Spectral Properties of Finite Quantum Hall Systems*

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Abstract

In this note we review spectral properties of magnetic random Schrödinger operators $H_\omega = H_0 + V_\omega + U_\ell + U_r$ defined on $L^2(\mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right], \mathrm{d}x\,\mathrm{d}y)$ with periodic boundary conditions along y. U_ℓ and U_r are two confining potentials for $x \leq -\frac{L}{2}$ and $x \geq \frac{L}{2}$ respectively and vanish for $-\frac{L}{2} \leq x \leq \frac{L}{2}$. We describe the spectrum in two energy intervals and we classify it according to the quantum mechanical current of eigenstates along the periodic direction. The first interval lies in the first Landau band of the bulk Hamiltonian, and contains intermixed eigenvalues with a quantum mechanical current of $\mathcal{O}(1)$ and $\mathcal{O}\left(e^{-\gamma B (\log L)^2}\right)$ respectively. The second interval lies in the first spectral gap of the bulk Hamiltonian, and contains only eigenvalues with a quantum mechanical current of $\mathcal{O}(1)$.

1 Introduction

In this note we review recent results on the spectrum of a magnetic random Schrödinger operator H_ω which describes the dynamics of an electron lying on a cylinder of circumference L and which is confined along the cylinder axis by two smooth increasing potentials whose supports are separated by a distance L. We suppose our particle spinless, thus the Zeeman term in the Hamiltonian is neglected. The complete proofs of the theorems stated here can be found in [FM1] and [FM2].

First let us shortly recall previous results on random Schrödinger operators with magnetic field in the infinite two dimensional plane $\mathbb{R}^2$. We denote by H_0 the kinetic term $H_0 = (p - A)^2$, where A is the vector potential associated to a constant magnetic field B. The spectrum of H_0 is given by the Landau levels $\{(2n+1)B : n \in \mathbb{N}\}$. The bulk Hamiltonian is

$$H^b_\omega = H_0 + V_\omega \tag{1.1}$$

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where V_ω is a Anderson-like random potential. The spectrum of (1.1) is contained in Landau bands around each Landau level, $\sigma(H^b_\omega) \subset \bigcup_{n\geq 0}[(2n+1)B - V_0, (2n+1)B + V_0]$ where $V_0 = \|V_\omega\|$, and if $V_0 < B$ there are open spectral gaps $G_n \supseteq ((2n+1)B + V_0, (2n+3)B - V_0)$ $(n \in \mathbb{N})$. It is proven that near the band edges the spectrum of H^b_ω is almost surely pure point with exponentially localized eigenfunctions [DMP1], [DMP2], [CH], [BCH], [W]. There are no rigorous results for energies at the band centers, except for a special model where the impurities are point scatterers [DMP3], [DMP4].

We now add a wall potential, translation invariant along the $y-$direction, such that $U_\ell(x)$ is confining for $x \leq -\frac{L}{2}$ and $U_\ell(x) = 0$ for $x \geq -\frac{L}{2}$. We have a semi-infinite system with a Hamiltonian

$$H^{si}_\omega = H_0 + V_\omega + U_\ell \, . \tag{1.2}$$

The spectrum contains the interval $[B, +\infty)$. For this system one can show that, for energies in intervals inside the gaps of the bulk Hamiltonian, the average velocity $(\psi, v_y\psi)$ in the y direction, of an assumed eigenstate ψ does not vanish. Since the velocity v_y is the commutator between $-iy$ and the Hamiltonian, the Virial Theorem implies that an eigenstate cannot exist, and that therefore the spectrum is purely continuous inside the gaps of the bulk Hamiltonian [MMP], [F]. By Mourre theory one can show that the spectrum therein is purely absolutely continuous [FGW], [dBP].

