\xi,\varepsilon}(r)$ be the solutions of the system of equations (4.11) with $a(r) = a_{b,\xi,\varepsilon}(r)$, and initial conditions $q(0) = 0$, $q^*(0) = 1$. Then

$$\begin{aligned} q_{b,\xi,\varepsilon}(\varepsilon) &= \sinh b\varepsilon, & q_{b,\xi,\varepsilon}(2\varepsilon) &= \frac{1}{2}(1-\xi)\sinh 2b\varepsilon, \\ q^*_{b,\xi,\varepsilon}(\varepsilon) &= \cosh b\varepsilon, & q^*_{b,\xi,\varepsilon}(2\varepsilon) &= 1 + (1-\overline{\xi})\sinh^2 b\varepsilon. \end{aligned} \tag{4.14}$$

Let $p_{b,\xi,\varepsilon}(r,\lambda)$ and $p^*_{b,\xi,\varepsilon}(r,\lambda)$ be the solutions of the system of equations (3.1) with $a(r) = a_{b,\xi,\varepsilon}(r)$, and initial conditions $p_{b,\xi,\varepsilon}(0,\lambda) = 0$ and $p^*_{b,\xi,\varepsilon}(0,\lambda) = 1$.

To estimate these solutions we use the following form of Gronwall's lemma: *if $\alpha(r)$ is a nonnegative integrable function such that for some $c, \beta(r) \geqslant 0$ we have $\alpha(r) \leqslant c\int_0^r \alpha(s)ds + \beta(r)$, then $\alpha(r) \leqslant ce^{cr}\int_0^r \beta(s)ds + \beta(r)$.*

In the following estimates we assume that $\lambda \in \mathbb{C}$ is fixed. We write "*const*" for a constant, different in different inequalities, which depends on λ, but is independent of ε, r and b provided $0<\varepsilon,r,b<1$. First, we use Gronwall's lemma and (3.1) to show that $|p_{b,\xi,\varepsilon}(r,\lambda)|+|p^*_{b,\xi,\varepsilon}(r,\lambda)|\leqslant 1+r(|\lambda|+b)e^{(|\lambda|+b)r}<const.$

Then we apply this form of Gronwall's lemma the second time in order to obtain, using (3.1) and the previous estimate, that

$$|p_{b,\xi,\varepsilon}(r,\lambda)|<const\ \left(\frac{1}{2}|\lambda|e^{|\lambda|r}br^2+br\right)<const\cdot br.$$

Using the same form of Gronwall's lemma the third time, we obtain

$$|p_{b,\xi,\varepsilon}(r,\lambda)-q_{b,\xi,\varepsilon}(r)|+|p^*_{b,\xi,\varepsilon}(r,\lambda)-q^*_{b,\xi,\varepsilon}(r)|<const\cdot br^2 \tag{4.15}$$

by (3.1), (4.11) and the previous estimate. This implies

$$|p^*_{b,\xi,\varepsilon}(r,\lambda)-q^*_{b,\xi,\varepsilon}(r)|<const\cdot b^2r^3 \tag{4.16}$$

by (3.1) and (4.11).

We define

$$\varepsilon_n=\frac{1}{\log^2 n},\qquad b_n=\frac{\log^2 n}{\sqrt{n}}$$

for $n\geqslant 3$. Also we define ξ_n as a unique complex number such that

$$|\xi_n|=1,\qquad |1-\xi_n|=\frac{1}{\log n}\qquad\text{and}\qquad \mathrm{Im}\xi_n>0.$$

Note that

$$\xi_n=1+\frac{i}{\log n}+O\Big(\frac{1}{\log^2 n}\Big)_{n\to\infty}. \tag{4.17}$$

