Transport in a slowly perturbed convective cell flow.

A. P. Itin[1], R. de la Llave[2], A. I. Neishtadt[1], and A. A. Vasiliev[1]

[1] *Space Research Institute, Russian Academy of Sciences*
Profsoyuznaya str. 84/32, 117810 Moscow, Russia
[2] *University of Texas at Austin, Dep. of Mathematics*

Abstract

We study transport properties in a simple model of two-dimensional roll convection under a slow periodic (period of order $1/\varepsilon \gg 1$) perturbation. The problem is considered in terms of conservation of the adiabatic invariant. It is shown that the adiabatic invariant is well conserved in the system. It results in almost regular dynamics on large time scales (of order $\sim \varepsilon^{-3}\ln\varepsilon$) and hence, fast transport. We study both generic systems and an example having some symmetry.

Corresponding author: A.A.Vasiliev. E-mail: valex@iki.rssi.ru

Transport properties of convective flows are of interest both for theoretical studies and for applied science, like studies of air and ocean pollution. We consider a popular model of a roll convection flow under a slow time-dependent perturbation. We show that under some general assumptions quasi-regular non-diffusive transport of a passive impurity is possible, resulting in fast propagation of the impurity.

1 Introduction

Transport in time-dependent two-dimensional flows is a subject of intensive investigation, both experimental and theoretical (see, for example, [1], [2], [3], [4], [5]). Streamlines in such flows may be very sensitive to initial conditions, making possible chaotic advection [6] of passive impurities.

In this paper, we study transport properties in a simple model of two-dimensional roll convection under a slow perturbation. A quantitative comparison of a similar model with the experimental data is given in [1]. The perturbation makes all the structure of rolls perform slow periodic lateral motion. The period of the perturbation is of order ε^{-1}, where $0 < \varepsilon \ll 1$ is a small parameter. This system was considered, for example, in [3] in the case of simple sinusoidal perturbation. Here, we consider a perturbation of general kind, and show that it can produce some different transport properties.

In the case of small ε, the problem can be considered in terms of conservation of the adiabatic invariant. It is known (see [7], [8], [9]) that when a perturbed phase trajectory (a streamline) crosses a separatrix of the unperturbed system, the value of the adiabatic invariant undergoes a jump. We evaluate this jump in the considered system, and show that these jumps for subsequent separatrix crossings are phase-correlated. This correlation breaks down only when the phase describing a separatrix crossing gets into a relatively small interval. This phenomenon makes diffusion of the adiabatic invariant happen only in a very small set. Good conservation of the adiabatic invariant in the system results in nearly regular transport on large time-scales (of order $\sim \varepsilon^{-3}\ln\varepsilon$). In case of a generic perturbation, a drop of passive impurity (e.g. dye) put initially into one convective cell, spreads over distance of order ε^{-3}. We emphasize that this spreading is very different from the usual diffusions described by the heat equation. In particular, the diameter of the covered by the dye grows nearly linearly with time.

The paper is organized as follows. In Section 2, we obtain the main equations and present typical phase portraits of the unperturbed (frozen) system. In Section 3, properties of diffusion of the adiabatic invariant due to separatrix crossings are discussed. The next Section 4 deals with transport properties of the system. In Section 5, two examples are considered, one of the sinusoidal perturbation, and another of more general kind. In

Appendix, the formula for the jump of the adiabatic invariant at a separatrix crossing is derived.

2 Main equations and typical phase portraits

Consider a two-dimensional flow defined by a time-dependent stream function

$$\psi(x, y, t) = \psi_0 \sin\left(x + \tilde{G}(\varepsilon t)\right) \sin y. \tag{1}$$

This stream function describes an oscillatory (time-periodic) roll convection [2] within rigid boundaries at the top $y = \pi$ and the bottom $y = 0$. We assume that $\tilde{G}(\varepsilon t)$ is a 2π-periodic function of the "slow time" $\tau = \varepsilon t, 0 < \varepsilon \ll 1$. All the structure of rolls performs slow periodic (with period $2\pi/\varepsilon$) motion along the x-axis according to the function $\tilde{G}$. The value of ψ_0 gives the rotation frequency near the center of a roll at a frozen value of τ. We assume that this frequency is much larger than the frequency of the slow oscillations ε. Without loss of generality, we take $\psi_0 = 1$.

