Fibre Bundles in Quantum Physics

R. N. Sen*
Clare Hall, Cambridge University, Cambridge CB3 9AL, England

G. L. Sewell†
Department of Physics, Queen Mary, University of London,
Mile End Road, London E1 4NS, England

Abstract

The theory of fibre bundles provides a natural setting for the description of macroscopic quantum systems, wherein their classical and quantum features are represented by actions on the base manifolds and the fibres, respectively, of the relevant bundles. We provide realizations of this picture in the description of (a) quasi-particle excitations of many-body systems, especially those in superfluid Helium, (b) the interplay between the microscopic and macroscopic dynamics in certain irreversible processes, such as that of a laser, and (c) local thermodynamic equilibrium. In particular, (b) involves the treatment of a dynamical system which is defined on a vector bundle.

PACS: 2.90.+p, 03.65.Fd, 05.70.Ln, 02.20 Mp

*On sabbatical leave from the Department of Mathematics, Ben-Gurion University, 84105 Beer Sheva, Israel. Electronic mail: rsen@math.bgu.ac.il
†Electronic mail: g.l.sewell@qmw.ac.uk

1 Introduction

As pointed out by Bohr, the description of a quantum system, Σ, must have a classical component: otherwise, we could not tell our friends what we have observed! A natural way of expressing this basic fact is to represent the classical and essentially quantum features of Σ by actions on the base space M and the fibres Y, respectively, of a fibre bundle B. In fact, this picture has already been employed in Refs. 1,2,3 for the description of quasi-particle excitations, such as phonons, magnons, etc., in many-particle systems. In the simplest cases, M was the space of Galilei boosts and Y the single particle Hilbert representation space. The natural action of the Galilei group on the bundle $M \times Y$ permitted a simple description of the quasi-particle spectrum and yielded the noteworthy result that the energy, E, and momentum, $\mathbf{p}$ of a phonon in superfluid Helium transformed, under boosts of velocity $\mathbf{v}$, according to Landau's formula

$$\mathbf{p} \to \mathbf{p}; \quad E \to E + \mathbf{p} \cdot \mathbf{v}.$$

The object of this article is to enlarge the above picture in two ways. Firstly, we extend the actions on the bundle from groups to semigroups, in order to accommodate a description of the irreversible dynamics of open dissipative systems; and secondly we obtain a picture of the interplay between the macroscopic (classical) and microscopic (quantum) dynamics of many-particle systems in terms of these actions.

We shall organize our material as follows. In Sec. 2, we present a simple sketch of the theory of fibre bundles. In Sec. 3, we formulate the action thereon of both groups and semigroups, and in Sec. 4, we apply the resultant theory to the description of quasi-particle excitations in many-particle systems. Here, the key action is that of the Galilei semigroup, as given by the restriction of the Galilei group to non-negative time translations, and this serves to extend the results of Refs. 1,2,3 to quasi-particles with finite lifetimes.

The rest of the article is devoted to a demonstration that the interplay between the macroscopic and microscopic properties of many-particle quantum systems may naturally be expressed in terms of actions on fibre bundles. Since dynamics is perceived as a group or semigroup action but studied by infinitesimal (Hamiltonian) methods, we start in Sec. 5 by obtaining the infinitesimal form of the action of a one-parameter group or semigroup on a product bundle, provided that the action is fibre-preserving. We continue in Sec. 6 with a brief sketch of the operator algebraic description of macroscopic quantum systems, including a formulation of local thermodynamic equilibrium conditions. We then turn, in Sec. 7, to a class of open systems, such as the laser model, in which the microscopic evolution is piloted by the macroscopic dynamics (Refs. 4,5). Here we show explicitly that the macroscopic-cum-microscopic dynamics of such a system corresponds to the action of a one-parameter semigroup of transformations of the bundle $B = M \times \mathfrak{A}$, where M, the base, is the space of its macroscopic (classical) variables and $\mathfrak{A}$, the fibre, is the $C^\star$-algebra of its microscopic (quantum) observables. Likewise, in Sec. 8, we show that the local thermodynamic equilibrium conditions of Sec. 6 can also be naturally expressed in terms of actions on a fibre bundle.

We conclude in Sec. 9 with a brief discussion of the picture presented here and of its possible further ramifications.

Since some readers might not be conversant with both fibre bundles and operator-algebraic quantum theory, we have included brief sketches of these topics in Secs. 2 and 5 respectively.

2 Fibre Bundles

A *fibre bundle* $\mathcal{B} = \{B, X, p, G, Y\}$ is, first and foremost, a topological space B which is locally, but not (necessarily) globally, a product. The notation and terminology are suggestive; one can visualize a fibre bundle as a bunch of fibres "tied together" by X, the base space. Y is the (abstract) fibre, and $p : B \to X$ a map called the *projection* and, for $x \in X$, the inverse image $Y_x = p^{-1}(x)$ is called the fibre over x. All fibres Y_x are homeomorphic to each other and to Y. G is a topological group, called the group of the bundle, which has an effective action on the fibre.

The space $B = X \times Y$ is a bundle with base X, fibre Y and projection $p(x, y) = x$. It is called a *trivial bundle* because it is globally a product of the base and the fibre, and G consists only of the identity. The Euclidean plane is also a product, but one needs a coordinate system to display it as a product $X \times Y$. For this reason, it is said to be *equivalent to the product*, and is sometimes called *trivializable*. However, the mathematical literature is not consistent about the use of these terms.

A simple example of a nontrivial bundle is the *Möbius band*, which may be constructed as follows. Take the rectangular strip B_0 in $\mathbb{R}^2$ delimited by the points

$$(a, 1), (a, -1), (-a, -1) \text{ and } (-a, 1).$$

It is the topological product $B_0 = [-a, a] \times [-1, 1]$. Now give this strip a twist and glue the short edges together; mathematically, identify the point $[a, -1]$ with $[-a, 1]$, and $[a, 1]$ with $[-a, -1]$. The twist flips the fibres, which destroys the global product structure. The Möbius band is a nontrivial bundle, with a circle S_1 as base and a real interval, here $[-1, 1]$, as fibre. Observe that if glueing had been done without the twist, we would have obtained the cylinder $S_1 \times [-1, 1]$, which is again a trivial bundle.

The local product (or local triviality) condition in the bundle $\{B, X, p, G, Y\}$ is expressed as follows. There is an open cover $\{V_\alpha\}_{\alpha \in A}$ of X and, for each $\alpha \in A$, a fibre-preserving homeomorphism $\phi_a : p^{-1}(V_\alpha) \to V_\alpha \times Y$; a map $\psi : B_1 \to B_2$ between two bundles or sub-bundles B_1, B_2 is called fibre-preserving if $b, b' \in B_1$, $p_1(b) = p_1(b')$ implies that $p_2(\psi(b)) = p_2(\psi(b'))$. The maps ϕ_a are called local trivializations. The local trivializations are glued together to form the bundle, and this is where the group of the bundle G comes in. We shall omit the details, and refer the reader to the standard text of Steenrod (Ref. 6).

The fact that the Möbius band is locally trivial means that, for any open arc C in S_1, there is a fibre-preserving homeomorphism between $p^{-1}(C)$ and $C \times [-1, 1]$. S_1 can be covered by two such arcs C_1 and C_2. In this case the topological group G is the two-element group $\{e, g\}$, where e is the identity and g acts to flip the fibres on one of the two components of the intersection $C_1 \cap C_2$.

A bundle is equivalent to the product if either 1) the base X is paracompact and contractible, or if 2) the base is paracompact and the group G is contractible. Therefore any Banach bundle (a bundle of Banach spaces) on a paracompact and contractible base is equivalent to the product. A stronger result holds for Hilbert bundles,[7] because the group which preserves the inner product structure of an infinite-dimensional Hilbert space (called the "big" unitary group) is contractible (Ref. 8). Therefore any Hilbert bundle on a paracompact base is equivalent to the product. All Hilbert and Banach bundles that we shall consider will be equivalent to the product, and may be written as $X \times \mathfrak{K}$, where $\mathfrak{K}$ is a Hilbert space or a Banach space. Their interest would lie in the fact that they bring together two different structures, the differential or symplectic structure of the base space and the linear inner-product or normed space structure of the fibre.