Let $a(r)$ be defined by

$$a(r)=\sum_{n=3}^{\infty}a_{b_n,\xi_n,\varepsilon_n}(r-r_n),$$

where $a_{b,\xi,\varepsilon}(\cdot)$ is defined by (4.13), and r_n are as follows. We fix any $\lambda_0\in H$. Then we choose $r_2=0$ and each r_n-r_{n-1} to be large enough so that

$$\frac{p^*(r_n+2\varepsilon_n,\lambda_0)}{p^*(r_n,\lambda_0)}=1+\frac{i}{n\log n}+O\left(\frac{1}{n\log^2 n}\right)_{n\to\infty}. \tag{4.18}$$

For any $\lambda_0\in H$ this is possible since $|p(r_n,\lambda)|\to 0$ exponentially as r_{n-1} is fixed and $(r_n-r_{n-1})\to\infty$. Therefore we can use (4.16) and the fact that

$$q^*_{b,\xi,\varepsilon}(2\varepsilon,\lambda)=1+(1-\overline{\xi})b^2\varepsilon^2+O(b^4\varepsilon^4)_{b\varepsilon\to\infty}$$

by (4.14).

We have that $p^*(r,\lambda)$ is constant for $r\in[r_n+2\varepsilon_n,r_{n+1}]$, in particular, $p^*(r_n+2\varepsilon_n,\lambda)=p^*(r_{n+1},\lambda)$. Hence (4.18) imply that

$$\left|\frac{p^*(r_{n+1},\lambda_0)}{p^*(r_n,\lambda_0)}\right|-1=O\left(\frac{1}{n\log^2 n}\right)_{n\to\infty}$$

and so the limit $\lim_{n\to\infty}|p^*(r_n,\lambda_0)| = |\Pi(\lambda_0)|$ converges, since the series $\sum\limits_{n=3}^{\infty}\frac{1}{n\log^2 n}$ converges. Thus statements (I–V) of Krein's theorem hold by (1.4) and (3.6).

If each $r_n - r_{n-1}$ is large enough, then the sum that defines $a(r)$ is a sum of the functions with disjoint support. Then $\|a(r)\|^p_{L^p} = 2\sum_{n=3}^{\infty} n^{-p/2}\log^{2p-2} n$, and so $a(r)\in L^p[0,\infty)$ if and only if $p>2$. In particular, this means that $a(r)$ is uniformly integrable and so part 2 of Theorem 3 implies (4.9).

To complete the proof note that the limit $\lim_{n\to\infty}p^*(r_n,\lambda_0)$ does not exists because of (4.18), since the series $\sum_{n=3}^{\infty}\frac{1}{n\log n}$ diverges. At the same time it is easy to see by (4.18) that

$$\lim_{n\to\infty} p^*(r_{n+1},\lambda_0) - p^*(r_n,\lambda_0) = 0,$$

and so for any complex θ of absolute value one there is a sequence $t^\theta_n\to\infty$, which is a subsequence of r_n, such that $\lim_{n\to\infty}p^*(t^\theta_n,\lambda_0) = \theta|\Pi(\lambda_0)|$. Note that $|\Pi(\lambda)|$ is well defined for any $\lambda\in H$ since $\lim_{r\to\infty}p^*(r,\lambda) = |\Pi(\lambda)|$ converges by (6.2). Also using (6.2) we can define a function $\Pi(\lambda)$, which is analytic in H, by

$$\Pi(\lambda) = |\Pi(\lambda_0)|^{-1}\lim_{n\to\infty} p^*(t^1_n,\lambda)\overline{p^*(t^1_n,\lambda_0)}.$$

Then $\lim_{n\to\infty}p^*(t^\theta_n,\lambda) = \theta\Pi(\lambda)$ for any $\lambda\in H$ because of (6.2). □

Proposition 4.2. *If $r_n - r_{n-1}$ are large enough in the proof of Teorem 2, then for all $\lambda\in H$ we have (4.18) as well as estimates*

$$\left|\frac{p(r,\lambda)}{p^*(r_n,\lambda)}\right| < \frac{const}{\sqrt{n}\log n} \tag{4.19}$$

for $r_n + 2\varepsilon_n \leqslant r \leqslant r_{n+1}$, and

$$\left|\frac{p^*(r,\lambda)}{p^*(r_n,\lambda)} - 1\right| < \frac{const}{n}, \qquad \left|\frac{p(r,\lambda)}{p^*(r_n,\lambda)}\right| < \frac{const}{\sqrt{n}} \tag{4.20}$$

for $r_n \leqslant r \leqslant r_n + 2\varepsilon_n$.