The motion of a passive impurity or a test particle in a flow defined by a stream function ψ is described by the following equations of motion:

$$\dot{x} = \frac{\partial \psi}{\partial y} \quad \dot{y} = -\frac{\partial \psi}{\partial x}. \tag{2}$$

Hence, (1) defines a Hamiltonian system with a time-dependent Hamiltonian function ψ. Following [3], we make a canonical transformation $(y, x) \mapsto (\bar{y}, \bar{x})$ to the coordinate system $(\bar{x}, \bar{y})$ moving with the rolls. The transformation is defined by the generating function

$$F(\bar{y}, x, \tau) = \bar{y}\left(x + \tilde{G}(\tau)\right). \tag{3}$$

In terms of the new variables $(\bar{y}, \bar{x})$, the Hamiltonian function is

$$H(\bar{y}, \bar{x}, \tau) = \sin\bar{x}\sin\bar{y} + \frac{\partial F}{\partial t} = \sin\bar{x}\sin\bar{y} + \varepsilon\bar{y}G(\tau), \tag{4}$$

where $G(\tau) = d\tilde{G}/d\tau$. The Hamiltonian (4) is a sum of the unperturbed Hamiltonian $H_0 = \sin x \sin y$ (bars over x and y are from now on omitted) and the small slowly time-dependent perturbation $\varepsilon H_1 = \varepsilon y G(\varepsilon t)$.

In what follows, we will consider often systems described by the Hamiltonian (4) at $\tau = \tilde{\tau} = \text{const}$ for different values of $\tilde{\tau}$. We call such a system the system frozen at $\tau = \tilde{\tau}$.

Consider phase portraits of the system with Hamiltonian (4) frozen at different values of $\tilde{\tau}$. If $G(\tilde{\tau}) = 0$, the phase portrait is the same as one of the unperturbed system with Hamiltonian H_0 (see Fig. 1a). All the phase trajectories except for the separatrices are

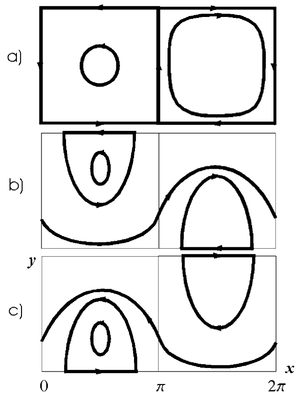

Figure 1: Phase portraits of the system with Hamiltonian (4) frozen at different values of τ: a) $G(\tau) = 0$; b) $G(\tau) > 0$; c) $G(\tau) < 0$.

closed and are confined within square separatrix cells. If $G(\tilde{\tau}) > 0$, the phase portrait is qualitatively different (Fig. 1b). Some trajectories are closed and correspond to finite motion within the cells confined by separatrices. Other trajectories are not closed and correspond to unbounded motion in the positive direction along the x-axis. If we choose $\tilde{\tau}$ such that $G(\tilde{\tau}) < 0$, the picture is similar to the previous one, but the channel for unbounded motion is open in the negative direction along the x-axis (see Fig. 1c).

It is straightforward to find the area $S^*(\tau)$ bounded by a separatrix loop in the system frozen at a certain value of τ. One obtains:

$$S^*(\tau) = \pi^2 - 2\pi|G(\tau)\varepsilon\ln\varepsilon| + O(\varepsilon). \tag{5}$$

Consider now a phase trajectory on a phase portrait frozen at a certain value of τ. If the trajectory is closed, the area S inside of it is connected with the action I of the system by a simple relation $S = 2\pi I$ (Fig. 1a). If the trajectory is not closed (Fig. 1b) and crosses the line $x = 0$ at $y \leq \pi/2$, we still have $2\pi I = S$, where S is the area bounded by the trajectory and the segments of the lines $y = 0, x = 0, x = 2\pi$. If it crosses the line

$x = 0$ at $y > \pi/2$, we put $2\pi I = 2\pi^2 - S$. Defined in this way, I is a continuous function of the coordinates.

In system (4) with $\tau = \varepsilon t$, the action I is an adiabatic invariant of motion (see, for example, [10]). Far from the separatrices, it undergoes variations of order ε^2.

In the following, we will use the so called improved adiabatic invariant $J = I + \varepsilon u(x, y, \tau)$ rather than I. The function u is defined in (17) in the Appendix. Far from the separatrices variations of J are of order ε^3. As the slow time τ grows, the area bounded by a separatrix $S^*(\tau)$ slowly changes. On the other hand, a value of the adiabatic invariant associated with a certain phase trajectory stays well-preserved. Accordingly, phase trajectories of (4) can cross separatrices. If, for instance, $S^*(\tau)$ slowly diminishes, some initially closed trajectories leave the region bounded by the separatrix and become unbounded. Conversely, if $S^*(\tau)$ grows grows slowly, unbounded trajectories become captured, until at the moment when $G(\tau) = 0$ the frozen system does not have any more any unbounded trajectory. This means that a passive impurity particle repeatedly changes modes of its motion from a flight mode to a captured mode. Note, that if the initial value J is small enough, the phase trajectory is captured in a cell forever (in accordance with Arnold's theorem on eternal adiabatic invariance [11]). The range of J where the motion from one cell to another is possible, is in the main approximation $\mathcal{J} = (\pi/2 - J_f, \pi/2)$, where J_f can be found from (5) to be:

$$J_f = \varepsilon |\ln \varepsilon| \max_{0 \le \tau \le 2\pi} |G(\tau)| \,.$$

3 Changes of the adiabatic invariant at separatrix crossings

Consider in the adiabatic approximation (i.e. neglecting variations of J along the motion) a phase trajectory with the value of $J \in \mathcal{J}$. On a period of function $G(\tau)$ consider time moments τ_i such that $S^*(\tau_i) = 2\pi J$. In the adiabatic approximation $J = \text{const}$, and at a moment τ_i the trajectory crosses a separatrix of the system (4) frozen at $\tau = \tau_i$.