3 Actions of Groups and Semigroups on Product Bundles

3.1 Principal and Associated Bundles; the Bundle Structure Theorem

A bundle $\mathcal{B} = \{B, X, p, G, G\}$ is called a *principal bundle* if the fibre is the same as the group, and the group of the bundle acts upon the fibre by left-translations. A bundle $\mathcal{B}' = \{B', X, p, G, Y\}$ is said to be *associated* with the principal bundle $\mathcal{B}$ if the group G acts effectively upon the topological space Y. Note that changing the fibre changes the total space, and therefore the projection. However, we continue to use the same letter to denote the projection in all associated bundles as well. As the context will always be clear, there will be no confusion.

An important result in fibre bundle theory, known as the *bundle structure theorem*, has the following corollary (see Steenrod, Ref. 6, Secs. 7.4, 7.5): If G is a Lie group and H a (topologically) closed subgroup of G, then G is a principal bundle over the space G/H of left-cosets with respect to the natural projection $p : G \to G/H$ which sends an element of $g \in G$ to its left-coset gH, with group and fibre H: $\mathcal{G} = \{G, G/H, p, H, H\}$.

3.2 Actions of Groups

Any group acts upon itself by left-translations. Therefore any Lie group G has a natural action upon the principal bundle $\mathcal{G} = \{G, G/H, p, H, H\}$. An important special case is that of an associated Hilbert bundle $\mathcal{B} = \{B, G/H, p, H, \mathfrak{H}\}$, but since the group H will generally have many possible actions on the Hilbert space $\mathfrak{H}$, we shall have to choose a particular action, or representation $D(H)$ of H upon $\mathfrak{H}$, and write the associated bundle accordingly:

$$\mathcal{G}_D = \{B, G/H, p, D(H), \mathfrak{H}\}. \tag{1}$$

However, if G is a Lie group, then G/H is necessarily paracompact, and any Hilbert bundle over the base G/H is necessarily equivalent to the product,

$$\mathcal{G}_D \equiv G/H \times \mathfrak{H}. \tag{2}$$

The action of G upon $\mathcal{G}$ (i.e. upon G) by left-translations is known, and so therefore is its action upon $\mathcal{G}_D$, as given by Eq. (1). We have to derive an explicit formula for this action, and then to rework this formula when the total space of $\mathcal{G}_D$ is given in the product form.

It turns out to be simpler to do it directly. This has the further advantage that the resulting formulae are valid for the action of the group upon *any* product bundle, for example a bundle of Banach spaces or operator algebras. We shall therefore work with the generic product bundle

$$B = G/H \times \mathfrak{F}, \tag{3}$$

where the only demand on $\mathfrak{F}$ is that it be a complete normed linear space, and determine the actions of the group G on it.

Let $g \in G,\ x \in G/H,\ \phi \in \mathfrak{F}$ and $b = (x, \phi) \in B$. Write

$$gb = g(x, \phi) = (gx, u(g, x)\phi), \tag{4}$$

where $u(g, x)$ is an appropriate linear map upon $\mathfrak{F}$ which depends upon g and x. $u(g, x)$ has to satisfy the condition

$$u(g'g, x) = u(g', gx)u(g, x) \tag{5}$$

that arises from the associativity of multiplication in G:

$$(g'g) \cdot b = g'(g \cdot b). \tag{6}$$

Observe that Eq. (6) has to hold even if G is only a semigroup.

The action of G on G/H being known, it only remains to determine the quantity $u(g, x)$ which appears in Eq. (4). $u(g, x)$ must be a linear transformation of $\mathfrak{F}$, and it must satisfy Eq. (5). The method of solving Eq. (5) was devised by Wigner in 1939 (Ref. 9). It is as follows:

Let

$$\eta : G/H \longrightarrow G \tag{7}$$

be an *algebraic cross-section* of the bundle $\mathcal{G}$, i.e. a map which satisfies

$$p \circ \eta = \mathrm{id}, \tag{8}$$

where p is the projection in the bundle $\mathcal{G}$ and id is the identity map. An important theorem in fibre bundle theory states that a *continuous* cross-section of a principal bundle $\mathcal{G}$ exists if and only if the bundle is equivalent to the product, $\mathcal{G} \equiv G/H \times H$, so that in general the map η cannot be continuous. We shall presently return to this point. Given η, define the map

$$k : G \times G/H \to G \tag{9}$$

by

$$k(g, x) = \eta(gx)^{-1} g \eta(x). \tag{10}$$

Then $k(g, x)$ obviously satisfies

$$k(g'g, x) = k(g', gx)k(g, x), \tag{11}$$

and it is easily seen that $k(g, x) \in H$, i.e. k is actually a map

$$k : G \times G/H \longrightarrow H. \tag{12}$$

Finally, choose a linear action $D(H)$ of H on $\mathfrak{F}$, and set

$$u(g, x) = D(k(g, x)). \tag{13}$$

The continuity of the action (4) remains to be established. It follows from the definition of the quotient topology that the action of G on G/H by left translations is continuous. According to a theorem of von Neumann, a unitary Hilbert-space representation of a locally compact group is strongly continuous if it is weakly measurable. If η is a measurable map, so is $k(g, x)$ defined by Eq. (10). Thus, for unitary representations, measurability of η suffices to ensure the continuity of $D(k(g, x))$, and therefore of the action (4). In practice, η can always be chosen to be piecewise-continuous. Finally, it is again a standard result in the theory of infinite-dimensional unitary group representations that the passage from one section η to another is equivalent to a change of basis in $\mathfrak{H}$. (Detailed treatments of infinite-dimensional unitary group representations may be found in Refs. 9,10,11.)

The arguments of the previous paragraph do not apply if the fibre $\mathfrak{F}$ is not a Hilbert space and the group H does not act upon it as a unitary representation. In this case the problem of continuity has to be studied separately. In the examples we shall consider, the map η will always be continuous, so that the continuity of $D(H)$ would be a sufficient requirement.

The map k of Eq. (10) is sometimes called a $(G, G/H, H)$ cocycle in the mathematics literature, and sometimes a "Wigner rotation" of G into H in the physics literature. In Wigner's language, H is the "little group".

3.3 Actions of Semigroups

The results of Sec. 3.2 were based on the bundle structure theorem for Lie groups, and the theory of unitary representations on locally compact groups. Neither is available for an arbitrary semigroup G. However, for physical applications one is mainly interested in a very restricted class of semigroups, those that arise from symmetry groups like the Galilei or the Poincaré group by dropping the requirement of time reversibility. In these cases one may mimic the general theorems by ad hoc constructions. We shall illustrate the procedure for the "Galilei semigroup".

Let $g = (b, \mathbf{a}, \mathbf{v}, R)$ be an element of the inhomogeneous Galilei group G, where b is a time translation, $\mathbf{a}$ is a space translation, $\mathbf{v}$ is a (velocity) boost, and R is a rotation. The multiplication law in the group is

$$g'g = (b', \mathbf{a}', \mathbf{v}', R')(b, \mathbf{a}, \mathbf{v}, R) = (b' + b, \mathbf{a}' + R'\mathbf{a} + \mathbf{v}'b, \mathbf{v}' + R'\mathbf{v}, R'R), \tag{14}$$

the identity element is $e = (0, 0, 0, 1)$ and the inverse of g is

$$g^{-1} = (-b, -R^{-1}(\mathbf{a} - b\mathbf{v}), R^{-1}\mathbf{v}, R^{-1}). \tag{15}$$

We define the Galilei semigroup G^+ by the condition

$$b \geq 0. \tag{16}$$

Then the generic element g does not have an inverse, but group elements with $b = 0$ are invertible,

$$(0, \mathbf{a}, \mathbf{v}, R)^{-1} = (0,\ -R^{-1}\mathbf{a},\ R^{-1}\mathbf{v},\ R^{-1}), \tag{17}$$

and G^+ has an identity which is the same as the identity of the group. Clearly, the one-parameter subsemigroup T^+ of time translations has an identity. Therefore, by the Hille-Yosida theorem (Ref. 12), $T^+(b)$ will be strongly differentiable and will have an infinitesimal generator in any bounded strongly continuous Hilbert space representation. We shall confine our search for representations to these cases.