This gives, in particular, a constructive proof of (3.5) and (4.9).

Proof. We can demonstrate that (4.19) and (4.20) hold for $\lambda=\lambda_0$ using estimates

$$\left|q^*_{b,\xi,\varepsilon}(r,2\varepsilon)\right| < const\cdot b\varepsilon|1-\xi|$$

and, for $0\leqslant r\leqslant 2\varepsilon$,

$$|q_{b,\xi,\varepsilon}(r,\lambda)| < const\cdot b\varepsilon, \qquad \left|q^*_{b,\xi,\varepsilon}(r,\lambda) - 1\right| < const\cdot b^2\varepsilon^2$$

which follows from 4.12 and 4.14.

We also can show (4.20) and (4.18) for all $\lambda\in H$ if the sequence r_n is chosen as follows. It is easy to see that estimates like (4.15) and (4.16) can be established uniformly in λ in a compact subsets of $\mathbb{C}$. Also $|p(r_n,\lambda)|\to 0$ uniformly in λ in a compact subsets of H as r_{n-1} is fixed and $(r_n - r_{n-1})\to\infty$. Thus for any compact subset A of of H there is a sequence r^A_n such that (4.18), (4.19) and (4.20) hold for $r_n = r^A_n$, and also for r_n that is any subsequence of r^A_n.

We can represent H as an increasing union of compact subsets H_k. Without loss of generality we can assume that $r_n^{H_{k+1}}$ is a subsequence of $r_n^{H_k}$ for each k. Then we define r_n by the "diagonal process" $r_n = r_n^{H_n}$. $\square$

Conjecture 4.3. We conjecture that if $a(r)$ is a *real-valued* function, and conditions (I–V) of Krein's theorem hold, then $\Pi(\lambda)$ is unique in the following sense: if $t_n \to \infty$ and $\lim_{n\to\infty} p(t_n, \lambda) = 0$, then the limit $\lim_{n\to\infty} p^*(t_n, \lambda) = \Pi(\lambda)$ converges uniformly on compact subsets of H. If true, this conjecture implies that the original form of Krein's theorem holds if $a(r)$ is real and uniformly integrable (see Theorem 3).

Conjecture 4.4. We conjecture that if $a(r) \in L^1_{loc}$ is real, and conditions (I–V) of Krein's theorem hold, then $\Pi(\lambda)$ is the limit in average of $p^*(t_n, \lambda)$, that is,

$$\Pi(\lambda) = \lim_{r\to\infty} \frac{1}{r} \int_0^r p^*(s, \lambda) ds$$

uniformly on compact subsets of H. Here $a(r) \in L^1_{loc}$ if $\sup_{r\geqslant 0} \int_r^{r+1} |a(s)| ds < \infty$. If true, this conjecture also implies the uniqueness of $\Pi(\lambda)$. Note that in the situation of Theorem 2 the limit in average of $p^*(t_n, \lambda)$ does not exists if $r_{n+1} - r_n$ are large enough. These two conjectures may be related to the results of [D2].

5. Sakhnovich's theorem.

In [S1–S5] L. A. Sakhnovich introduced and studied matrix analogs of Krein's system. He considered a system of canonical differential equations

$$\frac{d}{dr} Y(r, \lambda) = \boldsymbol{i} \lambda J \mathcal{H}(r) Y(r, \lambda), \qquad r \geqslant 0,$$

that can be transformed by a change of variables to a system

$$\begin{aligned} \frac{d}{dr} P_1(r, \lambda) &= \boldsymbol{i} \lambda D P_1(r, \lambda) + A_1(r)\, P_1(r, \lambda) + A_2^*(r)\, P_2(r, \lambda) \\ \frac{d}{dr} P_2(r, \lambda) &= A_2(r)\, P_1(r, \lambda) \end{aligned} \tag{5.1}$$