It is known from the general theory of separatrix crossing [7], [8], [9] that at a separatrix crossing the value of the adiabatic invariant undergoes a jump. In the system (4), one obtains in the main approximation the following formula for the value ΔJ of this jump (see Appendix):

$$2\pi \Delta J = -2\varepsilon \frac{\partial S^*}{\partial \tau}|_{\tau=\tau_*} \ln(2 \sin \pi \xi), \tag{6}$$

where the value of $\partial S^*/\partial \tau$ is calculated at the moment τ_* of the separatrix crossing in the adiabatic approximation. The value ξ in (6) is defined as follows. Consider the last passage near a saddle point before crossing the lower separatrix (i.e. the one defined by $H = 0$). Let h_0 be equal to the value h_c of the Hamiltonian at the point where the phase trajectory crosses the vertex bisecting the angle between incoming and outgoing separatrices of the saddle point (see Fig. 8 in Appendix). Then

$$\xi = \left| \frac{h_0}{\varepsilon \partial S^*/\partial\tau|_{\tau=\tau_*}} \right|.$$

To consider the crossing of the upper separatrix $H = \varepsilon\pi G$, one should put $h_0 = h_c - \varepsilon\pi G$ and use the same formula for ξ.

The value ξ strongly depends on initial conditions: a small of order ε variation of initial conditions results generally in variation of ξ of order 1. Hence, this value of the adiabatic jump at each separatrix crossing can be considered as a random variable t its distribution should be treated as uniform on the segment $(0, 1)$ (see [9], [12], see also [13] for numerically found distribution of ξ in a similar problem). We want to argue that, in many cases of interest, the values of the jumps at subsequent crossings can be considered as independent random variables.

From (5) and (6) one obtains

$$\Delta J = 2\varepsilon^2 \ln\varepsilon \frac{\partial G}{\partial\tau}|_{\tau=\tau_*} \ln(2\sin\pi\xi). \tag{7}$$

For a typical function $G(\tau)$, the interval between two successive separatrix crossings is a value of order 1, i.e. the corresponding time interval $\Delta t \sim 1/\varepsilon$. Consider two successive separatrix crossings. Let J_1 and J_2 be values of the adiabatic invariant before the first and the second crossing respectively, and let ξ_1, ξ_2 be the corresponding values of ξ. In the main approximation, we have the following map:

$$\begin{aligned} J_2 &= J_1 + \Delta J(\xi_1) && (8) \\ \xi_2 &= \xi_1 + \frac{1}{2\pi}\int_0^{\Delta t} \omega(J_2, \varepsilon t)dt \qquad \bmod 1, && (9) \end{aligned}$$

where $\omega(J,\tau) = 2\pi/T(J,\tau)$, and $T(J,\tau)$ is a period of fast motion: a period of rotation for a captured phase trajectory, or a period needed to cover 2π distance along the x-axis for an unbounded phase trajectory. For this period we have in the main approximation (see, for example, [9]):

$$T(J,\tau) = 4|\ln|2\pi J - S^*(\tau)||. \tag{10}$$

From (7), (9) and (10) we find the estimate

$$\frac{d\xi_2}{d\xi_1} - 1 \sim (|\ln\varepsilon| \sin \pi\xi_1)^{-1} . \tag{11}$$

We see that if $\xi_1 \ll |\ln\varepsilon|^{-1}$ or $(1 - \xi_1) \ll |\ln\varepsilon|^{-1}$, then $|d\xi_2/d\xi_1| \gg 1$, meaning that the changes in ξ_1 and ξ_2 are not correlated. At other values of ξ_1 in the interval $(0, 1)$ the correlation is strong. The map M:$(J, \xi) \mapsto (J', \xi')$ on a period of the perturbation is a composition of several maps of the form (9). If none of these maps put ξ close enough to the ends of the interval $(0, 1)$, the changes in ξ and ξ' are correlated. Otherwise, according to the Chirikov's criterion, changes in ξ and ξ' are not correlated.

