IMBEDDED SINGULAR CONTINUOUS SPECTRUM FOR SCHRÖDINGER OPERATORS

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ABSTRACT. We construct examples of potentials $V(x)$ satisfying $|V(x)| \leq \frac{h(x)}{1+x}$, where the function $h(x)$ is growing arbitrarily slowly, such that the corresponding Schrödinger operator has imbedded singular continuous spectrum. This solves one of the fifteen "twenty-first century" problems for Schrödinger operators posed by Barry Simon in [22]. The construction also provides the first example of a Schrödinger operator for which Möller wave operators exist but are not asymptotically complete due to the presence of singular continuous spectrum.

1. INTRODUCTION

Let H_V denote one-dimensional Schrödinger operator defined on the half-axis by differential expression

$$H_V = -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

and some self-adjoint boundary condition at zero. The operator (1.1) describes a charged particle, such as an electron, in electric field $V(x)$. When $V(x)$ is decaying quickly, one expects the spectral and dynamical properties of H_V to remain close to those of free operator H_0. Recall that with the operator (1.1) one can associate in a canonical way a spectral measure μ (see, for example, [5, 1]) which contains much information about the quantum system. The classical result in this direction, going back to the beginning of the century, is that if $V \in L^1$, the spectral measure on the positive semi-axis remains purely absolutely continuous. In one dimension, absolutely continuous spectrum corresponds to the ballistic rate of propagation of the particle. A natural question to ask is what are the critical rates of decay at which some changes in the spectral and dynamical properties of H_V may happen. As far as $V(x)$ has any decay at all, the essential spectrum coincides with $[0, \infty)$, but the quality of the spectrum and dynamics may change. In 1928, Wigner and von Neumann [25] showed that there exist potentials $V(x)$ satisfying $|V(x)| \leq \frac{C}{x+1}$ such that H_V has positive imbedded eigenvalue $E = 1$. This is a purely quantum resonance phenomenon, since V can be chosen arbitrarily small and bound state originates from long range correlations in the potential rather than usual confining effect. Naboko [15] and Simon [23] provided constructions which show that much more drastic changes are possible if potential decays arbitrarily slower than Coulomb. Namely, for any positive function $h(x)$ which grows at infinity, there exist potentials V such that $|V(x)| \leq \frac{h(x)}{1+x}$

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and H_V has a dense set of positive eigenvalues. Due to some recent results [2, 19], it is known that the absolutely continuous spectrum on the positive semi-axis is preserved for $|V(x)| \leq C(1+x)^{-1/2-\epsilon}$ and in fact for $V \in L^2$ [7]. More precisely, for such potentials the absolutely continuous part of the spectral measure, $\mu^{\rm ac}$, gives positive weight to any subset of $(0, \infty)$ of positive Lebesgue measure. Therefore, Naboko and Simon examples have dense point spectrum imbedded into the absolutely continuous spectrum.

In [22], Simon has posed fifteen problems on Schrödinger operators, one of which asks whether Schrödinger operators with potentials satisfying $|V(x)| \leq C(1+|x|)^{-1/2-\epsilon}$ can also have imbedded singular continuous spectrum. Remling [20] and later Christ-Kiselev [3] provided estimates on the dimension of the support of the possible singular component of the measure for $|V(x)| \leq C(1+x)^{-\alpha}$. Remling [21] and Kriecherbauer and Remling [11] constructed power-decaying potentials which have decaying solutions on the sets of Hausdorff dimension exactly as predicted by [20, 3]. However these constructions are not sufficient to infer the presence of the singular continuous component. The problem of controlling singular continuous spectrum is difficult since one has to study a set of energies which is uncountable but has Lebesgue measure zero. Because of that, in most known situations, the existence of singular continuous spectrum is proven "not by what it is, but by what it is not": by elimination of possibilities to have absolutely continuous or point spectrum. For instance, see the classical example of Pearson [16] or genericity of singular continuous spectrum results of Simon [24]. However, when singular continuous spectrum is imbedded, such path is obviously ruled out. To the best of our knowledge, up until now there were no constructive examples of Schrödinger operators with imbedded singular continuous spectrum in any setting. We notice that the fact that potentials leading to such operators exist follows from the inverse spectral theory (see, e.g. [13, 14]). This classical result has been recently improved by Denisov [8] to give existence of potentials leading to imbedded singular continuous spectrum in the L^2 class. Killip and Simon [9] have subsequently found necessary and sufficient conditions for a measure to be a spectral measure of a free Jacobi matrix with l^2 perturbation. However these interesting results provide only indirect information on what a potential leading to singular continuous spectrum looks like and do not seem promising in terms of obtaining results for power decaying potentials.

There is an additional point of view on the problem of imbedded singular continuous spectrum for potentials which decay faster than L^2 which in our opinion makes the question even more interesting. Recall the definition of wave operators

$$\Omega_\pm f = s - \lim_{t \mp \infty} e^{iH_V t} e^{-iH_0 t} f, \tag{1.2}$$

where the limit is taken in the strong L^2 sense. Existence of wave operators implies that the absolutely continuous spectrum of H_V fills the whole positive semi-axis, and provides much more precise information about long-time perturbed dynamics. The wave operators are called asymptotically complete (see, e.g. [18]) if the range of $\Omega_\pm$ coincides with the orthogonal complement of the subspace spanned by eigenvectors of the operator H_V. An alternative equivalent characterization is that the range of the wave operators is equal to the absolutely continuous subspace $\mathcal{H}_{\rm ac}(H_V)$ of the operator H_V, and that the singular continuous spectrum $\sigma_{\mathrm{sc}}(H_V)$ is empty. The intuitive meaning of asymptotic completeness is that the dynamics of the perturbed operator can be divided into two well-understood parts: scattering states traveling to infinity in a way similar to the free evolution, and bound states which remain confined in a certain sense for all times. On the other hand, the singular continuous spectrum is less understood, and corresponds to an intermediate situation: states which do not remain bounded in time, but travel slower, may exhibit recurrent phenomena and do not look like scattering states (see, for example, [12] and discussion in [4] for more detailed explanation and further references). A significant effort in quantum mechanical scattering theory was devoted to proving asymptotic completeness, and thus absence of singular continuous spectrum, in a wide variety of situations (see, e.g. [6] for a review of some examples). However, to the best of our knowledge, there are no examples of Schrödinger operators for which wave operators exist but are not asymptotically complete due to the presence of singular continuous spectrum. In the problem that we consider, a classical result is the existence of wave operators for $V \in L_1$, and in this class wave operators are also asymptotically complete. However recently in [4], the existence of (modified) wave operators for $V \in L_p$, $p < 2$ was established. Moreover, usual Möller wave operators (1.2) exist if $\lim_{x\to\infty} \int_0^x V(y)\,dy$ exists. Therefore, an example of (conditionally integrable) potential in L^p, $p < 2$ with imbedded singular continuous spectrum would also provide the first known example of a Schrödinger operator for which wave operators exist but fail to be asymptotically complete due to the presence of the singular continuous spectrum.

