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Rigorous Solution of the Gardner Problem

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Abstract: We prove rigorously the well-known result of Gardner about the typical fractional volume of interactions between N spins which solve the problem of storing a given set of p random patterns. The Gardner formula for this volume in the limit $N, p \to \infty$, $p/N \to \alpha$ is proven for all values of α. Besides, we prove a useful criterion of the factorisation of all correlation functions for a class of spin glass model.

1. Introduction

The spin glass and neural network theories are of considerable importance and interest for a number of branches of theoretical and mathematical physics (see [M-P-V] and references therein). Among many topics of interest the analysis of the different models of neural network dynamics is one of the most important.

The neural network dynamics is defined as

$$\sigma_i(t+1) = \operatorname{sign}\{\sum_{j=1, j\neq i}^{N} J_{ij}\sigma_j(t)\} \quad (i = 1, \ldots, N), \tag{1.1}$$

where $\{\sigma_j(t)\}_{j=1}^N$ are the Ising spins and the interaction matrix $\{J_{ij}\}$ (not necessarily symmetric) depends on the concrete model, but usually it satisfies the conditions

$$\sum_{j=1, j\neq i}^{N} J_{ij}^2 = NR \quad (i = 1, \ldots, N), \tag{1.2}$$

where R is some fixed number which could be taken equal to 1.

The main problem of the neural network theory is to introduce an interaction in such a way that some chosen vectors $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$ (patterns) are the fixed points of the dynamics (1.1). This implies the conditions:

$$\xi_i^{(\mu)} \sum_{j=1, j\neq i}^{N} J_{ij}\xi_j^{(\mu)} > 0 \quad (i = 1, \ldots, N). \tag{1.3}$$

Usually, to simplify the problem the patterns $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$ are chosen i.i.d. random vectors with i.i.d. components $\xi_i^{(\mu)}$ $(i = 1, \ldots, N)$, assuming values ±1 with probability $\frac{1}{2}$.

Sometimes condition (1.3) is not sufficient to have $\boldsymbol{\xi}^{(\mu)}$ as the end points of the dynamics. To have some "basin of attraction" (that is some neighbourhood of $\boldsymbol{\xi}^{(\mu)}$, starting from which we for sure arrive in $\boldsymbol{\xi}^{(\mu)}$) one should introduce some positive parameter k and impose the conditions:

$$\xi_i^{(\mu)} \sum_{j=1, j\neq i}^{N} \tilde{J}_{ij}\xi_j^{(\mu)} > k \quad (i = 1, \ldots, N). \tag{1.4}$$

Gardner [G] was the first who solved a kind of inverse problem. She asked the questions: for which $\alpha = \frac{p}{N}$ the interaction $\{J_{ij}\}$, satisfying (1.2) and (1.4) exists? What is the typical fractional volume of these interactions? Since all condition (1.2) and (1.4) are factorised with respect to i, this problem after a simple transformation should be replaced by the following. For the system of $p \sim \alpha N$ i.i.d. random patterns $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$ with i.i.d. $\xi_i^{(\mu)}$ $(i = 1, \ldots, N)$ assuming values ±1 with probability $\frac{1}{2}$, consider

$$\Theta_{N,p}(k) = \sigma_N^{-1} \int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J} \prod_{\mu=1}^{p} \theta(N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) - k), \tag{1.5}$$

where the function $\theta(x)$, as usually, is zero in the negative semi-axis and 1 in the positive and σ_N is the Lebesgue measure of N-dimensional sphere of radius $N^{1/2}$. Then, the question of interest is the behaviour of $\frac{1}{N}\log\Theta_{N,p}(k)$ in the limit $N, p \to \infty$, $\frac{p}{N} \to \alpha$. Gardner [G] had solved this problem by using the so-called replica trick, which is completely non-rigorous from the mathematical point of view but sometimes very useful in the physics of spin glasses (see [M-P-V] and references therein). She obtained that for any $\alpha < \alpha_c(k)$, where

$$\alpha_c(k) \equiv (\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u+k)^2 e^{-u^2/2} du)^{-1}, \tag{1.6}$$

there exists

$$\begin{aligned} &\lim_{N,p\to\infty, p/N\to\alpha} E\{\log\Theta_{N,p}(k)\} = \mathcal{F}(\alpha, k) \\ &\equiv \min_{q: 0\leq q\leq 1} \left[\alpha E\left\{\log H\left(\frac{u\sqrt{q}+k}{\sqrt{1-q}}\right)\right\} + \frac{1}{2}\frac{q}{1-q} + \frac{1}{2}\log(1-q)\right], \end{aligned} \tag{1.7}$$

where u is the Gaussian random variable with zero mean and variance 1, and here and below we denote by the symbol $E\{...\}$ the averaging with respect to all random parameters of the problem. And $\frac{1}{N}\log\Theta_{N,p}(k)$ tends to minus infinity for $\alpha \geq \alpha_c(k)$.

At the present paper we give the rigorous mathematical proof of the Gardner results. As far as we know, it is one of the first cases, when the problem of spin glass theory can be completely (i.e. for all parameters α and k) solved in the rigorous mathematical way. It can be explained by the fact, that in the Gardner problem the so-called replica symmetry solution is true for all α and k, while, e.g. in the Hopfield and Sherrington-Kirkpatrick models the replica symmetry solution is valid only for small enough α or for high temperatures (see [M-P-V] for the physical theory and [S1], [S2], [T1], [T2] for the respective rigorous results). The same situation holds, unfortunately, with a problem similar to the Gardner one, the so-called Gardner-Derrida [D-G] problem. Also only the case of small enough α was studied rigorously for this model (see [T3]).

We solve the Gardner problem in three steps which are Theorems 1, 2 and 3 below. At the first step we prove some general statement. We study an abstract situation, where the energy function (the Hamiltonian) and the configuration space are convex and prove that in this case all the correlation functions become factorised in the thermodynamic limit. Usually this factorisation means that the ground state and the Gibbs measure are uniquely defined. In fact, physicists understood this fact during a rather long time, but in the rigorous mathematical way it was not proved before.

The proof of Theorem 1 is based on the application of the theorem of classical geometry, known since the nineteenth century as the Brunn-Minkowski theorem. This theorem studies the intersections of a convex set with the family of parallel hyper-planes (see the proof of Theorem 1 for the exact statement). We only need to prove some corollary from this theorem (Proposition 1), which allows us to have N- independent estimates. As a result we obtain the rigorous proof of the general factorisation property of all correlation functions (see (2.8)). Everybody who is familiar with the theory of spin glasses knows that decay of correlations is the key point in the derivation of self-consistent equations.

The second step is the derivation of self-consistent equations for the order parameters of our model. In fact Theorem 1 provides all the necessary to express the free energy in terms of the order parameters, but the problem is that we are not able to produce the equations for these parameters in the case, when the "randomness" is not included in the Hamiltonian, but is connected with the integration domain. That is why we use a rather common trick in mathematics: substitute θ-functions by some smooth functions which depend on the small parameter ε and tend, as $\varepsilon \to 0$, to θ-function. We choose for these purposes $\mathrm{H}(x\varepsilon^{-1/2})$, where H is the erf-function (see definition (2.11)). But the particular form of these smoothing functions is not very important for us. The most important fact is, that they are not zero in any point and so, taking their logarithms, we can treat them as a part of our Hamiltonian.

The proof of Theorem 2 is based on the the application to the Gardner problem of the so-called cavity method, the rigorous version of which was proposed in [P-S] and developed in [S1], [P-S-T1], [P-S-T2]. But in the previous papers ([P-S],[P-S-T1], [P-S-T2]) we assumed the factorisation of the correlation functions in the thermodynamic limit and on the basis of this fact derived the replica symmetry equation for the order parameters (to be more precise, we assumed that the order parameter possesses the self-averaging property and obtained from this fact the factorisation of the correlation function). Here, due to Theorem 1, we can prove the asymptotical factorisation property, which allows us to finish completely the study of the Gardner model.

Our last step is the limiting transition $\varepsilon \to 0$, i.e. the proof that the product of αN θ-functions in (1.5) can be replaced by the product of $\mathrm{H}(\frac{x}{\sqrt{\varepsilon}})$ with the small difference, when ε is small enough. Despite our expectations, it is the most difficult step from the technical point of view. It is rather simple to prove, that the expression (1.7) is an upper bound or $\log \Theta_N, p(k)$. But the estimate from below is much more complicated. The problem is that to estimate the difference between the free energies corresponding to two Hamiltonians we, as a rule, need to have them defined in the common configuration space, or, at least, we need to know some a priori bounds for some Gibbs averages. In the case of the Gardner problem we do not possess this information. This leads to rather serious (from our point of view) technical problems (see the proof of Theorem 3 and Lemma 4).

The paper is organised as follows. The main definitions and results are formulated in Sec.2. The proof of these results are given in Sec.3. The auxiliary results (lemmas and propositions) which we need for the proof are formulated in the text of Sec.3 and their proofs are given in Sec.4.

2. Main Results

As it was mentioned above, we start from the abstract statement, which allows us to prove the factorisation of all correlation functions for some class of models.

Let $\{\Phi_N(\boldsymbol{J})\}_{N=1}^{\infty}$ $(\boldsymbol{J} \in \mathbf{R}^N)$ be a system of convex functions which possess the third derivatives, bounded in any compact. Consider also a system of convex domains $\{\Gamma_N\}_{N=1}^{\infty}$ $(\Gamma_N \subset \mathbf{R}^N)$ whose boundaries consist of a finite number (may be depending on N) of smooth pieces. We remark here, that for the Gardner problem we need to study Γ_N which is the intersection of αN half-spaces but in Theorem 1 (see below) we consider a more general sequence of convex sets. Define the Gibbs measure and the free energy, corresponding to $\Phi_N(\boldsymbol{J})$ in Γ_N:

$$\begin{gathered}\langle \ldots \rangle_{\Phi_N} \equiv \Sigma_N^{-1} \int_{\Gamma_N} d\boldsymbol{J} (\ldots) \exp\{-\Phi_N(\boldsymbol{J})\}, \\ \Sigma_N(\Phi_N) \equiv \int_{\Gamma_N} d\boldsymbol{J} \exp\{-\Phi_N(\boldsymbol{J})\}, \quad f_N(\Phi_N) \equiv \tfrac{1}{N} \log \Sigma_N(\Phi_N).\end{gathered} \tag{2.1}$$

Denote

$$\begin{gathered}\tilde{\Omega}_N(U) \equiv \{\boldsymbol{J} : \Phi_N(\boldsymbol{J}) \le NU\}, \quad \Omega_N(U) \equiv \tilde{\Omega}_N(U) \cap \Gamma_N, \\ \mathcal{D}_N(U) \equiv \tilde{\mathcal{D}}_N(U) \cap \Gamma_N,\end{gathered} \tag{2.2}$$

where $\tilde{\mathcal{D}}_N(U)$ is the boundary of $\tilde{\Omega}_N(U)$. Then define

$$f_N^*(U) = \frac{1}{N} \log \int_{J \in \mathcal{D}_N(U)} d\boldsymbol{J} e^{-NU}.$$

Theorem 1. *Let the functions $\Phi_N(\boldsymbol{J})$ satisfy the conditions:*

$$\frac{d^2}{dt^2} \Phi_N(\boldsymbol{J} + t\mathbf{e})|_{t=0} \ge C_0 > 0, \tag{2.3}$$

with any direction $\mathbf{e} \in \mathbf{R}^N$, $|\mathbf{e}| = 1$ *and uniformly in any set* $|\boldsymbol{J}| \le N^{1/2} R_1$,

$$\Phi_N(\boldsymbol{J}) \ge C_1(\boldsymbol{J}, \boldsymbol{J}), \quad as \ (\boldsymbol{J}, \boldsymbol{J}) > NR^2, \tag{2.4}$$

and for any $U > U_{min} \equiv \min_{J \in \Gamma_N} N^{-1} \Phi_N(\boldsymbol{J}) \equiv N^{-1} \Phi_N(\boldsymbol{J}^*)$

$$|\nabla \Phi_N(\boldsymbol{J})| \le N^{1/2} C_2(U), \quad as \quad \boldsymbol{J} \in \tilde{\Omega}_N(U) \tag{2.5}$$

with some positive N*-independent* $C_0, C_1, C_2(U)$ *and* $C_2(U)$ *continuous in* U.

Assume also, that there exists some finite N*-independent* C_3*, such that*

$$f_N(\Phi_N) \ge -C_3. \tag{2.6}$$

Then

$$|f_N(\Phi_N) - f_N^*(U_*)| \le O(\frac{\log N}{N}), \quad \left(U_* \equiv \frac{1}{N} \langle \Phi_N \rangle_{\Phi_N}\right). \tag{2.7}$$

Moreover, for any $\mathbf{e} \in \mathbf{R}^N$ $(|\mathbf{e}| = 1)$ *and any natural* p

$$\langle (\dot{\boldsymbol{J}}, \mathbf{e})^p \rangle_{\Phi_N} \le C(p) \quad (\dot{J}_i \equiv J_i - \langle J_i \rangle_{\Phi_N}) \tag{2.8}$$

with some positive N*-independent* $C(p)$.

Let us remark that the main conditions here are, of course, the condition that the domain Γ_N and the Hamiltonian Φ_N are convex (2.3). Condition (2.4) and (2.5) are not very restrictive, because they are fulfilled for the most part of Hamiltonians. The bound (2.6) in fact is the condition on the domain Γ_N. This condition prevents Γ_N to be too small. In the application to the Gardner problem the existence of such a bound is very important, because in this case we should study just the question of the measure of Γ_N, which is the intersection of αN random half-spaces with the sphere of radius $N^{1/2}$. But from the technical point of view for us it is more convenient to check the existence of the bound from below for the free energy, than for the volume of the configuration space (see the proof of Theorem 3 below).

Theorem 1 has two rather important for us corollaries.

Corollary 1. *Under conditions (2.3)- (2.6) for any* $U > U_{min}$

$$f_N^*(U) = \min_{z>0} \{f_N(z\Phi_N) + zU\} + O(\frac{\log N}{N}). \tag{2.9}$$

This corollary is a simple generalisation of the so called spherical model which becomes rather popular in the resent time (see, e.g. the review paper [K-K-P-S] and references therein). It allows us to substitute the integration over the level surface of the function Φ_N by the integration over the whole space, i.e. to substitute the "hard condition" $\Phi_N = UN$ by the "soft one" $\langle \Phi_N \rangle_{\Phi_N} = UN$. It is a common trick which often is very useful in statistical mechanics.

The second corollary gives the most important and convenient form of the general property (2.8):

Corollary 2. *Relations (2.8) imply that uniformly in* N

$$\frac{1}{N^2} \sum \langle \dot{J}_i \dot{J}_j \rangle_{\Phi_N}^2 \le \frac{C}{N}.$$

To found the free energy of the model (1.5) and to derive the replica symmetry equations for the order parameters we introduce the "regularised" Hamiltonian, depending on the small parameter $\varepsilon > 0$

$$\mathcal{H}_{N,p}(\boldsymbol{J}, k, h, z, \varepsilon) \equiv -\sum_{\mu=1}^{p} \log \mathrm{H}\left(\frac{k - (\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right) + h(\boldsymbol{h}, \boldsymbol{J}) + \frac{z}{2}(\boldsymbol{J}, \boldsymbol{J}), \tag{2.10}$$

where the function $\mathrm{H}(x)$ is defined as

$$\mathrm{H}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt \tag{2.11}$$

and $\boldsymbol{h} = (h_1, ..., h_N)$ is an external random field with independent Gaussian h_i with zero mean and variance 1, which we need from the technical reasons.

The partition function for this Hamiltonian is

$$Z_{N,p}(k, h, z, \varepsilon) = \sigma_N^{-1} \int d\boldsymbol{J} \exp\{-\mathcal{H}_\varepsilon(\boldsymbol{J}, h, z, \varepsilon)\}. \tag{2.12}$$

We denote also by $\langle \ldots \rangle$ the corresponding Gibbs averaging and

$$f_{N,p}(k, h, z, \varepsilon) \equiv \frac{1}{N} \log Z_{N,p}(k, h, z, \varepsilon). \tag{2.13}$$

Theorem 2. *For any $\alpha, k \geq 0$ and $z > 0$ the functions $f_{N,p}(k, h, z, \varepsilon)$ are self-averaging in the limit $N, p \to \infty$, $\alpha_N \equiv \frac{p}{N} \to \alpha$:*

$$E\left\{(f_{N,p}(k, h, z, \varepsilon) - E\{f_{N,p}(k, h, z, \varepsilon)\})^2\right\} \to 0 \tag{2.14}$$

and, if ε is small enough, $\alpha < 2$ and $z \leq \varepsilon^{-1/3}$, then there exists

$$\begin{aligned} \lim_{N,p\to\infty, \alpha_N\to\alpha} E\{f_{N,p}(k, h, z, \varepsilon)\} &= F(\alpha, k, h, z, \varepsilon), \\ F(\alpha, k, h, z, \varepsilon) \equiv \max_{R>0} \min_{0\leq q\leq R} &\left[\alpha E\left\{\log \mathrm{H}\left(\frac{u\sqrt{q} + k}{\sqrt{\varepsilon + R - q}}\right)\right\}\right. \\ &\left. + \frac{1}{2}\frac{q}{R-q} + \frac{1}{2}\log(R-q) - \frac{z}{2}R + \frac{h^2}{2}(R-q)\right], \end{aligned} \tag{2.15}$$

where u is a Gaussian random variable with zero mean and variance 1.

Let us note that the bound $\alpha < 2$ is not important for us, because for any $\alpha > \alpha_c(k)$ ($\alpha_c(k)$ is defined by (1.6) and $\alpha_c(k) < 2$ for any k) the free energy of the problem (1.5) tends to $-\infty$, as $N \to \infty$ (see Theorem 3 for the exact statement). The bound $z < \varepsilon^{-1/3}$ also is not a restriction for us. We could need to consider $z > \varepsilon^{-1/3}$ only if, applying (2.9) to the Hamiltonian (2.10), we obtain that the point of minimum $z_{min}(\varepsilon)$ in (2.9) does not satisfy this bound. But it is shown in Theorem 3, that for any $\alpha < \alpha_c(k)$ $z_{min}(\varepsilon) < \overline{z}$ with some finite $\overline{z}$ depending only on k and α.

We start the analysis of $\Theta_{N,p}(k)$, defined in (1.5), from the following remark.

Remark 1. Let us note that $\Theta_{N,p}(k)$ can be zero with nonzero probability (e.g., if for some $\mu \neq \nu$ $\boldsymbol{\xi}^{(\mu)} = -\boldsymbol{\xi}^{(\nu)}$). Therefore we cannot, as usually, just take $\log \Theta_{N,p}(k)$. To avoid this difficulty, we take some large enough M and replace below the log- function by the function $\log_{(MN)}$, defined as

$$\log_{(MN)} X = \log \max \left\{X,\ e^{-MN}\right\}. \tag{2.16}$$

Theorem 3. *For any $\alpha \leq \alpha_c(k)$ $N^{-1} \log_{(MN)} \Theta_{N,p}(k)$ is self-averaging in the limit $N, p \to \infty$, $p/N \to \alpha$*

$$E\left\{\left(N^{-1} \log_{(MN)} \Theta_{N,p}(k) - E\{N^{-1} \log_{(MN)} \Theta_{N,p}(k)\}\right)^2\right\} \to 0$$

and for M large enough there exists

$$\lim_{N,p\to\infty, p/N\to\alpha} E\{N^{-1} \log_{(MN)} \Theta_{N,p}(k)\} = \mathcal{F}(\alpha, k), \tag{2.17}$$

where $\mathcal{F}(\alpha, k)$ is defined by (1.7).