Now T^+ is also topologically closed. The following facts are readily established:

1. For any closed subsemigroup H^+ of G^+, the factor space $M = G^+/H^+$ is defined and is identical with G/H.
2. If H^+ is a normal subsemigroup of G^+, then M is a group, and G^+ is equivalent to the product bundle $M \times H^+$.
3. If the above condition is satisfied, then there exists a self-evident cross-section of M in G^+ which is continuous, and invertible in G^+.

In this case the action of the semigroup G^+ on the bundle $M \times \mathfrak{H}$ may be constructed exactly as in the case of the group.

We shall exhibit the above procedures by a simple example:

3.3.1 An Example

Let G^+ be the Galilei semigroup, and H^+ the subsemigroup of time translations. Then $M = G^+/H^+$ is a group, the semidirect product of the group of boosts V and the Euclidean group E_3 in three dimensions. Write $x \in M$ explicitly as

$$x = (\mathbf{z}; \mathbf{w}; \rho), \tag{18}$$

where $\mathbf{z}$ is a space translation, $\mathbf{w}$ a boost and ρ a rotation. The map $\eta : M \to G^+$ defined by

$$\eta(x) = (0, \mathbf{z}, \mathbf{w}, \rho), \tag{19}$$

is a continuous cross-section which is invertible in G^+, and G^+ acts on M as follows:

$$\begin{aligned} gx &= (b, \mathbf{a}, \mathbf{v}, R) \cdot (\mathbf{z}; \mathbf{w}; \rho) \\ &= (\mathbf{a} + R\mathbf{z} - (\mathbf{v} + R\mathbf{w})\, b;\ \mathbf{v} + R\mathbf{w}; R\rho). \end{aligned} \tag{20}$$

The cocycle k takes the simple form

$$k(g, x) = (b, 0, 0, 1). \tag{21}$$

Finally, we choose a bounded representation, D, of H^+ on $\mathfrak{H}$:

$$D(b, 0, 0, 1)\phi = D(k(g, x))\phi = e^{(i\alpha - \beta)b}\phi, \tag{22}$$

where $\phi \in \mathfrak{H}$, $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. Hence, by Eq. (12),

$$u(k(g,x))\phi = e^{(i\alpha-\beta)b}\phi. \tag{23}$$

Equations (20) and (23) provide a representation of G^+ on the bundle $M \times \mathfrak{H}$. To see this representation in a more transparent form, we look at the fibre $(0;0;1)$, in a more familiar notation:

$$\begin{aligned} g(x_0,\phi) = \; & (t, \mathbf{x}, \mathbf{v}, R)\cdot(0;0;1) \\ = \; & ((\mathbf{x}-\mathbf{v}t;\mathbf{v};R),\ e^{(i\alpha-\beta)t}\phi). \end{aligned} \tag{24}$$

The representation "decays" as t increases.

4 Elementary Excitations as Bundle Representations

The description of elementary excitations by bundle representations of the Galilei and Poincaré groups has been discussed at length in earlier works (Refs. 2,3). Therefore we shall confine ourselves, in the following, to one particular representation of the Galilei group which seems to describe (stable) nonrelativistic zero-mass excitations, and consider how the representation changes if the Galilei group is replaced by the Galilei semigroup.

4.1 Case of the Galilei Group

Here G is the Galilei group and $H = T \times E_3$, the direct product of the subgroup of time translations T and the Euclidean group E_3. In this case the base space M is just the space of the boosts, and there is a natural invertible cross-section:

$$\begin{aligned} x &= \mathbf{w}, \\ \eta(x) &= (0,0,\mathbf{w},1). \end{aligned} \tag{25}$$

Easy computations yield the formulae

$$\begin{aligned} gx &= \mathbf{v}+R\mathbf{w}, \\ k(g,x) &= (b,\ \mathbf{a}-(\mathbf{v}+R\mathbf{w})\,b,\ 0,\ R). \end{aligned} \tag{26}$$

To construct the bundle representation, we recall that irreducible unitary representations of E_3 are characterized by the invariants $\{\mathbf{p}^2, n\}$, where, loosely speaking, $\mathbf{p}^2$ is the square of the momentum and n is the helicity (a nonnegative integer). The irreducible representation space is $L_2(S_2, dS)$, where S_2 is the two-sphere and dS is the rotation-invariant measure on it. For our purposes, it is convenient to use *not* the irreducible unitaries, but the following one, U, on $\mathfrak{H} = L_2(\mathbf{p}, d^3p)$, which is the direct integral of copies of $L_2(S_2, dS)$ over two-spheres of all radii.

$$U(\mathbf{a}, R)\,\phi(\mathbf{p}) = e^{in\sigma(R,\omega)}e^{-i\mathbf{a}\cdot\mathbf{p}}\ \phi\circ R^{-1}(\mathbf{p}). \tag{27}$$

In the above, $\sigma(R,\omega)$ is an (O_3, S_2, O_2)-cocycle or Wigner rotation, where O_3, O_2 are the orthogonal (rotation) groups in three and two dimensions respectively, $S_2 = O_3/O_2$ is the 2-sphere, and ω a point on S_2. All component irreducibles in Eq. (27) have the same helicity. Finally, we represent $T \times E_3$ on $\mathfrak{H}$ as follows:

$$U(b, \mathbf{a}, R)\,\phi(\mathbf{p}) = e^{iE(\mathbf{p}^2,n)\,b}\,U(\mathbf{a}, R)\,\phi(\mathbf{p}). \tag{28}$$

In the above, E is a positive, but otherwise arbitrary function of $\mathbf{p}^2$, and n is fixed. By Eqs. (13) and (26), the bundle representations are obtained by taking $D(k(g,x))\phi = U(b, \mathbf{a} - (\mathbf{v} + R\mathbf{w})b), R)\phi$.

On the fibre $\mathbf{w} = 0$, the bundle representation takes the explicit form

$$(b, \mathbf{a}, \mathbf{v}, R)(0, \phi) = (\mathbf{v}, e^{i\Lambda}(\phi \circ R^{-1})\,\mathbf{p}), \tag{29}$$

where

$$\Lambda = -\mathbf{a}\cdot\mathbf{p} + b(E + \mathbf{v}\cdot\mathbf{p}) + n\sigma(R,\omega). \tag{30}$$

In the above, we have suppressed the arguments in E. Formulae (29) and (30) show that under a boost,

$$\begin{aligned} \mathbf{p} &\to \mathbf{p}, \\ E &\to E + \mathbf{p}\cdot\mathbf{v}. \end{aligned} \tag{31}$$

In their analysis of irreducible unitary representations of the Galilei group, Inönü and Wigner (Ref. 13) had concluded that these representations had no physical interpretation because they did not admit either localizable states (since $\mathbf{p}^2$ was constant in an irreducible, not enough momenta were available to form a delta-function) or states that were—like the photon—localizable in velocity space. The first argument would not apply to a *reducible* representation which contains all values of $\mathbf{p}^2$ with the same weight. However, the invariance group imposes no relationship between E and $\mathbf{p}^2$, and, as the second of Eqs. (31) shows, in any linear representation, for any value of $\mathbf{p}^2$, the quantity E takes—owing to the boosts—all values between $-\infty$ and $+\infty$. This alone would rule out the possibility of a physical interpretation for these representations.