Our main goal in this paper is to prove the following

Theorem 1.1. *Given any positive function $h(x)$ tending to infinity as x grows, there exist potentials $V(x)$ such that $|V(x)| \leq \frac{h(x)}{1+x}$ and the operator H_V has non-empty singular continuous spectrum.*

It is easy to see that without loss of generality one can assume that $h(x)$ is monotone. For the rest of the paper, we are going to operate under this assumption, which is convenient for technical reasons. We are going to provide a fairly explicit construction of such potentials. As a corollary of such construction we are also going to prove

Theorem 1.2. *Given any positive function $h(x)$ tending to infinity as x grows, there exist potentials $V(x)$ such that $|V(x)| \leq \frac{h(x)}{1+x}$, and the wave operators for the operator H_V exist but are not asymptotically complete.*

Our examples can be easily generalized to any dimension by looking at spherically symmetric potentials.

Let us outline briefly the idea of construction. Controlling singular continuous spectrum is hard because unlike point spectrum, one has to control in a certain sense an uncountable set of energies, and there is no simple criteria (like L^2) for what behavior of solutions implies existence of singular continuous component. It turns out that one can reduce matters to keeping track on each step of only finite number of energies, but the price one has to pay is having to control rather precisely norms of the solutions at these energies. The control has to be much more precise than in constructions of Naboko and Simon. At the heart of the construction is a lemma which allows instead of one imbedded eigenvalue in Wigner-von Neumann-type construction to get two, each having an eigenfunction (normalized at the origin) with the L^2 norm squared exactly twice the L^2 norm squared of the eigenfunction in a usual Wigner-von Neumann construction in a certain asymptotic regime. The L^2 norm squared of the eigenfunction is the inverse of the point weight the measure μ assigns to a given eigenvalue. Thus the lemma will allow to "spread out" evenly the weight of one point mass to two nearby points. This allows to carry out Cantor-like construction, building a sequence of potentials which have point measure components approximating singular continuous measure. In the limit, one gets the result.

The methods of this paper can be generalized to the discrete Schrödinger operators, by using the discrete analog of the Prüfer transform (see, e.g. [10]). The results are parallel to the continuous case. We plan to address these applications elsewhere.

2. A Splitting Lemma

For the rest of the paper, let us fix for simplicity some boundary condition at the origin, for example Dirichlet. Let us denote by μ the spectral measure corresponding to this boundary condition. Throughout the paper, we are going to use notation C for universal constants (not necessarily the same), more precisely, for the constants which do not depend in any way on the step of the inductive construction. A starting point is the following elementary observation.

Lemma 2.1. *Let E be an eigenvalue of H_V, and $u(x,E)$ be the corresponding eigenfunction normalized by the condition $u'(0,E)=1$. Then $\mu(E) = \|u(x,E)\|^{-2}_{L^2(\mathbb{R}^+)}$.*

Proof. Denote $v(x,E)$ the solution satisfying orthogonal boundary condition $v(x,E) = 1$, $v'(x,E)=0$. Recall that the Weyl m-function is defined for $z \in \mathbb{C}\setminus\mathbb{R}$ as a unique complex number $m(z)$ such that $v(x,z)+m(z)u(x,z) \in L^2$ (assuming we are in the limit point case). We have (see e.g. [5], Chapter 9, Section 3)

$$m(z) - m(z_0) = \int\limits_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{1}{t-z_0} \right) d\mu(t), \tag{2.1}$$

where z_0 is some fixed complex number. From (2.1) it follows that

$$\mu(E) = \lim_{\epsilon\to 0} \epsilon |m(E+i\epsilon)| = \lim_{\epsilon \to 0} \epsilon \,\mathrm{Im}\, m(E+i\epsilon).$$

On the other hand,

$$\mathrm{Im}\, m(E+i\epsilon) = \epsilon \|v(x,E+i\epsilon) + m(E+i\epsilon)u(x,E+i\epsilon)\|^2_{L^2(\mathbb{R}^+)}$$

(see, e.g. [5]). Multiplying both sides of the above equality by ϵ and passing to the limit $\epsilon \to 0$, we get

$$\mu(E) = \mu(E)^2 \|u(x,E)\|^2_{L^2},$$

which is exactly the claim of the lemma. □

Given the energy $E = k^2$, it will be convenient to introduce Prüfer variables R and θ, $R^2 = (u')^2 + k^2u^2$ and $\theta = \tan^{-1}(ku/u')$. Then it is easy to see that

$$(\log R(x,k)^2)' = \frac{1}{k}V(x)\sin 2\theta(x,k) \tag{2.2}$$

$$\theta(x,k)' = k - \frac{1}{k}V(x)(\sin\theta)^2. \tag{2.3}$$

There are two essential new ideas which play important role in the result: the Splitting Lemma, and a relation between L^2 norm of R and derivative of the Prüfer angle θ in energy. The purpose of this section is to prove the first key lemma which is at the heart of construction, the Splitting Lemma. To clarify its meaning, let us formulate first a simplified version of this result. Fix an arbitrary positive energy k_0 (we are going to work with values of quasimomentum k throughout the paper, but still call those "energies" for simplicity). Fix a parameter g, which we think of as large. Define $\tilde{g} = \exp(g^{3/4})$. Fix another parameter, f, which we think of as small. (It is useful to keep in mind that in the actual construction, we will be able to choose g as large as we want and then after that to choose f as small as we want). Let $\delta k = \tilde{g}^{-1}f$, and denote k_1, k_2, $k_2 > k_1$, the ends of the interval of length δk centered at k_0. Denote $\theta_i(x) = \theta(x,k_i)$ and $\delta\theta = \theta_2 - \theta_1$. Furthermore, define a small angle α by a condition $f\sin\alpha = \delta k$. We are going to use notation

$$x_\beta = \min\{x : \delta\theta(x) = \beta\}.$$

Define a potential $V(x)$ on $[0,\infty)$ by

$$V(x) = \begin{cases} -2fk_0\sin(\theta_1+\theta_2), & 0 \le x \le x_{\pi-\alpha} \\ 2fk_0\sin(\theta_1+\theta_2), & x_{\pi-\alpha} \le x \le f^{-1}g \\ -\min\left(\delta k, \frac{g}{2x}\right)\frac{k_0}{4}(\sin 2\theta_1 + \sin 2\theta_2), & x > f^{-1}g. \end{cases} \tag{2.4}$$