For $\alpha > \alpha_c(k)$ $E\{N^{-1} \log_{(MN)} \Theta_{N,p}(k)\} \to -\infty$, as $N \to \infty$ and then $M \to \infty$.

We would like to mention here that the self-averaging of $N^{-1} \log \Theta_{N,p}(k)$ was proven in ([T4]), but our proof of this fact is necessary for the proof of (2.17).

3. Proof of the Main Results

Proof of Theorem 1 For any $U > 0$ consider the set $\Omega_N(U)$ defined in (2.2) Since $\Phi_N(\boldsymbol{J})$ is a convex function, the set $\Omega_N(U)$ is also convex and $\Omega_N(U) \subset \Omega_N(U')$, if $U < U'$. Let

$$\begin{gathered} V_N(U) \equiv \text{mes}(\Omega_N(U)), \quad S_N(U) \equiv \text{mes}(\mathcal{D}_N(U)), \\ F_N(U) \equiv \textstyle\int_{\boldsymbol{J}\in\mathcal{D}_N(U)} |\nabla \Phi_N(\boldsymbol{J})|^{-1} dS_J. \end{gathered} \tag{3.1}$$

Here and below the symbol mes(...) means the Lebesgue measure of the correspondent dimension.

Then it is easy to see that the partition function Σ_N can be represented in the form

$$\begin{aligned} \Sigma_N = \int_{U>U_{min}} e^{-NU} F_N(U) dU &= N^{-1} \int_{U>U_{min}} e^{-NU} \frac{d}{dU} V_N(U) dU \\ &= \int_{U>U_{min}} e^{-NU} V_N(U) dU. \end{aligned} \tag{3.2}$$

Here we have used the relation $F_N(U) = N^{-1} \frac{d}{dU} V_N(U)$ and the integration by parts.

Besides, for a chosen direction $\mathbf{e} \in \mathbf{R}^N$ ($|\mathbf{e}| = \mathbf{1}$), and any real c consider the hyper-plane

$$\mathcal{A}(c, \mathbf{e}) = \left\{\boldsymbol{J} \in \mathbf{R}^N : (\boldsymbol{J}, \mathbf{e}) = N^{1/2} c\right\}$$

and denote

$$\begin{aligned} \Omega_N(U,c) &\equiv \Omega_N(U) \cap \mathcal{A}(c,\mathbf{e}), \quad V_N(U,c) \equiv \mathrm{mes}(\Omega_N(U,c)), \\ \mathcal{D}_N(U,c) &\equiv \mathcal{D}_N(U) \cap \mathcal{A}(c,\mathbf{e}), \quad F_N(U,c) \equiv \int_{\boldsymbol{J}\in\mathcal{D}_N(U,c)} |\nabla \Phi_N(\boldsymbol{J})|^{-1} dS_J. \end{aligned} \tag{3.3}$$

Then, since $F_N(U,c) = N^{-1}\frac{\partial}{\partial U} V_N(U,c)$, we obtain

$$\begin{aligned} \Sigma_N = \int dcdU e^{-NU} F_N(U,c) = \int dcdU e^{-NU} V_N(U,c), \\ \langle (\boldsymbol{J},\mathbf{e})^p \rangle_{\Phi_N} = \frac{N^{p/2} \int dcdU c^p e^{-NU} V_N(U,c)}{\int dcdU e^{-NU} V_N(U,c)}. \end{aligned} \tag{3.4}$$

Denote

$$s_N(U) \equiv \frac{1}{N} \log V_N(U), \quad s_N(U,c) \equiv \frac{1}{N} \log V_N(U,c). \tag{3.5}$$

Then relations (3.2), (3.4) give us

$$\begin{aligned} \Sigma_N = N \int \exp\{N(s_N(U) - U)\} dU, \\ \left\langle (\dot{\boldsymbol{J}},\mathbf{e})^p \right\rangle_{\Phi_N} = N^{p/2} \left\langle (c - \langle c \rangle_{(U,c)})^p \right\rangle_{(U,c)}, \end{aligned} \tag{3.6}$$

where

$$\langle ... \rangle_{(U,c)} \equiv \frac{\int dUdc (...) \exp\{N(s_N(U,c) - U)\}}{\int dUdc \exp\{N(s_N(U,c) - U)\}}. \tag{3.7}$$

Then (2.7) and (2.8) can be obtained by the standard Laplace method, if we prove that $s_N(U)$ and $s_N(U,c)$ are concave functions and they are strictly concave in the neighbourhood of the points of maximum of the functions $(s_N(U) - U)$ and $(s_N(U,c) - U)$. To prove this we apply the theorem of Brunn-Minkowski from classical geometry (see e.g. [Ha]) to the functions $s_N(U)$ and $s_N(U,c)$. To formulate this theorem we need some extra definitions.

Definition 1. *Consider two bounded sets in* $\mathcal{A}, \mathcal{B} \subset \mathbf{R}^N$. *For any positive* α *and* β

$$\alpha\mathcal{A} \times \beta\mathcal{B} \equiv \{\mathbf{s} : \mathbf{s} = \alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}\}.$$

$\alpha\mathcal{A} \times \beta\mathcal{B}$ *is the Minkowski sum of* $\alpha\mathcal{A}$ *and* $\beta\mathcal{B}$.

Definition 2. *The one-parameter family of bounded sets* $\{\mathcal{A}(t)\}_{t_1^* \le t \le t_2^*}$ *is a convex one- parameter family, if for any positive* $\alpha < 1$ *and* $t_{1,2} \in [t_1^*, t_2^*]$ *they satisfy the condition*

$$\mathcal{A}(\alpha t_1 + (1-\alpha)t_2) \supset \alpha\mathcal{A}(t_1) \times (1-\alpha)\mathcal{A}(t_2).$$

Theorem of Brunn-Minkowski *Let* $\{\mathcal{A}(t)\}_{t_1^* \le t \le t_2^*}$ *be some convex one-parameter family. Consider* $R(t) \equiv (\mathrm{mes}\mathcal{A}(t))^{1/N}$. *Then* $\frac{d^2R(t)}{dt^2} \le 0$ *and* $\frac{d^2R(t)}{dt^2} \equiv 0$ *for* $t \in [t_1', t_2']$ *if and only if all the sets* $\mathcal{A}(t)$ *for* $t \in [t_1', t_2']$ *are homothetic to each other.*

For the proof of this theorem see, e.g., [Ha].

To use this theorem for the proof of (2.7) let us observe that the family $\{\mathit{\Omega}_N(U))\}_{U>U_{min}}$ is a convex one-parameter family and then, according to the Brunn-Minkowski theorem, the function $R(U)=(V_N(U))^{1/N}$ is a concave function. Thus, we get that $s_N(U)$ is a concave function:

$$\frac{d^2}{dU^2}s_N(U)=\frac{d^2}{dU^2}\log R(U)=\frac{R''(U)}{R(U)}-\left(\frac{R'(U)}{R(U)}\right)^2\le-\left(\frac{R'(U)}{R(U)}\right)^2.$$

But $\frac{R'(U)}{R(U)}=\frac{d}{dU}s_N(U)>1$ for $U<U^*$, and even if $\frac{d}{dU}s_N(U)=0$ for $U>U^*$, we obtain that $\frac{d}{dU}(s_N(U)-U)=-1$. Thus, using the standard Laplace method, we get

$$\begin{gathered}f_N(\mathit{\Phi}_N)=s_N(U^*)-U^*+O(\frac{\log N}{N})=\frac{1}{N}\log V_N(U^*)-U^*+O(\frac{\log N}{N}),\\ U_*\equiv\tfrac{1}{N}\langle\mathit{\Phi}_N\rangle_{\mathit{\Phi}_N}=U^*+o(1).\end{gathered}\tag{3.8}$$

Using condition (2.5), and taking $\boldsymbol{J}^*$, which is the minimum point of $\mathit{\Phi}_N(\boldsymbol{J})$, we get

$$\begin{aligned}V_N(U^*)&\ge N^{-1}\int_{\boldsymbol{J}\in\mathcal{D}_N(U^*)}|(\boldsymbol{J}-\boldsymbol{J}^*,\nabla\mathit{\Phi}_N(\boldsymbol{J}))||\nabla\mathit{\Phi}_N(\boldsymbol{J})|^{-1}dS_{\boldsymbol{J}}\\ &\ge S_N(U^*)\frac{U^*-U_{min}}{\max_{\boldsymbol{J}\in\mathcal{D}_N(U^*)}|\nabla\mathit{\Phi}_N(\boldsymbol{J})|}=N^{-1/2}S_N(U^*)C(U^*).\end{aligned}\tag{3.9}$$

On the other hand, for any $U<U^*$

$$\begin{aligned}\frac{S_N(U)}{N^{1/2}V_N(U)}&\ge\min_{\boldsymbol{J}\in\mathcal{D}_N(U)}|\nabla\mathit{\Phi}_N(\boldsymbol{J})|\frac{F_N(U)}{N^{1/2}V_N(U)}\\ &\ge N^{1/2}\min_{\boldsymbol{J}\in\mathcal{D}_N(U)}\frac{U-U_{min}}{|\boldsymbol{J}-\boldsymbol{J}^*|}\frac{d}{dU}s_N(U)\ge\tilde{C}\frac{d}{dU}s_N(U)>\tilde{C}.\end{aligned}\tag{3.10}$$

Here we have used (3.3) and (2.4). Thus the same inequality is valid also for $U=U^*$. Inequalities (3.10) and (3.9) imply that

$$\frac{1}{N}\log S_N(U^*)=\frac{1}{N}\log V_N(U^*)+O(\frac{\log N}{N}).$$

Combining this relation with (3.8) we get (2.7).

Let us observe also that for any (U_0,c_0) and (δ_U,δ_c) the family $\{\mathit{\Omega}_N(U_0+t\delta_U,c_0+t\delta_c\}_{t\in[0,1]}$ is a convex one-parameter family and then, according to the Brunn-Minkowski theorem the function $R_N(t)\equiv V^{1/N}(U_0+t\delta_U,c_0+t\delta_c)$ is concave. But since in our consideration $N\to\infty$, to obtain that this function is strictly concave in some neighbourhood of the point (U^*,c^*) of maximum of $s_N(U,c)-U$, we shall use some corollary from the theorem of Brunn- Minkowski:

Proposition 1. *Consider the convex set* $\mathcal{M}\subset\mathbf{R^N}$ *whose boundary consists of a finite number of smooth pieces. Let the convex one-parameter family* $\{\mathcal{A}(t)\}_{t_1^*\le t\le t_2^*}$ *be given by the intersections of* $\mathcal{M}$ *with the parallel the hyper-planes* $\mathcal{B}(t)\equiv\{\boldsymbol{J}:(\boldsymbol{J},\mathbf{e})=tN^{1/2}\}$. *Suppose that there is some smooth piece* $\mathcal{D}$ *of the boundary* *of* $\mathcal{M}$*, such that for any* $\boldsymbol{J} \in \mathcal{D}$ *the minimal normal curvature satisfies the inequality* $N^{1/2}\kappa_{min}(\boldsymbol{J}) > K_0$*, and the Lebesgue measure* $S(t)$ *of the intersection* $\mathcal{D} \cap \mathcal{A}(t)$ *satisfies the bound*

$$S(t) \geq N^{1/2} V(t) C(t), \tag{3.11}$$

where $V(t)$ *is the volume of* $\mathcal{A}(t)$*. Then* $\frac{d^2}{dt^2} V^{1/N}(t) \leq -K_0 C(t) V^{1/N}(t)$*.*

One can see that, if we consider the sets $\mathcal{M}, \mathcal{M}', \mathcal{A}, \mathcal{B}(t) \subset \mathbf{R}^{N+1}$

$$\begin{gathered}\mathcal{M} \equiv \mathcal{M}' \cap \mathcal{A}, \;\; \mathcal{M}' \equiv \{(\boldsymbol{J}, U) : \, NU \geq \varPhi_N(\boldsymbol{J}), \;\; \boldsymbol{J} \in \varGamma_N\}, \\ \mathcal{A} \equiv \{(\boldsymbol{J}, U) : \, \delta_U((\boldsymbol{J}, \mathbf{e}) - N^{1/2} c_0) - N^{1/2} \delta_c (U - U_0) = 0\}, \\ \mathcal{B}(t) \equiv \{(\boldsymbol{J}, U) : \, \delta_c((\boldsymbol{J}, \mathbf{e}) - N^{1/2} c_0) + N^{1/2} \delta_U (U - U_0) = N^{1/2} t\},\end{gathered}$$

then $\varOmega_N(U_0 + t\delta_U, c_0 + t\delta_c) = \mathcal{M} \cap \mathcal{B}(t)$ (without loss of generality we assume that $\delta_c^2 + \delta_U^2 = 1$). Conditions (2.3) and (2.5) guarantee that the minimal normal curvature of $\mathcal{D}'_N(U) \equiv \{(\boldsymbol{J}, \varPhi_N(\boldsymbol{J})), \, \boldsymbol{J} \in \varGamma_N\}$ satisfies the inequality $N^{1/2}\kappa_{min}(\boldsymbol{J}) > \tilde{K}$ for $\boldsymbol{J} \in \mathcal{D}_N(U)$, if $|U - U^*| < \varepsilon$ with small enough but N-independent ε. Besides, similarly to (3.10)

$$\frac{\mathrm{mes}\mathcal{D}_N(U, c)}{N^{1/2} V_N(U, c)} \geq C_3 \frac{d}{dU} s_N(U, c).$$

Thus we get that

$$\frac{d}{dU} s_N(U, c) \geq \frac{1}{2} \Rightarrow \frac{d^2}{dt^2} s_N(U + t\sin\varphi, c + t\cos\varphi)\Big|_{t=0} \leq -C_4. \tag{3.12}$$

Remark 2. If $\varGamma_N = \mathbf{R}^N$, then conditions of Theorem 1 guarantee that $\frac{d}{dU} s_N(U, c) \geq$ *const*, when $(U, c) \sim (U^*, c^*)$ and so Proposition 1 and (3.10) give us that

$$s_N(U, c) - U - (s_N(U^*, c^*) - U^*) \leq -\frac{\tilde{C}_0}{2}((c - c^*)^2 + (U - U^*)^2). \tag{3.13}$$

which implies immediately (2.8). But in the general case, the proof is more complicated.

Let us introduce the new variables $\rho \equiv ((U - U^*)^2 + (c - c^*)^2)^{1/2}$, $\varphi \equiv \arcsin \frac{U - U^*}{((U - U^*)^2 + (c - c^*)^2)^{1/2}}$ and let $\tilde{\phi}_N(\rho, \varphi) \equiv \phi_N(U, c) \equiv s_N(U^* + U, c^* + c) - U - s_N(U^*, c^*) + U^*$. We shall prove now that

$$\tilde{\phi}_N(N^{-1/2}, \varphi) \leq -\frac{K}{N}, \tag{3.14}$$

where K does not depend on φ, N. Consider the set

$$\varLambda = \left\{(U, c) : \frac{d}{dU} s_N(U, c) < \frac{1}{2}\right\}.$$

One can see easily, that if $(U', c') \in \varLambda$, then $(U, c') \in \varLambda$ for any $U > U'$ and $\frac{d}{dU}\phi_N(U, c') < -\frac{1}{2}$. That is why it is clear, that $(U^*, c^*) \notin \varLambda$ (but it can belong to the boundary $\partial\varLambda$). Denote

$$\varphi^* \equiv \inf_{\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left\{\overline{r}(N^{-1/2}\sin\varphi, N^{-1/2}\cos\varphi) \cap \varLambda \neq \emptyset\right\},$$

where $\overline{r}(U,c)$ is the set of all points of the form (U^*+tU, c^*+tc), $t\in[0,1]$). Then for any $\varphi<\varphi^*$ we can apply (3.12) to obtain that

$$\tilde\phi_N(N^{-1/2},\varphi)\le -\frac{C_4}{2N}. \tag{3.15}$$

Assume that $-\frac{\pi}{4}\le\varphi^*\le\frac{\pi}{4}$. Let us remark that, using (2.5), similarly to (3.9) one can obtain that for all (U,c): $|U-U^*|\le N^{-1/2}$ and $|c-c^*|\le N^{-1/2}$

$$\frac{d}{dU}s_N(U,c)\le \min|\nabla\varPhi_N(\boldsymbol{J})|^{-1}\frac{S_N(U,c)}{V_N(U,c)}\le C_5. \tag{3.16}$$

Choose $d\equiv\frac{C_4}{4C_5}$. Then for all $\varphi^*\le\varphi\le\varphi_d\equiv\arctan(\tan\varphi^*+dN^{-1/2})$, using (3.15) and (3.16), we have got

$$\begin{aligned}&\tilde\phi_N(N^{-1/2},\varphi)=\phi_N(N^{-1/2}\sin\varphi,N^{-1/2}\cos\varphi)\\ &\le\phi_N(N^{-1/2}\sin\varphi-\frac{d}{N},N^{-1/2}\cos\varphi)+\frac{C_5d}{N}\le-\frac{C_4}{4N}+O(N^{-3/2}).\end{aligned} \tag{3.17}$$

For $\frac{\pi}{4}\ge\varphi>\varphi_d$, according to the definition of φ^* and φ_d, there exists $\rho_1<1$ such that

$$\begin{aligned}&(N^{-1/2}\rho_1\sin\varphi-\tfrac{d\rho_1}{N},\ N^{-1/2}\rho_1\cos\varphi)\in\varLambda\\ &\Rightarrow(N^{-1/2}\rho_1\sin\varphi-\tfrac{td\rho_1}{N},\ N^{-1/2}\rho_1\cos\varphi)\in\varLambda\ \ (t\in[0,1]).\end{aligned}$$

Therefore, using that $\tilde\phi_N(\rho,\varphi)$ is a concave function of ρ, we get

$$\begin{aligned}&\tilde\phi_N(N^{-1/2},\varphi)\le\rho_1^{-1}\tilde\phi_N(N^{-1/2}\rho_1,\varphi)\\ &=\rho_1^{-1}\phi_N(N^{-1/2}\rho_1\sin\varphi,N^{-1/2}\rho_1\cos\varphi)\\ &\le\rho_1^{-1}\phi_N(N^{-1/2}\rho_1\sin\varphi-\frac{d\rho_1}{N},N^{-1/2}\rho_1\cos\varphi)-\frac{d}{2N}\le-\frac{d}{2N}.\end{aligned} \tag{3.18}$$

And finally, if $|\varphi|>\frac{\pi}{4}$, denote

$$\mathcal{L}_\phi\equiv\overline{r}(N^{-1/2}\sin\varphi,N^{-1/2}\cos\varphi)\cap\varLambda,\ l_\phi=N^{1/2}\ \mathrm{mes}\{\mathcal{L}_\phi\}.$$

Then, using that for $(U,c)\in\mathcal{L}_\phi$

$$\frac{d}{d\rho}\tilde\phi_N(N^{-1/2}\rho,\varphi)\le N^{-1/2}\cos\frac{\pi}{4}\frac{d}{dU}\phi_N(U,c)<-\frac{1}{2}N^{-1/2}\cos\frac{\pi}{4},$$

and for $(U,c)\notin\mathcal{L}_\phi$ we can apply (3.12), we have got

$$\tilde\phi_N(N^{-1/2},\varphi)\le-\frac{(1-l_\phi)^2C_4}{2N}-\frac{l_\phi}{2(2N)^{1/2}}\le-\frac{K}{N}. \tag{3.19}$$

Inequalities (3.15)-(3.19) prove (3.14) for $|\varphi|<\frac{\pi}{2}$. For the rest of φ the proof is the same.