Finally we can add a second wall potential U_r such that $U_r(x) = 0$ for $x \leq \frac{L}{2}$ and which is confining for $x \geq \frac{L}{2}$. So the particle is confined between $x = -\frac{L}{2}$ and $x = \frac{L}{2}$. The Hamiltonian has the form

$$H_\omega = H_0 + V_\omega + U_\ell + U_r \, . \tag{1.3}$$

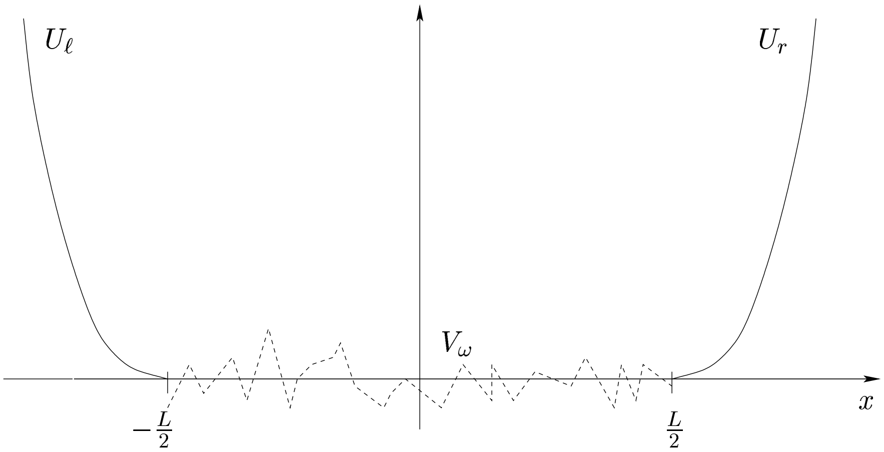

Figure 1: *The potentials along the x-axes.*

Additionally we make the $y-$direction periodic of length L (L a large parameter), this correspond to a finite, macroscopic cylindrical geometry.

Other models with confinement in the $x-$direction but without p.b.c. along y have been studied. The first consists of a parabolic channel where $U_\ell + U_r$ is replaced by a parabolic confinement γx^2, in this case it is shown that if the perturbation (V_ω in our case) is small enough in a suitable sense and satisfies a weak decay condition in the $y-$direction, there exists intervals of absolutely continuous spectrum [EJK]. The second model, more close to ours, consists to take two step walls of finite height for U_ℓ and U_r. In this case an initial state localized in energy in the spectral gap of the bulk Hamiltonian and near the left (resp. right) wall has a positive (resp. negative) velocity up to a finite time, limited by tunneling effect between the two walls [C].

The physical interest of our model is related to the integral quantum Hall effect [PG]. For the explanation of this effect Halperin [H] pointed out the importance of the boundary diamagnetic currents. Since many features of the integral quantum Hall effect can be described in the framework of one particle random magnetic Schrödinger operators it is important to understand their spectral properties for finite but macroscopic samples with boundaries.

2 The model

The family of random Schrödinger operators that we want to study is

$$H_\omega = p_x^2 + (p_y - Bx)^2 + V_\omega + U_r + U_\ell \qquad \omega \in \Omega \tag{2.1}$$

these are densely defined self-adjoint operators acting in the Hilbert space $L^2(\mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right], \mathrm{d}x\,\mathrm{d}y)$ with periodic boundary conditions along y: $\psi\left(x, -\frac{L}{2}\right) = \psi\left(x, \frac{L}{2}\right)$, $x \in \mathbb{R}$. $H_0 = p_x^2 + (p_y - Bx)^2$ is the kinetic Hamiltonian written in Landau gauge, its spectrum consists in infinitely degenerate eigenvalues

$$\sigma(H_0) = \sigma_{ess}(H_0) = \{(2n+1)B; n \in \mathbb{N}\} \ . \tag{2.2}$$