where $r \in [0, \infty)$, $\lambda \in \mathbb{C}$ and D, $P_1(r, \lambda)$, $P_2(r, \lambda)$, $A_1(r)$, $A_2(r)$ are $m \times m$ matrices, and the initial conditions are

$$P_1(0, \lambda) = P_2(0, \lambda) = I_m,$$

the $m \times m$ identity matrix. It is assumed that $A_1(r) = -A_1^*(r)$, and D is a constant diagonal matrix with positive diagonal entries. Functions A_1, A_2 are locally integrable.

There is a Borel matrix valued measure τ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2} d\tau(\lambda) < \infty \tag{5.2}$$

and the map $\mathcal{U} : L^2_{[0,\infty)} \to L^2_\tau$ defined by

$$\mathcal{U}f(\lambda) = \int_0^\infty f(r)P_1(r,\lambda)dr \tag{5.3}$$

is an isometry.

The following important result was proved by L. A. Sakhnovich in [S2–S4].

Theorem. *The following five statements are equivalent*

(I)

$$\int_{\mathbb{R}} \frac{\log\det\tau'(\lambda)}{1+\lambda^2}d\lambda > -\infty \tag{5.4}$$

where τ' is the density of the absolutely continuous component of τ with respect to the Lebesgue measure on $\mathbb{R}$.

(II) *There exists at least one λ in the domain*

$$H = \{\lambda : Im\lambda > 0\}$$

such that

$$\int_0^\infty |P_1(r,\lambda)|^2 dr < \infty, \tag{5.5}$$

where $|\cdot|$ is a matrix norm.

(III) *There exists at least one $\lambda \in H$ such that*

$$\liminf_{r\to\infty} |P_2(r,\lambda)| < \infty. \tag{5.6}$$

(IV) *The integral* (5.5) *converges uniformly on compact subsets of H.*

(V) *There exists a matrix valued function $\Pi(\lambda)$, analytic in H, such that on any compact subset of H there is a uniform limit*

$$\Pi(\lambda) = \lim_{n\to\infty} P_2(r_n,\lambda) \tag{5.7}$$

for a sequence $r_n \to \infty$.

Parts (III,V) of this theorem were not stated in [S2–S4] correctly in that it was written as if (I,II,IV) implied the existence of the limit $\lim_{r\to\infty} P_2(r,\lambda)$. Since Krein's system is a particular case of Sakhnovich's system, the counterexamples of Section 4 applies to this situation also. It is easy to construct a "true" matrix-valued counterexample along the lines of Section 4.

In Section 4 we prove that, in general, $\Pi(\lambda)$ can not be defined uniquely. In Section 6 we show that if $A_2(r) \in L^2[0,\infty)$, then the finite limit $\Pi(\lambda) = \lim_{r\to\infty} P_2(r,\lambda)$ exists, and so $\Pi(\lambda)$ is unique.

Below we give a corrected part of the proof of Sakhnovich's theorem. Following the lines of [S2–S4], we will show that statements (II–V) are equivalent. An alternative approach can be found in [T2].

A proof of a part of Sakhnovich's theorem. One can see that

$$P_2^*(r,\lambda)P_2(r,\lambda)-P_1^*(r,\lambda)P_1(r,\lambda)=2\mathrm{Im}\lambda\int_0^r P_1^*(s,\lambda)DP_1(s,\lambda)ds, \tag{5.8}$$

and so (II) and (III) are equivalent.