Now let us derive (2.8) (for $p=2$) from (3.14). Choose $\rho^*=\frac{4}{K}$ and remark, that since $\tilde\phi_N(\rho,\varphi)$ is a concave function of ρ, we have got that for $\rho>N^{-1/2}\rho^*$

$$\frac{1}{2}\frac{d}{d\rho}\tilde\phi_N(\rho,\varphi)\Big|_{\rho=N^{-1/2}\rho^*}\le\frac{d}{d\rho}[\tilde\phi_N(\rho,\varphi)+\frac{2}{N}\log\rho]<-\frac{K}{2N^{1/2}}.$$

Thus, using the Laplace method, one can obtain that

$$\frac{\int_{\rho>N^{-1/2}\rho^*} d\rho\,\rho^2 e^{N\tilde{\phi}_N(\rho,\varphi)}}{\int_{\rho>N^{-1/2}\rho^*} d\rho e^{N\tilde{\phi}_N(\rho,\varphi)}} \le \frac{(\rho^*)^2}{N}\frac{\frac{d}{d\rho}\tilde{\phi}_N(\rho,\varphi)}{\frac{d}{d\rho}[\tilde{\phi}_N(\rho,\varphi)+\frac{2}{N}\log\rho]}\Bigg|_{\rho=N^{-1/2}\rho^*} \le \frac{2(\rho^*)^2}{N}.$$

So, we have for any φ

$$\int d\rho\,\rho^2 e^{N\tilde{\phi}_N(\rho,\varphi)} \le \frac{(\rho^*)^2}{N}\int_{\rho<N^{-1/2}\rho^*} d\rho\, e^{N\tilde{\phi}_N(\rho,\varphi)}$$

$$+\frac{2(\rho^*)^2}{N}\int_{\rho>N^{-1/2}\rho^*} d\rho\, e^{N\tilde{\phi}_N(\rho,\varphi)} \le 2\frac{(\rho^*)^2}{N}\int d\rho e^{N\tilde{\phi}_N(\rho,\varphi)}.$$

This relation proves (2.8) for $p=2$, because of the inequalities

$$\langle (c-\langle c\rangle_{(U,c)})^2\rangle_{(U,c)} \le \langle (c-c^*)^2\rangle_{(U,c)} \le \frac{\int d\phi \int d\rho\,\rho^2 e^{N\tilde{\phi}_N(\rho,\varphi)}}{\int d\phi\int d\rho e^{N\tilde{\phi}_N(\rho,\varphi)}} \le \frac{2(\rho^*)^2}{N}.$$

For other values of p the proof of (2.8) is similar.

Proof of Theorem 2

For our consideration below it is convenient to introduce also the Hamiltonian

$$\overline{\mathcal{H}}_{N,p}(\boldsymbol{J},\overline{x},h,z,\varepsilon) \equiv \frac{1}{2\varepsilon}\sum_{\mu=1}^{p}(N^{-1/2}(\boldsymbol{\xi^{(\mu)}},\boldsymbol{J})-x^{(\mu)})^2+h(\boldsymbol{h},\boldsymbol{J})+\frac{z}{2}(\boldsymbol{J},\boldsymbol{J}). \quad (3.20)$$

Evidently

$$\mathcal{H}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon) = -\log\int_{x^{(\mu)}>k} d\overline{x}\exp\{\overline{\mathcal{H}}_{N,p}(\boldsymbol{J},\overline{x},h,z,\varepsilon)\} + \frac{p}{2}\log(2\pi\varepsilon)$$

and so $\langle\tilde{F}(\boldsymbol{J})\rangle = \langle\tilde{F}(\boldsymbol{J})\rangle_{\overline{\mathcal{H}}_{N,p}}$ for any $\tilde{F}(\boldsymbol{J})$. Therefore below we denote $\langle\ldots\rangle$ both averaging with respect to $\mathcal{H}_{N,p}$ and $\overline{\mathcal{H}}_{N,p}$.

Lemma 1. *Define the matrix* $X_N^{\mu,\nu} = \frac{1}{N}\sum_{i=1}^{N}\xi_i^{(\mu)}\xi_i^{(\nu)}$. *If the inequalities*

$$||X_N|| \le (\sqrt{\alpha}+2)^2, \quad \frac{1}{N}(\mathbf{h},\mathbf{h}) \le 2, \quad (3.21)$$

are fulfilled, then the Hamiltonian $\mathcal{H}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon)$ *satisfies conditions (2.3), (2.4), (2.5) and (2.6) of Theorem 1 and therefore*

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle\dot{J}_i\dot{J}_j\rangle\langle J_i\rangle\langle J_j\rangle \le \frac{C(z,\varepsilon)}{N}, \quad \frac{1}{N^2}\sum_{i,j=1}^{N}\langle\dot{J}_i\dot{J}_j\rangle^2 \le \frac{C(z,\varepsilon)}{N}, \quad (3.22)$$

where $\dot{J}_i \equiv J_i - \langle J_i\rangle$.

Moreover, choosing $\varepsilon_N \equiv N^{-1/2}\log N$ *we have got that there exist* N*-independent* C_1 *and* C_2*, such that*

$$\mathrm{Prob}\left\{\max_i\langle\theta(J_i-N^{1/2}\varepsilon_N)\rangle > e^{-C_1\log^2 N}\right\} \le e^{-C_2\log^2 N}. \quad (3.23)$$

Remark 3. According to the result of [S-T] and to a low of large numbers, P_N-the probability that inequalities (3.21) are fulfilled, is more than $1 - e^{-constN^{2/3}}$

Remark 4. Let us note that since the Hamiltonian (2.10) under conditions (3.21) satisfies (2.3), (2.4) and (2.6), we can choose R_0 large enough to have

$$\sigma_N^{-1} \int_{\Gamma_N} \theta(|\boldsymbol{J}| - N^{1/2} R_0) e^{-\mathcal{H}_{N,p}} d\boldsymbol{J} \le (R_0)^N e^{-NC_1 R_0^2} < e^{-NC_3 - N}$$
$$\Rightarrow \big\langle \theta(|\boldsymbol{J}| - N^{1/2} R_0) \big\rangle \le e^{-N},$$

so in all computations below we can use the inequality $|\boldsymbol{J}| \le N^{1/2} R_0$ with the error $O(e^{-N\,\mathrm{const}})$.

Remark 5. Let us note, that sometimes it is convenient to use (3.22) in the form

$$E\Big\{ \Big\langle \Big(N^{-1} \sum_i^N \dot{J}_i^{(1)} \dot{J}_i^{(2)} \Big)^2 \Big\rangle^{(1,2)} \Big\} \le \frac{C(z,\varepsilon)}{N},$$
$$E\Big\{ \Big\langle \Big(N^{-1} \sum_i^N \dot{J}_i \langle J_i \rangle \Big)^2 \Big\rangle \Big\} \le \frac{C(z,\varepsilon)}{N}.$$

Here and below we put an upper index to J_i to show that we take a few replicas of our Hamiltonians and the upper index indicate the replica number. We put also an upper index $\langle .. \rangle^{(1,2)}$ to stress that we consider the Gibbs measure for two replicas. The last relations means, in particularly, that

$$\frac{1}{N} \sum \dot{J}_i^{(1)} \dot{J}_i^{(2)} \to 0, \;\; \frac{1}{N} \sum \dot{J}_i \langle J_i \rangle \to 0, \;\; as \;\; N \to \infty \tag{3.24}$$

in the Gibbs measure and the probability.

We start the proof of Theorem 2 from the proof of the self-averaging property (2.14). of $f_{N,p}(h,z,\varepsilon)$. Using the idea, proposed in [P-S] (see also [S-T]), we write

$$f_{N,p}(h,z,\varepsilon) - E\{f_{N,p}(k,h,z,\varepsilon)\} = \frac{1}{N} \sum_{\mu=0}^{p} \varDelta_\mu,$$

where

$$\varDelta_\mu \equiv E_\mu \left\{ (\log Z_{N,p}(k,h,z,\varepsilon)) \right\} - E_{\mu+1} \left\{ (\log Z_{N,p}(h,z,\varepsilon)) \right\},$$

the symbol $E_\mu\{..\}$ means the averaging with respect to random vectors $\boldsymbol{\xi}^{(1)}, ..., \boldsymbol{\xi}^{(\mu)}$ and
$E_0 \left\{ \log Z_{N,p}(k,h,z,\varepsilon) \right\} = \log Z_{N,p}(h,z,\varepsilon)$. Then, in the usual way,

$$E\left\{ \varDelta_\mu \varDelta_\nu \right\} = 0 \; (\mu \ne \nu),$$

and therefore

$$E\left\{ (f_{N,p}(h,z,\varepsilon) - E\{f_{N,p}(k,h,z,\varepsilon)\})^2 \right\} = \frac{1}{N^2} \sum_{\mu=0}^{p} E\{\varDelta_\mu^2\}. \tag{3.25}$$

But

$$\begin{aligned} E\{\Delta_{\mu-1}^2\} \le E\{(E_{\mu-1}\{(\log Z_{N,p}(k,h,z,\varepsilon))\} \\ E_{\mu-1}\left\{(\log Z_{N,p-1}^{(\mu)}(k,h,z,\varepsilon)))^2\right\} \le E\{(\Delta_\mu')^2\}, \end{aligned} \tag{3.26}$$

where

$$\Delta_\mu' \equiv \log Z_{N,p}(k,h,z,\varepsilon) - \log Z_{N,p-1}^{(\mu)}(k,h,z,\varepsilon),$$

with $Z_{N,p-1}^{(\mu)}(k,h,z,\varepsilon)$ being the partition function for the Hamiltonian (2.10), where in the r.h.s. we take the sum with respect to all upper indexes except μ. Denoting by $\langle ... \rangle_{p-1}^{(\mu)}$ the correspondent Gibbs averaging and integrating with respect to $\overline{x}$, we get:

$$\Delta_\mu' = \sqrt{\varepsilon} \log \left\langle \mathrm{H}\left(\frac{k - (\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right)\right\rangle_{p-1}^{(\mu)}. \tag{3.27}$$

But evidently

$$\begin{aligned} 0 &\ge \log \left\langle \mathrm{H}\left(\varepsilon^{-1/2}(k - (\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})N^{-1/2})\right)\right\rangle_{p-1}^{(\mu)} \\ &\ge \left\langle \log \mathrm{H}\left(\varepsilon^{-1/2}(k - (\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})N^{-1/2})\right)\right\rangle_{p-1}^{(\mu)} \\ &\ge -\,\mathrm{const}\, \left\langle (N\varepsilon)^{-1}(\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})^2\right\rangle_{p-1}^{(\mu)} + \mathrm{const} \end{aligned} \tag{3.28}$$

Thus,

$$E\{(\Delta_\mu')^2\} \le \mathrm{const}\, E\left\{\left\langle (N\varepsilon)^{-1}(\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})^2\right\rangle_{p-1}^{(\mu)} \left\langle (N\varepsilon)^{-1}(\boldsymbol{\xi}^{(\boldsymbol{\mu})}, \boldsymbol{J})^2\right\rangle_{p-1}^{(\mu)}\right\}.$$

But since $\langle ... \rangle_{p-1}^{(\mu)}$ does not depend on $\boldsymbol{\xi}^{(\boldsymbol{\mu})}$ we can average with respect to $\boldsymbol{\xi}^{(\boldsymbol{\mu})}$ inside $\langle ... \rangle_{p-1}^{(\mu)}$. Hence, we obtain

$$E\{(\Delta_\mu')^2\} \le \mathrm{const}\, \varepsilon^{-2} E\left\{\left\langle N^{-1}(\boldsymbol{J}, \boldsymbol{J})\right\rangle_{p-1}^{(\mu)} \left\langle N^{-1}(\boldsymbol{J}, \boldsymbol{J})\right\rangle_{p-1}^{(\mu)}\right\} \le \mathrm{const}\,. \tag{3.29}$$

Inequalities (3.25)-(3.28) prove (2.14).

Define the order parameters of our problem

$$R_{N,p} \equiv \frac{1}{N}\sum_{i=1}^N \langle J_i^2 \rangle, \quad q_{N,p} \equiv \frac{1}{N}\sum_{i=1}^N \langle J_i \rangle^2 \tag{3.30}$$

To prove the self-averaging properties of $R_{N,p}$ and $q_{N,p}$ we use the following general lemma:

Lemma 2. *Consider the sequence of convex random functions* $\{f_n(t)\}_{n=1}^\infty$ *(*$f_n''(t) \ge 0$*) in the interval* (a,b)*. If functions* f_n *are self-averaging (*$E\{(f_n(t) - E\{f_n(t)\})^2\} \to 0$*, as* $n \to \infty$ *uniformly in* t*) and bounded (*$|E\{f_n(t)\}| \le C$ *uniformly in* n*,* $t \in (a,b)$*), then for almost all* t

$$\lim_{n\to\infty} E\{[f_n'(t) - E\{f_n'(t)\}]^2\} = 0, \tag{3.31}$$

i.e. the derivatives $f_n'(t)$ *are also self-averaging ones for almost all* t.

In addition, if we consider another sequence of convex functions $\{g_n(t)\}_{n=1}^{\infty}$ ($g_n'' \geq 0$) *which are also self-averaging* ($E\{(g_n(t) - E\{g_n(t)\})^2\} \to 0$, *as* $n \to \infty$ *uniformly in* t), *and* $|E\{f_n(t)\} - E\{g_n(t)\}| \to 0$, *as* $n \to \infty$, *uniformly in* t, *then for all* t, *which satisfy (3.31)*

$$\lim_{n\to\infty} |E\{f_n'(t)\} - E\{g_n'(t)\}| = 0, \qquad \lim_{n\to\infty} E\{[g_n'(t) - E\{g'(t)\}]^2\} = 0. \tag{3.32}$$

For the proof of this lemma see [P-S-T2]. On the basis of Lemma 2, in Sec.4 we prove

Proposition 2. *Denote* $R_{N,p-1}$, $q_{N,p-1}$ *the analogs of* $R_{N,p}$, $q_{N,p}$ *(see definition (3.30)) for* $H_{N,p-1}$. *Then for any convergent subsequence* $E\{f_{N_m,p_m}(k,h,z,\varepsilon)\}$ *for almost all* z *and* h R_{N_m,p_m}, q_{N_m,p_m} *we have got*

$$\begin{gathered} E\{(R_{N_m,p_m} - \overline{R}_{N_m,p_m})^2\},\ E\{(q_{N_m,p_m} - \overline{q}_{N_m,p_m})^2\} \to 0, \\ |\overline{R}_{N_m,p_m} - \overline{R}_{N_m,p_m-1}|,\ |\overline{q}_{N_m,p_m} - \overline{q}_{N_m,p_m-1}| \to 0 \ \ as\ \ k \to \infty, \end{gathered} \tag{3.33}$$

where

$$\overline{R}_{N,p} = E\{R_{N,p}\}, \quad \overline{q}_{N,p} = E\{q_{N,p}\} \tag{3.34}$$

and

$$E\left\{\left\langle\left(N_m^{-1}\sum_{i=1}^{N_m} J_i^2 - \overline{R}_{N_m,p_m}\right)^2\right\rangle\right\} \to 0,\ as\ N_m \to \infty. \tag{3.35}$$

Our strategy now is to choose an arbitrary convergent subsequence $f_{N_m,p_m}(k,h,z,\varepsilon)$, by applying to it the above proposition, to show that its limit for all h, z coincides with the r.h.s. of (2.15). Then this will mean that there exists the limit $f_{N,p}(h,z,\varepsilon)$ as $N, p \to \infty$, $\frac{p}{N} \to \alpha$. But in order to simplify formulae below we shall omit the subindex m for N and p.

Now we formulate the main technical point of the proof of Theorem 2.

Lemma 3. *Consider* $H_{N,p-1}$ *and denote by* $\langle\ldots\rangle_{p-1}$ *the respective Gibbs averages. For any* $\varepsilon_1 > 0$ *and* $0 \leq k_1 \leq 2k$ *define*

$$\phi_N(\varepsilon_1, k_1) \equiv \varepsilon_1^{1/2}\left\langle \mathrm{H}\left(\frac{k_1 - N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})}, \langle \boldsymbol{J}\rangle_{p-1})}{\sqrt{\varepsilon_1}}\right)\right\rangle_{p-1}, \tag{3.36}$$

$$\phi_{0,N}(\varepsilon_1, k_1) \equiv \varepsilon_1^{1/2}\mathrm{H}\left(\frac{k_1 - N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})}, \langle \boldsymbol{J}\rangle_{p-1})}{\sqrt{U_{N,p-1}(\varepsilon_1)}}\right), \tag{3.37}$$

where $U_{N,p-1}(\varepsilon_1) \equiv \overline{R}_{N,p-1} - \overline{q}_{N,p-1} + \varepsilon_1$. *Then,*

$$\begin{gathered} E\left\{(\phi_N(\varepsilon_1,k_1) - \phi_{0,N}(\varepsilon_1,k_1))^2\right\} \to 0, \\ E\left\{(\log\phi_N(\varepsilon_1,k_1) - \log\phi_{0,N}(\varepsilon_1,k_1))^2\right\} \to 0, \\ E\left\{\left(\tfrac{d}{d\varepsilon_1}\log\phi_N(\varepsilon_1,k_1) - \tfrac{d}{d\varepsilon_1}\log\phi_{0,N}(\varepsilon_1,k_1)\right)^2\right\} \to 0, \\ E\left\{\left(\tfrac{d}{dk_1}\log\phi_N(\varepsilon_1,k_1) - \tfrac{d}{dk_1}\log\phi_{0,N}(\varepsilon_1,k_1)\right)^2\right\} \to 0, \end{gathered} \tag{3.38}$$

and $N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})}, \langle \boldsymbol{J}\rangle_{p-1})$ *converges in distribution to* $\sqrt{\overline{q}_{N,p}}u$*, where* u *is a Gaussian random variable with zero mean and variance 1.*

Besides, if we denote

$$\begin{aligned} &t^{(\mu)} \equiv N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\mu})}, J) - x^{\mu}, \quad \dot{t}^{(\mu)} \equiv t^{(\mu)} - \langle t^{(\mu)}\rangle \\ &\tilde{U}_N \equiv \frac{1}{\varepsilon^2 N}\sum_{\mu=1}^{p}\langle (t^{(\mu)})^2\rangle, \quad \tilde{q}_N \equiv \frac{1}{\varepsilon^2 N}\sum_{\mu=1}^{p}\langle t^{(\mu)}\rangle^2, \end{aligned} \tag{3.39}$$

then $\tilde{U}_N$ *and* $\tilde{q}_N$ *are self-averaging quantities and for* $\mu \neq \nu$

$$\begin{aligned} E\left\{\langle \dot{t}^{(\mu)}\dot{t}^{(\nu)}\rangle^2\right\} \to 0,\ E\left\{\langle ((t^{(\mu)})^2 - \langle (t^{(\mu)})^2\rangle)((t^{(\mu)})^2 - \langle (t^{(\mu)})^2\rangle)\rangle^2\right\} \to 0, \\ E\left\{\langle (t^{(\mu)})^4\rangle\right\} \leq \ \textit{const},\ E\left\{\langle (t^{(\mu)})^4 (t^{(\nu)})^4\rangle\right\} \leq \text{const}. \end{aligned} \tag{3.40}$$

Now we are ready to derive the equations for $\overline{q}_{N,p}$ and $\overline{R}_{N,p}$. From the symmetry of the Hamiltonian (3.20) it is evident that $\overline{q}_{N,p} = E\{\langle J_1\rangle^2\}$ and $\overline{R}_{N,p} = E\{\langle J_1^2\rangle\}$. The integration with respect J_1 is Gaussian. So, if we denote

$$t_1^{(\mu)} \equiv t^{(\mu)} - N^{-1/2}\xi_1^{(\mu)}J_1,$$

we get

$$\langle J_1\rangle = -(z+\alpha_N/\varepsilon)^{-1}\left(\frac{1}{\varepsilon N^{1/2}}\sum_{\mu=1}^{p}\xi_1^{(\mu)}\langle t_1^{(\mu)}\rangle + hh_1\right).$$