The two confining walls are assumed twice differentiable, strictly monotonic and satisfy

$$c_1|x + \tfrac{L}{2}|^{m_1} \le U_\ell(x) \le c_2|x + \tfrac{L}{2}|^{m_2} \qquad \text{for } x \le -\tfrac{L}{2} \tag{2.3}$$

$$c_1|x - \tfrac{L}{2}|^{m_1} \le U_r(x) \le c_2|x - \tfrac{L}{2}|^{m_2} \qquad \text{for } x \ge \tfrac{L}{2} \tag{2.4}$$

for some constants $0 < c_1 < c_2 < \infty$ and $2 \le m_1 < m_2 < \infty$. Moreover $U_\ell(x) = 0$ for $x \ge -\frac{L}{2}$ and $U_r(x) = 0$ for $x \le \frac{L}{2}$. We could allow steeper confinements but the present polynomial conditions turn out to be technically convenient. The random potential V_ω consists of a sum of local perturbations located at the sites of a finite lattice $\Lambda = \mathbb{Z}^2 \cap \left[X \times \left[-\frac{L}{2}, \frac{L}{2}\right]\right]$ where X will be defined latter. Thus

$$V_\omega(x,y) = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x-n, y-m) \quad \omega \in \Omega \tag{2.5}$$

where the coupling constants $X_{n,m}$ are i.i.d. random variables with common bounded probability density $h \in C^2([-1,1])$. The local potential V satisfies $V \in C^2$,

$0 \le V(x,y) \le V_0 < \infty$, $\operatorname{supp} V \subset \mathbb{B}\left(\mathbf{0}, \frac{1}{4}\right)$ (the open ball centred at $(0,0)$ of radius $\frac{1}{4}$). $\Omega = [-1,1]^{\Lambda}$ is the set of all possibles realizations, we will denote by $\mathbb{P}_{\Lambda}$ the product measure defined on Ω. Clearly for all $\omega \in \Omega$ we have $\|V_{\omega}\| \le V_0$. We will assume that $V_0 \ll B$, that is, we work in a strong magnetic field regime or, equivalently, in a weak disorder regime.

Our first result concerns the study of $\sigma(H_{\omega})$ in the energy interval $\Delta_{\varepsilon} = [B+\varepsilon, B+V_0]$ that lies inside the first Landau band of the infinite bulk system. In this case the interval X, that defines the support of the random potential along the $x-$direction, is $\left[-\frac{L}{2} + \log L, \frac{L}{2} - \log L\right]$: we leave a thin strip of size $\log L$ without random potential along each confining wall.

The second result is about $\sigma(H_{\omega})$ inside the first spectral gap of the infinite bulk system, more precisely in the energy interval $\Delta = (2B-\delta, 2B+\delta) \subset (B+V_0+\varepsilon, 3B-V_0-\varepsilon)$. In this case the random potential fills the whole space in between the confining walls, that means $X = \left[-\frac{L}{2}, \frac{L}{2}\right]$.

Since our system is confined the spectrum is made of discrete eigenvalues. There exists a natural classification of the eigenvalues via the quantum mechanical current along the periodic direction. If ψ satisfies the eigenvalue equation $H_{\omega}\psi = E\psi$ the current is defined (here) as

$$J_E \equiv (\psi, v_y \psi) \tag{2.6}$$

where $v_y = 2(p_y - Bx)$ is the velocity operator in the $y-$direction. Thanks to J_E we can classify the eigenvalues in two classes: the first consists on those which have $|J_E| > C$ with C a positive constant uniform in L, the second consists on those for which $|J_E| < \epsilon(L)$ with $\epsilon(L) \to 0$ as $L \to \infty$ (we stress that here L is finite but macroscopic, the limit means that $\epsilon(L)$ is infinitesimally small with L). The physical meaning of this classification is briefly discussed at the end of section 3.

The main idea of our approach is to first look at some easier Hamiltonians and then link them together to get properties on the full Hamiltonian H_{ω}. In what follows we do not analyze these easier Hamiltonians but we just introduce the minimal notations and properties (see [FM1] and [FM2] for the details).