Clearly, by (5.8), (IV–V) implies (II–III). So we have to show that (II–III) implies (IV–V). Now assume that (II–III) hold for some $\lambda=\lambda_0\in H$. By (5.1) and (5.8), the family $\{|P_2(r,\lambda)|: r\geqslant 0, \lambda\in S\}$ is uniformly bounded from below for any compact $S\subset H$. By (5.6) and Montel's theorem, there exists a sequence $r_n\to\infty$ such that there is a uniformly convergent limit (5.7). Thus (V) holds, and so does (IV) because of (5.8). □

Note that

$$P_2^*(r,\lambda_0)P_2(r,\lambda)-P_1^*(r,\lambda_0)P_1(r,\lambda)=2(\overline{\lambda_0}-\lambda)\int_0^r P_1^*(s,\lambda_0)DP_1(s,\lambda)ds, \tag{5.9}$$

and so, if (I–V) hold, then

$$\Pi(\lambda)=2(\overline{\lambda_0}-\lambda)\big(\Pi^*(\lambda_0)\big)^{-1}\int_0^r P_1^*(s,\lambda_0)DP_1(s,\lambda)ds \tag{5.10}$$

provided $\lim_{n\to\infty}P_1^*(r_n,\lambda_0)P_1(r_n,\lambda)=0$.

6. Some convergence results.

All the results in this section apply to Krein's system if we set $m=1$, $D=1$, $A_1(r)=0$, $a(r)=-A_2(r)$, $p(r,\lambda)=P_1(r,\lambda)$ and $p^*(r,\lambda)=P_2(r,\lambda)$.

Note that, even under conditions 1 and 2 of the following theorem, the limit $\lim_{n\to\infty}P_2(r_n,\lambda)$ may not exist by Remark 6.1.

In the next theorem we assume that the matrix norm is $|M|=\sqrt{\mathrm{Tr}M^*M}$. A function $f(r)$ is called uniformly integrable if $\lim\limits_{\varepsilon\to 0}\Big(\sup\limits_{r\geqslant 0}\int_r^{r+\varepsilon}|f(s)|ds\Big)=0$.

Theorem 3. (1) *Suppose that the equivalent conditions (I–V) of Sakhnovich's theorem hold, $t_n\to\infty$ and $\lim_{n\to\infty}P_1(t_n,\lambda_0)=0$ for λ in a nonempty open subset S of H. Then, for an analytic function $\Pi(\lambda)$ on H, the limits*

$$\lim_{n\to\infty}P_2^*(t_n,\xi)P_2(t_n,\lambda)=\Pi^*(\xi)\Pi(\lambda) \quad \textit{and} \quad \lim_{n\to\infty}P_1(t_n,\lambda)=0 \tag{6.1}$$

converge uniformly on compact subsets of $H\times H$ and H respectively. In particular, $\lim_{n\to\infty}|P_2(t_n,\lambda)|=|\Pi(\lambda)|$.

(2) *Suppose that $A_2(r)$ is uniformly integrable, and conditions (I–V) of Sakhnovich's theorem hold. Then, for an analytic function $\Pi(\lambda)$ on H, the limits*

$$\lim_{r\to\infty}P_2^*(r,\xi)P_2(r,\lambda)=\Pi^*(\xi)\Pi(\lambda) \quad \textit{and} \quad \lim_{r\to\infty}P_1(r,\lambda)=0 \tag{6.2}$$

converge uniformly on compact subsets of $H\times H$ and H respectively. In particular, $\lim_{r\to\infty}|P_2(r,\lambda)|=|\Pi(\lambda)|$.

(3) *Suppose that* $A_2(r) \in L^2[0,\infty)$. *Then conditions (I–V) of Sakhnovich's theorem hold and, moreover, the limits*

$$\lim_{r\to\infty} P_2(r,\lambda) = \Pi(\lambda) \quad \text{and} \quad \lim_{r\to\infty} P_1(r,\lambda) = 0 \tag{6.3}$$

converge uniformly on compact subsets of H.

Remark 6.1. This result is sharp in the sense that there is a real C^∞ coefficient $A_2(r)$, which is in L^p for any $p > 2$, such that statements (I–V) of Sakhnovich's theorem do not hold.