It can be concluded that in the process of iterations of a point (J, ξ) by the map M, its images lie on a smooth curve (a level set of the adiabatic invariant) until ξ is mapped too close (at a distance smaller than $1/|\ln\varepsilon|$) to one of the ends of the interval $(0, 1)$. Then, there is a jump in ξ and J uncorrelated with the previous ones. After that, the point continues its motion along another smooth curve, level set of the adiabatic invariant.

The value ξ is mapped into these narrow intervals at the ends of $(0, 1)$ with a probability of order $1/|\ln\varepsilon|$. Hence, a jump of the adiabatic invariant uncorrelated with the previous one occurs after $N \sim |\ln\varepsilon|$ separatrix crossings. Therefore, the time interval between quasi-random jumps of the adiabatic invariant $\Delta J \sim \varepsilon^2|\ln\varepsilon|$ is $\delta t \sim |\ln\varepsilon|/\varepsilon$. After N_1 quasi-random jumps the value of J changes by $\delta J \sim \sqrt{N_1}\Delta J$. After $N_1 \sim \varepsilon^{-2}$ jumps, which needs time $t \sim \varepsilon^{-3}|\ln\varepsilon|$, the change δJ is of order $\varepsilon|\ln\varepsilon|$. In a time interval of this order, the value of J covers all the range $\mathcal{J}$.

To summarize, the motion in terms of the variable J can be described as follows. On a time interval of length $\delta t \sim |\ln\varepsilon|/\varepsilon$, the value of J along the trajectory stays approximately the same and oscillates with an amplitude of order $\varepsilon^2|\ln\varepsilon|$ around a certain mean value. Then an uncorrelated with the previous one jump of J of order $\varepsilon^2|\ln\varepsilon|$ occurs. On the following time interval of order $|\ln\varepsilon|/\varepsilon$, the value of J oscillates around the new mean value. On time scales of order $\varepsilon^{-3}|\ln\varepsilon|$ and larger, this results in diffusion of the adiabatic invariant J. This diffusion is very slow, and its coefficient can be estimated as $D \sim \varepsilon^5|\ln\varepsilon|$.

4 Transport properties

We start with the adiabatic approximation and consider motion in the system at $J = \text{const}, J \in \mathcal{J}$. Let τ_1, τ_2 be two consecutive moments of slow time τ such that $S^*(\tau_1) = S^*(\tau_2) = 2\pi J$ and $S^*(\tau) < 2\pi J$, if $\tau_1 < \tau < \tau_2$. Assume for definiteness that $G(\tau) > 0$, if $\tau_1 \leq \tau \leq \tau_2$. Then on the interval of slow time (τ_1, τ_2), the phase point moves in the positive direction of the x-axis (see Section 2). In terms of the convective flow, this means that the test particle moves in the x-axis direction from one convective cell to another.

Estimate the distance $d_{1,2}$ covered by the phase point during the slow time interval

(τ_1, τ_2). The distance 2π along the x-axis is covered in time T, where T is given by (10). Hence, for the distance $d_{1,2}$ we find

$$d_{1,2}(J) = \varepsilon^{-1} \int_{\tau_1}^{\tau_2} \frac{\pi d\tau}{2|\ln(2\pi J - S^*(\tau))|}. \tag{12}$$

To evaluate the integral, introduce

$$\bar{J} = \frac{2\pi J - \pi^2}{\varepsilon|\ln\varepsilon|}\ ; \qquad \bar{S}(\tau) = \frac{S^*(\tau) - \pi^2}{\varepsilon|\ln\varepsilon|}. \tag{13}$$

It follows from (5) that $\bar{J} \sim 1, \bar{S} \sim 1$. Substituting (13) in (12), we find in the main approximation

$$d_{1,2}(J) = \frac{\pi}{2\varepsilon|\ln\varepsilon|}(\tau_2 - \tau_1). \tag{14}$$

To find the total shift of the phase point during the slow time period $(0, 2\pi)$, one has to find on this period all the intervals $\Delta\tau_i$ where $S^*(\tau) < 2\pi J$, and sum up expressions similar to (14) taking into account the signs of $G(\tau)$ on these intervals. Thus we obtain for the total shift

$$d(J) = \frac{\pi}{2\varepsilon|\ln\varepsilon|} \sum_i \mathrm{sign}\,(G(\tau)|_{\tau\in\Delta\tau_i})\,|\Delta\tau_i|, \tag{15}$$

where $|\Delta\tau_i|$ is the length of the interval $\Delta\tau_i$. Note, that for a typical function $G(\tau)$ and a typical value of J, the distance $d(J)$ is not zero and is a large value of order $\varepsilon^{-1}|\ln\varepsilon|^{-1}$. This means that the test particle moves in the flow in a certain direction along the x-axis at a mean velocity of order $|\ln\varepsilon|^{-1}$.