3 Main results

For the analysis of $\sigma(H_{\omega})$ in Δ_{ε} we need to know some properties of the Hamiltonian

$$H_{\alpha}^{0} = H_0 + U_{\alpha} \qquad \alpha = \ell, r \tag{3.1}$$

called *pure edge Hamiltonian.* Its spectrum is given by

$$\sigma(H_{\alpha}^{0}) = \left\{E_{nk}^{\alpha}; n \in \mathbb{N}, k \in \tfrac{2\pi}{L}\mathbb{Z}\right\} \tag{3.2}$$

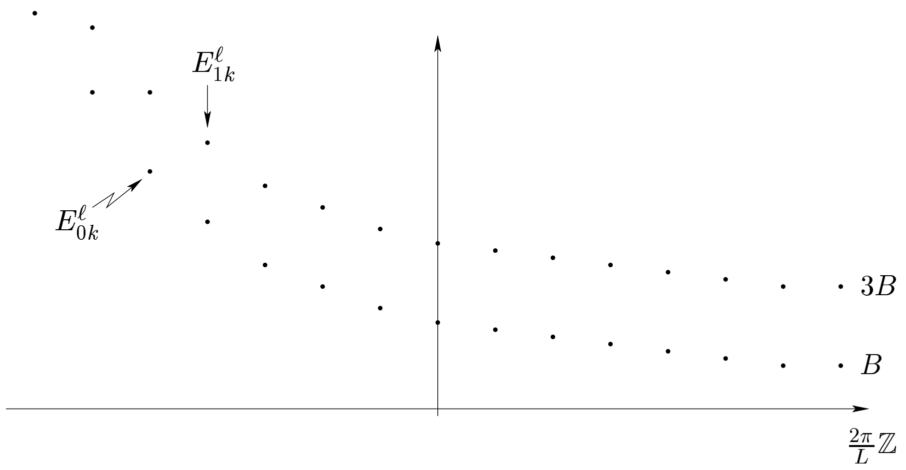

Figure 2: *The spectrum of H_ℓ^0 lies on monotonic decreasing branches. That of H_r^0 lies on similar, but monotonic increasing, branches. The spectral branches are given by the dispersion relation for $L=\infty$.*

The quantum mechanical currents associated to the eigenfunctions ψ^α_{nk} whose eigenvalue is in Δ_ε (we have $n=0$) satisfy

$$|J^\alpha_{0k}| = |(\psi^\alpha_{0k}, v_y \psi^\alpha_{0k})| > C \tag{3.3}$$

with $C>0$ a numerical constant independent of L [FM1]. Moreover we will assume the following

Hypothesis 1. *Fix $\varepsilon>0$. There exist $L(\varepsilon)$ and $d(\varepsilon)>0$ such that for all $L>L(\varepsilon)$*

$$\operatorname{dist}\left(\sigma(H^0_\ell)\cap\Delta_\varepsilon, \sigma(H^0_r)\cap\Delta_\varepsilon\right) \geq \frac{d(\varepsilon)}{L} \ . \tag{3.4}$$

This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed in a system with two boundaries backscattering favors localization and has a tendency to destroy currents. Remark that this hypothesis can be verified by taking two symmetric confining potentials $U_\ell(x)=U_r(-x)$ and adding a suitable flux line along the cylinder axes (see in particular Appendix C in [FM2]).

We also need to know some properties of the *bulk Hamiltonian*

$$H_b = H_0 + V_\omega \ . \tag{3.5}$$

Its essential spectrum is given by the Landau levels and the whole spectrum is contained in the Landau bands $\sigma(H_b)\subset\bigcup_{n\geq 0}[(2n+1)B-V_0,(2n+1)B+V_0]$. We will suppose that the (discrete) spectrum in Δ_ε fullfills the

Hypothesis 2. *Fix any $\varepsilon>0$. There exist $\mu(\varepsilon)$ a strictly positive constant and $L(\varepsilon)$ such that for all $L>L(\varepsilon)$ one can find a set of realizations of the random potential*

Ω' *with* $\mathbb{P}_\Lambda(\Omega') \geq 1 - L^{-\theta}$, $\theta > 0$, *with the property that if* $\omega \in \Omega'$ *the eigenstates corresponding to* $E^b_\beta \in \sigma(H_b) \cap \Delta_\varepsilon$ *satisfy*

$$|\psi^b_\beta(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \qquad , \qquad |\partial_y \psi^b_\beta(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \tag{3.6}$$

for some $\bar{y}_\beta$ *depending on* ω *and* L.