Also this result is sharp in a more delicate sense: by Theorem 2 there exists a coefficient $A_2(r)$, which is again in L^p for any $p > 2$, such that $\lim_{r\to\infty} P_1(r,\lambda) = 0$, statements (I–V) of Sakhnovich's theorem hold, but the limit $\lim_{r\to\infty} P_2(r,\lambda)$ does not exist. In fact, we show that $\Pi(\lambda)$ can not be defined uniquely, but only up to a constant factor even though the limit $\lim_{r\to\infty} |P_2(r,\lambda)| = |\Pi(\lambda)|$ exists by part 2 of Theorem 3.

Note that in statements 2 and 3 there is no restriction on the coefficient $A_1(r)$, except for the usual assumption of local integrability.

It was proved in [S2,S3] that if $A_2(r) \in L^1[0,\infty)$, then conditions (I–V) of Sakhnovich's theorem hold, and the limits (6.3) converge uniformly on compact subsets of $H \cup \mathbb{R}$ and H respectively. This fact and statement 3 of Theorem 3 were formulated in [K] for Krein's system. Also, for Krein's system statements 2 and 3 of Theorem 3 are related to the results of [D2].

Proof of 1. Let $\lambda_0 \in S$. Then for any $\lambda \in H$ we have that

$$\lim_{n\to\infty} P_2^*(t_n,\lambda_0) P_2(t_n,\lambda) = \int_0^\infty P_1^*(s,\lambda_0) D P_1(s,\lambda) ds$$

by (5.8), since $P_1^*(t_n,\lambda_0)P_1(t_n,\lambda) = o\big(P_2^*(t_n,\lambda_0)P_2(t_n,\lambda)\big)_{n\to\infty}$ by (5.9). Then we define $\Pi(\lambda_0)$ as any limit point of the sequence $\{P_2(t_n,\lambda_0)\}_{n\geqslant 1}$, and the analytic function $\Pi(\lambda)$ by (5.10). By (5.8), the family of analytic functions $\{P_1(t_n,\lambda_0)\}_{n\geqslant 1}$ is locally uniformly bounded and so relatively compact. Then our assumptions imply $\lim_{n\to\infty} P_1(t_n,\lambda) = 0$. □

Proof of 2. Let us assume that $\limsup_{r\to\infty} |P_1(r,\lambda)| > 0$ for some $\lambda \in H$. Then there is a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} |P_1(t_n,\lambda)| = \delta > 0$. Relation (5.8) implies that $\lim_{n\to\infty} |P_2(t_n,\lambda)| = \gamma > \delta$. From (5.1) we have

$$\frac{d}{dr} \log\left(|P_1(r,\lambda)|^2 + |P_2(r,\lambda)|^2\right) \leqslant 4|A_2(r)|. \tag{6.4}$$

Hence there exist $\varepsilon > 0$ and $C > 0$ such that $|P_1(r,\lambda)|^2 + |P_2(r,\lambda)|^2 < C$ for any $r \in [t_n, t_n+\varepsilon]$, $n \geqslant 1$, because of the uniform integrability of $A_2(r)$. We also have from (5.1) that $\frac{d}{dr}|P_2(r,\lambda)|^2 \leqslant 2|A_2(r)||P_1(r,\lambda)|^2$. Therefore we can conclude from (5.8) that there are $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that $|P_1(r,\lambda)| > \delta_1$ for any $r \in [t_n, t_n+\varepsilon_1]$. This is a contradiction with (5.5), and so $\lim_{r\to\infty} P_1(r,\lambda) = 0$ for any $\lambda \in H$.