Hence,

$$\begin{aligned} (z+\alpha_N/\varepsilon)^2 E\left\{\langle J_1\rangle^2\right\} = \frac{1}{\varepsilon^2 N}E\left\{\sum_{\mu,\nu=1}^{p}\xi_1^{(\mu)}\xi_1^{(\nu)}\langle t_1^{(\mu)}\rangle\langle t_1^{(\nu)}\rangle\right\} \\ +h^2 + \frac{2h}{\varepsilon_N^{1/2}}E\left\{\sum_{\mu=1}^{p}h_1\xi_1^{(\mu)}\langle t_1^{(\mu)}\rangle\right\} + o(1), \end{aligned} \tag{3.41}$$

and similarly

$$\begin{aligned} (z+\alpha_N/\varepsilon)^2 E\left\{\langle J_1^2\rangle\right\} = (z+\alpha_N/\varepsilon) + \frac{1}{\varepsilon^2 N}\sum_{\mu,\nu=1}^{p}E\left\{\xi_1^{(\mu)}\xi_1^{(\nu)}\langle t_1^{(\mu)}t_1^{(\nu)}\rangle\right\} \\ +h^2 + \frac{2h}{\varepsilon_N^{1/2}}\sum_{\mu=1}^{p}E\left\{h_1\xi_1^{(\mu)}\langle t_1^{(\mu)}\rangle\right\} + o(1). \end{aligned} \tag{3.42}$$

Now to calculate the r.h.s. in (3.41) and (3.42) we use the formula of "integration by parts" which is valid for any function f with bounded third derivative

$$\begin{aligned} E\left\{\xi_1^{(\mu)}f\left(\xi_1^{(\mu)}N^{-1/2}\right)\right\} \\ = \frac{1}{N^{1/2}}E\left\{f'\left(\xi_1^{(\mu)}N^{-1/2}\right)\right\} + \frac{1}{N^{3/2}}E\left\{f'''\left(\zeta(\xi_1^{(\mu)})\xi_1^{(\mu)}N^{-1/2}\right)\right\}, \end{aligned} \tag{3.43}$$

where $|\zeta(\xi_1^{(\mu)})| \le 1$. Thus, using this formula and the second line of (3.40), we get:

$$\begin{aligned} &(z+\alpha_N/\varepsilon)^2\overline{q}_{N,p} = \tilde{q}_N \\ &+\frac{1}{N^2\varepsilon^4}\sum_{\mu\ne\nu} E\left\{\left\langle \dot{t}_1^{(\mu)}(t_1^{(\mu)}J_1 - \langle t_1^{(\mu)}J_1\rangle)\right\rangle \left\langle \dot{t}_1^{(\nu)}(t_1^{(\nu)}J_1 - \langle t_1^{(\nu)}J_1\rangle)\right\rangle\right\} \\ &+\frac{2}{N^2\varepsilon^4}\sum_{\mu\ne\nu} E\left\{\left\langle \dot{t}_1^{(\mu)}(t_1^{(\mu)}J_1 - \langle t_1^{(\mu)}J_1\rangle)(t_1^{(\nu)}J_1 - \langle t_1^{(\nu)}J_1\rangle)\right\rangle \langle t_1^{(\nu)}\rangle\right\} \\ &+\frac{1}{N^2\varepsilon^4}\sum_{\mu\ne\nu} E\left\{\left\langle \dot{t}_1^{(\mu)}(t_1^{(\nu)}J_1 - \langle t_1^{(\nu)}J_1\rangle)\right\rangle \left\langle \dot{t}_1^{(\nu)}(t_1^{(\mu)}J_1 - \langle t_1^{(\mu)}J_1\rangle)\right\rangle\right\} \\ &+h^2 + \frac{2h^2}{\varepsilon^2 N}\sum_{\mu} E\left\{\left\langle \dot{t}_1^{(\mu)}(t_1^{(\mu)}J_1 - \langle t_1^{(\mu)}J_1\rangle)\dot{J}_1\right\rangle\right\} + o(1). \end{aligned} \tag{3.44}$$

Substituting $t_1^{(\mu)}$ by $t^{(\mu)}$ and using the symmetry of the Hamiltonian with respect to J_i, we obtain e.g. for the first sum in (3.44):

$$\begin{aligned} &\frac{1}{N^2}\sum_{\mu\ne\nu} E\left\{\left\langle \dot{t}_1^{(\mu)}(t_1^{(\mu)}J_1 - \langle t_1^{(\mu)}J_1\rangle)\right\rangle \left\langle \dot{t}_1^{(\nu)}(t_1^{(\nu)}J_1 - \langle t_1^{(\nu)}J_1\rangle)\right\rangle\right\} \\ &= \frac{1}{N^3}\sum_{i=1}^{N}\sum_{\mu,\nu=1}^{p} E\left\{\left\langle \dot{t}^{(\mu)}(t^{(\mu)}(\dot{J}_i + \langle J_i\rangle) - \langle t^{(\mu)}(\dot{J}_i + \langle J_i\rangle)\rangle\right\rangle\right. \\ &\qquad \left.\cdot\left\langle \dot{t}^{(\nu)}(\dot{J}_i + \langle J_i\rangle) - \langle t^{(\nu)}(\dot{J}_i + \langle J_i\rangle)\rangle\right\rangle\right\} + o(1) \\ &= \frac{1}{N^3}\sum_{i=1}^{N}\sum_{\mu,\nu=1}^{p} E\left\{\langle J_i\rangle^2\langle(\dot{t}^{(\mu)})^2\rangle\langle(\dot{t}^{(\nu)})^2\rangle\right\} + o(1) = \overline{q}_{N,p}(\tilde{U}_N - \tilde{q}_N)^2 + o(1). \end{aligned}$$

Here we have used the relation (3.24), which allows us to get rid from the terms containing $\dot{J}_i$ and the self-averaging properties of $\overline{q}_{N,p}$, $\tilde{U}_N$ and $\tilde{q}_N$. Transforming in a similar way the other sums in the r.h.s. of (3.44) and using also relations (3.40) to get rid from the terms, containing $\langle \dot{t}^{(\mu)}\dot{t}^{(\nu)}\rangle$, we get finally:

$$\begin{aligned} &(z+\alpha_N/\varepsilon)^2\overline{q}_{N,p} = \tilde{q}_N + 2(\overline{R}_{N,p} - \overline{q}_{N,p})\tilde{q}_N(\tilde{U}_N - \tilde{q}_N) \\ &+\overline{q}_{N,p}(\tilde{U}_N - \tilde{q}_N)^2 + h^2(1 + 2(\tilde{U}_N - \tilde{q}_N)(\overline{R}_{N,p} - \overline{q}_{N,p})) + o(1). \end{aligned} \tag{3.45}$$

Similarly we obtain

$$\begin{aligned} &(z+\alpha_N/\varepsilon)^2\overline{R}_{N,p} = (z+\alpha_N/\varepsilon) + \tilde{U}_N + \overline{R}_{N,p}(\tilde{U}_N^2 - \tilde{q}_N^2) \\ &-2\overline{q}_{N,p}\tilde{q}_N(\tilde{U}_N - \tilde{q}_N) + h^2(1 + 2(\tilde{U}_N - \tilde{q}_N)(\overline{R}_{N,p} - \overline{q}_{N,p})) + o(1). \end{aligned} \tag{3.46}$$

Considering (3.45) and (3.46) as a system of equation with respect to $\overline{R}_{N,p}$ and $\overline{q}_{N,p}$, we get

$$\overline{q}_{N,p} = \frac{\tilde{q}_N + h^2}{(z+\Delta_N)^2} + o(1), \quad \overline{R}_{N,p} - \overline{q}_{N,p} = \frac{1}{z+\Delta_N} + o(1), \tag{3.47}$$

where we denote for simplicity

$$\Delta_N \equiv \frac{\alpha}{\varepsilon} - \tilde{U}_N + \tilde{q}_N. \tag{3.48}$$

Now we should find the expressions for $\tilde{q}_N$ and $\tilde{U}_N$.

From the symmetry of the Hamiltonian (2.10) it is evident that

$$\begin{aligned}\tilde{q}_N &= \alpha_N E\left\{\frac{1}{\varepsilon^2}\left\langle N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})},\boldsymbol{J}) - x^{(p)}\right\rangle^2\right\}\\ &= \alpha_N E\left\{\left[\frac{d}{dk_1}\log\int_{x>0}dx\left\langle \exp\{-\frac{1}{2\varepsilon_1}(N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})},\boldsymbol{J}) - x - k_1)^2\}\right\rangle_{p-1}\right]^2\right\}\Bigg|_{k_1=k}\\ &= \alpha_N E\left\{\left[\frac{d}{dk_1}\log\phi_N(k_1,\varepsilon_1)\right]^2\right\}\Bigg|_{k_1=k}.\end{aligned} \tag{3.49}$$

Therefore, using Lemma 3, we derive:

$$\tilde{q}_N = \alpha_N E\left\{\left[\frac{d}{dk_1}\log \mathrm{H}\left(\frac{\sqrt{\overline{q}_{N,p}}u + k_1}{\sqrt{U_{N,p}}}\right)\right]^2\right\} = \frac{\alpha_N}{U_{N,p}}E\left\{\mathrm{A}^2\left(\frac{\sqrt{\overline{q}_{N,p}}u + k_1}{\sqrt{U_{N,p}}}\right)\right\}. \tag{3.50}$$

Here and below we denote

$$\mathrm{A}(x) \equiv -\frac{d}{dx}\log \mathrm{H}(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}\mathrm{H}(x)}, \tag{3.51}$$

where the function $\mathrm{H}(x)$ is defined by (2.11). Similarly

$$\begin{aligned}\tilde{U}_N &= \alpha_N E\left\{\frac{1}{\varepsilon^2}\langle (N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})},\boldsymbol{J}) - x^{(p)})^2\rangle\right\}\\ &= 2\alpha_N E\left\{\frac{d}{d\varepsilon_1}\log\int_{x>0}dx\langle \exp\{-\frac{1}{2\varepsilon_1}(N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{p})},\boldsymbol{J}) - x - k_1)^2\}\rangle_{p-1}\right\}\Bigg|_{\varepsilon_1=\varepsilon}\\ &= 2\alpha_N E\left\{\frac{d}{d\varepsilon_1}\log\phi_p(k_1,\varepsilon_1)\right\}\Bigg|_{\varepsilon_1=\varepsilon}.\end{aligned} \tag{3.52}$$

Now, using Lemma 3 and Lemma 1, we derive:

$$\begin{aligned}\tilde{U}_N &= 2\alpha_N E\left\{\frac{d}{d\varepsilon_1}\log \varepsilon_1^{-1/2}\mathrm{H}\left(\frac{\sqrt{\overline{q}_{N,p}}u + k_1}{\sqrt{U_{N,p}}}\right)\right\}\Bigg|_{\varepsilon_1=\varepsilon}\\ &= \frac{\alpha_N}{\varepsilon} + \frac{\alpha_N}{U_{N,p}^{3/2}}E\left\{(k + \sqrt{\overline{q}_{N,p}}u)\mathrm{A}\left(\frac{\sqrt{\overline{q}_{N,p}}u + k_1}{\sqrt{U_{N,p}}}\right)\right\}.\end{aligned} \tag{3.53}$$

Thus, from (3.45), (3.46), (3.50) and (3.53) we obtain the system of equations for $\overline{R}_{N,p}$ and $\overline{q}_{N,p}$

$$\begin{aligned}\overline{q}_{N,p} &\equiv (\overline{R}_{N,p} - \overline{q}_{N,p})^2\left[\frac{\alpha}{U_{N,p}}E\left\{\mathrm{A}^2\left(\frac{\sqrt{\overline{q}_{N,p}}u + k}{\sqrt{U_{N,p}}}\right)\right\} + h^2\right] + \tilde{\varepsilon}_N\\ &\frac{\alpha}{U_{N,p}^{3/2}}E\left\{(\sqrt{\overline{q}_{N,p}}u + k)\mathrm{A}\left(\frac{\sqrt{\overline{q}_{N,p}}u + k}{\sqrt{U_{N,p}}}\right)\right\}\\ &= z + \frac{\overline{q}_{N,p}}{(\overline{R}_{N,p} - \overline{q}_{N,p})^2} - \frac{1}{\overline{R}_{N,p} - \overline{q}_{N,p}} - h^2 + \tilde{\varepsilon}'_N,\end{aligned} \tag{3.54}$$

where $\tilde{\varepsilon}_N, \tilde{\varepsilon}'_N \to 0$, as $N, p \to \infty$, $\alpha_N \to \alpha$.

Proposition 3. *For any* $\alpha < 2$ *there exists* $\varepsilon^*(\alpha, k)$ *such that for any* $\varepsilon \le \varepsilon^*$ *and* $z < \varepsilon^{-1/3}$ *the solution of the system (3.54) tends as* $\tilde{\varepsilon}_N, \tilde{\varepsilon}'_N \to 0$ *to* (R^*, q^*) *which gives the unique point of* $\max_R \min_q$ *in the r.h.s. of (2.15).*

On the basis of this proposition we conclude that for almost all z, h there exist the limits

$$\lim_{m\to\infty} E\left\{\frac{d}{dz} f_{N_m,p_m}(k,h,z,\varepsilon)\right\} = R^*(\alpha,k,h,z,\varepsilon),$$
$$\lim_{m\to\infty} E\left\{\frac{d}{dh} f_{N_m,p_m}(k,h,z,\varepsilon)\right\} = h(R^*(\alpha,k,h,z,\varepsilon) - q^*(\alpha,k,h,z,\varepsilon)).$$

But since the r.h.s. here are continuous functions of z, h we derive that for any convergent subsequence $f_{N_m,p_m}(k,h,z,\varepsilon)$ the above limits exist for all z, h. Besides, choosing subsequence $f_{N'_m,p'_m}(k,h,z,\varepsilon)$ which converges for any rational α, we obtain that for any N'_m, p'_m such that $\alpha_m = \frac{p'_m}{N'_m} \to \alpha_1$ (α_1 is a rational number) and p''_m such that $\alpha'_m = \frac{p''_m}{N'_m} \to 0$

$$\begin{aligned}
&E\left\{f_{N'_m,p'_m}(\alpha_k,k,h,z,\varepsilon)\right\} - E\{f_{N'_m,p''_m}(\alpha'_k,k,h,z,\varepsilon)\} = \\
&\frac{1}{N'_m}\sum_{i=0}^{p'_m-p''_m} E\left\{\log Z_{N'_m,p''_m-i}(k,h,z,\varepsilon) - \log Z_{N'_m,p'_m-i-1}(k,h,z,\varepsilon)\right\} \\
&\qquad \to \frac{1}{N'_m}\sum_{i=0}^{p'_m-p''_m} E\left\{\log \mathrm{H}\left(\frac{\sqrt{q_{N'_m,p'_m-i}}u+k}{\sqrt{U_{N'_m,p'_m-i}}}\right)\right\} \\
&\qquad \to \int_0^{\alpha_1} E\left\{\log \mathrm{H}\left(\frac{\sqrt{q^*(\alpha)}u+k}{\sqrt{R^*(\alpha)+\varepsilon-q^*(\alpha)}}\right)\right\} d\alpha.
\end{aligned}$$

Thus, for all rational α there exists

$$\lim_{m\to\infty} E\left\{f_{N_m,p_m}(k,h,z,\varepsilon)\right\} = F(\alpha,k,h,z,\varepsilon),$$

where $F(\alpha,k,h,z,\varepsilon)$ is defined by (2.15). But since the free energy is obviously monotonically decreasing in α, we obtain, that for any convergent subsequence the limit of the free energy coincides with the r.h.s. of (2.15). Hence, as it was already mentioned after Proposition 2, there exist a limit which coincides with the r.h.s. of (2.15). Theorem 2 is proven.

Proof of Theorem 3. For any $z > 0$ let us take h small enough and consider

$$\Theta_{N,p}(k,h,z) \equiv \sigma_N^{-1}\int_{\Omega_N} d\boldsymbol{J}\exp\{-\frac{z}{2}(\boldsymbol{J},\boldsymbol{J}) - h(\mathbf{h},\boldsymbol{J})\},$$

where

$$\Omega_{N,p} \equiv \left\{\boldsymbol{J}:\ N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\nu})},\boldsymbol{J}) \ge k,\ (\nu = 1,\dots,p)\right\}.$$

To obtain the self-averaging of $N^{-1}\log_{(MN)}\Theta(k,h,z)$ and the expression for $E\{N^{-1}\log_{(MN)}\Theta(k,h,z)\}$ we define also the interpolating Hamiltonians, corresponding partition functions and free energies:

$$\mathcal{H}^{(\mu)}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon)\equiv-\sum_{\nu=\mu+1}^{p}\log\mathrm{H}\left(\frac{k-N^{-1/2}(\boldsymbol{\xi^{(\nu)}},\boldsymbol{J})}{\sqrt{\varepsilon}}\right)+\frac{z}{2}(\boldsymbol{J},\boldsymbol{J})+h(\mathbf{h},\boldsymbol{J}), \tag{3.55}$$

$$\begin{gathered}Z^{(\mu)}_{N,p}(k,h,z,\varepsilon)\equiv\sigma_N^{-1}\int_{\Omega^{(\mu)}_{N,p}}d\boldsymbol{J}\exp\{-\mathcal{H}^{(\mu)}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon)\},\\ f^{(\mu)}_{N,p}(k,h,z,\varepsilon,M)\equiv\tfrac{1}{N}\log_{(MN)}Z_{N,p}(k,h,z,\varepsilon),\end{gathered} \tag{3.56}$$

where

$$\Omega^{(\mu)}_{N,p}\equiv\left\{\boldsymbol{J}:\ N^{-1/2}(\boldsymbol{\xi^{(\mu')}},\boldsymbol{J})\geq k,\ (\mu'=1,\dots,\mu)\right\}.$$

According to Theorem 2, for large enough M with probability more than $(1-O(N^{-1}))$

$$f^{(0)}_{N,p}(k,h,z,\varepsilon,M)=f_{N,p}(k,h,z,\varepsilon),\quad f^{(p)}_{N,p}(k,h,z,\varepsilon)=\frac{1}{N}\log_{(MN)}\Theta(k,h,z),$$

where $f_{N,p}(k,h,z,\varepsilon)$ is defined by (2.13). Hence,

$$\begin{gathered}f_{N,p}(k,h,z,\varepsilon,M)-\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k,h,z)=\frac{1}{N}\sum_{\mu=1}^{p}\tilde{\Delta}^{(\mu)},\\ \tilde{\Delta}^{(\mu)}\equiv\log_{(MN)}Z^{(\mu-1)}_{N,p}-\log_{(MN)}Z^{(\mu)}_{N,p}.\end{gathered} \tag{3.57}$$

Below in the proof of Theorem 3 we denote by $x^{(\mu)}\equiv N^{-1/2}(\boldsymbol{\xi^{(\mu)}},\boldsymbol{J})$, by the symbol $\langle\dots\rangle_\mu$ the Gibbs averaging corresponding to the Hamiltonian $\mathcal{H}^{(\mu)}_{N,p}$ in the domain $\Omega^{(\mu-1)}_{N,p}$ and by $Z^{(\mu,\mu)}_{N,p}$ the correspondent partition function.Denote also

$$T_\mu(x)\equiv\left\langle\theta(x^{(\mu)}-x)\right\rangle_\mu,\quad X_\mu\equiv\left\langle x^{(\mu)}\right\rangle_\mu.$$

To proceed further, we use the following lemma:

Lemma 4. *If the inequalities (3.21) are fulfilled and there exists N,μ,ε-independent D such that*

$$\frac{1}{N}\left\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\right\rangle_\mu\geq D^2, \tag{3.58}$$

then there exist N,μ,ε-independent K_1,C_1^*,C_2^*,C_3^*, such that for $|X_\mu|\leq\log N$

$$\begin{gathered}T_\mu(k+2\varepsilon^{1/4})\geq C_1^*e^{-C_2^*X_\mu^2},\\ T_\mu(k-2\varepsilon^{1/4})-T_\mu(k+2\varepsilon^{1/4})\leq\varepsilon^{1/4}C_3^*\end{gathered} \tag{3.59}$$

with probability $P^{(\mu)}_N\geq(1-K_1N^{-3/2})$.