In the *bundle* representations constructed above, the boosts are represented not on the fibres but on the base space. Specializing formulae (29) and (30) to the case $\mathbf{a} = 0$, $R = 1$ and setting $\sigma(1,\omega) = 0$ (which can be done without loss of generality), we find that

$$(b, 0, \mathbf{v}, 1)(0, \phi) = (\mathbf{v}, e^{ib(E+\mathbf{p}\cdot\mathbf{v})}\phi(\mathbf{p})), \tag{32}$$

i.e. on any given fibre, we are able to assign a value of E to a given $\mathbf{p}^2$, because this assignment will not be disturbed by the boosts. That is, we may choose, arbitrarily, a *dispersion law* $E = E(\mathbf{p}^2)$. Since all values of $\mathbf{p}$ occur with the same weight, it becomes possible to construct localizable states, which may be interpreted as excitations corresponding to the chosen dispersion law. Finally, Eqs. (31) show that these excitations have *zero nonrelativistic mass.*

Eqs. (31) are the transformations laws, under boosts, of the excitations posited by Landau in his explanation of superfluidity (Refs. 14,15).

4.2 Case of the Galilei Semigroup

For the Galilei semigroup G^+, we shall construct a representation that is as close as possible to the above, without being identical with it. We shall take $H^+ = T^+ \times E_3$, so that G^+/H^+ will again be the space of the boosts. The natural cross-section will again be given by the second of Eqs. (25), and formulae (26) for $g\mathbf{x}$ and $k(g,\mathbf{x})$ will remain valid, with the proviso that $b \geq 0$. We shall represent E_3 on $\mathfrak{H}$ as before, i.e. by formula (27). However, the subsemigroup of time translations will be represented, not by a character $\exp i\epsilon b$, but by a quantity $\exp(i\epsilon - \gamma)b$, where $\gamma \geq 0$. Eq. (28) will (cf. Eqs. (22,23)) accordingly change to the following formula for the relevant representation, Δ, of H^+.

$$\Delta(b, \mathbf{a}, R)\, \phi(\mathbf{p}) = e^{(iE(\mathbf{p}^2,n) - \Gamma(\mathbf{p}^2,n))\, b}\, U(\mathbf{a}, R)\, \phi(\mathbf{p}). \tag{33}$$

where Γ is a positive function, n is a fixed positive integer and $b \geq 0$. Formally, the passage from the group to the semigroup is equivalent to replacing E by $E + i\Gamma$ in the representation $U(b, \mathbf{a}, R)$. Clearly, $\mathbf{p}, E$ and Γ will transform under boosts as

$$\begin{aligned} \mathbf{p} &\to \mathbf{p}, \\ E &\to E + \mathbf{p}\cdot\mathbf{v}, \\ \Gamma &\to \Gamma. \end{aligned} \tag{34}$$

The quantity $\Gamma(\mathbf{p}^2)$ may be interpreted as the inverse lifetime of the excitation with momentum $\mathbf{p}$. For large values of $\Gamma(\mathbf{p}^2)$, the excitations become very unstable, and perhaps no longer experimentally detectable. As the last equation in (34) shows, properties of the excitations that depend on Γ are invariant under the boosts.

Since the representation Δ is bounded and continuous, and since the semigroup has an identity, the semigroup of time translations has an infinitesimal generator.

5 Dynamics on Banach Bundles

Symmetries are usually studied via group rather than Lie algebra actions, because they avoid the problems associated with unbounded operators. By contrast, dynamics is usually studied via the infinitesimal generator of the one-parameter group or semigroup of time translations—the Hamiltonian—because the Hamiltonian represents the total energy of the system. In quantum mechanics, the Hamiltonian is an operator on a Hilbert space. In classical mechanics, it is a complete vector field on a manifold of states. From physical considerations, one would expect a one-parameter group or semigroup of time translations of a Banach bundle to possess an infinitesimal generator, which would act like a linear transformation on the fibres and a vector field on the base space. To the best of the authors' knowledge, such mathematical objects have not yet been studied, and there is no generalized "Stone-Whitney theorem" which asserts their existence for dynamics on vector bundles. We therefore begin—assuming that time translations indeed act as a one-parameter group or semigroup of fibre-preserving maps on a vector bundle—by defining the two constituents of this generator, one acting on the base space and the other on the fibres. The dynamical systems these pairs define will be seen in Sec. 7 to be realized by certain quantum mechanical models.

Let Γ be a dynamical system (B,T), where $B = X \times Y$ is a Banach bundle and $T = \{T_t \,|\, t \in \mathbb{R} \text{ or } \mathbb{R}^+\}$ is a one-parameter group or semigroup of fibre-preserving transformations of B. These maps define a one-parameter group or semigroup of base maps $\{S_t \,|\, t \in \mathbb{R} \text{ or } \mathbb{R}^+\}$, and the action of T on B is analogous to that of g in Eq. (4), i.e.

$$T_t(x,y) = (S_t x, V(t,x)y) \equiv (x_t, y_t) \;\forall\; x \in X,\; y \in Y,\; t \in \mathbb{R} \text{ or } \mathbb{R}^+, \tag{35}$$

where $V(t,x)$ is a linear transformation of Y. Then it follows immediately from the semigroup properties of S and T that

$$\begin{aligned} S_{t+t'}x &= S_t(S_{t'}x), \\ V(t+t',x) &= V(t',S_t x)V(t,x). \end{aligned} \tag{36}$$

Furthermore, assuming that $S_t x$ and $V(t,x)y$ are differentiable with respect to t in the natural topologies of X and Y respectively, it follows from Eqs. (35) and (36) that the equations of motion for x_t and y_t are

$$\frac{dx_t}{dt} = F(x_t) \tag{37}$$

and

$$\frac{dy_t}{dt} = L(x_t)\, y_t, \tag{38}$$

where

$$F(x) = \lim_{h\to 0} h^{-1}(S_h - I)x \;\forall\; x \in X \tag{39}$$

and

$$L(x)y = \lim_{h\to 0} h^{-1}(V(h,x) - I)y \;\forall\; x \in X,\; y \in Y. \tag{40}$$

Thus, Eqs. (37) and (38) signify that the flow $x \to x_t$ in the base space X is not only autonomous but also serves to drive the flow $y \to y_t$ in the fibres.

6 The Generic Operator Algebraic Quantum Model

6.1 Observables, States, Dynamics

In a standard way (cf. Refs. 15,16,17,18,19), we idealize a macroscopic quantum system, Σ, as an infinitely extended assembly of interacting particles living in a space X, which may be either a Euclidean space $\mathbb{R}^d$ or a lattice $\mathbb{Z}^d$. The generic model of Σ is given (Ref. 21) in the Schrödinger picture,[22] by an algebra of observables $\mathfrak{A}$, a state space Ω of expectation functionals on $\mathfrak{A}$, and a one-parameter group or semigroup τ of transformations of Ω representing the dynamics of the system. The model conforms to the following specifications:

I. $\mathfrak{A}$ is the $C^\star$-algebra of observables. This has a local structure, which is as follows:

a) Each bounded region Λ of X carries a $C^\star$-subalgebra $\mathfrak{A}_\Lambda$, comprising the bounded operators in a local Hilbert space $\mathfrak{H}_\Lambda$, whose self-adjoint elements are the observables in Λ.

b) The algebras $\mathfrak{A}_\Lambda$ satisfy the conditions of isotony and local commutativity, namely

$$\mathfrak{A}_\Lambda \subset \mathfrak{A}_{\Lambda'} \text{ if } \Lambda \subset \Lambda'$$

and

$$[a, a'] = 0 \text{ if } a \in \mathfrak{A}_\Lambda,\ a' \in \mathfrak{A}_{\Lambda'} \text{ and } \Lambda \cap \Lambda' = \emptyset.$$

Furthermore, space translations correspond to a homomorphism, σ, of the additive group X into $\mathrm{Aut}(\mathfrak{A})$, and the local algebras $\mathfrak{A}_\Lambda$ transform covariantly under $\sigma(X)$, i.e.

$$\sigma(x)\mathfrak{A}_\Lambda = \mathfrak{A}_{\Lambda+x}.$$

The unbounded observables for the region Λ are the unbounded self-adjoint operators in $\mathcal{H}_\Lambda$ (Ref. 23).

c) $\mathfrak{A}$ is the norm-closure of the union of the local algebras $\mathfrak{A}_\Lambda$. We shall denote this union, which is a subalgebra of $\mathfrak{A}$, by $\mathfrak{A}_{\mathrm{loc}}$.