The definition of V involves the Prüfer angles θ_i, and θ_i in their turn depend on V through (2.3). This ambiguity is easily resolved, however, by substituting the expression (2.4) for V into (2.3) and solving the resulting nonlinear system for θ_i. It is not difficult to see from (2.3) that this resulting system has unique piecewise smooth global solution by standard ODE existence and uniqueness theorem. The functions $\theta_i(x)$ are smooth apart from three points where V is not smooth, and where θ_i' (or θ_i'') may jump. We can then define V in terms of these θ_i by (2.4) recipe, and then solutions of (2.3) will coincide with θ_i by uniqueness. For the rest of the paper, we are going to consider on each step V defined in terms of Prüfer angles at a finite number of energies without further explanation. We are going to ignore for now the issue whether V is well-defined in a sense that $x_{\pi-\alpha} < f^{-1}g$; this will be verified later for all g large enough. We notice that the constant f in (2.4) can be replaced by an appropriate slowly varying function with no difference for the result. We choose a particular representation (2.4) to simplify technicalities. Then we have

Lemma 2.2. *Assume that g is sufficiently large, and f is sufficiently small. Then for a potential $V(x)$ given by (2.4), both k_1 and k_2 are eigenvalues. Moreover,*

$$\|R(x,k_i)\|^2_{L^2(\mathbb{R}^+)} = 2f^{-1}(1 + O(fe^g, \tilde{g}^{-1/2}g)) \tag{2.5}$$

provided that $R(x,k_i)$ are normalized by a condition $R(0,k_i) = 1$, $i = 1,2$.

We are not going to prove Lemma 2.2, but directly a more advanced version which is going to be needed in the construction. The difference is that we will need to handle not just a pair of eigenvalues, but 2^{n-1} pairs of eigenvalues on the nth step simultaneously. However, we are going to use this more transparent formulation to make a few clarifying remarks. First of all, observe that $|V(x)| \leq h(x)/(x+1)$ provided that f is sufficiently small. Indeed, we basically need to ensure that

$$2k_0 f \leq h(f^{-1}g)/(f^{-1}g+1),$$

or $2k_0(g+f) \leq h(f^{-1}g)$. This clearly holds if for fixed g we take f sufficiently small. Next, consider a Wigner-von Neumann-type potential $V_0(x)$ defined by

$$V_0(x) = \begin{cases} -2k_0 f \sin 2\theta_0, & 0 \leq x \leq f^{-1}g \\ -\min\left(\delta k, \frac{g}{2x}\right) \frac{k_0}{2} \sin 2\theta_0, & x > f^{-1}g. \end{cases}$$

It is fairly straightforward to check using (2.2), (2.3) that V_0 leads to an eigenvalue at k_0 with $\|R(x,k_0)\|^2_{L^2(\mathbb{R}^+)} = f^{-1}(1+O(fg,e^{-g}))$. Moreover, one cannot get a smaller (in the main term) norm for an eigenvalue at k_0 with a potential V satisfying $|V(x)| \leq f$. By Lemma 2.1, this means that we cannot have a point mass bigger than $f(1+O(fg,e^{-g}))$ in the spectral measure for such potentials. Thus Lemma 2.2 provides a way, using potential satisfying the same upper bound as V_0, to split the eigenvalue k_0 into two nearby eigenvalues with equal weights without loss of the total weight (in the asymptotic regime). The fact that it can be done is not surprising, but the construction is not trivial. For example, taking just $V = \frac{1}{2}(V_1+V_2)$, where V_i is given by the same expression as V_0 at k_i, seems to lead to a loss of a constant factor in the total mass of two eigenvalues and thus does not work for the Splitting Lemma. Moreover, one cannot hope for a coefficient smaller than 2 in front of f^{-1} in (2.5). As the construction below will show, that would have led by splitting to spectral measures giving arbitrary large weight to a fixed finite interval. But the spectral measure μ is well known to satisfy $\int \frac{d\mu(t)}{1+|t|^2} < C$, with C uniform for uniformly bounded V. Therefore, the factor 2 in the Splitting Lemma is optimal.

After these preliminary remarks, let us consider the general case. On the nth step of our construction, we are going to look at 2^n energies $k_j^{(n)}$, $j = 1, \dots 2^n$, ordered in increasing order. We are going to assume that for every n, all these energies lie in a fixed compact interval away from zero; moreover, we are going to assume that

$$2\max_{j,j'}|k_j^{(n)} - k_{j'}^{(n)}| \leq \min(k_j^{(n)}) \tag{2.6}$$

for every j, n. This assumption is made for technical convenience and will be easy to satisfy on each step of the inductive construction. We denote $\theta_j^{(n)} = \theta(x, k_j^{(n)})$, $R_j^{(n)} = R(x, k_j^{(n)})$, $\delta k_j^{(n)} = k_{2j}^{(n)} - k_{2j-1}^{(n)}$, and $\delta\theta_j^{(n)} = \theta_{2j}^{(n)} - \theta_{2j-1}^{(n)}$. Then $V^{(n)}(x)$ is going to be defined by $V^{(n)}(x) = \sum_{j=1}^{2^{n-1}} V_j^{(n)}(x)$, where