Remark 6. Similarly to Remark 4 one can conclude that, if $Z^{(\mu,\mu)}_{N,p}>e^{-MN}$, then there exists ε,N,μ-independent R_0, such that we can use the inequality $|\boldsymbol{J}|\leq N^{1/2}R_0$ with the error $O(e^{-N\,\mathrm{const}})$.

Remark 7. Denote $\tilde{D}_\mu^2$ the l.h.s. of (3.58). Then

$$\begin{gathered} 4\tilde{D}_\mu^2\langle\theta(|\dot{\boldsymbol{J}}|-2\tilde{D}_\mu N^{1/2})\rangle_\mu \le N^{-1}\left\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\right\rangle_\mu = \tilde{D}_\mu^2 \\ \Rightarrow \langle\theta(|\dot{\boldsymbol{J}}|-2\tilde{D}_\mu N^{1/2})\rangle_\mu \le \tfrac{1}{4} \\ \Rightarrow Z_N^{(\mu,\mu)} \le \tfrac{4}{3}\sigma_N^{-1}\int_{|\dot{\boldsymbol{J}}|<2\tilde{D}_\mu N^{1/2}} \exp\{-\tfrac{z}{2}(\boldsymbol{J},\boldsymbol{J})-h(\mathbf{h},\boldsymbol{J})\} \le \tfrac{4}{3}(2\tilde{D}_\mu)^N e^{2hNR_0} \end{gathered}$$

Thus, the inequality $Z_{N,p}^{(\mu,\mu)} > e^{-MN}$ implies that $\tilde{D}_\mu \ge \frac{1}{2}\exp\{-M-2hR_0\} \equiv D^2$.

Let us prove self-averaging property of $f_{N,p}^{(p)}(k,h,z,\varepsilon,M)$, using Lemma 4. Similarly to (3.25) we write

$$f_{N,p}^{(p)}(k,h,z,\varepsilon,M) - E\{f_{N,p}^{(p)}(k,h,z,\varepsilon,M)\} = \frac{1}{N}\sum_{\nu=0}^{p-1}\Delta_\nu,$$

where

$$\Delta_\nu \equiv E_\nu\{f_{N,p}^{(p)}(k,h,z,\varepsilon,M)\} - E_{\nu+1}\{(f_{N,p}^{(p)}(k,h,z,\varepsilon,M)\},$$

Then $E\{\Delta_\nu\Delta_{\nu'}\} = 0$, $(\nu \ne \nu')$ and therefore

$$E\{(f_{N,p}^{(p)}(k,h,z,\varepsilon,M) - E\{f_{N,p}^{(p)}(k,h,z,\varepsilon,M)\})^2\} = \frac{1}{N^2}\sum_{\nu=0}^{p-1}E\{\Delta_\nu^2\}, \tag{3.60}$$

where similarly to (3.26)

$$E\{\Delta_\nu^2\} \le E\{\overline{\Delta}_\nu^2\}, \tag{3.61}$$

with

$$\overline{\Delta}_\nu \equiv \log_{(MN)} Z_{N,p}^{(p)} - \log_{(MN)} Z_{N,p}^{(p,\nu+1)},$$

where $Z_{N,p}^{(p,\nu)}$ is the partition function, corresponding to the Hamiltonian $\mathcal{H}_{N,p}^{(p)}$ in the domain $\Omega_{N,p}^{(p,\nu)}$ which differs from $\Omega_{N,p}^{(p)}$ by the absence of the inequality for $\mu' = \nu$. Therefore for $\nu \le p-1$

$$\begin{aligned} &E\{|\overline{\Delta}_\nu|^2\} = E\{|\overline{\Delta}_{p-1}|^2\} \\ &= E\{\theta(Z_{N,p}^{(p,p)} - e^{-MN})|\log_{(MN)} Z_{N,p}^{(p)} - \log_{(MN)} Z_{N,p}^{(p,p)}|^2\} \\ &+E\{\theta(e^{-MN} - Z_{N,p}^{(p,p)})|\log_{(MN)} Z_{N,p}^{(p)} - \log_{(MN)} Z_{N,p}^{(p,p)}|^2\}. \end{aligned} \tag{3.62}$$

But the second term in the r.h.s. is zero, because $Z_{N,p}^{(p)} \le Z_{N,p}^{(p,p)}$ and thus $Z_{N,p}^{(p,p)} \le e^{-MN}$ implies $Z_{N,p}^{(p)} \le e^{-MN}$, and so $\log_{(MN)} Z_{N,p}^{(p)} = \log_{(MN)} Z_{N,p}^{(p,p)} = -MN$. Then, denoting by χ_μ the indicator function of the set, where $Z^{(\mu,\mu)} > e^{-MN}$, and the inequalities (3.59) are fulfilled, on the basis of Lemma 4, we obtain that

$$\begin{aligned} E\{\overline{\Delta}_\nu^2\} &= E\{\theta(Z_{N,p}^{(p,p)} - e^{-MN})\log^2_{(M)}\left\langle\theta(x^{(p)}-k)\right\rangle_p\} \\ &\le (MN)^2[E\{\theta(Z_{N,p}^{(p,p)} - e^{-MN})\theta(|X_p| - \log N)\} \\ &+E\{\theta(Z_{N,p}^{(p,p)} - e^{-MN})(1-\chi_p)\theta(\log N - |X_p|)\}] \\ &+E\left\{\theta(Z_{N,p}^{(p,p)} - e^{-MN})\chi_p\theta(\log N - |X_p|)\log^2\exp\{-C_1^*X_\mu^2\}\right\} \\ &\le (MN)^2[e^{-\log^2 N/2R_0^2}) + K_1N^{-3/2}] + 2(R_0^2C_1^*)^2 \le 2M^2K_1N^{1/2}. \end{aligned} \tag{3.63}$$

Here we have used that, according to the definition of the function $\log_{(MN)}$ (see (2.16), $|\log_{(MN)} \langle\theta(x^{(p)}-k)\rangle_p| \le MN$. Besides, we used the standard Chebyshev inequality, according to which

$$P_\mu(X) \equiv \mathrm{Prob}\{X_\mu \ge X\} \le e^{-X^2/2R_0^2}. \tag{3.64}$$

Relations (3.60), (3.61) and (3.63) prove the self-averaging property of $\frac{1}{N}\log_{(MN)} \Theta_{N,p}(k,h,z)$.

Now let us prove that $\tilde{\Delta}^{(\mu)}$, defined in (3.57), for any μ satisfies the bound

$$\begin{aligned} |E\{\tilde{\Delta}^{(\mu)}\}| = |E\{\theta(Z_{N,p}^{(\mu,\mu)} - e^{-MN})[\log_{(MN)} \langle \mathrm{H}((k-x^{(\mu)})\varepsilon^{-1/2})\rangle_\mu \\ -\log_{(MN)} \langle\theta(x^{(\mu)}-k)\rangle_\mu]\} \le \varepsilon^\lambda K, \end{aligned} \tag{3.65}$$

with some positive N,μ,ε-independent λ, K. We remark here, that similarly to (3.62) $Z_{N,p}^{(\mu-1)}, Z_{N,p}^{\mu} \le Z_{N,p}^{\mu,\mu}$ and so, if $Z_{N,p}^{\mu,\mu} < e^{-MN}$, then $\log_{(MN)} Z_{N,p}^{(\mu-1)} = \log_{(MN)} Z_{N,p}^{(\mu)} = MN$.

Using the inequalities

$$\mathrm{H}(-\varepsilon^{-1/4})\theta(x-\varepsilon^{1/4}) \le \mathrm{H}\left(-\frac{x}{\varepsilon^{1/2}}\right) \le \varepsilon_1 + \theta(x+\varepsilon^{1/4}) \tag{3.66}$$

with $\varepsilon_1 \equiv \mathrm{H}(\varepsilon^{-1/4})$, we get

$$\begin{aligned} \log \mathrm{H}(-\varepsilon^{-1/4}) - E\{\theta(Z_{N,p}^{(\mu,\mu)} - e^{-MN})\log(1+r_1(k,\varepsilon))\} \\ \le E\{\tilde{\Delta}^{(\mu)}\} \le E\{\theta(Z_{N,p}^{(\mu,\mu)} - e^{-MN})\log(1+r_2(k,\varepsilon))\}, \end{aligned} \tag{3.67}$$

where

$$r_1(k,\varepsilon) \equiv \frac{T_\mu(k) - T_\mu(k+\varepsilon^{1/4})}{T_\mu(k+\varepsilon^{1/4})}, \quad r_2(k,\varepsilon) \equiv \frac{T_\mu(k-\varepsilon^{1/4}) - T_\mu(k) + \varepsilon_1}{T_\mu(k)}.$$

But by the virtue of Lemma 4, one can get easily that, if $|X_\mu| \le \log N$, then with probability $P_N^{(\mu)} \ge (1 - K_1 N^{-3/2})$

$$r_{1,2}(k,\varepsilon) \le \varepsilon^{1/4} C e^{CX_\mu^2}$$

with some N,μ-independent C. Therefore, choosing $\lambda \equiv \frac{1}{8}R_0^2(1+2CR_0^2)^{-1}$ and $L^2 \equiv 2\lambda|\log\varepsilon|$, for small enough ε we can write similarly to (3.63)

$$\begin{aligned} E\left\{\theta(Z_{N,p}^{(\mu,\mu)} - e^{-MN})\log_{(MN)}(1+r_{1,2}(k,\varepsilon))\right\} \le (MN)P_\mu(\log N) \\ +K_1N^{-3/2}(MN) + \textstyle\int \theta(\log N - |X|)\log(1+\varepsilon^{1/4}Ce^{CX^2})dP_\mu(X) \\ = \varepsilon^{1/4}Ce^{CL^2} + C\textstyle\int\theta(|X|-L)X^2dP_\mu(X) + o(1) \\ \le \varepsilon^{1/4}Ce^{CL^2} + 2CL^2P(L) \le K(C,R_0)\varepsilon^\lambda, \end{aligned}$$

where $P_\mu(X)$ is defined and estimated in (3.64) and we have used that, according to definition (2.16), $-MN \le \log_{(MN)}\theta\langle(x^{(\mu)}-k)\rangle_\mu, \log_{(MN)}\langle\theta(x^{(\mu)}-k\pm\varepsilon^{1/4})\rangle_\mu \le 0$ and therefore always $|\log_{(MN)}(1+r_{1,2}(k,\varepsilon)| \le MN$.

Using the bound

$$|\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k,h,z) - \frac{1}{N}\log_{(MN)}\Theta_{N,p}(k,0,z)| \le 2hR_0,$$

representation (3.57) and self-averaging property of $\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k,h,z)$, we obtain that with probability $P_N \geq 1 - O(N^{-1/2})$

$$\begin{aligned} F(\alpha,k,0,z,\varepsilon) + O(\varepsilon^{\lambda}) + O(h) &\leq \frac{1}{N}\log_{(MN)}\Theta_{N,p}(k,0,z) \\ &\leq F(\alpha,k,0,z,\varepsilon) + O(\varepsilon^{\lambda}) + O(h). \end{aligned}$$

Now we are going to use Corollary 1 to replace the integration over the whole space by the integration over the sphere of the radius $N^{1/2}$. But since Theorem 2 is valid only for $z < \varepsilon^{-1/3}$, we need to check, that $\min_z\{F(\alpha,k,0,z,\varepsilon) + \frac{z}{2}\}$ takes place for z, satisfying this bound.

Proposition 4. *For any $\alpha < \alpha_c(k)$ there exists ε-independent $\overline{z}(k,\alpha)$ such that $z_{min} < \overline{z}(k,\alpha)$.*

Then, using 2.9, we have got that with the same probability for $\alpha \leq \alpha_c(k)$

$$\begin{aligned} \min_z\{F(\alpha,k,0,z,\varepsilon) + \frac{z}{2}\} + O(\varepsilon^{\lambda}) + O(h) &\leq \frac{1}{N}\log_{(MN)}\Theta_{N,p}(k) \\ \leq \min_z\{F(\alpha,k,0,z,\varepsilon) + \frac{z}{2}\} &+ O(\varepsilon^{\lambda}) + O(\delta) + O(h). \end{aligned} \tag{3.68}$$

Thus,

$$\lim_{N\to\infty} E\left\{\left(\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k) - E\{\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k)\}\right)^2\right\} \leq O(\varepsilon^{2\lambda}) + O(h), \tag{3.69}$$

and since ε, h are arbitrarily small numbers (3.69) proves the self-averaging property of $\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k)$. Besides, averaging $\frac{1}{N}\log_{(MN)}\Theta_{N,p}(k)$ with respect to all random variables and taking the limits $h, \varepsilon \to 0$, we obtain (2.15) from (3.69).

The last statement of Theorem 3 follows from that proven above, if we note that $\log_{(MN)}\Theta_{N,p}(k)$ is a monotonically decreasing function of α and, on the other hand, the r.h.s. of (2.17) tends to $-\infty$ as $\alpha \to \alpha_c(k)$

Hence, we have finished the proof of Theorem 3.

4. Auxiliary Results

Proof of Proposition 1 Let us fix $t \in (t_1^*, t_2^*)$ take some small enough δ and consider $\mathcal{D}^{\delta}(t)$ which is the set of all $\boldsymbol{J} \in \mathcal{A}(t) \cap \mathcal{D}$ whose distance from the boundary of $\mathcal{D}$ is more than $d = N^{1/2}\max\{\delta, 2K_0\delta\}$. Now for any $\boldsymbol{J}_0 \in \mathcal{D}^{\delta}(t)$ consider $(\tilde{\boldsymbol{J}}, \phi(\tilde{\boldsymbol{J}}))$ - the local parametrisation of $\mathcal{D}$ with the points of the $(N-1)$-dimensional hyper-plane $\mathcal{B} = \{\tilde{\boldsymbol{J}} : (\tilde{\boldsymbol{J}}, \tilde{\mathbf{n}}) = 0\}$, where $\tilde{\mathbf{n}}$ is the projection of the normal $\mathbf{n}$ to $\mathcal{D}$ at the point $\boldsymbol{J}_0$ on the hyper-plane $\mathcal{B}(t)$. We chose the orthogonal coordinate system in $\mathcal{B}$ in such a way that $\tilde{J}_1 = (\boldsymbol{J}, \mathbf{e}) = N^{1/2}t$. Denote $\tilde{\boldsymbol{J}}_0 = P\boldsymbol{J}_0$ (P is the operator of the orthogonal projection on $\mathcal{B}$). According to the standard theory of the Minkowski sum (see e.g.[Ha]), the boundary of $\frac{1}{2}\mathcal{A}(t) \times \frac{1}{2}\mathcal{A}(t+\delta)$ consists of the points

$$\boldsymbol{J}' = \frac{1}{2}\boldsymbol{J} + \frac{1}{2}\boldsymbol{J}^{(\delta)}(\boldsymbol{J}), \tag{4.1}$$

where $\boldsymbol{J}$ belongs to the boundary of $\mathcal{A}(t)$ and the point $\boldsymbol{J}^{(\delta)}(\boldsymbol{J})$ (belonging to the boundary of $\mathcal{A}(t+\delta)$) is chosen in such a way that the normal to the boundary of $\mathcal{A}(t+\delta)$ at this point coincides with the normal $\mathbf{n}$ to the boundary of $\mathcal{A}(t)$ at the point $\boldsymbol{J}$. Denote $\tilde{\mathcal{D}}(\frac{1}{2})$ the part of the boundary of $\frac{1}{2}\mathcal{A}(t) \times \frac{1}{2}\mathcal{A}(t+\delta)$ for which in representation (4.1) $\boldsymbol{J} \in \mathcal{D}^{\delta}(t)$. Now for $\boldsymbol{J}_0 \in \mathcal{D}^{\delta}(t)$ let us find the point $\boldsymbol{J}^{(\delta)}(\boldsymbol{J}_0)$. Since by construction $\frac{\partial}{\partial \tilde{J}_i}\phi(\tilde{\boldsymbol{J}}_0) = 0$ $(i = 2, \ldots, N-1)$, we obtain for $\tilde{\boldsymbol{J}}^{(\delta)}(\boldsymbol{J}_0) \equiv P\boldsymbol{J}^{(\delta)}(\boldsymbol{J}_0)$ the system of equations

$$\frac{\partial}{\partial \tilde{J}_i}\phi(\tilde{\boldsymbol{J}}^{(\delta)}) = 0, \ (i = 2, \ldots, N-1)$$

and $\tilde{J}_1^{(\delta)} = N^{1/2}(t+\delta)$. Then we get

$$\tilde{J}_i^{(\delta)} = \tilde{J}_i^0 + \delta N^{1/2}(D_{11}^{-1})^{-1}(D^{-1})_{i,1} + o(\delta) \ (i = 2, \ldots, N-1), \tag{4.2}$$

where the matrix $\{D_{i,j}\}_{i,j=1}^{N-1}$ consists of the second derivatives of the function $\phi(\tilde{\boldsymbol{J}})$ $(D_{i,j} \equiv \frac{\partial^2}{\partial \tilde{J}_i \partial \tilde{J}_j}\phi(\tilde{\boldsymbol{J}}))$. Thus, it was mentioned above, the point $\boldsymbol{J}_1 \equiv (\frac{1}{2}(\tilde{\boldsymbol{J}}_0 + \tilde{\boldsymbol{J}}^{(\delta)}), \frac{1}{2}(\phi(\tilde{\boldsymbol{J}}_0)) + \phi(\tilde{\boldsymbol{J}}^{(\delta)})) \in \tilde{\mathcal{D}}(\frac{1}{2})$. Consider also the point $\boldsymbol{J}_1' \equiv (\frac{1}{2}(\tilde{\boldsymbol{J}}_0 + \tilde{\boldsymbol{J}}^{(\delta)}), \phi(\frac{1}{2}(\tilde{\boldsymbol{J}}_0 + \tilde{\boldsymbol{J}}^{(\delta)}))) \in \mathcal{A}(t + \frac{1}{2}\delta) \cap \mathcal{D}$. Then,

$$\begin{gathered} |\boldsymbol{J}_1 - \boldsymbol{J}_1'| = \phi\left(\tfrac{1}{2}(\tilde{\boldsymbol{J}}_0 + \tilde{\boldsymbol{J}}^{(\delta)})\right) - \tfrac{1}{2}\left(\phi(\tilde{\boldsymbol{J}}_0) + \phi(\tilde{\boldsymbol{J}}^{(\delta)})\right) = \\ \tfrac{\delta^2}{2}N\left((D_{1,1}^{-1})^2 \textstyle\sum_{i,j=2}^{N-1} D_{i,j}D_{i,1}^{-1}D_{j,1}^{-1} + 2D_{1,1}^{-1}\sum_{i=2}^{N-1} D_{i,1}D_{i,1}^{-1} + D_{1,1}\right) + o(\delta^2) \\ = N\delta^2(D_{1,1}^{-1})^{-1} + o(\delta^2). \end{gathered}$$