Now we take into consideration the diffusive change of J due to the separatrix crossings. As it was found in the previous section, on time scales larger than $\sim \varepsilon^{-3}|\ln\varepsilon|$ the value of J along a trajectory covers in the course of diffusion all the interval $\mathcal{J}$. Except for a small measure on this interval, it may be assumed that the values of J are uniformly distributed on $\mathcal{J}$. Therefore, to find the mean shift on large time scales larger than $\sim \varepsilon^{-3}|\ln\varepsilon|$, one has to integrate $d(J)$ over $\mathcal{J}$. It is straightforward to see that this integral in the main approximation is proportional to the integral of $G(\tau)$ over the slow time period. This latter integral is zero, because $G(\tau) = d\tilde{G}/d\tau$ and $\tilde{G}(\tau)$ is a periodic function. Hence, the total mean shift on such time scales is also zero.

Consider generic periodic function $G(\tau)$. It has extrema of different absolute values. Let $G(\tau_{max}) = \max_{0\le\tau\le2\pi} G(\tau)$ and $G(\tau_{min}) = \min_{0\le\tau\le2\pi} G(\tau)$. We have $G(\tau_{max}) > 0$, $G(\tau_{min}) < 0$. Assume for definiteness, that $|G(\tau_{max})| > |G(\tau_{min})|$. Denote the corresponding values of $S^*(\tau)$: $S^*(\tau_{max}) = S_{max}$, $S^*(\tau_{min}) = S_{min}$. Generally, $|S_{max} - S_{min}| \sim \varepsilon|\ln\varepsilon|$. Let $S_{max} < 2\pi J_0 < S_{min}$. A phase trajectory with initial value of the adiabatic invariant J_0 moves from one convective cell to another only in the positive direction of x-axis till the value of J along this trajectory satisfies $S_{max} < 2\pi J < S_{min}$. The value of J changes due to diffusion of the adiabatic invariant, and it takes time of order $\delta t \sim \varepsilon^{-3}|\ln\varepsilon|$ for J to leave the interval $(S_{max}/2\pi, S_{min}/2\pi)$. Before this happens the phase trajectory shifts at a large distance of order ε^{-3} in the positive direction. Afterwards, it moves in both directions, and on time scales larger than $\varepsilon^{-3}|\ln\varepsilon|$ the mean shift is zero.

Put a drop of passive impurity (e.g. dye) into one convective cell of the flow in such a way that it covers all the range $(0, \pi/2)$ in initial values of the adiabatic invariant. In the process of evolution a large part of this drop, that with $J < S_{max}/2\pi$, stays forever captured inside the initial cell. The part of the drop with initial values of $J \in \mathcal{J}$ over a period time of order $\varepsilon^{-3}|\ln\varepsilon|$ spreads along an interval of length $\sim \varepsilon^{-3}$ around the initial cell. On time intervals of this order and smaller, the distance between the rightmost and the leftmost points reached by parts of the drop grows nearly linearly in time. An example of such behavior is studied in the second part of the next section.

5 Two examples

5.1 $G(\tau) = B\cos\tau$

This case, though not a generic one, was considered in the previous studies (see [1], [2], [4], [3]). The function $G(\tau) = B\cos\tau$ possesses non-generic symmetry properties. In particular, $|G(\tau_{max})| = |G(\tau_{min})|$. Moreover, for any $J \in \mathcal{J}$, on a period of the slow time the total shift $d(J)$ given by (15) is zero in the main approximation. This fact makes transport properties to be different from those of the general case described in the previous section.

Consider an adiabatic trajectory with a value of the adiabatic invariant $J \in \mathcal{J}$. On the period of the slow time $(-\pi/2, 3\pi/2)$ the equation $S^*(\tau) = 2\pi J$ has four roots τ_i, $i = 1, 2, 3, 4$ (see Fig. 2). On the slow time interval (τ_1, τ_2) the phase point drifts to the right and covers the distance of $d_{1,2} = \pi(\tau_2 - \tau_1)(2\varepsilon|\ln\varepsilon|)^{-1}$ in the main approximation (see (14)). On the slow time interval (τ_3, τ_4) the phase point drifts to the left and covers the distance $d_{3,4} = \pi(\tau_4 - \tau_3)(2\varepsilon|\ln\varepsilon|)^{-1}$ in the main approximation. Obviously, $\tau_4 - \tau_3 = \tau_2 - \tau_1$ and we find that the total shift is zero in the main approximation. On the other parts of the period the phase point is captured in a cell. Hence, in the adiabatic approximation the phase point cannot leave an interval between certain right and left borders on the x-axis.