(2.7)
$$V_j^{(n)}(x) = \begin{cases} -2f_j^{(n)} k_j^{(n-1)} \sin(\theta_{2j-1}^{(n)} + \theta_{2j}^{(n)}), & 0 \leq x \leq x^{(n)}_{j,\pi-\alpha_j^{(n)}} \\ 2f_j^{(n)} k_j^{(n-1)} \sin(\theta_{2j-1}^{(n)} + \theta_{2j}^{(n)}), & x^{(n)}_{j,\pi-\alpha_j^{(n)}} \leq x \leq (f_j^{(n)})^{-1} g \\ -\min\left(\delta k_j^{(n)}, \frac{g_n}{2x}\right) \frac{k_j^{(n-1)}}{4} (\sin 2\theta_{2j-1}^{(n)} + \sin 2\theta_{2j}^{(n)}), & x > (f_j^{(n)})^{-1} g. \end{cases}$$

Here $\alpha_j^{(n)}$ is defined by $f_j^{(n)} \sin \alpha_j^{(n)} = \delta k_j^{(n)}$, and

$$x_{j,\beta}^{(n)} = \min\{x : \delta\theta_j^{(n)}(x) = \beta\}.$$

Notice that the value of g_n is going to be the same for all j. Also, as before, we will denote $\tilde{g}_n = \exp(g_n^{3/4})$. The parameters g_n, $f_j^{(n)}$ and $\delta k_j^{(n)}$ are to be chosen on the nth step of construction. The energies $k_j^{(n-1)}$ are given (from the previous step). On the nth step, each $k_j^{(n-1)}$ splits into two eigenvalues $k_{2j-1}^{(n)}, k_{2j}^{(n)}$; that is, the interval $(k_{2j-1}^{(n)}, k_{2j}^{(n)})$ is centered at $k_j^{(n-1)}$ and has length $\delta k_j^{(n)}$. Throughout the paper, we will assume the relationship $f_j^{(n)} = \tilde{g}_n \delta k_j^{(n)}$. In the actual construction, $V^{(n)}(x)$ will be given by (2.7) only to the right of some value of $x = x_n$, and $R(x_n, k_j^{(n)}) \neq 1$. However in this section we will consider $V^{(n)}(x)$ defined by (2.7) on $[0, \infty)$ and we will assume that $R(0,k) = 1$ and $\theta_j^{(n)}(0,k) = 0$ for all k. Let us introduce one more parameter that we need, $a_n = \min_{j,j'} |k_j^{(n-1)} - k_{j'}^{(n-1)}|$ (essentially, in the construction process we will have $a_n = \min_j(\delta k_j^{(n-1)})$). To keep notation compact, henceforth in this section we are going to omit the index "n" for most variables, including V (but with an exception of k's, where members of two different steps participate explicitly in a construction).

Lemma 2.3 (Splitting Lemma). *Let $V(x)$ be given by* (2.7)*, and assume that*

$$4\sum_j f_j k_j^{(n-1)} < \frac{1}{3}a, \quad g >> 1, \quad 2^n g a^{-2} \sum_{j=1}^{2^{n-1}} f_j << 1, \quad \delta k_j < \frac{1}{12}a, \tag{2.8}$$

for $j = 1, \dots, 2^{n-1}$. Then for any $l = 1, \dots, 2^{n-1}$,

$$\|R(x, k_{2(l-1)+i}^{(n)})\|_{L^2(\mathbb{R}^+)}^2 = 2f_l^{-1}\left(1 + O\left(2^n e^g a^{-2} \sum_{j=1}^{2^{n-1}} f_j, \tilde{g}^{-1/2} g\right)\right), \tag{2.9}$$

where $i = 1, 2$.

Remark. The condition $g >> 1$ means that g needs to be greater than some universal constant, the value of which can be derived from the proof. The same interpretation applies to the other condition in (2.8) involving "<<". Notice also that the fourth condition basically follows from the first two and $\delta k_j = \tilde{g}^{-1} f_j$, but we state it separately for convenience.

We are going to fix $l = 1$, $i = 1$ and prove the result for this case; other l, i are treated in exactly the same way. Write $V = V_1 + W$, where $V_1 = V_1^{(n)}$ and W is the rest of the potential. Direct substitution of (2.7) into (2.2), (2.3) and some trigonometry leads to the following equations valid for $x < f_1^{-1} g$:

$$(\log R^2(x, k_1^{(n)}))' = -\frac{k_1^{(n-1)}}{k_1^{(n)}} \tilde{f}_1(x)(\cos\delta\theta_1 - \cos(\theta_2 + 3\theta_1)) + \frac{W(x)}{k_1^{(n)}} \sin 2\theta_1 \tag{2.10}$$

$$\begin{aligned}(\delta\theta_1)' &= \delta k_1 \left(1 - \frac{V(x)}{2k_1^{(n)}k_2^{(n)}}(1+\cos 2\theta_2)\right) + \\ &+ \left(\frac{k_1^{(n-1)}}{k_1^{(n)}}\tilde{f}_1(x)(1-\cos 2(\theta_1+\theta_2)) - \frac{W(x)}{k_1^{(n)}}\sin(\theta_1+\theta_2)\right)\sin\delta\theta_1;\end{aligned} \tag{2.11}$$

here $\tilde{f}_1(x) = \pm f_1$ with a change from $+$ to $-$ at $x = x_{1,\pi-\alpha_1}$. For the rest of this section, we are also going to omit the index "1" in $x_{1,\beta}$ and write simply x_β for x such that $\delta\theta_1(x_\beta) = \beta$.

Remark. Equations (2.10) and (2.11) are fairly complicated, but not all terms are of equal importance. At the heart of the matter is the following simpler nonlinear dynamical system:

$$\begin{gathered}(\log R^2)' = -\tilde{f}_1(x)\cos\delta\theta_1 \\ (\delta\theta_1)' = \delta k_1 + \tilde{f}_1(x)\sin\delta\theta_1.\end{gathered} \tag{2.12}$$

In some sense, we will show that the rest of terms in (2.10), (2.11) produce only small corrections to the behavior of this system. It is quite instructive (and technically simple) to prove a version of the Splitting Lemma for the system (2.12), but to save the space, we are going to treat directly the system (2.10), (2.11).