But $(D_{1,1}^{-1})^{-1} \geq \lambda_{min}$, where λ_{min} is the minimal eigenvalue of the matrix D. Therefore, since

$$\lambda_{min} = \min_{(\tilde{\boldsymbol{J}},\tilde{\boldsymbol{J}})=1}(D\tilde{\boldsymbol{J}},\tilde{\boldsymbol{J}}) \geq \min_{(\tilde{\boldsymbol{J}},\tilde{\boldsymbol{J}})=1}\frac{(D\tilde{\boldsymbol{J}},\tilde{\boldsymbol{J}})}{(1+\tilde{J}_1^2(\mathbf{n},\mathbf{e})^2)^{3/2}} \geq \kappa_{min} \geq K_0N^{-1/2}, \tag{4.3}$$

we obtain that

$$|\boldsymbol{J}_1 - \boldsymbol{J}_1'| \geq \delta^2 K_0 N^{1/2}. \tag{4.4}$$

Besides, since by construction $\frac{\partial}{\partial \tilde{J}_i}\phi(\tilde{\boldsymbol{J}}_0) = 0$ and $\frac{\partial}{\partial \tilde{J}_i}\phi(\tilde{\boldsymbol{J}}^{(\delta)}) = 0$, we get that the tangent hyper-plane of the boundary $\frac{1}{2}\mathcal{A}(t) \times \frac{1}{2}\mathcal{A}(t+\delta)$ at the point $\boldsymbol{J}_1$ is orthogonal to $(\boldsymbol{J}_1 - \boldsymbol{J}_1')$. So, in fact, we have proved that the distance between $\mathcal{D}^{\delta}(t+\frac{1}{2}\delta)$ and $\tilde{\mathcal{D}}(\frac{1}{2})$ is more than $\delta^2 K_0 N^{1/2}$. Thus, denoting by $\tilde{S}(\frac{1}{2}) \equiv \mathrm{mes}\tilde{\mathcal{D}}(\frac{1}{2})$, we obtain that

$$V(t+\frac{1}{2}\delta) - \tilde{V}(\frac{1}{2}) \geq \delta^2 N^{1/2} K_0 \tilde{S}(\frac{1}{2}) + o(\delta^2) = \delta^2 N^{1/2} K_0 S(t) + o(\delta^2). \tag{4.5}$$

Here we have used that $\tilde{S}(\frac{1}{2}) = S(t) + o(1)$, as $\delta \to 0$, because the boundary $\mathcal{D}$ is smooth. Therefore, denoting $\tilde{V}(\tau)$ the volume of $\tau\mathcal{A}(t) \times (1-\tau)\mathcal{A}(t+\delta)$ and using (4.5), we get

$$\begin{aligned}
&2V^{1/N}(t+\frac{1}{2}\delta)-V^{1/N}(t)-V^{1/N}(t+\delta)\\
&\geq 2\left(\tilde V(\frac{1}{2})+\delta^2N^{-1/2}K_0S(t)\right)^{1/N}-\tilde V^{1/N}(0)-\tilde V^{1/N}(1)+o(\delta^2)\\
&=2\tilde V^{1/N}(\frac{1}{2})-\tilde V^{1/N}(0)-\tilde V^{1/N}(1)+\frac{2\delta^2K_0S(t)}{N^{1/2}\tilde V^{1-1/N}(\frac{1}{2})}+o(\delta^2)\\
&\geq\frac{2\delta^2K_0S(t)}{N^{1/2}V^{1-1/N}(t+\frac{1}{2}\delta)}+o(\delta^2)=2\delta^2K_0C(t)V^{1/N}(t)+o(\delta^2).
\end{aligned}$$

Here we have used the inequality $2\tilde V^{1/N}(\frac{1}{2})-\tilde V^{1/N}(0)-\tilde V^{1/N}(1)\geq 0$, which follows from the Brunn-Minkowski theorem and the relation $V(t+\frac{1}{2}\delta)=V(t)+o(1)$ (as $\delta\to 0$). Then, sending $\delta\to 0$, we obtain the statement of Proposition 1.

Proof of Lemma 1 Since $\log\mathrm{H}(x)$ is a concave function of x, $\mathcal{H}_{N,p}(\boldsymbol{J},h,z,\varepsilon)$ is the convex function of $\boldsymbol{J}$, satisfying (2.3). Since $\log\mathrm{H}(x)<0$ for any x, (2.4) is also fulfilled. To prove (2.5) let us write

$$\begin{aligned}
|\nabla\mathcal{H}_{N,p}(\boldsymbol{J})|^2&\leq\frac{3}{N\varepsilon}\sum_{i,\mu,\nu}\xi_i^{(\mu)}\xi_i^{(\nu)}A_\mu A_\nu+3h^2(\mathbf{hh})+3z^2(\boldsymbol{J},\boldsymbol{J})\\
&\leq\operatorname{const}\varepsilon^{-1}\left[\sum_\mu A_\mu^2+z^2(\boldsymbol{J},\boldsymbol{J})+h^2(\mathbf{hh})\right]\\
\leq\operatorname{const}\varepsilon^{-1}&\left[pC^*-\sum_\mu\log\mathrm{H}\left(k-\frac{N^{-1/2}(\boldsymbol{J},\boldsymbol{\xi^{(\mu)}})}{\sqrt{\varepsilon}}\right)+h^2+z^2(\boldsymbol{J},\boldsymbol{J})\right],
\end{aligned}\tag{4.6}$$

where we denote for simplicity $A_\mu\equiv\mathrm{A}\left(k-\frac{N^{-1/2}(\boldsymbol{J},\boldsymbol{\xi^{(\mu)}})}{\sqrt{\varepsilon}}\right)$, with the function $A(x)$ defined in (3.51). The second inequality in (4.6) is based on the first line of (3.21), the third inequality is valid by the virtue of the bound $\frac{1}{2}A^2(x)\leq-\log H(x)+C^*$, with some constant C^*, and the last inequality is valid due to the second line of (3.21).

Taking into account (2.4) one can conclude also, that for any U there exists some N-independent constant $C(U)$, such that $(\boldsymbol{J},\boldsymbol{J})\leq NC(U)$, if $\mathcal{H}_{N,p}(\boldsymbol{J})\leq NU$. Thus, we can derive from (4.6) that under conditions (3.21) (2.5) is fulfilled. Besides, due to the inequality $\log H(x)\geq C_1^*-\frac{1}{2}x^2$, it is easy to obtain that

$$f_{N,p}(k,h,z,\varepsilon)\geq C_1^*+\frac{1}{N}\log\det(\varepsilon^{-2}X+zI),$$

so (2.6) is also fulfilled.

Hence, we have proved that under conditions (3.21) the norm of the matrix $\mathcal{D}\equiv\{\langle\dot J_i\dot J_j\rangle\}_{i,j=1}^N$ is bounded by some N-independent $C(z,\varepsilon)$. Then with the same probability

$$N^{-1}\sum_{i,j=1}^N\langle\dot J_i\dot J_j\rangle^2=N^{-1}\mathrm{Tr}\mathcal{D}^2\leq C(z,\varepsilon),$$

which implies (3.22).

To prove (3.23) let us observe that

$$\langle\theta(|J_N|-N^{1/2}\varepsilon_N)\rangle=\langle\theta(|c|-\varepsilon_N)\rangle_{(U,c)}, \tag{4.7}$$

where $\langle\ldots\rangle_{(U,c)}$ is defined in (3.3)- (3.7) with $\mathbf{e}=(0,\ldots,0,1)$. For the function $s_N(U,c)$, defined by (3.5), we get

$$\begin{aligned}\langle\frac{\partial}{\partial c}s_N(U,0)\rangle_{(U,0)}&=N^{-1/2}\frac{\int\frac{\partial}{\partial J_N}\mathcal{H}_{N.p}(\boldsymbol{J})\exp\{-\mathcal{H}_{N.p}(\boldsymbol{J})\}|_{J_N=0}dJ_1\ldots dJ_{N-1}}{\exp\{-\mathcal{H}_{N.p}(\boldsymbol{J})\}|_{J_N=0}dJ_1\ldots dJ_{N-1}}\\&=\frac{hh_N}{N^{1/2}}+\frac{1}{N\varepsilon}\sum_{\mu=1}^{p}\xi_N^{(\mu)}\langle A_\mu\rangle\Big|_{J_N=0}.\end{aligned} \tag{4.8}$$

But since $\langle A_\mu\rangle|_{J_N=0}$ does not depend on $\xi_N^{(\mu)}$, by using the standard Chebyshev inequality, we obtain that

$$\mathrm{Prob}\left\{|\langle\frac{\partial}{\partial c}s_N(U,0)\rangle_{(U,0)}|>\varepsilon_N\right\}\le e^{-C_1N\varepsilon_N^2}=e^{-C_1\log^2N}. \tag{4.9}$$

On the other hand, since $s_N(U,c)$ is a concave function of U,c satisfying (3.13), denoting $\phi_N(U,c)\equiv s_N(U,c)-U-(s_N(U^*,c^*)-U^*)$ for any $(U,c)\sim(U^*,c^*)$ one can write

$$C_0[(U-U^*)^2+(c-c^*)^2]\le-\frac{\partial}{\partial c}\phi_N(U,c)(c-c^*)-\frac{\partial}{\partial U}\phi_N(U,c)(U-U^*). \tag{4.10}$$

Multiplying this inequality by $e^{N\phi_N(U,c)}$ and integrating with respect to U, we obtain for $c=0$

$$C_0(c^*)^2\le c^*\langle\frac{\partial}{\partial c}s_N(U,0)\rangle_{(U,0)}+O(N^{-1}).$$

Therefore, taking into account (4.9), we get that, if 3.21 is fulfilled, then

$$\mathrm{Prob}\left\{|c^*|>\frac{\varepsilon_N}{2}\right\}\le e^{-C_1\log^2N}. \tag{4.11}$$

But, using the Laplace method, we get easily

$$\langle\theta(|c-c^*|-\frac{\varepsilon_N}{2})\rangle_{(U,c)}\le e^{-CN\varepsilon_N^2}\le e^{-C\log^2N}.$$

Combining this inequality with (4.7) and using the symmetry with respect to $J_1,\ldots,J_N$, we obtain (3.23).

Proof of Proposition 2 Applying Lemma 2 to the sequences f_{N_m,p_m} and f_{N_m,p_m-1} as a functions of z, we obtain immediately relations (3.33) for R_{N_m,p_m} for all z, where the limiting free energy $f(z,h)$ has continuous first derivative with respect to z. Besides, since for all $\lambda\in(-1,1)$ and arbitrarily small $\delta>0$

$$\begin{aligned}\lambda E\left\{\delta^{-1}\left(f_{N_m,p_m}(z-\delta)-f_{N_m,p_m}(z-2\delta)\right)\right\}&\le E\left\{\log\left\langle\exp\left\{\lambda N_m^{-1}(\boldsymbol{J},\boldsymbol{J})\right\}\right\rangle\right\}\\&\le\lambda E\left\{\left(\delta^{-1}(f_{N_m,p_m}(z+2\delta)-f_{N_m,p_m}(z+\delta)\right)\right\},\end{aligned}$$

we obtain that $E\left\{\log\left\langle\exp\{\lambda(N_m^{-1}(\boldsymbol{J},\boldsymbol{J})\}\right\rangle-\overline{R}_{N_m,p_m})\right\}\to 0$ for all such z and all $\lambda\in(-1,1)$. Using Remark 3, we can derive then that

$$f_m(\lambda)\equiv E\left\{\left\langle\exp\left\{\lambda(N_m^{-1}(\boldsymbol{J},\boldsymbol{J})-\overline{R}_{N_m,p_m})\right\}\right\rangle\right\}\to 1.$$

Then, since it follows from Remark 3 that $f_k^{(3)}(\lambda)$ is bounded uniformly in m and λ, we derive that $f_m''(\lambda)\to 0$ and, taking here $\lambda=0$, obtain (3.35).

To derive relations (3.33) for q_{N_m,p_m} we consider f_{N_m,p_m} and f_{N_m,p_m-1} as a functions of h, derive from Lemma 2 that

$$E\left\{\left(N_m^{-1}(\mathbf{h},\langle\boldsymbol{J}\rangle_{N_m,p_m})-E\left\{N_m^{-1}(\mathbf{h},\langle\boldsymbol{J}\rangle_{N_m,p_m})\right\}\right)^2\right\}\to 0$$

and therefore

$$E\left\{\left(N_m^{-1}(\mathbf{h},\langle\boldsymbol{J}\rangle_{N_m,p_m})-E\left\{N_m^{-1}(\mathbf{h},\langle\boldsymbol{J}\rangle_{N_m,p_m})\right\}\right)N_m^{-1}(\langle\boldsymbol{J}\rangle_{N_m,p_m},\langle\boldsymbol{J}\rangle_{N_m,p_m})\right\}\to 0.$$

Integrating it with respect to h_i, we get

$$\begin{aligned}&E\left\{(q_{N_m,p_m}-\overline{q}_{N_m,p_m}-(R_{N_m,p_m}-\overline{R}_{N_m,p_m}))q_{N_m,p_m}\right\}\\&\qquad=\frac{2}{N_m^2}\sum_{i,j=1}^{N_m}E\left\{\langle J_i\rangle\langle\dot{J}_i\dot{J}_j\rangle\langle J_j\rangle\right\}.\end{aligned}$$

Using relations (3.22) and (3.27) we derive now (3.33) for q_{N_m,p_m}.

Proof of Lemma 3 Let us note that, by the virtue of Lemma 1, computing $\phi_N(\varepsilon_1,k_1)$, $\phi_{0,N}(\varepsilon_1,k_1)$ with probability more than $(1-e^{-C_2\log^4 N})$ we can restrict all the integrals with respect to $\boldsymbol{J}$ by the domain

$$\Omega_N=\left\{|J_i|\le\varepsilon_N N^{1/2},\ (i=1,\ldots,N),(\boldsymbol{J},\boldsymbol{J})\le NR_0^2\right\}$$

In this case the error for $\phi_N(\varepsilon_1,k_1)$ and $\phi_{0,N}(\varepsilon_1,k_1)$ will be of the order $O(Ne^{-C_1\log^2 N})$. So below in the proof of Lemma 3 we denote by $\langle\ldots\rangle_{p-1}$ the Gibbs measure, corresponding to the Hamiltonian $H_{N,p-1}$ in the domain Ω_N. In this case the inequalities (3.22) are also valid, because their l.h.s., comparing with those, computing in the whole $\mathbf{R}^N$, have the errors of the order $O(N^2e^{-C_1\log^2 N})$.

We start from the proof of the first line of (3.38). To this end consider the functions

$$\begin{aligned}&F_N(t)\equiv\left\langle\theta(N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\mu})},\boldsymbol{J})-t)\right\rangle_{p-1},\\&F_{0,N}(t)=\mathrm{H}\left(U_{N,p}^{-1/2}(0)\left(N^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\mu})},\langle\boldsymbol{J}\rangle_{p-1})-t\right)\right),\\&\psi_N(u)\equiv\left\langle\exp\left\{iu(\boldsymbol{\xi}^{(\boldsymbol{\mu})},\dot{\boldsymbol{J}})N^{-1/2}\right\}\right\rangle_{p-1},\\&\psi_{0,N}(u)\equiv\exp\left\{-\tfrac{u^2}{2}(R_{N,p-1}-q_{N,p-1})\right\}.\end{aligned}\tag{4.12}$$

Take $L\equiv\frac{\pi}{4\varepsilon_N}$. According to the Lyapunov theorem (see [Lo]),

$$\max_t|F_N(t)-F_{0,N}(t)|\le\frac{2}{\pi}\int_{-L}^{L}u^{-1}du|\psi_N(u)-\psi_{0,N}(u)|+\frac{\text{const}}{L}.\tag{4.13}$$

Since evidently

$$\begin{aligned}\phi_N(\varepsilon_1,k_1)&=\varepsilon_1^{1/2}\int \mathrm{H}(\varepsilon_1^{-1/2}(k_1-t))dF_N(t),\\ \phi_{0,N}(\varepsilon_1,k_1)&=\varepsilon_1^{1/2}\int \mathrm{H}(\varepsilon_1^{-1/2}(k_1-t))dF_{0,N}(t),\end{aligned}$$

we obtain

$$|\phi_N(\varepsilon_1,k_1)-\phi_{0,N}(\varepsilon_1,k_1)|\le \max_t|F_N(t)-F_{0,N}(t)|\,\mathrm{const}\,. \tag{4.14}$$

Thus, using (4.13), we have got

$$\begin{aligned}E\left\{|\phi_N(\varepsilon_1,k_1)-\phi_{0,N}(\varepsilon_1,k_1)|^2\right\}&\le\ \mathrm{const}\,(\frac{1}{L}+I_1+I_2),\\ I_1\equiv E\left\{\int_1^1 u^{-2}|\psi_N(u)-\psi_{0,N}(u)|^2du\right\},&\\ I_2\equiv\int_{1<|u^{(1)}|,|u^{(2)}|\le L}du^{(1)}du^{(2)}E&\left\{(\psi_N(u^{(1)})-\psi_{0,N}(u^{(1)}))\right.\\ &\left.\cdot(\overline{\psi}_N(u^{(2)})-\overline{\psi}_{0,N}(u^{(2)}))\right\}.\end{aligned}\tag{4.15}$$

Consider

$$\begin{aligned}I_2^{(1)}\equiv E_p&\left\{\int_{1<|u^{(1)}|,|u^{(2)}|<L}du^{(1)}du^{(2)}\psi_N(u^{(1)})\overline{\psi}_N(u^{(2)})\right\}\\ =\textstyle\int_{1<|u^{(1)}|,|u^{(2)}|<L}&du^{(1)}du^{(2)}\left\langle\textstyle\prod_{j=1}^N\cos N^{-1/2}\left(u^{(1)}\dot{J}_j^{(1)}-u^{(2)}\dot{J}_j^{(2)}\right)\right\rangle_{p-1}.\end{aligned}\tag{4.16}$$

We would like to prove that one can substitute the product of $\cos(a_i)$ in (4.16) by the product of $\exp\{-a_i^2/2\}$. So we should estimate

$$\begin{aligned}\varDelta\equiv E\Big\{\textstyle\int_{1<|u^{(1)}|,|u^{(2)}|<L}du^{(1)}du^{(2)}&\Big\langle\Big[\textstyle\prod_{j=1}^N\cos N^{-1/2}\left(u^{(1)}\dot{J}_j^{(1)}-u^{(2)}\dot{J}_j^{(2)}\right)\\ &-\exp\left\{-\tfrac{1}{2N}\sum\left(u^{(1)}\dot{J}_j^{(1)}-u^{(2)}\dot{J}_j^{(2)}\right)^2\right\}\Big]\Big\rangle_{p-1}\Big\}.\end{aligned}\tag{4.17}$$

Let us denote

$$\begin{aligned}g(\tau)\equiv\sum_i\Big(&\log\cos N^{-1/2}\tau\left(u^{(1)}\dot{J}_j^{(1)}-u^{(2)}\dot{J}_j^{(2)}\right)\\ &+\frac{\tau^2}{2N}\sum\left(u^{(1)}J_j^{(1)}-u^{(2)}J_j^{(2)}\right)^2\Big).\end{aligned}$$

