Optimal stability and instability results for a class of nearly integrable Hamiltonian systems

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Abstract: We consider a nearly integrable, non-isochronous, a-priori unstable Hamiltonian system with a (trigonometric polynomial) $O(\mu)$-perturbation which does not preserve the unperturbed tori. We prove the existence of Arnold diffusion with diffusion time $T_d = O((1/\mu)\log(1/\mu))$ by a variational method which does not require the existence of "transition chains of tori" provided by KAM theory. We also prove that our estimate of the diffusion time T_d is optimal as a consequence of a general stability result proved via classical perturbation theory. [1]

Keywords: Arnold diffusion, variational methods, shadowing theorem, perturbation theory, nonlinear functional analysis

AMS subject classification: 37J40, 37J45.

Riassunto: *Risultati ottimali di stabilità e di instabilità per una classe di sistemi Hamiltoniani quasi-integrabili.* Consideriamo sistemi Hamiltoniani quasi-integrabili, non-isocroni, a-priori instabili soggetti ad una pertubazione $O(\mu)$ (un polinomio trigonometrico) che non preserva i tori imperturbati. Mediante un metodo variazionale dimostriamo l'esistenza di orbite di diffusione con tempo di diffusione $T_d = O((1/\mu)\log(1/\mu))$. Il nostro approccio non richiede l'esistenza di "catene di tori KAM di transizione". Proviamo inoltre l'ottimalità della nostra stima sul tempo di diffusione, in conseguenza di un risultato generale di stabilità dimostrato mediante la teoria classica delle perturbazioni.

1 Introduction

We outline in this Note some recent results on Arnold's diffusion obtained in [6] where we refer for complete proofs. We consider nearly integrable *non-isochronous* Hamiltonian systems described by

$$\mathcal{H}_\mu = \frac{I^2}{2} + \frac{p^2}{2} + (\cos q - 1) + \mu f(I, p, \varphi, q, t), \tag{1.1}$$

where $(\varphi, q, t) \in \mathbf{T}^d \times \mathbf{T}^1 \times \mathbf{T}^1$ are the angle variables, $(I, p) \in \mathbf{R}^d \times \mathbf{R}^1$ are the action variables and $\mu \geq 0$ is a small real parameter. The Hamiltonian system associated with $\mathcal{H}_\mu$ writes

$$\dot\varphi = I + \mu \partial_I f, \quad \dot I = -\mu \partial_\varphi f, \quad \dot q = p + \mu \partial_p f, \quad \dot p = \sin q - \mu \partial_q f. \tag{$\mathcal{S}_\mu$}$$

The perturbation f is assumed, as in [11], to be a trigonometric polynomial of order N in φ and t, namely

$$f(I, p, \varphi, q, t) = \sum_{|(n,l)| \leq N} f_{n,l}(I, p, q) exp^{\mathrm{i}(n\varphi + lt)}. \tag{1.2}$$

$(\mathcal{S}_\mu)$ describes a system of d "rotators" weakly coupled with a pendulum through a small periodically time dependent perturbation term. The unperturbed Hamiltonian system $(\mathcal{S}_0)$ is completely integrable and, in particular, the energy $I_i^2/2$ of each rotator is a constant of the motion. The problem of *Arnold diffusion* in this context is whether, for $\mu \neq 0$, there exist motions whose net effect is to transfer $O(1)$-energy among the rotators. A natural complementary question regards the time of stability (or instability) for

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the perturbed system: what is the minimal time to produce an $O(1)$-exchange of energy, if any takes place, among the rotators?