From (2.11) and $\delta\theta_1(0) = 0$, we can write

$$\delta\theta_1 = \delta k_1 \int_0^x e^{\int_y^x \xi(s)\,ds}(1 + O(|V(y)|))\,dy, \tag{2.13}$$

where

$$\xi(y) = \frac{k_1^{(n-1)}}{k_1^{(n)}}\left(\tilde{f}_1(y)(1-\cos 2(\theta_1+\theta_2)) - \frac{W(y)}{k_1^{(n-1)}}\sin(\theta_1+\theta_2)\right)\frac{\sin\delta\theta_1}{\delta\theta_1}. \tag{2.14}$$

We need the following technical lemma

Lemma 2.4. *Under the assumptions of the Splitting Lemma, we have for* $y < x \leq f_1^{-1}g$

$$\int_y^x \xi(s)\,ds = \int_y^x \tilde{f}_1(s)\frac{\sin\delta\theta_1}{\delta\theta_1}\,ds(1+O(\delta k_1)) + O(\mathcal{E}_n), \tag{2.15}$$

where $\mathcal{E}_n = 2^n(\sum_j f_j)ga^{-2}$.

Proof. Let us consider one of the terms entering into ξ,

$$\int_y^x V_j(s)\sin(\theta_1+\theta_2)\frac{\sin\delta\theta_1}{\delta\theta_1}\,ds, \tag{2.16}$$

where $2 \leq j \leq 2^{n-1}$ is arbitrary. Recall that $V_j(s) = \tilde{f}_j(s)\sin(\theta_{2j-1}+\theta_{2j})$ for $x \leq (f_j)^{-1}g$, where $\tilde{f}_j = \pm f_j$ with only at most one jump (we do not assume apriori that $x_{j,\pi-\alpha_j} < (f_j)^{-1}g$, but will show this below). For $x > (f_j)^{-1}g$, $V_j(s) = \tilde{f}_j(s)(\sin(2\theta_{2j-1}) + \sin(2\theta_{2j}))$, where $|\tilde{f}_j(s)| \leq (k_j^{(n-1)}g)/2s$, $|\tilde{f}_j'(s)| \leq (k_j^{(n-1)}g)/(2s^2)$. Notice that by assumption (2.8) we have

$$|(\theta_{2j-1}+\theta_{2j}-\theta_1-\theta_2)'| \geq \frac{a}{2}, \quad |(2\theta_{2j-1}-\theta_1-\theta_2)'| \geq \frac{a}{2}, \quad |(2\theta_{2j}-\theta_1-\theta_2)'| \geq \frac{a}{2}. \tag{2.17}$$

Substituting the expression for V_j into (2.16) and using formula for the product of sines, we reduce matters to estimation of the integrals of type

$$I_{\pm}(x,y) = \int_y^x \tilde{f}_j \cos(2\theta_{2j} \pm (\theta_1+\theta_2))\frac{\sin\delta\theta_1}{\delta\theta_1}\,ds.$$

Consider the "$-$" case, and integrate by parts integrating

$$\cos(2\theta_{2j} - (\theta_1+\theta_2))(2\theta_{2j} - (\theta_1+\theta_2))'.$$

Observing that $|\theta_j''(s)| \leq C(|V(s)| + |V'(s)|)$ for any j, $|(\delta\theta_1)'(s)| \leq C(|V(s)| + \delta k_1)$ and using the inequality (2.17) for the derivative of the argument of cosine, we obtain

$$|I_-(x,y)| \leq Ca^{-1}\left(f_j + \int_y^x \left(|\tilde{f}_j'(s)| + |\tilde{f}_j(s)|(|V(s)|+\delta k_1) + |\tilde{f}_j(s)|(|V(s)|+|V'(s)|)a^{-1}\right)ds\right),$$

where C is a universal constant. The discontinuities of $\tilde{f}_j$ are taken into account in off-integral term. Since $|V(s)|, |V'(s)| \leq C\sum_l |\tilde{f}_l(s)|$, $\delta k_1 < f_1$ and $a \leq C$, we arrive at a bound valid for all $y < x \leq f_1^{-1}g$

$$|I_-(x,y)| \leq C\left(f_j a^{-1} + a^{-2}\int_y^x |\tilde{f}_j(s)|\sum_l |\tilde{f}_l(s)|\,ds\right).$$

The estimate on I_+ is similar, but involves a constant independent of n instead of a (arising from the minimal possible size of $k_j^{(n)}$). The contribution of the term $\tilde{f}_1(s)\cos(2(\theta_1+\theta_2))$ in (2.14) is bounded similarly to I_+. Summing up all bounds, we obtain at most

$$C\left(\sum_j f_j a^{-1} + a^{-2}\int_y^x \left(\sum_j |\tilde{f}_j(s)|\right)^2 ds\right). \tag{2.18}$$

We can compute explicitly that $\|\tilde{f}_j\|_{L^2}^2 \leq 2f_j g$, and therefore the total bound does not exceed $Ca^{-2}2^n(\sum_j f_j)g$. Notice that the first term in (2.18) is included into this error since a is less than a fixed constant and $g > 1$. This is exactly the error term in (2.15); the origin of the main term is clear. □

Remark. The error $O(\delta k_1)$ in the main term of (2.15) gives after integration $O(g\delta k_1) = O(\tilde{g}^{-1}gf_1)$ which can also be subsumed into $\mathcal{E}_n$.

Next we show that, provided that the error terms in (2.15) are small, we have $x_{\pi-\alpha_1} < f_1^{-1}g$, and thus our potential is well defined.

Lemma 2.5. *For sufficiently large g and for sufficiently small $\mathcal{E}_n$, we have*

$$x_{\pi-\alpha_1} \leq \pi f_1^{-1}\log(2\tilde{g})(1+C\mathcal{E}_n). \tag{2.19}$$

Remark. Similarly to Lemma 2.3, by sufficiently large (or small) we mean that there exist universal constants such that if the value of g (respectively $\mathcal{E}_n$) exceeds (is less than) these constants, the result holds.