Such a behavior is illustrated by computer simulations of an exact, not adiabatic, trajectory (Fig. 3). Note, that the trajectory covers large distances (many convective cells) moving almost regularly at a constant mean velocity. One can also see from Fig. 3 that the value of the adiabatic invariant indeed is nearly constant in spite of a large

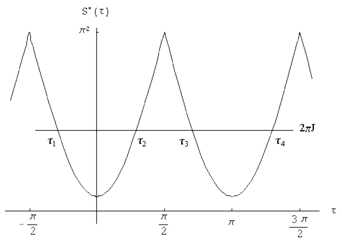

Figure 2: Plot of the function $S^*(\tau)$ in example 5.1.

number of separatrix crossings. The variations of J (see the previous section) are of order $\varepsilon^2|\ln\varepsilon|$ leading to variations of $d_{1,2}$ and $d_{3,4}$ of an order $(\ln\varepsilon)^{-1}$. Hence, the borders of the region where the phase point is kept may change by one convective cell. This, however, does not result in significant diffusion on the x-axis on time intervals smaller than $\sim \varepsilon^{-3}|\ln\varepsilon|$ (see Section 3).

Consider a drop of dye put into one convective cell in such a way that its particles cover all the range of J from 0 to $\pi/2$. In the main approximation, evolution of this drop goes as follows. A large part of the drop, with values of $J \in (0, \pi/2 - J_f)$ stays eternally captured inside the initial cell. The other part of the drop spreads linearly in time over a large interval of the length $d \approx \pi^2(2\varepsilon|\ln\varepsilon|)^{-1}$, which takes the time $t \approx \pi/\varepsilon$. Then this part of the drop squeezes back to the initial cell and so on. On large time intervals of order $\varepsilon^{-3}|\ln\varepsilon|$ the area accessible for the dye slowly grows as a result of the diffusion of the adiabatic invariant.

In Fig. 4, time evolution of the length Δx of interval covered by the points put initially into one cell and equidistantly distributed in J, is shown. One can see that this evolution is close to regular.

5.2 $G(\tau) = B(1 + \cos\tau)\cos(\tau + \sin\tau)$

This case corresponds to $\tilde{G}(\tau) = B\sin(\tau + \sin\tau)$, which is a simple example of the perturbation representing main generic properties of the system. A plot of the function $S^*(\tau)$ is shown in Fig. 5. Consider a trajectory with a certain initial value of the adiabatic

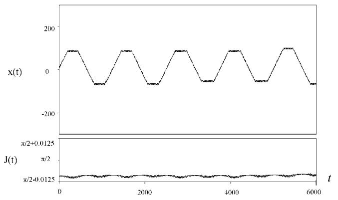

Figure 3: Plots of the functions $x(t)$ and $J(t)$ obtained by numerical integration of system (4) with the perturbation of example 5.1. $\varepsilon = 0.005, B = 0.6$.

invariant J on a time interval not long enough for a significant change of J (see Section 3). If the equation $S^*(\tau) = 2\pi J$ has only two roots τ_1, τ_2 on the interval $(-\pi, \pi)$, the phase point drifts only in the positive direction of the x-axis (see Section 4). Numerical simulation of such behavior is shown in Fig. 6a. When $\tau \in (\tau_1 + 2\pi k, \tau_2 + 2\pi k)$, k is an integer, the phase point performs regular drift to the right, otherwise it is kept in a convective cell. If the equation $S^*(\tau) = 2\pi J$ has six roots $\tau_3, ..., \tau_8$ on the interval $(-\pi, \pi)$, the phase point can drift in both directions. The results of computer simulations in this case are shown in Fig. 6b. One can also see in Fig. 6 that the adiabatic invariant J is nearly constant in the both cases.

The time evolution of a drop of dye in this case was described in the end of Section 4.

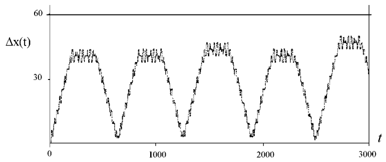

Figure 4: Time evolution of Δx in example 5.1. $\varepsilon = 0.005, B = 0.6$.

For computer simulations, a set of initial points was chosen in one convective cell, with values of J equidistantly distributed on the interval $(0, \pi/2)$. In Fig. 7, time evolution of the distance Δx between leftmost and rightmost phase points is shown. A close to regular dynamics can be seen. Deviations from regularity are due to slow diffusion of the adiabatic invariant described in Section 3.

6 Acknowledgements

This work was supported in part by grants RFBR 00-01-00538, CRDF RM1-184 and INTAS 97-10771. R. L. was partially supported by NSF grants.

7 Appendix. Jump of the adiabatic invariant at a separatrix crossing

In this appendix, we calculate in the main approximation the variation of the improved adiabatic invariant J at a separatrix crossing. In the calculation, we follow the general method developed in [7], [8], [9].

The improved adiabatic invariant is

$$J = J(x, y, \tau) = I + \varepsilon u, \tag{16}$$

where

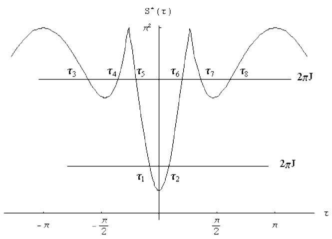

Figure 5: Plot of the function $S^*(\tau)$ in example 5.2.