The mechanism proposed in [2] to prove the existence of Arnold diffusion and thereafter become classical, is the following one. The unperturbed Hamiltonian system $(\mathcal{S}_0)$ admits a continuous family of d-dimensional partially hyperbolic invariant tori $\mathcal{T}_\omega = \{\varphi \in \mathbf{T}^d, I = \omega,\ q = p = 0\}$ possessing stable and unstable manifolds $W_0^s(\mathcal{T}_\omega) = W_0^u(\mathcal{T}_\omega) = \{\varphi \in \mathbf{T}^d, I = \omega,\ (p^2/2) + (\cos q - 1) = 0\}$. The method used in [2] to produce unstable orbits relies on the construction, for $\mu \neq 0$, of *"transition chains"* of perturbed partially hyperbolic tori $\mathcal{T}_\omega^\mu$ close to $\mathcal{T}_\omega$ connected one to another by heteroclinic orbits. Therefore in general the first step is to prove the persistence of such hyperbolic tori $\mathcal{T}_\omega^\mu$ for $\mu \neq 0$ small enough, and to show that its perturbed stable and unstable manifolds $W_\mu^s(\mathcal{T}_\omega^\mu)$ and $W_\mu^u(\mathcal{T}_\omega^\mu)$ split and intersect transversally ("splitting problem"). The second step is to find a transition chain of perturbed tori: this is a difficult task since, for general non-isochronous systems, the surviving perturbed tori $\mathcal{T}_\omega^\mu$ are separated by the gaps appearing in KAM constructions. Two perturbed invariant tori $\mathcal{T}_\omega^\mu$ and $\mathcal{T}_{\omega'}^\mu$ could be too distant one from the other, forbidding the existence of an heteroclinic intersection between $W_\mu^u(\mathcal{T}_\omega^\mu)$ and $W_\mu^s(\mathcal{T}_{\omega'}^\mu)$: this is the famous *"gap problem"*. In [2] this difficulty is bypassed by the peculiar choice of the perturbation $f(I, \varphi, p, q, t) = (\cos q - 1) f(\varphi, t)$, whose gradient vanishes on the unperturbed tori $\mathcal{T}_\omega$, leaving them *all* invariant also for $\mu \neq 0$. The final step is to prove, by a "shadowing argument", the existence of a true diffusion orbit, close to a given transition chain of tori, for which the action variables I undergo a drift of $O(1)$ in a certain time T_d called the *diffusion time*.

The first paper proving Arnold diffusion in presence of perturbations not preserving the unperturbed tori was [11]. Extending Arnold's analysis, it is proved in [11] that, if the perturbation is a trigonometric polynomial in the angles φ, then, in some regions of phase space, the "density" of perturbed invariant tori is high enough for the construction of a transition chain.

Regarding the shadowing problem, geometrical method, see e.g. [11], [14], [12], [13], and variational ones, see e.g. [8], have been applied, in the last years, in order to prove the existence of diffusion orbits shadowing a given transition chain of tori and to estimate the diffusion time. We also quote the important paper [7] which, even if dealing only with the Arnold's model perturbation, has obtained, using variational methods, very good time diffusion estimates and has introduced new ideas for studying the shadowing problem. For isochronous systems new variational results concerning the shadowing and the splitting problem have been obtained in [3], [4] and [5].

In this Note we describe an *alternative mechanism*, proposed in [6], to produce diffusion orbits. This method is not based on the existence of transition chains of tori, namely it avoids the KAM construction of the perturbed hyperbolic tori, proving directly the existence of a drifting orbit as a local minimum of an action functional, see Theorem 2.1. At the same time this variational approach achieves the optimal diffusion time $T_d = O((1/\mu)\ln(1/\mu))$, see (2.2). We also prove that our time diffusion estimate is the optimal one as a consequence of a general stability result, Theorem 2.2, proved via classical perturbation theory. As in [11] our diffusion orbit will not connect any two arbitrary frequencies of the action space, even if we manage to connect more frequencies than in [11], proving the drift also in some regions of phase space where transition chains might not exist. Clearly if the perturbation is chosen as in Arnold's example we can drift in all phase the space with no restriction, see Theorem (2.3).

Actually our variational shadowing technique is not restricted to the a-priori unstable case, but would allow, in the same spirit of [3], [4] and [5], once a "splitting property" is somehow proved, to get diffusion orbits with the best diffusion time (in terms of some measure of the splitting).

In conclusion the results and the method described in this Note constitute a further step in a research line, started in [3]-[4] and [5], whose aim is to find new mechanisms for proving Arnold diffusion. We expect that these variational methods could be suitably refined in order to prove the existence of drifting orbits in the whole phase space, and also for generic analytic perturbations. Another possible application of these methods could regard infinite dimensional Hamiltonian systems where the existence of "transition chains of infinite dimensional hyperbolic tori" is far for being proved.