Since V_ω is random we expect that eigenfunctions with energies in Δ_ε (not too close to the Landau levels where the localization length diverges) are exponentially localized on a scale $\mathcal{O}(1)$ with respect to L. Inequalities (3.6) are a weaker version of this statement, and have been checked for the special case where the random potential is a sum of rank one perturbations [FM3] using the methods of Aizenman and Molchanov [AM]. The main consequence of Hypothesis 2 is that a state satisfying (3.6) does not carry any appreciable current (contrary to the eigenstates of H^0_α) in the sense that

$$J^b_\beta = (\psi^b_\beta, v_y \psi^b_\beta) = \mathcal{O}\left(e^{-\mu(\varepsilon)L}\right) . \tag{3.7}$$

We are now ready to state our first result on the eigenvalues lying in $\Delta_\varepsilon = [B + \varepsilon, B + V_0]$.

Theorem 1. *Fix* $\varepsilon > 0$ *and assume that* $(H1)$ *and* $(H2)$ *are fulfilled. Assume* $B > 4V_0$. *Then there exists a numerical constant* $\gamma > 0$ *and an* $\bar{L} \geq L(\varepsilon)$ *such that for all* $L > \bar{L}$ *one can find a set* $\hat{\Omega} \subset \Omega$ *of realizations of the random potential* V_ω *with* $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-s}$ $(s \gg 1)$ *such that for any* $\omega \in \hat{\Omega}$, $\sigma(H_\omega) \cap \Delta_\varepsilon$ *is the union of three sets* $\Sigma_\ell \cup \Sigma_b \cup \Sigma_r$, *each depending on* ω *and* L, *and characterized by the following properties:*

a) $\mathcal{E}^\alpha_k \in \Sigma_\alpha$ $(\alpha = \ell, r)$ *are a small perturbation of* $E^\alpha_{0k} \in \sigma(H^0_\alpha) \cap \Delta_\varepsilon$ *with*

$$|\mathcal{E}^\alpha_k - E^\alpha_{0k}| \leq e^{-\gamma B(\log L)^2}, \qquad \alpha = \ell, r . \tag{3.8}$$

b) For $\mathcal{E}^\alpha_k \in \Sigma_\alpha$ *the current* $\mathcal{J}^\alpha_k$ *of the associated eigenstate satisfies*

$$|\mathcal{J}^\alpha_k - J^\alpha_{0k}| \leq e^{-\gamma B(\log L)^2}, \qquad \alpha = \ell, r . \tag{3.9}$$

c) Σ_b *contains the same number of energy levels as* $\sigma(H_b) \cap \Delta_\varepsilon$ *and* $(p \gg 1)$

$$\mathrm{dist}(\Sigma_b, \Sigma_\alpha) \geq L^{-p}, \qquad \alpha = \ell, r . \tag{3.10}$$

d) The current associated to each level $\mathcal{E}_\beta \in \Sigma_b$ *satisfies*

$$|\mathcal{J}_\beta| \leq e^{-\gamma B(\log L)^2} . \tag{3.11}$$

We now turn to the characterisation of eigenvalues lying in $\Delta = (2B - \delta, 2B + \delta)$. In this case the first easier Hamiltonian is

$$H_\alpha = H_0 + U_\alpha + V^\alpha_\omega \qquad \alpha = \ell, r \tag{3.12}$$

called *random edge Hamiltonian.* In (3.12) the random potential V_ω^α is the restriction of V_ω to $\Lambda_\alpha \subset \Lambda$, where Λ_α is a strip of size $\sqrt{L}$ in the $x-$direction along the confining walls: $\Lambda_r = \mathbb{Z}^2 \cap \left[[\frac{L}{2} - \frac{3\sqrt{L}}{4} - 1, \frac{L}{2}] \times [-\frac{L}{2}, \frac{L}{2}]\right]$ and $\Lambda_\ell = \mathbb{Z}^2 \cap \left[[-\frac{L}{2}, -\frac{L}{2} + \frac{3\sqrt{L}}{4} + 1] \times [-\frac{L}{2}, \frac{L}{2}]\right]$. The spectrum of H_α is given by