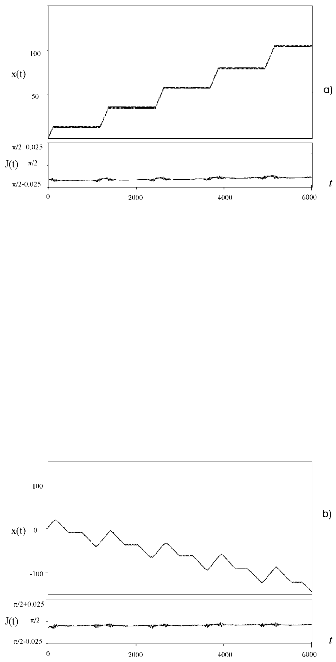

Figure 6: Plots of the functions $x(t)$ and $J(t)$ obtained by numerical integration of system (4) with the perturbation of example 5.2. $\varepsilon = 0.005, B = 0.6$. Figures a) and b) correspond to different initial conditions.

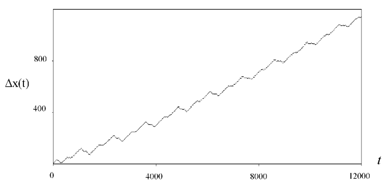

Figure 7: Time evolution of Δx in example 5.2. $\varepsilon = 0.005, B = 0.6$.

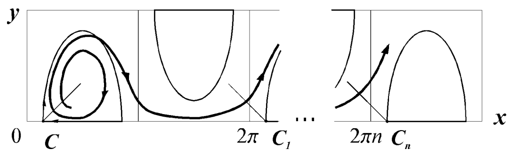

Figure 8: A schematical picture of separatrix crossing. Note, that actually the separatrix enclosed areas change and the turns of the trajectory stay roughly constant in size.

$$u = u(x, y, \tau) = \frac{1}{2\pi} \int_0^T \left(\frac{T}{2} - t \right) \frac{\partial H}{\partial \tau} dt. \tag{17}$$

The integral in (17) is computed along the unperturbed trajectory passing through the point (x, y) at the time moment $t = 0$; T is the period of motion on this trajectory. If the trajectory is not closed, T is the time necessary for a phase point on this trajectory to cover the distance of 2π along the x-axis.

Consider the motion in the system with Hamiltonian (4) frozen at a certain value of the slow time $\tau = \bar{\tau}$. As the unperturbed trajectory tends to the separatrix, the period T grows logarithmically. On a phase trajectory $H = h$ one obtains in the main approximation:

$$T = -2\ln|h| - 2\ln|\varepsilon\pi G| + b. \tag{18}$$

Here b is a constant which will not be important for the rest of the argument. In (18), it is assumed that $|h| < \varepsilon\pi|G|c^{-1}$, where c is a large positive constant. (For definiteness, we consider crossing the lower separatrix $H = 0$; for the upper separatrix $H = \varepsilon\pi G$ the final result obviously is the same). In agreement with the formula $T = 2\pi\partial I/\partial h$ we find for the adiabatic invariant I in the main approximation

$$2\pi I = S^* - 2h\ln|h| - 2h\ln|\varepsilon\pi G| + (2 + b)h. \tag{19}$$

We now compute the function u at a point of the vertex bisecting the angle between incoming and outgoing separatrices of a saddle point (see Fig. 8). From (17), (18) one obtains:

$$2\pi u = \frac{\theta}{2}\ln|h|, \qquad \theta = \frac{\partial S^*}{\partial \tau}. \tag{20}$$

Consider now a separatrix crossing in the exact system. Assume for definiteness, that on the relevant interval of the slow time $G(\tau) > 0$, $\theta(\tau) < 0$. Let a phase point be initially captured in a separatrix cell bounded by the separatrix $H = 0$ (see Fig. 8). The initial values of the Hamiltonian and the improved adiabatic invariant at $\tau = \tau_-$ are h_-, J_-. The phase point rotates close to the separatrix. As the area bounded by the separatrix decreases, the point comes closer to the separatrix with each turn. Denote by h_n, I_n, u_n values of the corresponding functions at time moments τ_n when the trajectory crosses the vertex bisecting the angle between incoming and outgoing separatrices of the saddle point C (see Fig. 8). We enumerate τ_n as follows: τ_0 is the time of the last crossing of the vertex before crossing the separatrix, other τ_n have negative numbers $\tau_0 > \tau_{-1} > ... > \tau_{-N} \geq \tau_-$. Here $N \gg 1$ is a large integer. Its exact value does not influence the result in the main approximation. After crossing the separatrix, the phase point shifts along the x-axis. At time moments τ_n, $1 \leq n \leq N$, the trajectory subsequently crosses the vertices near the saddles C_n such that the distance between C and C_n equals $2\pi n$ (see Fig. 8). The corresponding values of h, I, u are h_n, I_n, u_n.