Proof. Let us first estimate from above $x_{\pi/2}$. On the interval $[0, x_{\pi/2}]$ we have

$$2/\pi \leq (\sin\delta\theta_1)/\delta\theta_1 \leq 1.$$

Thus, according to (2.13), Lemma 2.4 and the remark after its proof, we have

$$\delta\theta_1(x) \geq e^{\frac{2}{\pi}f_1 x}\frac{\pi\delta k_1}{2f_1}(1-e^{-\frac{2}{\pi}f_1 x})(1-C\mathcal{E}_n) \tag{2.20}$$

for $x < (f_1)^{-1}g$ and such that $\delta\theta_1 \in [0,\pi/2]$. It is clear from (2.20) that if g is large enough, then $x_{\pi/2} < f_1^{-1}g$. Indeed, recall that $\delta k_1/f_1 = e^{-g^{3/4}}$. If $f_1^{-1}g$ were less than $x_{\pi/2}$, (2.20) would imply that

$$\delta\theta_1(f_1^{-1}g) \geq e^{\frac{2}{\pi}g-g^{3/4}}\frac{\pi}{2}\left(1-e^{-\frac{2}{\pi}g}\right)(1-C\mathcal{E}_n),$$

an obvious contradiction for g large enough and sufficiently small $\mathcal{E}_n$. Moreover, provided that $\mathcal{E}_n$ is small, we can estimate (assuming also $x_{\pi/2} \geq \frac{\pi}{2}(f_1)^{-1}\log 2$, or else we have a nice upper bound)

$$e^{\frac{2}{\pi}f_1 x_{\pi/2}} \leq 2\tilde{g}\left(1+C\mathcal{E}_n\right).$$

From this inequality it is easy to see that

$$x_{\pi/2} \leq \frac{\pi}{2}f_1^{-1}\log(2\tilde{g})(1+C\mathcal{E}_n). \tag{2.21}$$

Imagine for a moment that $\tilde{f}_1(x) = f_1$ for $x \leq x_\pi$. Notice that under a change of variables $\overline{\delta\theta}_1 = \pi - \delta\theta_1$, $\overline{x} = x_\pi - x$ the equation (2.11) transforms into the equation for $\overline{\delta\theta}_1$ of the same form as (2.11), with the same initial condition and same estimates available. In particular, the point $\overline{x}_{\pi/2}$ such that $\overline{\delta\theta}_1(\overline{x}_{\pi/2}) = \pi/2$ satisfies the same bound as $x_{\pi/2}$. But $\overline{\delta\theta}_1(\overline{x}) = \pi - \delta\theta_1(x_\pi - x)$, and thus $\overline{x}_{\pi/2} = x_\pi - x_{\pi/2}$. But $x_{\pi-\alpha_1} \leq x_\pi$, and the above observation together with (2.21) finishes the proof of the lemma. □

Next, we prove the following essential

Lemma 2.6 (Small Angle Lemma)**.** *Let γ be a small angle, in particular such that $\gamma < g^{-1}$. Denote x_γ the smallest value of x where $\delta\theta_1(x) = \gamma$. Then*

$$R(x_\gamma, k_1^{(n)})^2 = \frac{\delta k_1}{\gamma f_1 + \delta k_1}\left(1+O(\gamma^2 g, \mathcal{E}_n)\right). \tag{2.22}$$

Moreover,

$$\|R(x,k_1^{(n)})\|^2_{L^2(0,x_\gamma)} = f_1^{-1}\frac{1}{1+\tilde{g}^{-1}\gamma^{-1}}\left(1+O(\gamma^2 g,\mathcal{E}_n)\right). \tag{2.23}$$

Proof. Notice that on $[0, x_\gamma]$, we have $(\sin\delta\theta_1)/\delta\theta_1 = 1 + O(\gamma^2)$. Obesrve that by Lemma 2.5 we have that $x_\gamma < f_1^{-1}g$. By (2.13), Lemma 2.4 and the definition of potential we have that

$$\delta k_1 \int\limits_0^{x_\gamma} e^{f_1(x_\gamma - y)(1+O(\gamma^2))+O(\mathcal{E}_n)}(1 + O(|V(y)|)\,dy = \gamma.$$

Thus, we obtain

$$e^{f_1 x_\gamma}(1 - e^{-f_1 x_\gamma})f_1^{-1}\delta k_1(1 + O(\mathcal{E}_n, \gamma^2 g)) = \gamma. \tag{2.24}$$

We included the $O(|V(y)|)$ error into $\mathcal{E}_n$, since we can assume freely that $a < 1$ and $g > 1$. Thus

$$e^{f_1 x_\gamma} = \left(\frac{\gamma f_1}{\delta k_1} + 1\right)\left(1 + O(\gamma^2 g, \mathcal{E}_n)\right). \tag{2.25}$$

Now in the equation (2.10) for the amplitude, we have for $x < f_1^{-1}g$

$$\log(R(x, k_1^{(n)})^2) = -\int\limits_0^x \tilde{f}_1(y)\cos\delta\theta_1(y)\,dy + O(\mathcal{E}_n), \tag{2.26}$$

by an estimate directly analogous to the estimates of Lemma 2.4 (but easier). Therefore, for $x \le x_\gamma$, we have

$$R(x, k_1^{(n)})^2 = e^{-f_1 x}(1 + O(\gamma^2 g, \mathcal{E}_n)). \tag{2.27}$$

The first statement of the Lemma, estimate (2.22), follows directly from (2.27) and (2.25). Also, the L^2 norm of $R(x, k_1^{(n)})$ on $[0, x_\gamma]$ satisfies

$$\|R(x, k_1^{(n)})\|^2_{L^2(0,x_\gamma)} = f_1^{-1}(1 - e^{-f_1 x_\gamma})(1 + O(\gamma^2 g, \mathcal{E}_n)) = f_1^{-1}\frac{1}{1 + \tilde{g}^{-1}\gamma^{-1}}(1 + O(\gamma^2 g, \mathcal{E}_n)).$$

In the last step we used (2.22) and the fact that $\delta k_1 = \tilde{g}^{-1}f_1$. □

Next we need a lemma describing the solution of (2.10), (2.11) on the interval $[0, x_{\pi-\alpha_1}]$. Let $R(x, k_1^{(n)})$, $\delta\theta_1$ solve (2.10), (2.11) and assume that $\tilde{R}(x)$, $\tilde{\delta\theta}_1(x)$ solve a variant of a system (2.12)

$$(\log\tilde{R}^2(x))' = -f_1\cos\tilde{\delta\theta}_1(x) \tag{2.28}$$

$$(\tilde{\delta\theta}_1(x))' = \delta k_1 + f_1\sin\tilde{\delta\theta}_1(x) \tag{2.29}$$

for all x. Assume furthermore that the initial conditions for $\tilde{R}$, $\tilde{\delta\theta}_1$ at $x = 0$ are the same as for R, $\delta\theta$. Then the following lemma holds.