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2 Main results

For simplicity, even if not really necessary, when proving the existence of diffusion orbits, we assume f to be a purely spatial perturbation, namely $f(\varphi, q, t) = \sum_{|(n,l)|\le N} f_{n,l}(q) exp(\mathrm{i}(n\varphi + lt))$. The functions $f_{n,l}$ are assumed to be smooth.

Let us define the "resonant web" $\mathcal{D}_N$, formed by the frequencies ω "resonant with the perturbation",

$$\mathcal{D}_N := \left\{ \omega \in \mathbf{R}^d \;\Big|\; \exists (n,l) \in \mathbf{Z}^{d+1} \text{ s.t. } 0 < |(n,l)| \le N \text{ and } \omega n + l = 0 \right\} = \cup_{0<|(n,l)|\le N} E_{n,l} \tag{2.1}$$

where $E_{n,l} := \{\omega \in \mathbf{R}^d \mid \omega n + l = 0\}$. Let us also consider the Poincaré-Melnikov primitive

$$\Gamma(\omega, \varphi_0, \theta_0) := -\int_{\mathbf{R}} \Big[f(\omega t + \varphi_0, q_0(t), t + \theta_0) - f(\omega t + \varphi_0, 0, t + \theta_0) \Big] \, dt,$$

where $q_0(t) = 4 \arctan(\exp t)$ is the unperturbed separatrix of the pendulum satisfying $q_0(0) = \pi$.

The next theorem states that, for any connected component $\mathcal{C} \subset \mathcal{D}_N^c$, $\omega_I, \omega_F \in \mathcal{C}$, there exists a solution of $(\mathcal{S}_\mu)$ connecting a $O(\mu)$-neighborhood of $\mathcal{T}_{\omega_I}$ to a $O(\mu)$-neighborhood of $\mathcal{T}_{\omega_F}$, in a time-interval of length $T_d = O((1/\mu)|\log \mu|)$.

Theorem 2.1 *Let $\mathcal{C}$ be a connected component of $\mathcal{D}_N^c$, $\omega_I, \omega_F \in \mathcal{C}$ and let $\gamma : [0, L] \to \mathcal{C}$ be a smooth embedding such that $\gamma(0) = \omega_I$ and $\gamma(L) = \omega_F$. Assume that $\forall \omega := \gamma(s)$ $(s \in [0, L])$, $\Gamma(\omega, \cdot, \cdot)$ possesses a non-degenerate minimum $(\varphi_0^\omega, \theta_0^\omega)$. Then $\forall \eta > 0$ there exists $\mu_0 := \mu_0(\gamma, \eta) > 0$ and $C := C(\gamma) > 0$ such that $\forall 0 < \mu \le \mu_0$ there exists a solution $(\varphi_\mu(t), q_\mu(t), I_\mu(t), p_\mu(t))$ of $(\mathcal{S}_\mu)$ and two instants $\tau_1 < \tau_2$ such that $I_\mu(\tau_1) = \omega_I + O(\mu)$, $I_\mu(\tau_2) = \omega_F + O(\mu)$,*

$$|\tau_2 - \tau_1| \le \frac{C}{\mu} |\log \mu| \tag{2.2}$$

and $\mathrm{dist}(I_\mu(t), \gamma) < \eta$ *for all* $\tau_1 \le t \le \tau_2$.

We can also build diffusion orbits approaching the boundaries of $\mathcal{D}_N$ at distances as small as a certain power of μ: see for a precise statement Theorem 6.1 of [6]. Theorem 2.1 improves the corresponding result in [11] which enables to connect any two frequencies ω_I and ω_F belonging to the same connected component $\mathcal{C} \subset \mathcal{D}_{N_1}^c$ for $N_1 = 14dN$ and with $\mathrm{dist}(\{\omega_I, \omega_F\}, \mathcal{D}_{N_1}) = O(1)$. As already said such restriction of [11] arises because transition chains might not exist in the whole $\mathcal{C} \subset \mathcal{D}_N^c$.