Then

$$\begin{aligned}|\varDelta|&=\left|\int_{1<|u^{(1)}|,|u^{(2)}|<L}du^{(1)}du^{(2)}\langle e^{g(1)}-e^{g(0)}\rangle\right|\\ &\le\int_{|u^{(1)}|,|u^{(2)}|<L}du^{(1)}du^{(2)}\Big\langle|g(1)-g(o)|(e^{g(1)}+e^{g(0)})\\ &\qquad\exp\left\{-\tfrac{1}{2N}\sum\left(u^{(1)}\dot{J}_j^{(1)}-u^{(2)}\dot{J}_j^{(2)}\right)^2\right\}\Big\rangle_{p-1}.\end{aligned}\tag{4.18}$$

But since $g(0), g'(0), g''(0), g'''(0) = 0$,

$$\begin{aligned}|g(1)-g(0)| \le \frac{1}{6}|g^{(4)}(\zeta)| \le \frac{\text{const}}{N^2}\sum\left(u^{(1)}\dot J_j^{(1)} + u^{(2)}\dot J_j^{(2)}\right)^4 \\ \le\ \text{const}\,\varepsilon_N^2\left[\left(N^{-1}(\dot{\boldsymbol J}^{(1)},\dot{\boldsymbol J}^{(1)}) + N^{-1}(\dot{\boldsymbol J}^{(2)},\dot{\boldsymbol J}^{(2)})\right)\left(|u^{(1)}|^4+|u^{(2)}|^4\right)\right].\end{aligned}$$

Besides, using the inequality (valid for any $|x| \le \frac{\pi}{2}$)

$$\log\cos x + \frac{x^2}{2} \le \frac{x^2}{6},$$

we obtain that

$$|e^{g(0)} + e^{g(1)}| \le 2\exp\left\{\frac{1}{6N}\sum\left(u^{(1)}\dot J_j^{(1)} + u^{(2)}\dot J_j^{(2)}\right)^2\right\}.$$

Thus, we get from (4.18) $|\varDelta| \le\ \text{const}\,\varepsilon_N^2$. Hence, we have proved that

$$I_2^{(1)} = \int du^{(1)}du^{(2)}\left\langle \exp\left\{-\frac{1}{2}\sum_{l,m=1}^{2} A_{l,m}^{(1)}u^{(l)}u^{(m)}\right\}\right\rangle_{p-1}^{(1,2)} + O(\varepsilon_N^2), \qquad (4.19)$$

where

$$A_{l,l}^{(1)} = \frac{1}{N}(\dot{\boldsymbol J}^{(l)},\dot{\boldsymbol J}^{(l)}), \quad (l=1,2)\ \ A_{1,2}^{(1)} = \frac{1}{N}(\dot{\boldsymbol J}^{(1)},\dot{\boldsymbol J}^{(2)}).$$

Now, taking into account that Proposition 2 implies

$$\sum_{m,l=1,2} E\left\{\left\langle (A_{l,m}^{(1)} - A_{l,m})^2\right\rangle_{p-1}^{(1,2)}\right\} \to 0,\ (N\to\infty),$$

where $A_{l,m} = \delta_{l,m}(R_{N,p-1} - q_{N,p-1})$, we obtain immediately that

$$\begin{aligned}&\int_{1<|u^{(1)}|,|u^{(2)}|<L} du^{(1)}du^{(2)}E\left\{\psi_N(u^{(1)})\overline{\psi}_N(u^{(2)})\right\} \\ &= \int_{1<|u^{(1)}|,|u^{(2)}|<L} du^{(1)}du^{(2)}E\left\{\psi_{0,N}(u^{(1)})\overline{\psi}_{0,N}(u^{(2)})\right\} + o(1).\end{aligned}$$

By the same way one can prove also

$$\begin{aligned}&\Re\int_{1<|u^{(1)}|,|u^{(2)}|<L} du^{(1)}du^{(2)}E\left\{\psi_N(u^{(1)})\overline{\psi}_{0,N}(u^{(2)})\right\} \\ &= \int_{1<|u^{(1)}|,|u^{(2)}|<L} du^{(1)}du^{(2)}E\left\{\psi_{0,N}(u^{(1)})\overline{\psi}_{0,N}(u^{(2)})\right\} + o(1),\end{aligned}$$

which gives us that $I_2 = o(1)$. Similarly one can prove that $I_1 = o(1)$. Then, using (4.15), we obtain the first line of (3.38).

To prove the second line of (3.38) we denote by $A \equiv (\phi_N(\varepsilon_1, k_1))$, $B \equiv (\phi_{0,N}(\varepsilon_1, k_1))$, $\tilde{\varepsilon}_N \equiv E\{(A-B)^2\}$, $\tilde{L} \equiv |\log \tilde{\varepsilon}_N| \tilde{\varepsilon}_N^{-1/2}$ and write

$$\begin{gathered} E\left\{|\log A - \log B|^2\right\} \le E\Bigl\{\theta(\tilde{L} - A^{-1})\theta(\tilde{L} - B^{-1})(|\log A - \log B|^2\Bigr\} \\ +2E\Bigl\{(\theta(\tilde{L} - A^{-1}) + \theta(\tilde{L} - B^{-1})))(\log^2 A + \log^2 B)\Bigr\} \le \\ 4\tilde{L}^{-2}E\left\{(A-B)^2\right\} + 4|\log \tilde{L}|^{-2}E\left\{(\log^4 A + \log^4 B)\right\} \\ \le 4\tilde{\varepsilon}_N \tilde{L}^{-2} + |\log \tilde{L}|^{-2}\,\mathrm{const} \;\le\; \mathrm{const}\,|\log \tilde{L}|^{-3/2}. \end{gathered} \tag{4.20}$$

Here we have used the inequality

$$|\log A - \log B| \le |A - B|(A^{-1} + B^{-1}),$$

the first line of (3.38) and the fact that $E\{\log^4 A\}$, $E\{\log^4 B\}$ are bounded (it can be obtained similarly to (3.28)-(3.29)). Since we have proved above that $\tilde{\varepsilon}_N \to 0$, as $N \to \infty$, inequality (4.20) implies the second line of (3.38). The third and the fourth line of (3.38) can be derived in the usual way (see e.g. [P-S-T2]) from the second line by using the fact that functions $\log \phi_N(\varepsilon_1, k_1)$ and $\log \phi_{0,N}(\varepsilon_1, k_1)$ are convex with respect to ε_1^{-1} and k_1.

The convergence in distribution $N^{-1/2}(\boldsymbol{\xi^{(p)}}, \langle \boldsymbol{J} \rangle_{p-1}) \to \sqrt{q_{N,p}}u$ follows from the central limit theorem (see, e.g. the book [Lo]), because $\langle \boldsymbol{J} \rangle_{p-1}$ does not depend on $\boldsymbol{\xi^{(p)}}$ and the Lindenberg condition is fulfilled:

$$\frac{1}{N^2}\sum_i \langle J_i \rangle_{p-1}^4 \le \mathrm{const}\,\varepsilon_N^2.$$

Thus, to finish the proof of Lemma 3 we are left to prove (3.40). It can be easily done, e.g. for $\mu = p$ and $\nu = p-1$, if we in the same manner as above consider the functions

$$\begin{gathered} \phi_N^{(2)}(\varepsilon_1, \varepsilon_2, k_1, k_2) \equiv \int_{x_1, x_2 > 0} dx_1 dx_2 \Bigl\langle \exp\Bigl\{-\frac{1}{2\varepsilon_1}(N^{-1/2}(\boldsymbol{\xi^{(p)}}, \boldsymbol{J}) - x_1 - k_1)^2 \\ -\frac{1}{2\varepsilon_2}(N^{-1/2}(\boldsymbol{\xi^{(p)}}, \boldsymbol{J}) - x_2 - k_2)^2\Bigr\}\Bigr\rangle_{p-1} \end{gathered} \tag{4.21}$$

$$\begin{gathered} \phi_{0,N}^{(2)}(\varepsilon_1, \varepsilon_2, k_1, k_2) \\ \equiv (\varepsilon_1\varepsilon_2)^{1/2}\mathrm{H}\Bigl(\frac{N^{-1/2}(\boldsymbol{\xi^{(p)}}, \langle \boldsymbol{J} \rangle_{p-2}) - k_1}{U_{N,p-2}^{1/2}(\varepsilon_1)}\Bigr)\mathrm{H}\Bigl(\frac{N^{-1/2}(\boldsymbol{\xi^{(p-1)}}, \langle \boldsymbol{J} \rangle_{p-2}) - k_2}{U_{N,p-2}^{1/2}(\varepsilon_2)}\Bigr) \end{gathered} \tag{4.22}$$

and prove for them analog of relations (3.38). Then relations (3.40) will follow immediately. The self-averaging property for $\tilde{U}_N$ and $\tilde{q}_N$ follows from the fact that $\phi_{0,N}^{(2)}(\varepsilon_1, \varepsilon_2, k_1, k_2)$ is a product of two independent functions.

Proof of Proposition 3. It is easy to see, that equations (3.54) have the form

$$\frac{\partial F}{\partial q} = O(\tilde{\varepsilon}_N), \quad \frac{\partial F}{\partial R} = O(\tilde{\varepsilon}'_N), \tag{4.23}$$

where $F(q,R)$ is defined by the expression in the square brackets in the r.h.s. of (2.15).

Let us make the change of variables $s = q(R+\varepsilon-q)^{-1}$. Then equations (4.23) take the form

$$\frac{\partial \tilde{F}}{\partial s} = O(\overline{\varepsilon}_N), \quad \frac{\partial \tilde{F}}{\partial R} = O(\overline{\varepsilon}_N), \tag{4.24}$$

where $\overline{\varepsilon}_N = |\tilde{\varepsilon}_N| + |\tilde{\varepsilon}'_N|$ and

$$\begin{gathered}\tilde{F}(s,R) \equiv \alpha E\left\{\log \mathrm{H}\left(u\sqrt{s} + \frac{k\sqrt{1+s}}{\sqrt{\varepsilon+R}}\right)\right\} \\ +\frac{1}{2}\frac{s(R+\varepsilon)}{R-\varepsilon s} + \frac{1}{2}\log(R-\varepsilon s) - \frac{1}{2}\log(1+s) - \frac{z}{2}R + \frac{h^2}{2}\frac{R-\varepsilon s}{1+s}.\end{gathered} \tag{4.25}$$

Then (4.24) can be written in the form

$$\begin{aligned} f_1(s,R) &\equiv -\frac{\alpha}{s}E\left\{\mathrm{A}^2\right\} + \frac{(R+\varepsilon)^2}{(R-\varepsilon s)^2} - \frac{h^2}{s(s+1)}(R+\varepsilon) = O(\overline{\varepsilon}_N), \\ f_2(s,R) &\equiv \frac{\alpha k\sqrt{1+s}}{(R+\varepsilon)^{3/2}}E\left\{\mathrm{A}\right\} - \frac{\varepsilon s(s+1)}{(R-\varepsilon s)^2} + \frac{1}{R-\varepsilon s} + \frac{h^2}{s+1} - z = O(\overline{\varepsilon}_N), \end{aligned} \tag{4.26}$$

where the function $A(x)$ is defined by (3.51) and to simplify formulae we here and below omit the arguments of functions A and A'. But

$$\begin{gathered}\frac{\partial}{\partial s}f_1(s,R) = -\frac{\alpha}{s^2}E\left\{\left(u\sqrt{s} + \frac{k\sqrt{1+s}}{\sqrt{\varepsilon+R}}\right)\mathrm{A}'\mathrm{A}\right\} + \frac{\alpha}{s^2}E\left\{\mathrm{A}^2\right\} \\ +\frac{\alpha k}{s^2(1+s)^{1/2}(\varepsilon+R)^{1/2}}E\left\{\mathrm{A}'\mathrm{A}\right\} + \frac{2(R+\varepsilon)^2\varepsilon}{(R-\varepsilon s)^3} + \frac{h^2(2s+1)}{s^2(s+1)^2}(R+\varepsilon) > \frac{h^2}{s^2}R.\end{gathered} \tag{4.27}$$

Here we have used the inequality (we prove it below):

$$x\mathrm{A}'(x)\mathrm{A}(x) \le A^2(x), \tag{4.28}$$

which gives us that the sum of the first two terms in (4.27) is positive. Therefore we conclude, that equation $\frac{\partial \tilde{F}}{\partial s}(s,R) = 0$ for any R has a unique solution $s = s(R)$ and, if we consider the first of equations (4.24), then its solution $s_1(R)$ for any R behave like

$$s_1(R) = s(R) + O(\overline{\varepsilon}_N). \tag{4.29}$$

For $k = 0$ the second equation in (4.26) is quadratic with respect to $(R-\varepsilon s)$, and so we can easily obtain that the system (4.26) for $z < \varepsilon^{-1/3}$ has the unique solution. Consider now the case, when $k \ne 0$. Then the function $f_2(s,R)$ for $s >> 1$ behaves like

$$f_2(s,R) \sim \frac{s}{R+\varepsilon}D + \frac{1}{R-\varepsilon s} - z,$$

where

$$\begin{gathered} D \equiv \alpha\tilde{k}I_1(\tilde{k}) - \alpha I_2(\tilde{k}) + \sqrt{\alpha I_2(\tilde{k})},\ \tilde{k} = k(R+\varepsilon)^{-1/2} \\ I_{1,2}(\tilde{k}) \equiv \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{-\tilde{k}}^{\infty}(u+\tilde{k})^{1,2}e^{-u^2/2}du. \end{gathered}$$

Since D can be represented in the form

$$D = \frac{\alpha \mathrm{H}(-\tilde{k})}{\alpha \mathrm{H}(-\tilde{k}) + \sqrt{\alpha I_2(-\tilde{k})}} \left[(\tilde{k}\mathrm{A}(-\tilde{k}) + \tilde{k}^2 + 1 - 2\mathrm{H}(-\tilde{k})) + (2-\alpha)\mathrm{H}(-\tilde{k}) \right] > 0$$

(we have checked that $\tilde{k}\mathrm{A}(-\tilde{k}) + \tilde{k}^2 + 1 - 2\mathrm{H}(-\tilde{k}) \geq 0$ numerically), we get from (4.26) that the inequality $z \leq \varepsilon^{-1/3}$ implies that $s \leq$ const $R\varepsilon^{-1/3}$. On the other hand,

$$2\frac{\partial^2 \tilde{F}}{\partial R^2}(s,R) = -\frac{3\alpha k\sqrt{s+1}}{2(R+\varepsilon)^{5/2}} E\left\{\mathrm{A}\right\} - \frac{\alpha k^2(s+1)}{2(R+\varepsilon)^3} E\left\{\mathrm{A}'\right\} - \frac{R - 3\varepsilon s - 2s^2\varepsilon}{(R-\varepsilon s)^3}. \tag{4.30}$$

We would like to remark here, that the case when s is bounded and $R \sim \varepsilon$ is impossible for $k \neq 0$ due to the first equation in (4.26). Thus, for ε small enough we get

$$\frac{\partial^2 \tilde{F}}{\partial R^2}(s,R) < -\frac{1}{2(R-\varepsilon s)^2}. \tag{4.31}$$

Now, if we consider the function $\varphi(R) \equiv \tilde{F}(s(R),R)$ it is obviously concave and therefore the equation $\varphi'(R) = 0$ has the unique solution R^* which is a maximum of $\varphi(R)$. Besides, since in view of (4.31) $\varphi''(R) < 0$, $R(\tilde{\varepsilon})$ - the solution of equation $\varphi'(R) = \tilde{\varepsilon}$ has the form $R(\tilde{\varepsilon}) = R^* + O(\tilde{\varepsilon})$. But in view of (4.29) the second equation of (4.24) can be rewritten in the form

$$\varphi'(R) = O(\tilde{\varepsilon}_N) + O(\overline{\varepsilon}_N).$$

Therefore its solution tends to R^* as $\overline{\varepsilon}_N \to 0$.

Now to finish the proof of Proposition 3 we are left to prove (4.28). For $x \leq 0$ it is evidently fulfilled. For $x > 0$ let us write

$$\begin{aligned} x\mathrm{A}'(x)\mathrm{A}(x) &= x(\mathrm{A}(x) - x)\mathrm{A}^2(x) \leq x\tfrac{\sqrt{x^2+4}-x}{2}\mathrm{A}^2(x) \\ &= \frac{2x}{\sqrt{x^2+4}+x}\mathrm{A}^2(x) \leq \mathrm{A}^2(x), \end{aligned}$$

where we have used the well known inequality (see, e.g.,[A-S])

$$\mathrm{A}(x) \leq \frac{\sqrt{x^2+4}+x}{2}.$$

Proposition 3 is proven.

Proof of Proposition 4.
One can see easily that, if we want to study $\min_z\{F(\alpha,k,0,z,\varepsilon) + \frac{z}{2}\}$, then we should consider the system (4.26) with zeros in the r.h.s. and with the additional equation

$$\frac{\partial}{\partial z}F(\alpha,k,0,z,\varepsilon) = 1 \Leftrightarrow R = 1$$

Thus, we need to substitute $R = 1$, in the first equation. Since the l.h.s. of this equation for $\varepsilon = 0$ is an increasing function which tends to $1 - \alpha\alpha_c^{-1} > 0$, as $s \to \infty$, there exist the unique s^*, which is the solution of this equation.

Then, choosing ε small enough, it is easy to obtain, that $s(\varepsilon)$ is in some ε-neighbourhood of s^* and therefore $s(\varepsilon) \le \overline{s}(k,\alpha)$. Then, substituting this $s(\varepsilon)$ in the second equation, we get the ε-independent bound for z.

Proof of Lemma 4. Repeating conclusions (3.3)-(3.6) of the proof of Theorem 1, one can see that

$$\langle \theta(x^{(\mu)} - k) \rangle_\mu = \langle \theta(c - kN^{-1/2}) \rangle_{(U,c)}, \tag{4.32}$$

where $\langle \ldots \rangle_{(U,c)}$ are defined by (3.7) (see also (3.3), (3.5) for $\varGamma_N = \varOmega_{N,p}^{(\mu-1)}$, $\varPhi_N = \mathcal{H}_{N,p}^{(\mu)}$ and $c = N^{-1} \sum \xi_i^{(\mu)} J_i$. We denote $\phi_N^{(\mu)}(c,U) \equiv (s_N^{(\mu)}(c,U) - U - (s_N^{(\mu)}(c^*,U^*) - U^*))$, where $s_N^{(\mu)}(c,U)$ is defined by (3.5) and (c^*,U^*) is the point of maximum of the function $s_N^{(\mu)}(c,U) - U$.

Applying Theorem 1, we found that $s_N^{(\mu)}(c,U)$ is a concave function of (c,U) and it satisfies (3.14).

Denote

$$\varLambda_M \equiv \{(U,c) : N\phi_N^{(\mu)}(c,U) \ge M\}, \ \ \varPi_{c^*,\tilde{c}'} \equiv \{(U,c) : c^* \le c \le \tilde{c}'\}, \tag{4.33}$$

and let for any measurable $\mathcal{B} \subset \mathbf{R}^2$ $m(\mathcal{B}) \equiv \langle \chi_{\mathcal{B}}(c,U) \rangle_{(U,c)}$.

To prove Lemma 4 we use the following statement:

Proposition 5. *If the function $\phi_N^{(\mu)}(c,U)$ is concave and satisfies inequality (3.14), $\tilde{c}, \tilde{c}' > c^*$, and the constant $A \le -\frac{N^{1/2}}{2(\tilde{c}-c^*)} \max_U \phi_N^{(\mu)}(\tilde{c},U)$, then*

$$\frac{\langle \theta(c-\tilde{c}) e^{AN^{1/2}c} \rangle_{(U,c)}}{\langle \theta(c-\tilde{c}) \rangle_{(U,c)}} \le 2e^{\sqrt{N}A\tilde{c}}, \tag{4.34}$$

and for any $M < -4$

$$m(\overline{\varLambda}_M) \le \frac{1}{4}, \quad \frac{m(\overline{\varLambda}_M \cap \varPi_{c^*,\tilde{c}'})}{m(\varLambda_M \cap \varPi_{c^*,\tilde{c}'})} \le \frac{1}{4}. \tag{4.35}$$

The proof of this Proposition is given after the proof of Lemma 4.