II. The state space Ω and the dynamical group or semigroup τ are defined as follows (Ref. 21). $\Omega$is a folium[24] of positive, normalized, locally normal[25] linear functionals on $\mathfrak{A}$ that is stable under the dual of σ, while τ is a representation *either* of $\mathbb{R}$ *or* of $\mathbb{R}^+$ in the affine transformations of the state space Ω. The interpretation of the model is that the expectation value of an observable a at time t, subject tothe condition that ω is the initial state ω of the system, is $\langle \tau(t)\,\omega; a\rangle$, where $\langle \omega; a\rangle \equiv \omega(a)$. We assume that τ and Ω satisfy the following conditions:

a) τ is the infinite volume limit, in a sense we shall presently specify, of the one-parameter dynamical group or semigroup τ_Λ of the finite volume version, Σ_Λ, of Σ consisting of the particles of the given species confined to the bounded region Λ. Here τ_Λ is the predual of a one-parameter group of automorphisms or a semigroup of completely positive,[26] identity preserving transformations of $\mathfrak{A}_\Lambda$, depending on whether Σ_Λ is a conservative system or an open dissipative one. In either case, its form is obtained, by standard constructions, in terms of the interactions between the particles of the system. Denoting now by ω_Λ the restriction of a state ω of Σ to $\mathfrak{A}_\Lambda$, i.e. the state it induces in Σ_Λ, we express our condition that τ is an infinite volume limit of τ_Λ by the formula

$$\langle \tau(t)\,\omega; a\rangle = \lim_{\Lambda\uparrow X} \langle \tau_\Lambda(t)\,\omega; a\rangle \ \forall\ a \in \mathfrak{A}_{\mathrm{loc}},\ \omega \in \Omega,\ t \in \mathbb{R} \text{ or } \mathbb{R}^+.$$

b) Ω is the maximal locally normal folium of states that supports the dynamics given by this formula.

It follows from these specifications that Ω and τ, like τ_Λ, are determined by the interactions in the system. We note here that the norm-closed linear span $[\Omega]$ of Ω is a Banach space, whose dual is a $W^\star$-algebra $\hat{\mathfrak{A}}$ (Ref. 29) which in turn is just the weak closure of the representation, Π, of $\mathfrak{A}$ given by the direct sum of the GNS representations of this algebra for all states in the space Ω (Ref. 21). Correspondingly, the transformations $\tau^\star(t)$ of $\hat{\mathfrak{A}}$ dual to $\tau(t)$ form a one-parameter group of automormorphisms of $\hat{\mathfrak{A}}$ or a semigroup of completely positive, identity preserving linear transformations of $\hat{\mathfrak{A}}$, according to whether Σ is a conservative system or an open dissipative one.

6.2 Macroscopic Observables

These are *classical* nonlocalized observables (such as the global space averages of the translates of local ones) that are attached, not to $\mathfrak{A}$, but to certain representations of $\mathfrak{A}$. We define them as follows (cf. Ref. 30). Let π be a representation of $\mathfrak{A}$ in a Hilbert space $\mathfrak{H}$, such that the set, Ω_π, of states on $\mathfrak{A}$ corresponding to the density matrices in $\mathfrak{H}$ is a subset of Ω. Then the macroscopic observables associated with the representation π are defined to be the strong limits, $\hat{F}$, of uniformly bounded sequences $\{\pi(F_n) \mid n \in \mathbb{N}\}$, where the F_n's are elements of $\mathfrak{A}$ such that

$$\text{norm}\!-\!\lim_{n\to\infty} [F_n, A] = 0 \ \forall \ A \in \mathfrak{A}.$$

It follows easily from these specifications that the macroscopic observables $\hat{F}$ belong the centre of the von Neumann algebra $\pi(\mathfrak{A})''$. They therefore form an abelian $W^\star$-algebra, $\mathfrak{M}$, i.e. an algebra of functions on a classical phase space (Ref. 29). Evidently, they may be identified with functionals on the set of states Ω_π according to the formula

$$\hat{F}(\omega) = \lim_{n\to\infty} \omega(F_n) \ \forall \ \omega \in \Omega_\pi.$$

A prototype of a macroscopic observable is obtained by choosing

$$F_n = |\Lambda_n|^{-1} \int_{\Lambda_n} dx\, \sigma(x) f,$$

where $f \in \mathfrak{A}$ and $\{\Lambda_n\}$ is a increasing sequence of bounded spatial regions, whose union is X. $\hat{F}$ is then the global space average of $\pi(\sigma(x)f)$, i.e.

$$\hat{F} = s\!-\!\lim_{n\to\infty} |\Lambda_n|^{-1} \int_{\Lambda_n} dx\, \pi(\sigma(x)f).$$

We shall now provide a sketch of how the above general definition of $\hat{F}$ can be extended, for some representations π, to cases where the local observables F_n are unbounded, e.g., when they correspond to space averages over bounded regions of the locally conserved densities of standard thermodynamic observables (Refs. 14,20). Thus, we consider the case where $(\mathfrak{H}, \pi, \Phi)$ is the GNS representation of $\mathfrak{A}$ for a primary,[31] translationally invariant state ω. We assume that $\hat{\xi}$ is a Hermitian quantum field such that $\hat{\xi}(x)$ is a distribution-valued observable for the ball of center x and some x-independent radius a and that $\hat{\xi}$

is covariant with respect to space translations, i.e. $\sigma(x')\hat{\xi}(x) = \hat{\xi}(x + x')$. We define (formally) the derivation δ and the local observable F_n by the formulae

$$\delta = i \int_X dx\, [\pi(\hat{\xi}(x)), .],$$

where the dot on the right refers to the argument of δ and

$$F_n = |\Lambda|^{-1} \int_{\Lambda_n} dx\, \hat{\xi}(x).$$

We then make the following assumptions concerning $\hat{\xi}$.

a) $\omega(F_n)$ converges to a limit, $\hat{F}(\omega)$, as $n \to \infty$. In view of the assumed translational invariance of ω, this is equivalent to the condition that $\omega(\hat{\xi}(x))$ is well-defined.

b) There is a dense subset, $\mathfrak{B}$, of $\mathfrak{A}_{\rm loc}$ such that Φ lies in the domain of the (possibly unbounded) observable δB, for all $B \in \mathfrak{B}$. Here we note that it follows from the above specifications that, in view of the local commutativity property of $\mathfrak{A}$, δB reduces here to $i \int_{\Lambda_n} dx\, [\hat{\xi}(x), B]$ for n sufficiently large.

c) For each $A \in \mathfrak{A}$, the correlation function $\omega(A\hat{\xi}(x)) - \omega(A)\omega(\hat{\xi}(x))$ is of class L_1, in keeping with the standard cluster properties of primary states (Ref. 32).

It follows easily from these specifications that, for all $B \in \mathfrak{B}$, the expectations of the observables F_n in the states $\omega_B(.) := \omega(B^\star(.)B)/\omega(B^\star B)$ satisfy the condition

$$\lim_{n \to \infty} \omega_B(F_n) = \hat{F}(\omega)\ \forall\, B \in \mathfrak{B}.$$

This is equivalent to the statement that the sequence $\{\pi(F_n)\}$ converges to $\hat{F}(\omega)I$ as $n \to \infty$, and therefore signifies that the limit of the sequence is a macroscopic observable $\hat{F} := \hat{F}(\omega)I$.