$$\sigma(H_\alpha) = \{E_\kappa^\alpha; \kappa \in \mathbb{Z}\} \tag{3.13}$$

and E_κ^α are isolated eigenvalues with accumulation points at the Landau levels. The quantum mechanical currents J_κ^α associated to the energies in Δ satisfy

$$|J_\kappa^\alpha| = |(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| > C' \tag{3.14}$$

with $C' > 0$ a numerical constant independent of L [FM2]. By the way remark that, since the random variables in the Anderson potential are i.i.d., H_ℓ and H_r are two independent random Hamiltonians. Here, as before we suppose that Hypothesis 1 is fulfilled.

We now state our second result.

Theorem 2. *Let V_0 small enough, fix $\varepsilon > 0$ and let $0 < \delta \equiv \delta(V_0) < B - V_0 - \varepsilon$. Suppose that $(H1)$ holds. Then there exists $\mu > 0$, $\bar{L} \geq L(\varepsilon)$ such that if $L > \bar{L}$ one can find a set $\hat{\Omega} \subset \Omega$ of realizations of the random potential V_ω with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-\nu}$ ($\nu \gg 1$) such that for all $\omega \in \hat{\Omega}$ the spectrum of H_ω in $\Delta = (B-\delta, B+\delta)$ is the unions of two sets Σ'_ℓ and Σ'_r, each depending on ω and L, with the following properties:*

a) $\mathcal{E}_\kappa^\alpha \in \Sigma'_\alpha$ $(\alpha = \ell, r)$ are a small perturbation of $E_\kappa^\alpha \in \sigma(H_\alpha) \cap \Delta$ with

$$|\mathcal{E}_\kappa^\alpha - E_\kappa^\alpha| \leq e^{-\mu\sqrt{B}\sqrt{L}} . \tag{3.15}$$

b) For $\mathcal{E}_\kappa^\alpha \in \Sigma'_\alpha$ the current $\mathcal{J}_\kappa^\alpha$ of the associated eigenstate satisfies

$$|\mathcal{J}_\kappa^\alpha - J_\kappa^\alpha| \leq e^{-\mu\sqrt{B}\sqrt{L}} . \tag{3.16}$$

The idea of the proofs of Theorems 1 and 2 is to link the resolvent of the full Hamiltonian H_ω to those of the easier Hamiltonians $R_\ell^0(z)$ (resp. $R_\ell(z)$), $R_r^0(z)$ (resp. $R_r(z)$) and $R_b(z)$. This is achieved via a decoupling formula for the resolvent [BCD], [BG]. Using it we can do deterministic estimates on the norm difference between the projector $P_{H_\omega}(\Gamma)$, associated to H_ω into the disc with boundary Γ, and the projector associated to one of the easier Hamiltonians. This is done for suitable circles Γ in the complex plane and a suitable set $\hat{\Omega}$ of realizations of the random potential. Using Wegner estimates on H_b (resp. H_α) we control the probability of $\hat{\Omega}$ and show that it can be made large.

Our classification of the spectrum via the quantum mechanical current leads to a well defined notion of *extended edge states* and *localized bulk states.* The former are those belonging to Σ_α (resp. Σ'_α), they are small perturbations of the eigenvalues of $\sigma(H^0_\alpha)$ (resp. $\sigma(H_\alpha)$) and have a quantum mechanical current of order $\mathcal{O}(1)$ with respect to L. The latter are those belonging to Σ_b, and have a infinitesimal current with respect to L (of order $\mathcal{O}\left(e^{-\gamma B(\log L)^2}\right)$), they "arise" from the spectrum of H_b. It is interesting to note that our description leads, in the interval inside the first Landau band, to a spectrum in which extended edge and localized bulk states are intermixed and in some sense there is no "mobility edge". On the other hand in the interval inside the spectral gap there exists only extended edge states.

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