Let us choose any $\tilde{c} > c^*$ and $A = -\frac{N^{1/2}}{2(\tilde{c}-c^*)} \max_U \phi_N^{(\mu)}(\tilde{c},U)$. Using (4.34), we get

$$\begin{aligned} \left\langle e^{AN^{1/2}(c-\tilde{c})} \right\rangle_{(U,c)} &= \langle \theta(c_2 - c) \rangle_{(U,c)} + \frac{\langle \theta(c-\tilde{c}) e^{AN^{1/2}c} \rangle_{(U,c)}}{\langle \theta(c-\tilde{c}) \rangle_{(U,c)}} \langle \theta(c - c_2) \rangle_{(U,c)} \\ &\le \langle \theta(c_2 - c) \rangle_{(U,c)} + 2\langle \theta(c - c_2) \rangle_{(U,c)} \le 2 \end{aligned} \tag{4.36}$$

On the other hand, we shall prove below

Proposition 6. *For any $|A| \le O(\log N)$*

$$\begin{aligned} g(A) \equiv \log \left\langle \exp\{AN^{1/2}(c - \langle c \rangle)\} \right\rangle_{(U,c)} &= \log \left\langle \exp\{AN^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \dot{\boldsymbol{J}})\} \right\rangle_\mu \\ &= \tfrac{A^2}{2N} \left\langle (\dot{\boldsymbol{J}}, \dot{\boldsymbol{J}}) \right\rangle_\mu + R_N, \quad E\left\{R_N^4\right\} = O(A^{16}N^{-2}). \end{aligned} \tag{4.37}$$

It follows from this proposition that the probability to have for all $A_i = \pm 1, \ldots, \pm[\log N]$ the inequalities

$$e^{A_i^2 R_0^2} \geq \left\langle \exp\left\{A_i N^{1/2}(c - \langle c\rangle)\right\}\right\rangle_{(U,c)} \geq e^{A_i^2 D^2/4} \tag{4.38}$$

is more than $P_N' \geq 1 - O(N^{-3/2})$. Therefore, using that $\log\langle\exp\{AN^{1/2}(c - \langle c\rangle)\}\rangle_{(U,c)}$ is a convex function of A, and this function is zero for $A = 0$, one can conclude that with the same probability for any $A: \ 1 \leq |A| \leq \log N$

$$e^{2A^2 R_0^2} \geq \left\langle \exp\left\{AN^{1/2}(c - \langle c\rangle)\right\}\right\rangle_{(U,c)} \geq e^{A^2 D^2/8}. \tag{4.39}$$

The first of these inequalities implies, in particular, that for any $0 < L < \log N$

$$\begin{gathered}\langle\theta(\langle c\rangle - LN^{-1/2} - c)\rangle_{(U,c)} \\ \leq \max\nolimits_{A>0} \left\langle \exp\left\{AN^{1/2}(\langle c\rangle - LN^{-1/2} - c)\right\}\right\rangle_{(U,c)} \leq e^{-L^2/8R_0^2}.\end{gathered} \tag{4.40}$$

The same bound is valid for $\langle\theta(c - \langle c\rangle - LN^{-1/2})\rangle_{(U,c)}$. Thus, assuming that $\langle c\rangle > c^*$ and denoting $L_0 = \frac{1}{2}N^{1/2}(\langle c\rangle - c^*)$, $c_1 \equiv \langle c\rangle - 2L_0N^{-1/2} = c^*$, $c_2 \equiv \langle c\rangle - L_0N^{-1/2}$, $c_3 \equiv \langle c\rangle + L_0N^{-1/2}$ we can write

$$\begin{gathered}1 = \langle\theta(c_1 - c)\rangle_{(U,c)} + \langle\chi_{c_1,c_3}(c)\rangle_{(U,c)} + \langle\theta(c - c_3)\rangle_{(U,c)} \leq 4e^{-L_0^2/8R_0^2}. \\ \Rightarrow N|\langle c\rangle - c^*|^2 = 4L_0^2 \leq 16R_0^2.\end{gathered} \tag{4.41}$$

Here we have used (4.40) and the fact that since $\phi_N^{(\mu)}(U, c)$ is a concave function and (U^*, c^*) is the point of its maximum, we have for any $d > 0$ and $\tilde{c} > c^*$

$$\begin{gathered}\langle\chi_{\tilde{c},\tilde{c}+d}(c)\rangle_{(U,c)} \leq \langle\chi_{c^*,c^*+d}(c)\rangle_{(U,c)} \Rightarrow \\ \langle\chi_{c_2,\langle c\rangle}(c)\rangle_{(U,c)}, \langle\chi_{\langle c\rangle,c_3}(c)\rangle_{(U,c)} \leq \langle\chi_{c^*,c_2}(c)\rangle_{(U,c)} \leq \langle\theta(c^* - c)\rangle_{(U,c)} \leq e^{-L_0^2/8R_0^2}.\end{gathered} \tag{4.42}$$

The case $\langle c\rangle < c^*$ can be studied similarly. We would like to stress here, that Theorem 1 also allows us to estimate $N|\langle c\rangle - c^*|^2$, but this estimate can depend on ε.

Now let us come back to (4.36). In view of (4.39) for our choice of A

$$\begin{gathered}\frac{A^2D^2}{8} - AN^{1/2}(\tilde{c} - \langle c\rangle) \leq \log 2 \Rightarrow A \leq \frac{8N^{1/2}(\tilde{c} - \langle c\rangle) + 4D}{D^2} \\ \Rightarrow \max_U \phi_N^{(\mu)}(\tilde{c}, U) \geq -2\frac{7(\tilde{c} - \langle c\rangle)^2 + 3(\langle c\rangle - c^*)^2}{D^2} - \frac{4}{N} \geq -14\frac{(\tilde{c} - \langle c\rangle)^2}{D^2} - \frac{K_0}{N}\end{gathered} \tag{4.43}$$

with some N, μ, ε-independent K_0.

Let us take $L_1 = 8R_0$ and $\tilde{c} > \langle c\rangle + L_1N^{-1/2}$. Consider $\tilde{M}(\tilde{c}) \equiv N \max_U \phi_N^{(\mu)}(\langle c\rangle + 2(\tilde{c} - \langle c\rangle), U)$

If $\tilde{M}(\tilde{c}) < -4$, consider the sets

$$\Pi_1 \equiv \{(U, c) : c > \tilde{c}\}, \quad \Pi_2 \equiv \{(U, c) : \langle c\rangle - L_1N^{-1/2} \leq c \leq \tilde{c}\}. \tag{4.44}$$

Applying (4.35) and (4.40), we get

$$\begin{aligned} &m(\Pi_1\cup\Pi_2)\ge\frac{3}{4},\ m(\Lambda_{\tilde M(\tilde c)})\ge\frac{3}{4}\\ &\Rightarrow m(\Lambda_{\tilde M(\tilde c)}\cap(\Pi_1\cup\Pi_2))\ge\frac{1}{2}\ge m(\overline{\Lambda}_{\tilde M(\tilde c)}\cup(\overline{\Pi}_1\cap\overline{\Pi}_2))\\ &\Rightarrow\langle\theta(c-\tilde c)\rangle_{(U,c)}\ge\frac{m(\Lambda_{\tilde M(\tilde c)}\cap\Pi_1)}{m(\Lambda_{\tilde M(\tilde c)}\cap(\Pi_1\cup\Pi_2))+m(\overline{\Lambda}_{\tilde M(\tilde c)}\cup(\overline{\Pi}_1\cap\overline{\Pi}_2))}\\ &\ge\frac{m(\Lambda_{\tilde M(\tilde c)}\cap\Pi_1)}{2(m(\Lambda_{\tilde M(\tilde c)}\cap\Pi_1)+m(\Lambda_{\tilde M(\tilde c)}\cap\Pi_2))}\ge\frac{1}{2(1+e^{-\tilde M(\tilde c)}S_2S_1^{-1})}, \end{aligned}\tag{4.45}$$

where we denote by $S_{1,2}$ the Lebesgue measure of $\Lambda_{\tilde M(\tilde c)}\cap\Pi_{1,2}$, and use the fact that $0\ge N\phi_N^{(\mu)}(U,c)\ge\tilde M(\tilde c)$.

Consider the point $(\langle c\rangle+2(\tilde c-\langle c\rangle),U_1)$, found from the condition $N\phi_N^{(\mu)}(\langle c\rangle+2(\tilde c-\langle c\rangle),U_1)=\tilde M(\tilde c)$ and two points $(\tilde c,U_2)$, $(\tilde c,U_3)$ which belong to the boundary of $\Lambda_{\tilde M(\tilde c)}$. Since $\Lambda_{\tilde M(\tilde c)}$ is a convex set, if we draw two straight lines through the first and the second and the first and the third points and denote by T the domain between these lines, then $T\cap\Pi_1\subset\Lambda_{\tilde M(\tilde c)}\cap\Pi_1$ and $\Lambda_{\tilde M(\tilde c)}\cap\Pi_2\subset T\cap\Pi_2$. Therefore

$$\frac{S_1}{S_2}\ge\frac{(\tilde c-\langle c\rangle)^2}{(2(\tilde c-\langle c\rangle)+L_1)^2-(\tilde c-\langle c\rangle)^2}\ge\frac{1}{8}.\tag{4.46}$$

Thus, we derive from (4.45):

$$\langle\theta(c-\tilde c\rangle_{(U,c)}\ge\frac{e^{\tilde M(\tilde c)}}{2e^{\tilde M(\tilde c)}+16}.\tag{4.47}$$

If $\tilde M(\tilde c)>-4$, let us chose $c_1>c^*$, which satisfies condition $N\max_U\phi_N^{(\mu)}(2c_1,U)=-4$ $(c_1>\langle c\rangle+2(\tilde c-\langle c\rangle))$. Replacing in the above consideration $\Lambda_{\tilde M(\tilde c)}$ by Λ_{-4}, we finish the proof of the first line of (3.59).

To prove the second line of (3.59) we choose any $c_1>c^*+L_1N^{-1/2}$, which satisfies the condition $N\max_U\phi_N^{(\mu)}(2c_1,U)<-4$, denote $d=2\varepsilon^{1/4}N^{-1/2}$ and write similarly to (4.45)

$$\begin{aligned} \langle\chi_{c^*,c^*+d}(c)\rangle_{(U,c)}&\le\frac{m(\Lambda_{-4}\cap\Pi_{c^*,c^*+d})+m(\overline{\Lambda}_{-4}\cap\Pi_{c^*,c^*+d})}{m(\Lambda_{-4}\cap\overline{\Pi}_{c^*,c^*+d})}\\ &\le\frac{5m(\Lambda_{-4}\cap\Pi_{c^*,c^*+d})}{4m(\Lambda_{-4}\cap\overline{\Pi}_{c^*,c^*+d})}\\ \le\frac{5e^4\tilde S_2}{4\tilde S_1}\le\frac{5e^4}{4}&\frac{(c_1-c^*)^2-(c_1-c^*-d)^2}{(c_1-c^*-d)^2}\le\varepsilon^{1/4}C_3^*, \end{aligned}\tag{4.48}$$

where we denote by $\tilde S_{1,2}$ the Lebesgue measures of $\Lambda_{-4}\cap\overline{\Pi}_{c^*,c^*+d}$ and $\Lambda_{-4}\cap\Pi_{c^*,c^*+d}$ respectively. Now, using the first line of (4.42), we obtain the second line of (3.59). Lemma 4 is proven.

Proof of Proposition 5

Let us introduce new variables $\rho\equiv\sqrt{(c-c^*)^2+(U-U^*)^2}$,

$\varphi \equiv \arcsin \frac{U-U^*}{\sqrt{(c-c^*)^2+(U-U^*)^2}}$. Then $\phi_N^{(\mu)}(\rho,\varphi)$ for any φ is a concave function of ρ. Let $r(\varphi)$ be defined from the condition $N\phi_N^{(\mu)}(r(\varphi),\varphi) = M$. Consider $\phi_M(\rho,\varphi) \equiv r^{-1}(\varphi)\cdot\phi_N^{(\mu)}(r(\varphi),\varphi)\rho$. Since $\phi_N^{(\mu)}(\rho,\varphi)$ is concave, we obtain that

$$\begin{array}{c} \phi_N^{(\mu)}(\rho,\varphi) \geq \phi_M(\rho,\varphi),\, 0 \leq \rho \leq r(\varphi), \\ \phi_N^{(\mu)}(\rho,\varphi) \leq \phi_M(\rho,\varphi),\, \rho \geq r(\varphi). \end{array} \tag{4.49}$$

Thus, denoting by R the l.h.s. of the first inequality in (4.35), we get

$$\begin{aligned} R &\leq \frac{\int d\varphi \int_{\rho>r(\varphi)} d\rho \exp\{N\phi_N^{(\mu)}(\rho,\varphi)\}}{\int d\varphi \int_{\rho<r(\varphi)} d\rho \exp\{\phi_N^{(\mu)}(\rho,\varphi)\}} \\ &\leq \frac{\int d\varphi \int_{\rho>r(\varphi)} d\rho \exp\{N\phi_M(\rho,\varphi)\}}{\int d\varphi \int_{\rho<r(\varphi)} d\rho \exp\{N\phi_M(\rho,\varphi)\}} \leq \frac{(1-M)e^M}{1-(1-M))e^M} \leq \frac{1}{4}. \end{aligned}$$

For the second inequality in (4.35) the proof is the same. To obtain (4.34) let us remark first that due to the choice of A the function $\phi_{\tilde{c}}(\rho,\varphi) \equiv \phi_N^{(\mu)}(\rho,\varphi) + N^{-1/2}A\rho\cos\varphi$ for any φ is a concave function of ρ, whose derivative at the point $\rho = \rho_\varphi \equiv \tilde{c}|\cos\varphi|^{-1}$ satisfies the condition

$$\frac{d}{d\rho}\phi_{\tilde{c}}(\rho_\varphi,\varphi) \leq \frac{d}{d\rho}\phi_N^{(\mu)}(\rho_\varphi,\varphi) - \frac{1}{2}\frac{\phi_N^{(\mu)}(\rho_\varphi,\varphi)}{\rho_\varphi} \leq \frac{1}{2}\frac{d}{d\rho}\phi_N^{(\mu)}(\rho_\varphi,\varphi).$$

Thus, for any φ we can write

$$\frac{\int_{\rho>\rho_\varphi} d\rho e^{N\phi_N^{(\mu)}(\rho,\varphi)} e^{AN^{1/2}(\cos\varphi\rho-\tilde{c})}}{\int_{\rho>\rho_\varphi} e^{N\phi_N^{(\mu)}(\rho,\varphi)}} \leq \frac{|\frac{d}{d\rho}\phi_N^{(\mu)}(\rho_\varphi,\varphi) + AN^{-1/2}\cos\varphi|^{-1}}{|\frac{d}{d\rho}\phi_N^{(\mu)}(\rho_\varphi,\varphi)|^{-1}} \leq 2.$$

This inequality implies (4.34).

Proof of Proposition 6. To prove Proposition 6 we use the method, developed in [P-S-T2]. Consider the function $g(A)$ defined by (4.37) and let us write the Taylor expansion up to the second order with respect to t for $g(tA)$ $(t \in [o,1])$. Then

$$\begin{aligned} R_N &= A^2 \textstyle\int_0^1 dt(1-t)g''(tA)dt - \frac{1}{2}A^2 g''(0) \\ &= A^3 \textstyle\int_0^1 dt(1-t) \int_0^t dt_1 N^{-3/2} \sum \xi_i^{(\mu)} \langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_i\rangle_{\mu,t_1} \\ &+A^2 \textstyle\int_0^1 dt(1-t)N^{-1} \sum_{i\neq j} \xi_i^{(\mu)}\xi_j^{(\mu)} \langle\dot{J}_i\dot{J}_j\rangle_{\mu,t} \equiv R_N^{(1)} + R_N^{(2)}, \end{aligned} \tag{4.50}$$

where we denote

$$\langle\ldots\rangle_{\mu,t} \equiv \frac{\left\langle(\ldots)\exp\{tAN^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\mu})},\boldsymbol{J})\}\right\rangle_\mu}{\left\langle\exp\{tAN^{-1/2}(\boldsymbol{\xi}^{(\boldsymbol{\mu})},\boldsymbol{J})\}\right\rangle_\mu}.$$

Let us estimate

$$\begin{gathered}E\{(R_N^{(1)})^4\} \le A^{12}N^{-6}\int_0^1 dt\Big(\sum_{i_1\ne i_2\ne i_3\ne i_4} E\{\xi_{i_1}^{(\mu)}\xi_{i_2}^{(\mu)}\xi_{i_3}^{(\mu)}\xi_{i_4}^{(\mu)}\\ \langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_2}\rangle_{\mu,t}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_3}\rangle_{\mu,t}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_4}\rangle_{\mu,t}\}\\ +6\sum_{i_1\ne i_2\ne i_3} E\{\xi_{i_2}^{(\mu)}\xi_{i_3}^{(\mu)}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}^2\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_2}\rangle_{\mu,t}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_3}\rangle_{\mu,t}\}\\ 3\sum_{i_1\ne i_2} E\{\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}^2\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_2}\rangle_{\mu,t}^2\}\\ 4\sum_{i_1\ne i_2} E\{\xi_{i_1}^{(\mu)}\xi_{i_2}^{(\mu)}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}^3\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_2}\rangle_{\mu,t}\} + \sum_{i_1} E\{\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}^4\}\Big).\end{gathered} \tag{4.51}$$

Now, using the formula of integration by parts (3.43), taking into account that in our case $\frac{\partial}{\partial\xi_i^{(\mu)}} = Ath^{-1}N^{-1/2}\frac{\partial}{\partial h_i}$, and then using integrations by parts with respect to the Gaussian variable h_i, one can substitute

$$\begin{gathered}E\{\xi_i^{(\mu)}\langle\ldots\rangle_{t,\mu}\} \to Ath^{-1}N^{-1/2}E\{h_i\langle\ldots\rangle_{t,\mu}\}\\ +N^{-3/2}A^3O(E\{\langle(\dot{J}_i)^2(\ldots)\rangle_{t,\mu}\}).\end{gathered} \tag{4.52}$$

Thus, for the first sum in (4.51), we obtain

$$\begin{gathered}E\{\Sigma_1\} \le h^{-4}A^{16}N^{-8}\int_0^1 dtE\left\{\left(\sum_{i_1} h_{i_1}\langle(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}})\dot{J}_{i_1}\rangle_{\mu,t}\right)^4\right\} + O(A^{18}N^{-3})\\ \le h^{-4}A^{16}N^{-2}\int_0^1 dtE\left\{\left(N^{-1}\sum_{i,j} h_ih_j\langle\dot{J}_i\dot{J}_j\rangle_{\mu,t}\right)^2\langle(N^{-1}(\dot{\boldsymbol{J}},\dot{\boldsymbol{J}}))^2\rangle^2\right\}\\ \le \text{ const}\, A^{16}N^{-2}.\end{gathered} \tag{4.53}$$

Here to estimate the errors term in (4.52) we use that, according to Theorem 1 (see (2.8)), for any fixed p $E\{\langle\dot{J}_i^p\rangle_{\mu,t}\}$ is bounded by N-independent constant.

Other sums in the r.h.s. of (4.51) and $E\{(R_N^{(1)})^4\}$ can be estimated similarly to (4.53).

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