Equilibrium States and Thermodynamical Variables. In the case where Σ is conservative, its equilibrium states at inverse temperature β comprise the convex set Ω_β of states that satisfy the Kubo-Martin-Schwinger (KMS) condition, whose general significance was first exposed by Haag, Hugenholtz and Winnink (Ref. 33). This takes the form

$$\omega(A_t B) = \omega(B A_{t+i\hbar\beta})\ \forall\ A, B \in \mathfrak{A},$$

where, in our Schrödinger picture (cf. Refs. 34,21), ω is the canonical extension of the state denoted by the same symbol, and A_t is the element $\tau^\star(t)\pi(A)$ of the von Neumann algebra generated by the corresponding GNS representation, π, of $\mathfrak{A}$. The extremal elements of Ω_β are precisely the primary ones and therefore may naturally be interpreted as the pure thermodynamic phases of the system (Refs. 35,36). Furthermore, it follows from their primary property that they may be indexed by an appropriate set of macroscopic observables (Ref. 30); and, in accordance with the demands of classical thermodynamics, we shall assume that the latter observables are the global space averages of locally conserved fields $\hat{q}(x) = (\hat{q}_1(x), .., \hat{q}_n(x))$, such as those representing densities of energy, mass, etc. We shall denote the pure equilibrium phase at inverse temperature β for which these macroscopic variables take the values $(q_1, . ., q_n)$ by $\omega_{\beta,q}$.

6.3 Local Equilibrium

The concept of local equilibrium is natural for a hydrodynamic, i.e. large scale, description of Σ, in which a point corresponds to cell containing an enormous number of particles in the microscopic picture. We shall now formulate this concept for the case where Σ is a conservative system, whose algebra of observables, $\mathfrak{A}$, is generated by a quantum field, ψ, that satisfies the canonical commutation or anticommutation relations, according to whether Σ consists of bosons or fermions. Thus,

$$[\psi(x), \psi^\star(y)]_\pm = \delta(x-y);\ [\psi(x), \psi(y)]_\pm = 0,$$

and the action of the space translational automorphisms is given by the formula

$$\sigma(x)\psi(y) = \psi(x+y).$$

We define $\gamma(L)$ to be the automorphism of $\mathfrak{A}$ representing the distance rescaling $x \to Lx$, i.e.,

$$\gamma(L)\psi(y) = L^{d/2}\psi(Ly),$$

the factor $L^{d/2}$ ensuring that γ preserves the canonical commutation and anticommutation relations. It follows easily from these definitions that $\{\gamma(L) \mid L \in \mathbb{R}^+\}$ form a group, with $\gamma(L_1)\gamma(L_2) = \gamma(L_1 L_2)$, and that γ and σ satisfy the important equation

$$\gamma(L)\sigma(x) = \sigma(Lx)\gamma(L). \tag{41}$$

Now let $\{\omega^{(L)} \mid L \in \mathbb{R}^+\}$ be a family of states indexed by L, where the parameter L is a characteristic length governing the spatial variations of $\omega^{(L)}$ in the following sense. For any local observable a, $\omega^{(L)}(\sigma(Lx)a)$ reduces to an L-independent function of x, at least for sufficiently large L. We define $\tilde{\omega}^{(L)}$ to be the version of $\omega^{(L)}$ in the scaling for which the unit of length is L, i.e.

$$\tilde{\omega}^{(L)}(a) = \omega^{(L)}(\gamma(L)a)\ \forall\ a \in \mathfrak{A}. \tag{42}$$

Our hydrodynamic picture of Σ will be based on the structure of this state in the large scale limit where $L \to \infty$.

We now note that, by Eqs. (41) and (42),

$$\tilde{\omega}^{(L)}(\sigma(x)a) = \omega^{(L)}(\gamma(L)\sigma(x)a) = \omega^{(L)}(\sigma(Lx)\gamma(L)a).$$

Hence, replacing a by $\gamma(L^{-1})a$ and defining

$$\alpha(L, x) = \sigma(x)\gamma(L^{-1}), \tag{43}$$

it follows that

$$\tilde{\omega}^{(L)}(\alpha(L, x)a) = \omega^{(L)}(\sigma(Lx)a). \tag{44}$$

In order to see the significance of this equation, we note that it follows from our definitions of the automorphisms σ, γ and α that

$$\alpha(L, x)\psi(y) = L^{-d/2}\psi(x + L^{-1}y). \tag{45}$$

Thus, for large L, the action of $\alpha(L, x)$ serves to localize the observables around the point x. Hence the left hand side of our formula for $\tilde{\omega}^{(L)}$ yields the state in the vicinity of x, according to the large scale description.

Suppose now that D is a fixed, bounded spatial region that is sufficiently large to contain an enormous number of particles, in the microscopic description. Thus, the subsystem in D is mesoscopic, in that, on the one hand, it contains a huge number of particles, while, on the other hand, the ratio of its span to the distance, L, vanishes in the limit $L \to \infty$. In other words, D corresponds to a *hydrodynamic point.*

Our local equilibrium condition is that the restriction of $\omega^{(L)}$ to the space translate, $D + Lx$, of D reduces to an equilibrium state corresponding to some position-dependent values, $\beta(x)$ and $q(x)$ of the thermodynamic variables β and q, in the limit $L \to \infty$, i.e. that, if a is an arbitrary observable for the region D, then

$$\lim_{L \to \infty} \omega^{(L)}(\sigma(Lx)a) = \omega_{\beta(x),q(x)}(a), \tag{46}$$

which is consistent with our earlier assertion that $\omega^{(L)}(\sigma(x)a)$ is L-independent for L sufficiently large. Evidently, $(\beta(x), q(x))$ is a classical field, representing the local thermodynamic variables in the hydrodynamic description. Further, since D is an arbitrary bounded region, the above local equilibrium condition extends to all elements a of the algebra $\mathfrak{A}$. Moreover, by Eq. (44), it takes a particularly simple form when expressed in terms of the rescaled state, $\tilde{\omega}^{(L)}$, namely

$$\lim_{L \to \infty} \tilde{\omega}^{(L)}(\alpha(L, x)a) = \omega_{\beta(x),q(x)}(a) \ \forall \ a \in \mathfrak{A}. \tag{47}$$

In other words, the state as described in the large scale limit reduces, at any spatial point x, to the equilibrium state $\omega_{\beta(x),q(x)}$. In fact, this condition has been proved to be fulfilled by Gibbs states of non-relativistic gravitational systems (Ref. 37).

Finally, this formulation of local equilibrium may easily be extended to dynamical situation in the case where the macroscopic equations of motion for the evolutes, $(\beta_t(x), q_t(x))$, of $(\beta(x), q(x))$ are invariant under the transformation $x \to Lx, t \to L^k t$, for some positive constant k, as in the cases of both diffusion or Eulerian hydrodynamics. For then, the space-time generalization of the above local equilibrium condition is simply

$$\lim_{L \to \infty} \langle \tau(L^k t)\omega^{(L)}; (\sigma(Lx)a) \rangle = \omega_{\beta_t(x),q_t(x)}(a). \tag{48}$$

Equivalently, defining the time-dependent state, $\tilde{\omega}_t^{(L)}$, in the hydrodynamic description by the formula

$$\tilde{\omega}_t^{(L)}(a) = \langle \tau(L^k t)\tilde{\omega}^{(L)}; \gamma(L)a \rangle, \tag{49}$$

the local equilibrium condition takes the form

$$\lim_{L \to \infty} \tilde{\omega}_t^{(L)}(\alpha(L, x)a) = \omega_{\beta_t(x),q_t(x)}(a) \ \forall \ a \in \mathfrak{A}. \tag{50}$$

This condition has been proved to be satisfied by various classical stochastic models (Ref. 38).