Then the proof of (6.1) follows from (5.9). ☐

Proof of 3. Our first aim is to show that the integral (5.5) converges for any $\lambda \in H$. Let us assume that $\int_0^\infty |P_1(r,\lambda)|^2 dr = \infty$ for some $\lambda \in H$. Then one can see that

$$|P_2(r,\lambda)|^2 \leqslant \Big(\int_0^r |A_2(s)||P_1(s,\lambda)|ds \Big)^2 = o\Big(\int_0^r |P_1(s,\lambda)|^2 ds \Big)_{r\to\infty},$$

which contradicts to (5.8). Thus the integral (5.5) converges for any $\lambda \in H$ and so

$$\Pi(\lambda) = \lim_{r\to\infty} P_2(r,\lambda) = I_m + \int_0^\infty A_2(r) P_1(r,\lambda) dr$$

holds, since $A_2(r) \in L^2[0,\infty)$. The rest of the proof follows from (5.8), (5.9). ☐

References

[A] N. I. Akhiezer, *The classical moment problem and some related questions in analysis.* Kemmer Hafner Publishing Co., New York 1965.

[AR] N. I. Akhiezer and A. M. Rybalko, *Continual analogues of polynomials orthogonal on a circle* (Russian), Ukrain. Mat. Z. **20** (1968), 3–24.

[At] F. V. Atkinson, *Discrete and continuous boundary problems.* Academic Press, New York – London 1964.

[D1] S. A. Denisov, *On the application of some of M. G. Krein's results to the spectral analysis of Sturm-Liouville operators,* J. Math. Anal. Appl. **261** (2001), 177–191.

[D2] S. A. Denisov, *To the spectral theory of Krein system,* Integral Equations Operator Theory **42** (2002), 166–173.

[D3] S. A. Denisov, *On the continuous analog of Rakhmanov's theorem for orthogonal polynomials,* preprint.

[D4] S. A. Denisov, *On the existence of the absolutely continuous component for the measure associated with some orthogonal system,* preprint.

[D5] S. A. Denisov, *On the coexistence of absolutely continuous and singular continuous components of the spectral measure for some Sturm-Liouville operators with square summable potential,* preprint.

[DKu] S. A. Denisov and S. Kupin, *On singular spectrum of Schrodinger operators with decaying potential,* preprint.

[G] L. Ya. Geronimus, *Orthogonal polynomials: Estimates, asymptotic formulas, and series of polynomials orthogonal on the unit circle and on an interval.* Authorized translation from Russian. Consultants Bureau, New York 1961.

[K] M. G. Krein, *Continuous analogues of propositions on polynomials orthogonal on the unit circle* (Russian), Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 637–640.

[R] A. M. Rybalko, *On the theory of continual analogues of orthogonal polynomials* (Russian), Teor. Funkcii Funkcional. Anal. i Prilozen. Vyp. 3 (1966) 42–60.

[S1] L. A. Sakhnovich, *Spectral analysis of a class of canonical differential systems* (Russian), Algebra i Analiz **10** (1998), 187–201; translation in St. Petersburg Math. J. **10** (1999), 147–158.

[S2] L. A. Sakhnovich, *On a class of canonical systems on half-axis,* Integral Equations Operator Theory **31** (1998), no. 1, 92–112.

[S3] L. A. Sakhnovich, *Spectral theory of canonical differential systems. Method of operator identities.* Operator Theory: Advances and Applications, 107. Birkhäuser Verlag, Basel, 1999.

[S4] L. A. Sakhnovich, *On the spectral theory of a class of canonical differential systems* (Russian), Funktsional. Anal. i Prilozhen. **34** (2000), 50–62, 96; translation in Funct. Anal. Appl. **34** (2000), 119–128.

[S5] L. A. Sakhnovich, *On the spectral theory of the generalized differential system of M. G. Krein* (Russian), Ukrain. Mat. Zh. **52** (2000), 717–721; translation in Ukrainian Math. J. **52** (2000), 821–826 (2001).

[Se] G. Szegö, *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

[T1] A. Teplyaev, *Continuous analogues of random polynomials that are orthogonal on the circle* (Russian), Teor. Veroyatnost. i Primenen. **39** (1994), 588–604; translation in Theory Probab. Appl. **39** (1994), 476–489 (1995).

[T2] A. Teplyaev, *Random Orthogonal on the Circle Polynomials and Their Continuous Analogues* (Russian), Diploma work at Leningrad (St.-Petersburg) State University, 1990.

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