It follows from (18) - (20), that in the main approximation the following expressions are valid:

$$\begin{aligned}
h_{n+1} &= h_n - \varepsilon\theta, \\
\tau_{n+1} &= \tau_n - \frac{\varepsilon}{2}\left[\ln|h_n| + 3\ln|h_n - \varepsilon\theta| + 4\ln\varepsilon\pi G - 2b\right], \\
2\pi I_n &= S^*(\tau_0) - \theta(\tau_0 - \tau_n) - 2h_n\ln|h_n| - 2h_n\ln\varepsilon\pi G + (2+b)h_n, \\
2\pi u_n &= \frac{\theta}{2}\ln|h_n|.
\end{aligned} \tag{21}$$

Here and below the value of θ is calculated at $\tau = \tau_*$, the time of separatrix crossing in the adiabatic approximation. Summing the above expressions (22) from $n = -N$ to $n = 0$, we find the change of the improved adiabatic invariant before the separatrix crossing in the main approximation:

$$2\pi(J_0 - J_{-N}) = \varepsilon\theta\left[\ln\xi - 2\xi\ln\xi + 2\xi - 2\ln(\sqrt{2\pi}/\Gamma(\xi))\right], \tag{22}$$

where $\xi = |h_0/(\varepsilon\theta)|$, $\Gamma(\cdot)$ is the gamma function. At the separatrix crossing we obtain:

$$2\pi(J_1 - J_0) = \varepsilon\theta\left[-\ln\xi - \ln(1-\xi) + 2(1-\xi)\ln(1-\xi) + 2\xi\ln\xi - 2\right]. \tag{23}$$

For the change of J after the separatrix crossing we find

$$2\pi(J_N - J_1) = \varepsilon\theta\left[\ln(1-\xi) - 2(1-\xi)\ln(1-\xi) - 2\xi + 2 - 2\ln(\sqrt{2\pi}/\Gamma(1-\xi))\right]. \tag{24}$$

Far from the separatrix, the variation of J is of order ε^3 on time periods of order $1/\varepsilon$. Hence, to obtain in the main approximation the jump of J at the separatrix crossing, one has to sum up expression (22) - (24). Thus, we find formula (6).

References

[1] T.H.Solomon and J.P.Gollub, Chaotic particle transport in time-dependent Rayleigh-Benard convection, Phys. Rev. A **38** (1988) 6280-6286.

[2] T.H.Solomon, S.Tomas, and J.L.Warner, Chaotic mixing of immiscible impurities in a two-dimensional flow, Phys.Fluids **10** (1998) 342-350.

[3] A.A.Chernikov, A.I.Neishtadt, A.V.Rogal'sky, and V.Z.Yakhnin, Lagrangian turbulence in nonstationary 2-D flows, Chaos **1** (1991) 206-211.

[4] P.Castiglione, A.Mazzino, P.Muratore-Ginanneschi, and A.Vulpiani, On strong anomalous diffusion, Physica D, **34** (1999) 75-93.

[5] P.Castiglione, R.Festa, and A.Mazzino, Different transport regimes in a spatially-extended recirculating background, Electronic preprint chao-dyn/9910029 20 Oct 1999.

[6] H.Aref, Stirring by chaotic advection, J.Fluid.Mech. **143** (1984) 1-21.

[7] A.V.Timofeev, On the constancy of an adiabatic invariant when the nature of the motion changes, Sov. Phys. JETP **48** (1978) 656-659.

[8] J.R.Cary, D.F.Escande, and J.Tennyson, Adiabatic invariant change due to separatrix crossing, Phys. Rev. **A34** (1986) 4256-4275.

[9] A.I.Neishtadt, Change of an adiabatic invariant at a separatrix, Sov.J.Plasma Phys. **12** (1986) 568-573.

[10] V.I.Arnold, V.V.Kozlov, and A.I.Neishtadt, *Mathematical aspects of classical and celestial mechanics (Encyclopaedia of mathematical sciences 3)* (Springer, Berlin, 1988).

[11] V.I.Arnol'd, Small denominators and problems of stability of motion in classical and celestial mechanics *Russ. Math. Surveys* **18** (1963) 85-192.

[12] A.I.Neishtadt, On the change in the adiabatic invariant on crossing a separatrix in systems with two degrees of freedom, PMM U.S.S.R. **51** (1987) 586-592.

[13] A.P.Itin, A.I.Neishtadt, and A.A.Vasiliev, Captures into resonance and scattering on resonance in dynamics of a charged relativistic particle in magnetic field and electrostatic wave, Physica D **141** (2000) 281-296.