7 Bundle Dynamical Model of Macroscopic-cum-Microscopic Evolution

In the generic model of a many-particle system, as formulated in the previous Section, the microscopic observables are quantum mechanical variables, whereas the macroscopic ones are classical. This dichotomy is essential for the reconciliation of quantum microphysics with the empirical fact that the macroscopic dynamics of 'large' systems generally conforms to classical laws, such as those of hydrodynamics or heat conduction. Furthermore, it has been established that, at least in particular models (cf. Refs. 39,5), the macroscopic dynamics is governed by self-contained deterministic laws and acts so as to 'pilot' the microdynamics. We shall now show that the macroscopic-cum-microscopic evolution of such systems provides a concrete example of the bundle dynamical model of Sec. 5.

We base our treatment on the model Σ of Sec. 6, and for simplicity, we restrict our considerations to the case where this is an open dissipative system, i.e. where τ is a one-parameter semigroup. The treatment of the conservative case can be carried out analogously.

We assume that, as in the laser model of Refs. 4,5, τ induces macroscopic and microscopic evolutions that may be described as follows.

(I) Σ has a set of macroscopic observables that are functions on a phase space M and evolve according to a deterministic law given by a one-parameter semigroup $\{S_t \mid t \in \mathbb{R}^+\}$ of transformations $m \to S_t m = m_t$ of M. Thus, assuming that $S_t m$ is differentiable with respect to t, the macroscopic equation of motion takes the form

$$\frac{dm_t}{dt} = F(m_t), \tag{51}$$

where F is a vector field on M.

(II) The microscopic evolution of the system is piloted by its macroscopic dynamics in such a way that the equation of motion for the evolute, a_t, of an observable a at time t has the form

$$\frac{da_t}{dt} = L(m_t)a_t, \tag{52}$$

where $L(m_t)$ acts linearly on $\mathfrak{A}$. To be precise, the microscopic dynamics is given by a two-parameter family, $\{W(t,u;m) \mid t \geq u\}$, of completely positive, identity-preserving linear transformations of $\mathfrak{A}$, such that

$$a_t = W(t,u;m)a_u, \tag{53}$$

and

$$W(t,u;m)W(u,v;m) = W(t,v;m), \tag{54}$$

which is essentially a determinacy condition. Eqs. (51)–(54) then signify that $L(m_t)$ is the generator of the transformations W, i.e. that

$$L(m_t) = (\partial W(t,u;m)/\partial t)_{t=u} \tag{55}$$

and hence that

$$\frac{\partial}{\partial t}W(t,u;m) = L(m_t)W(t,u;m) \equiv L(S_t m)W(t,u;m), \ \forall \ m \in M, \ t \geq u. \tag{56}$$

(III) $L(m_t)$ is indeed a true generator, in that Eq. (56) determines W uniquely, subject to the condition that $W(t,t;m) \equiv I$.

We shall now show that, under the assumptions (I)-(III), the macroscopic-cum-microscopic dynamics of Σ reduces to that on a fibre bundle, as formulated in Sec. 5. To this end, we start by defining B to be the bundle $M \times \mathfrak{A}$ and $V(t,m)$ and T_t to be the transformations of $\mathfrak{A}$ and B, respectively, given by the formulae

$$V(t,m) = W(t,0;m) \ \forall \ m \in M, \ t \in \mathbb{R}^+ \tag{57}$$

and

$$T_t(m,a) = (S_t m, V(t,m)a) \ \forall \ m \in M, \ a \in \mathfrak{A}, \ t \in \mathbb{R}^+. \tag{58}$$

We then prove the following proposition, which establishes that the transformations T_t form a one-parameter semigroup that satisfies all the conditions of the bundle dynamical model of Sec. 5.

Proposition 7.1 *Under the above assumptions, V satisfies the bundle structure property* (sometimes called the *cocycle condition*)

$$V(t+u,m) = V(u,S_t m)V(t,m) \ \forall \ m \in M, \ t,u \in \mathbb{R}^+ \tag{59}$$

and T possesses the semigroup properties

$$T_t T_u = T_{t+u} \text{ and } T_0 = I \ \forall \ t,u \in \mathbb{R}^+. \tag{60}$$

Comment: This proposition establishes that the transformations T_t form a one-parameter semigroup that satisfies all the conditions of the bundle dynamical model of Sec. 5. In other words, the macroscopic-cum-microscopic dynamics of the system Σ are completely captured by that model.

Proof of Prop. 7.1: It follows from the replacement of m, t, u by $S_u m, t-u, 0$, respectively, in Eq. (56) that

$$\frac{\partial}{\partial t}W(t-u,0;S_u m) = L(S_{t-u}S_u m)W(t-u,0;S_u m),$$

which, in view of the semigroup property of S, is equivalent to

$$\frac{\partial}{\partial t}W(t-u,0;S_u m) = L(S_t m)W(t-u,0;S_u m). \tag{61}$$

Hence, $W(t-u,0;S_u m)$, as well as $W(t,u;m)$, is a solution of Eq. (56), and therefore, in view of the uniqueness assumption (III),

$$W(t-u,0;S_u m) = W(t,u;m) \ \forall \ m \in M, \ u, t \ (\geq u) \in \mathbb{R}^+.$$

On replacing t by $t+u$ in this equation, we see that

$$W(t,0;S_u m) = W(t+u,u;m)\ \forall\ m\in M,\ \ t,u\in\mathbb{R}^+. \tag{62}$$

The bundle structure formula (59) for V now follows immediately from this formula, together with Eqs. (54) and (57). Furthermore, this result, together with the semigroup property of S, implies that the transformations T, defined by Eq. (58), form a one-parameter semigroup.

8 Fibre Bundle Description of Local Equilibrium

We shall now show that local thermodynamic equilibrium (LTE), as formulated in Sec. 6.3, may be simply described in terms of actions on a fibre bundle. For notational simplicity, we denote by $\theta(x)$ the $(n+1)$-component hydrodynamic field $(\beta(x), q(x)) \equiv (\beta(x), q_1(x), .\,., q_n(x))$ specified in Sec. 6.3. Thus, the LTE formula (47) takes the form

$$\lim_{L\to\infty} \tilde{\omega}^{(L)}(\alpha(L,x)a)) = \omega_{\theta(x)}(a). \tag{63}$$

Evidently, θ is a mapping from X into $\mathbb{R}^{n+1}$ whose range is the Cartesian product of those of the thermodynamic variables $(\beta, q_1, .\,., q_n)$. We shall denote by Θ the space of kinematically possible forms of the field θ. For present purposes, it is not necessary to specify the topology of Θ. We shall assume that the dynamical group, τ, of Σ induces a phenomenological dynamics given by a semigroup $\{\phi(t)\,|\,t\in\mathbb{R}^+\}$ of transformations of Θ. Thus,

$$\phi(t)\theta = \theta_t \equiv (\beta_t, q_t), \tag{64}$$

where β_t, q_t are the time-dependent fields that appear in Eq. (50).

We now define the fibre bundle B, in the notation of Sec. 5, to be $X\times\mathfrak{A}$. Its fibre at the spatial point x therefore consists of elements a_x of $\mathfrak{A}$. We then define P to be the mapping $\theta\to P_\theta$ of Θ into the complex-valued functions on B given by the formula

$$P_\theta(x, a_x) = \omega_{\theta(x)}(a_x). \tag{65}$$

On comparing this equation with Eq. (63), we see immediately that P carries a full description of LTE in that, for each $\theta\in\Theta$, P_θ represents the local equilibrium corresponding to the hydrodynamic field θ at each spatial point x. Likewise, defining the evolute P^t of P to be the mapping $\theta\to {P^t}_\theta$ of Θ into the complex functions on B given by

$${P^t}_\theta = P_{\theta_t}, \tag{66}$$

we see from Eqs. (50) and (64) that P^t carries a full representation of the LTE at time t. Moreover, by Eqs. (64) and (66),

$${P^t}_\theta = P_{\phi(t)\theta}, \tag{67}$$

which signifies that the time-dependence of the local equilibrium profile is completely determined by the phenomenological semigroup ϕ.