Theorem 2.1 improves also the known time-estimates on the diffusion time. The first estimate on the diffusion time obtained by geometrical method in [11] is $T_d = O(\exp(1/\mu^2))$. In [14]-[12]-[13], still by geometrical methods, and in [8], by means of Mather's theory, the diffusion time has been proved to be just polynomially long in the splitting μ (the splitting angles between the perturbed stable and unstable manifolds $W_\mu^{s,u}(\mathcal{T}_\omega^\mu)$ at a homoclinic point are, by classical Poincaré-Melnikov theory, $O(\mu)$). We note that the variational method proposed by Bessi in [7] had already given, even if in the case of perturbations preserving all the unperturbed tori, the time diffusion estimate $T_d = O(1/\mu^2)$. For isochronous systems the estimate on the diffusion time $T_d = O((1/\mu)|\ln \mu|)$ has already been obtained in [3]-[4]. Very recently, in [13], the diffusion time has been estimated as $T_d = O((1/\mu)|\log \mu|)$ by a method which uses "hyperbolic periodic orbits"; however the result of [13] is of local nature: the previous estimate holds only for diffusion orbits shadowing a transition chain close to some torus run with diophantine flow.

Our next statement (a stability result) concludes this quest of the minimal diffusion time T_d : it proves the optimality of our estimate $T_d = O((1/\mu)|\log \mu|)$.

Theorem 2.2 *Let $f(I, \varphi, p, q, t)$ be as in (1.2), where the $f_{n,l}$ $(|(n,l)| \le N)$ are analytic functions. Then $\forall \kappa, \overline{r}, \widetilde{r} > 0$ there exist $\kappa_0, \mu_1 > 0$ such that $\forall\, 0 < \mu \le \mu_1$, any solution $(I(t), \varphi(t), q(t), p(t))$ of $(\mathcal{S}_\mu)$ with $|I(0)| \le \overline{r}$ and $|p(0)| \le \widetilde{r}$ satisfies*

$$|I(t) - I(0)| \le \kappa, \qquad \forall\ |t| \le \frac{\kappa_0}{\mu} \ln \frac{1}{\mu}. \tag{2.3}$$

Actually the proof of Theorem 2.2 contains much more information: in particular the stability time (2.3) is sharp only for orbits lying close to the separatrices. On the other hand the orbits lying far away from the separatrices are much more stable, namely exponentially stable in time according to Nekhoroshev type time estimates, see (2.4). Indeed the diffusion orbit of Theorem 2.1 is found (as a local minimum of an action functional) close to some pseudo-diffusion orbit whose (q, p) variables stay close to the separatrices of the pendulum (turning $O(1/\mu)$ times around them).

As a byproduct of the techniques developed in this paper we have the following result concerning "Arnold's example" [2]

Theorem 2.3 *Let $f(\varphi, q, t) := (1 - \cos q)\widetilde{f}(\varphi, t)$. Assume that for some smooth embedding $\gamma : [0, L] \to \mathbf{R}^d$, with $\gamma(0) = \omega_I$ and $\gamma(L) = \omega_F$, $\forall \omega := \gamma(s)$ $(s \in [0, L])$, $\Gamma(\omega, \cdot, \cdot)$ possesses a non-degenerate minimum $(\varphi_0^\omega, \theta_0^\omega)$. Then $\forall \eta > 0$ there exists $\mu_0 := \mu_0(\gamma, \eta) > 0$, and $C := C(\gamma) > 0$ such that $\forall 0 < \mu \le \mu_0$ there exists a heteroclinic orbit (η-close to γ) connecting the invariant tori $\mathcal{T}_{\omega_I}$ and $\mathcal{T}_{\omega_F}$. Moreover the diffusion time T_d needed to go from a μ-neighbourhood of $\mathcal{T}_{\omega_I}$ to a μ-neighbourhood of $\mathcal{T}_{\omega_F}$ is bounded by $T_d \le (C/\mu)|\log \mu|$.*