Lemma 2.7. *For every $x \le x_{\pi-\alpha_1}$, we have*

$$|\tilde{\delta\theta}_1(x) - \delta\theta_1(x)| \le C\tilde{g}^{-1}\tilde{\mathcal{E}}_n \tag{2.30}$$

$$\tilde{R}^2(x)/R^2(x, k_1^{(n)}) = 1 + O(\tilde{\mathcal{E}}_n), \tag{2.31}$$

where $\tilde{\mathcal{E}}_n = 2^n e^g a^{-2}\sum_j f_j$.

Proof. Denote $\delta\theta^*(x) = \widetilde{\delta\theta}_1(x) - \delta\theta_1(x)$. From (2.11), (2.29) it follows that for $x \leq f_1^{-1}g$,

$$\begin{aligned}(\delta\theta^*)'(x) &= \delta k_1 O(|V(x)|) + f_1 \frac{\sin\widetilde{\delta\theta}_1 - \sin\delta\theta_1}{\delta\theta^*}\delta\theta^* + \\ & \left(f_1\cos 2(\theta_1+\theta_2) + \frac{W(x)}{k_1^{(n)}}\sin(\theta_1+\theta_2)\right)\sin\delta\theta_1.\end{aligned}$$

Therefore,

$$\delta\theta^*(x) = \int_0^x e^{\int_y^x \eta(s)\,ds}\left(\delta k_1 O(|V(y)|) + \left(f_1\cos 2(\theta_1+\theta_2) + \frac{W(x)}{k_1^{(n)}}\sin(\theta_1+\theta_2)\right)\sin\delta\theta_1\right), \tag{2.32}$$

where

$$\eta(s) = 2f_1\frac{\sin\delta\theta^*}{\delta\theta^*}\cos(\delta\theta_1 + \widetilde{\delta\theta}_1).$$

Now all oscillatory terms in (2.32) (that is, all terms except the $\delta k_1 O(|V|)$ term) are estimated by integration by parts similarly to Lemma 2.4. For example, for any of the $2^{n-1}-1$ terms entering into $W(y)$, we have for $x \leq x_{\pi-\alpha_1}$

$$\left|\int_0^x e^{\int_y^x \eta(s)\,ds} V_j(y)\sin(\theta_1+\theta_2)\,dy\right| \leq \tag{2.33}$$

$$Ca^{-1}\tilde{g}\left(f_j + \int_0^x\left(|\tilde{f}_j'(y)| + |\eta(y)||\tilde{f}_j(y)| + |\tilde{f}_j(y)|O(|V(y)| + |V'(y)|)a^{-1}\right)dy\right),$$

where

$$\tilde{f}_j(y) = \begin{cases} -f_j, & 0 \leq x \leq x_{j,\pi-\alpha_j} \\ f_j, & x_{j,\pi-\alpha_j} \leq x \leq (f_j)^{-1}g \\ -\min(\delta k_j, \frac{g}{2x})\frac{k_j^{(n-1)}}{4}, & x > (f_j)^{-1}g. \end{cases} \tag{2.34}$$

Lemma 2.5 was used to estimate the exponential by $\tilde{g}$. We make a convention that the derivative $\tilde{f}_j'(y)$ does not involve δ functions from the two jumps of $\tilde{f}_j$; those jumps contribute to the first, off-integral term in the estimate. Notice that $|\eta(x)|, |V(x)|, |V'(x)| \leq C\sum_j \tilde{f}_j(x)$, where C is a universal constant. Thus the total bound in (2.33) does not exceed

$$Ca^{-1}\tilde{g}\left(f_j + a^{-1}\int_0^\infty \tilde{f}_j(y)\sum_l \tilde{f}_l(y)\,dy\right).$$

The term $f_1\cos 2(\theta_1+\theta_2)$ is estimated similary (but gives smaller error), while the term $\delta k_1 O(|V(y)|)$ integrated over $[0, x_{\pi-\alpha_1}]$ with the exponential gives at most

$(f_1)^{-1}\delta k_1 \tilde{g} \sum_j f_j$ by Lemma 2.5. Summing together all errors we obtain

$$|\delta\theta^*| \leq C\tilde{g}\left(a^{-1}\sum_l f_l + a^{-2}\int\limits_0^\infty \left(\sum_j \tilde{f}_j\right)^2 dy\right),$$

which leads to

$$|\delta\theta^*(x)| \leq C2^n \tilde{g} g a^{-2} \sum_j f_j \tag{2.35}$$

for $x \leq x_{\pi-\alpha_1}$. This proves (2.30), since $e^g \geq Cg(\tilde{g}^2)$ for some universal C and all g.

By (2.10), (2.28) and an argument parallel to Lemma 2.4,

$$\log R^2(x, k_1^{(n)}) - \log \tilde{R}^2(x) = -\int\limits_0^x f_1(\cos\delta\theta_1 - \cos\tilde{\delta\theta}_1)\,dy + O(\mathcal{E}_n).$$

Using (2.30) and Lemma 2.5 we obtain for $x \leq x_{\pi-\alpha_1}$

$$R^2(x, k_1^{(n)})/\tilde{R}^2(x) = 1 + O(\tilde{\mathcal{E}}_n), \tag{2.36}$$

which finishes the proof (we used that evidently $\mathcal{E}_n < \tilde{\mathcal{E}}_n$ and so the former error is absorbed by the latter). Also on the last step we generously exchanged $\tilde{g}^{-1}$ for $g^{3/4}$ to arrive at an error in (2.36). □

Now we are ready to proceed with the proof of Lemma 2.3.

Proof of the Splitting Lemma. The structure of the eigenfunctions obtained in Splitting Lemma is illustrated by Figure 1.