9 Concluding Remarks

The purpose of this paper has been to combine the geometric picture of fibre bundles with the operator algebraic one of quantum theory for the purpose of describing macroscopic quantum systems. The combination of these mathematical structures is quite natural in this context, since it offers the means of representing the complementary classical and quantum aspects of complex systems by actions on the bases and the fibres, respectively, of the relevant bundles. Here, we have shown that it serves to provide simple pictures here of quasi-particle excitations, of local thermodynamic equilibrium and of the interplay between the macroscopic (classical) dynamics and microscopic (quantum) dynamics of certain systems, e.g. lasers. In view of its apparent generality, we envisage that the fusion of geometric and operator algebraic concepts presented here should have wider ramifications for the theory of complex systems.

Finally, in view of the results of this paper, it would appear that the mathematical problem of establishing the existence of infinitesimal generators of one-parameter groups of bundle maps of vector bundles is of physical interest as well. We have not addressed this problem in its abstract form, but have given an explicit construction in Sec. 5 which is of fairly wide applicability.

[1] R. N. Sen, *Nonrelativistic Zero-Mass Systems*, Lecture Notes, Göttingen, 1974, pp. iii + 78.

[2] H.-J. Borchers and R. N. Sen, "Relativity groups in the presence of matter", Comm. Math. Phys. **42**, 101 (1975).

[3] R. N. Sen, "Theory of symmetry in the quantum mechanics of infinite systems", I. The state space and the group action, Physica **94A**, 39 (1978); II. Isotropic representations of the Galilei group and its central extensions, *ibid.* **94A**, 55 (1978).

[4] G. Alli and G. L. Sewell: "New methods and structures in the theory of the Dicke laser model", J. Math. Phys. **36**, 5598 (1995).

[5] F. Bagarello and G. L. Sewell: "New structures in the theory of the laser model II: microscopic dynamics and a nonequilibrium entropy principle", J. Math. Phys. **39**, 2730 (1998).

[6] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, 1951, 1972).

[7] That is, a fibre bundle in which the fibre is an infinite-dimensional complex Hilbert space. It should be noted that, in the mathematical literature, the term *Hilbert bundle* is often used to denote a somewhat larger class of spaces. See R. C. Fabec, *Fundamentals of Infinite-Dimensional Representation Theory*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2000, and the references cited therein.

[8] N. H. Kuiper, "The homotopy type of the unitary group of Hilbert space", Topology **3**, 19 (1965).

[9] E. P. Wigner, "Unitary representations of the inhomogeneous Lorentz group", Ann. Math. **40**, 149 (1939).

[10] A non-technical account is given in G. W. Mackey, *Induced Representations of Groups and*

Quantum Mechanics (Benjamin, New York, and Boringhieri, Torino, 1968).

[11] Full proofs may be found G. W. Mackey, *The Theory of Unitary Group Representations*, University of Chicago Press, Chicago, 1976.

[12] See E. Hille, "What is a Semigroup?", in: *Studies in Real and Complex Analysis*, Ed. I. I. Hirschman, Vol. 3 of

Studies in Mathematics (Mathematical Association of America and Prentice-Hall, Englewood Cliffs, NJ, 1965), and the references cited therein.

[13] E. Inönü and E. P. Wigner, "Representations of the Galilei Group", Nuovo Cimento **9**, 705-718 (1952).

[14] L. D. Landau and E. M. Lifschitz: *Statistical Physics* (Pergamon, London 1959).

[15] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity*, trans. from the Russian by P. C. Hohenberg (Benjamin, New York, 1955). Landau's original memoirs are reprinted in translation at the end of this book.

[16] D. Ruelle: *Statistical Mechanics* (Benjamin, New York, 1969).

[17] G. G. Emch: *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (Wiley, New York, 1972).

[18] O. Bratteli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics* (Springer-Verlag, Heidelberg, Vol. 1, 1979; Vol. 2, 1981).

[19] W. Thirring: *Quantum Mechanics of Large Systems* (Springer-Verlag, New York, 1980).

[20] G. L. Sewell: *Quantum Theory of Collective Phenomena* (Oxford University Press, Oxford, 1989).

[21] G. L. Sewell: "W*-dynamics of infinite quantum systems", Lett. Math. Phys. **6**, 209 (1982).

[22] For infinite systems, this is more generally applicable than the Heisenberg picture. The essential point here is that the Heisenberg picture of the dynamics of infinite systems corresponds to automorphisms of their algebras of observables, which exist only for particular models, e.g., those of lattice systems with finite range interactions. For details, see Ref. 21 and articles cited therein.

[23] G. L. Sewell: "Unbounded local observables in quantum statistical mechanics", J. Math. Phys. **11**, 1868 (1970).

[24] A folium is a norm-closed, convex set of states that is stable under the local modifications $\omega \rightarrow \omega_A(.) = \omega(A^\star(.)A)/\omega(A^\star A)$, for all elements A of $\mathfrak{A}$. It is thus a subset, possibly a proper one, of the *mathematical* state space, given by the positive, normalized, continuous linear functionals on $\mathfrak{A}$.

[25] The local normality condition signifies that the restriction of a state to a local algebra $\mathfrak{A}_\Lambda$ corresponds to a density matrix in $\mathcal{H}_\Lambda$, and ensures that the state cannot harbour an infinite number of particles in a bounded spatial region.

[26] A linear transformation ϕ of $\mathfrak{A}$ is completely positive if $\phi \otimes I$ is a positive transformation of $\mathfrak{A} \otimes M_n$, where M_n is the algebra of $n \times n$ matrices. It is a stronger property than positivity because of quantum interference, and is naturally required in the present physical context. See Refs. 27, 28.

[27] V. Gorini and A. Kossakowski: "N-level system in contact with a singular reservoir", J. Math.

Phys. **17**, 1298 (1976).

[28] G. Lindblad: "On the generators of quantum dynamical semigroups", Commun. Math. Phys. **48**, 119 (1976).

[29] J. Dixmier: *Les algèbres des opérateurs dans l'espace Hilbertien* (Gauthier-Villars, Paris, 1964.)

[30] K. Hepp: "Quantum theory of measurement and macroscopic observables", Helv. Phys. Acta **45**, 237-48 (1972).

[31] The state ω is primary if the centre of the von Neumann algebra $\pi(\mathfrak{A})''$ is trivial.

[32] D. Ruelle: "Symmetry breakdown in statistical mechanics", pp. 169-94 of *Cargèse Lectures*, Vol. 4, Ed. D. Kastler (Gordon and Breach, New York, 1969).

[33] R. Haag, N. M. Hugenholtz and M. Winnink: "On the equilibrium states in quantum statistical mechanics", Commun. Math. Phys. **5**, 215 (1967).

[34] D. A. Dubin and G. L. Sewell: "Time translations in the algebraic formulation of statistical mechanics", J. Math. Phys. **11**, 2990 (1970).

[35] G. G. Emch and H. J. F. Knops: "Pure thermodynamic phases as extremal KMS states", J. Math. Phys. **11**, 3008 (1970).

[36] G. G. Emch, H. J. F. Knops and E. Verboven: "Breaking of Euclidean symmetry with an application to the theory of crystallization", J. Math. Phys. **11**, 1655 (1970).

[37] H. Narnhofer and G. L. Sewell: "Equilibrium states of gravitational systems", Commun. Math. Phys. **71**, 1 (1980).

[38] A. De Masi, N. Ianiro, A. Pellegrenotti and E. Presutti: "A survey of the hydrodynamic behavior of many-particle systems", pp. 123-294 of *Studies in Statistical Mechanics*, Vol. XI, Ed. E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1984).

[39] F. Bagarello and G. Morchio: "Dynamics of mean field spin models from basic results in abstract differential equations", J. Stat. Phys. **66**, 849 (1992).