The method of proof of Theorem 2.1 (and Theorem 2.3) relies on a finite dimensional reduction of Lyapunov-Schmidt type, variational in nature, introduced in [1] and later extended in [3],[4] and [5] to the problem of Arnold diffusion. The diffusion orbit of Theorem 2.1 (and Theorem 2.3) is found as a local minimum of the action functional close to some pseudo-diffusion orbit whose (q, p) variables move along the separatrices of the pendulum. The pseudo-diffusion orbits, constructed by the Implicit Function Theorem, are true solutions of $(\mathcal{S}_\mu)$ except possibly at some instants θ_i, for $i = 1, \ldots, k$, when they are glued continuously at the section $\{q = \pi, \mod 2\pi\mathbf{Z}\}$ but the speeds $(\dot{q}_\mu(\theta_i), \dot{\varphi}_\mu(\theta_i)) = (p_\mu(\theta_i), I_\mu(\theta_i))$ may possibly have a jump, see lemma 3.1 of [6]. The time interval $T_s = \theta_{i+1} - \theta_i$ is heuristically the time required to perform a single transition during which the rotators can exchange $O(\mu)$-energy, i.e. the action variables vary of $O(\mu)$. During each transition we can exchange only $O(\mu)$-energy since the splitting is $O(\mu)$-large. Hence in order to exchange $O(1)$ energy the number of transitions required will be $k = O(1/\text{splitting}) = O(1/\mu)$.

We underline that the question of finding the optimal time and the mechanism for which we can avoid the construction of true transition chains of tori are deeply connected. Indeed the main reason for which our drifting technique avoids the construction of KAM tori is the following one: if the time to perform a simple transition T_s is, say, just $T_s = O(|\ln \mu|)$ then, on such short time intervals, it is easy to approximate the action functional with the unperturbed solutions living on the stable and unstable manifolds of the unperturbed tori $W^s(\mathcal{T}_\omega) = W^u(\mathcal{T}_\omega) = \{(\varphi, \omega, q, p) \mid p^2/2 + (\cos q - 1) = 0\}$, see lemma 5.6 of [6]. In this way we do not need to construct the true hyperbolic tori $\mathcal{T}_\omega^\mu$ close to $\mathcal{T}_\omega$ (actually for our approximation we only need the time for a single transition to be $T_s << 1/\mu$).

The fact that it is possible to perform a single transition in a very short time interval like $|\ln \mu|$ is not obvious at all. In [7] the time to perform a single transition, in the example of Arnold, is $O(1/\mu)$. This time separation arises in order to ensure that the variations of the action functional of the rotators are small compared with the (positive definite) second derivative of the Poincaré-Melnikov primitive at its minimum point. Unfortunately this time is too long to make directly our approximations of the action functional. The key observation that enables us to perform a single transition in a very short time interval concerns the behaviour the "gradient flow" of the unperturbed action functional of the rotators. In section 6 of [6] it is shown that the variations of the action of the rotators are small, even on time intervals $T_s << 1/\mu$, and do not "destroy" the minimum of the Poincaré-Melnikov primitive.

When trying to build a pseudo-diffusion orbit which performs single transitions in very short time intervals we encounter another difficulty linked with the ergodization time. The time to perform a single transition T_s must be sufficiently long to settle, at each instant θ_i, the projection of the pseudo-orbit on the torus sufficiently close to the minimum of the Poincaré-Melnikov function, i.e. the homoclinic point (in our method it is sufficient to arrive just $O(1)$-close, independently of μ, to the homoclinic point). This necessary request creates some difficulty since our pseudo-diffusion orbit may arrive $O(\mu)$-close in the action space to resonant hyperplanes of frequencies whose linear flow does not provide a dense enough net of the torus. The way in which this problem is overcome is discussed in [6]: we observe a phenomenon of "stabilization close to resonances" which forces the time T_s for some single transitions to increase . Anyway the total time required to cross these (finite number of) resonances is still $T_d = O((1/\mu)\log(1/\mu))$. This discussion enables us to prove optimal fast-Arnold diffusion in large regions of phase space and allows to improve the local diffusion results of [13].

As explained before we need, in order to prove Theorems 2.1 and 2.3, some estimates on the "ergodization time of the torus" for linear flows possibly resonant but only at a "sufficiently high order". The following result (Lemma 2.1) gives an answer which may be of independent interest. Let Γ be a lattice of $\mathbf{R}^l$, *i.e.* a discrete subgroup of $\mathbf{R}^l$ such that $\mathbf{R}^l/\Gamma$ has finite volume. $\forall \Omega \in \mathbf{R}^l$ the "ergodization time required by the flow $\{\Omega t\}$ to fill the torus $\mathbf{R}^l/\Gamma$ within δ" is defined as $T(\Gamma,\Omega,\delta) := \inf\{t \in \mathbf{R}_+ \mid \forall x \in \mathbf{R}^l \ d(x,[0,t]\Omega + \Gamma) \leq \delta\}$ (with $\inf E = +\infty$ if $E = \emptyset$). For $R > 0$ define also $\Gamma_R^* = \{p \in \mathbf{R}^l \mid 0 < |p| \leq R \ , \ \forall \gamma \in \Gamma \ p \cdot \gamma \in \mathbf{Z}\}$ and $\alpha(\Gamma,\Omega,R) = \inf\{|p \cdot \Omega| \mid p \in \Gamma_R^*\}$. The following result holds:

Lemma 2.1 *$\forall l \in \mathbf{N}$, $\exists a_l > 0$ such that, for all lattice $\Gamma \subset \mathbf{R}^l$, $\forall \Omega \in \mathbf{R}^l$, $\forall \delta > 0$, $T(\Gamma,\Omega,\delta) \leq (\alpha(\Gamma,\Omega,a_l/\delta))^{-1}$. Moreover $T(\Gamma,\Omega,\delta) \geq (1/4)\alpha(\Gamma,\Omega,1/4\delta)^{-1}$.*

A straighforward consequence of Lemma 2.1 is, for example, Theorem D of [10]: if Ω is a $C-\tau$ diophantine vector, i.e. there exist $C > 0$ and $\tau \geq l-1$ such that $|k \cdot \Omega| \geq C/|k|^\tau$, $\forall k \in \mathbf{Z}^l \setminus \{0\}$, then $T(\Omega,\delta) \leq a_l^\tau/(C\delta^\tau)$ (where $T(\Omega,\delta) := T(\mathbf{Z}^l,\Omega,\delta)$). Also Theorem B of [10] is an easy consequence of Lemma 2.1.

We conclude this Note discussing briefly the proof of Theorem 2.2, see section 7 of [6]. First we prove stability in the region "far from the separatrices of the pendulum" $\mathcal{E}_1 := \{(I,\varphi,q,p) \mid |E(q,p)| \geq \mu^c\}$, where $E(q,p) := p^2/2 + (\cos q - 1)$ and c is a suitable positive constant. In $\mathcal{E}_1$ we can write Hamiltonian $\mathcal{H}_\mu$ in action-angle variables (I,φ,Q,P,t) where $Q := Q(q,p)$ and $P := P(q,p)$ are the action-angle variables of the standard pendulum $E(q,p)$, i.e $E(q(Q,P),p(Q,P)) := K(P)$. In these variables the new Hamiltonian writes $\mathcal{H}_1 := I^2/2 + K(P) + \mu f_1(\varphi,Q,t,P,I)$, and, by a result of [9], it is analytic with an analyticity radius $r_1 \approx \mu^c$ (when μ goes to zero, the region $\mathcal{E}_1$ approximate closer and closer to the separatrices and the analyticity estimate deteriorates). It turns out that $\mathcal{H}_1$ is *steep* (actually for E positive it is even *quasi-convex*, see [16]) and then, for $c > 0$ small enough, we can apply the Nekhoroshev Theorem as proved in [15]. In this way we obtain exponential stability in the whole region $\mathcal{E}_1$, i.e.

$$|I(t) - I(0)| \leq const.\mu^b \qquad \forall\ |t| \leq T := const\frac{1}{\mu}\exp\left(\frac{1}{\mu}\right)^a \tag{2.4}$$

for two constants $a, b > 0$. Finally we study the behaviour of an orbit close to the separatrices of the pendulum, namely in the region $\mathcal{E}_1^c$. Roughly speaking, such an orbit will spend alternatively a time $T_S = O(|\ln\mu|)$ into a small O(1)-neighborhood of $(p,q) = (0,0)$ and a time $T_F = O(1)$ outside. In this second case we directly obtain $\Delta I := \int_s^{s+T_F} \dot{I}_\mu = O(\mu T_F) = O(\mu)$. In [6] it is proved that $\Delta I = O(\mu)$ also in the first case. This result is obtained performing one step of classical perturbation theory (as in [11]) after writing the pendulum in hyperbolic variables in a small neighborhood of the origin.

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