1. Consider first the interval $[0, x_{\pi/2}]$. The estimate for the norm of $R(x, k_1^{(n)})$ on the interval $[0, x_\gamma]$ for some small γ is provided by Lemma 2.6. By (2.22) and (2.26), the norm on the interval $[x_\gamma, x_{\pi/2}]$ does not exceed $(\frac{\gamma f_1}{\delta k_1} + 1)^{-1} x_{\pi/2}(1 + O(\gamma^2 g, \mathcal{E}_n))$. Choosing $\gamma = \tilde{g}^{-1/2}$, for example, and using the bound (2.21) on $x_{\pi/2}$, we find that

$$\|R(x, k_1^{(n)})\|^2_{L^2[x_\gamma, x_{\pi/2}]} = f_1^{-1} O(\tilde{g}^{-1/2} g^{3/4}, \mathcal{E}_n).$$

Thus, altogether,

$$\|R(x, k_1^{(n)})\|^2_{L^2[0, x_{\pi/2}]} = f_1^{-1}(1 + O(\mathcal{E}_n, \tilde{g}^{-1/2} g)). \tag{2.37}$$

2. Next, we estimate norm on $[x_{\pi/2}, x_{\pi-\alpha_1}]$. Consider an auxiliary simplified system (2.28), (2.29) of Lemma 2.7. By (2.30), we have

$$\tilde{\delta\theta}_1(x_{\pi/2}) = \pi/2 + O(\tilde{g}^{-1}\tilde{\mathcal{E}}_n), \;\; \tilde{\delta\theta}_1(x_{\pi-\alpha_1}) = \pi - \alpha_1 + O(\tilde{g}^{-1}\tilde{\mathcal{E}}_n).$$

Denote $\tilde{x}_\beta = \min\{x | \tilde{\delta\theta}_1(x) = \beta\}$. Notice that the system (2.28), (2.29) is symmetric in a sense that for $x \leq \tilde{x}_\pi$,

$$\tilde{\delta\theta}_1(x) = \pi - \tilde{\delta\theta}_1(\tilde{x}_\pi - x) \;\text{ and }\; \tilde{R}^2(x) = \tilde{R}^2(\tilde{x}_\pi - x).$$

Observe also that $\alpha_1 = \tilde{g}_1^{-1}(1 + O(\tilde{g}^{-2}))$. Thus the norm

$$\|\tilde{R}\|^2_{L^2[x_{\pi/2}, x_{\pi-\alpha_1}]} = \|\tilde{R}\|^2_{L^2[\tilde{x}_{\alpha_1(1+O(\tilde{\mathcal{E}}_n))}, \tilde{x}_{\pi/2(1+O(\tilde{\mathcal{E}}_n))}]}. \tag{2.38}$$

By identical arguments, Lemma 2.5 and Lemma 2.6 (Small Angle Lemma) apply to the system (2.28), (2.29) (with the only difference that the conclusion does not involve the error $\mathcal{E}_n$ coming from oscillatory terms which are absent in the simplified system). Therefore, the considerations close to the first part of the proof above apply to the norm of $\tilde{R}$ and give

$$\|\tilde{R}\|^2_{L^2[0,\tilde{x}_{\pi/2(1+O(\tilde{\mathcal{E}}_n))}]} = f_1^{-1}(1+O(\tilde{g}^{-1/2}g)). \tag{2.39}$$

By (2.23) of Lemma 2.6, we also have

$$\|\tilde{R}\|^2_{L^2[0,\tilde{x}_{\alpha_1(1+O(\tilde{\mathcal{E}}_n))}]} = \frac{1}{2}f_1^{-1}(1+O(\tilde{\mathcal{E}}_n,\tilde{g}^{-2}g)). \tag{2.40}$$

Therefore, the norm in (2.38) is equal to $\frac{1}{2}f_1^{-1}(1+O(\tilde{\mathcal{E}}_n,\tilde{g}^{-1/2}g))$. Now by the estimate (2.31) of Lemma 2.7 we have

$$\|R(x,k_1^{(n)})\|^2_{L^2[x_{\pi/2},x_{\pi-\alpha_1}]} = \frac{1}{2}f_1^{-1}(1+O(\tilde{\mathcal{E}}_n,\tilde{g}^{-1/2}g)). \tag{2.41}$$

In addition, by Lemma 2.6 and symmetry

$$\tilde{R}^2(x_{\pi-\alpha_1}) = \tilde{R}^2(x_{\alpha_1(1+O(\tilde{\mathcal{E}_n}))}) = \frac{1}{2}(1+O(\tilde{\mathcal{E}}_n,\tilde{g}^{-2}g)).$$

By Lemma 2.7, this implies

$$R(x_{\pi-\alpha_1},k_1^{(n)})^2 = \frac{1}{2}(1+O(\tilde{\mathcal{E}}_n,\tilde{g}^{-2}g)). \tag{2.42}$$

3. Next step is the consideration of $[x_{\pi-\alpha_1}, f_1^{-1}g]$ interval. On this interval, in the idealized system (2.12) the angle $\delta\theta_1$ stays constant, and since it is very close to π, this allows for a consistent decay in R equation. The main goal therefore is to control the additional perturbation terms, which are either small or oscillating. Define $\delta\overline{\theta}_1 = (\pi-\alpha_1)-\delta\theta_1$, then

$$(\delta\overline{\theta}_1)' = -\delta k_1(1+O(|V(x)|))+\sin(\delta\overline{\theta}_1+\alpha_1)\left(f_1(1-\cos 2(\theta_1+\theta_2))+\frac{W(x)}{k_1^{(n)}}\sin(\theta_1+\theta_2)\right).$$

Using the relationship $\delta k_1 = f_1\sin\alpha_1$, we arrive at

$$\begin{aligned}(\delta\overline{\theta}_1)' &= \left(f_1(1-\cos 2(\theta_1+\theta_2))+\frac{W(x)}{k_1^{(n)}}\sin(\theta_1+\theta_2)\right)\times\\ &\quad\times\left(\cos\alpha_1\frac{\sin\delta\overline{\theta}_1}{\delta\overline{\theta}_1}+\sin\alpha_1\frac{\cos\delta\overline{\theta}_1-1}{\delta\overline{\theta}_1}\right)\delta\overline{\theta}_1 -\\ &\quad \delta k_1 O(|V(x)|)+f_1\sin\alpha_1\cos 2(\theta_1+\theta_2)-\frac{W(x)}{k_1^{(n)}}\sin\alpha_1\sin(\theta_1+\theta_2),\end{aligned} \tag{2.43}$$

with the initial data $\delta\overline{\theta}_1(x_{\pi-\alpha_1}) = 0$. Denote $\zeta(x)$ the expression in front of $\delta\overline{\theta}_1$ in (2.43). The solution to (2.43) can be written in the following form:

$$\delta\overline{\theta}_1(x) = \delta k_1\int\limits_{x_{\pi-\alpha_1}}^{x} e^{\int_y^x\zeta(s)\,ds}\left(O(|V(y)|)-\cos 2(\theta_1+\theta_2)-\frac{W(y)}{f_1k_1^{(n)}}\sin(\theta_1+\theta_2)\right)dy. \tag{2.44}$$

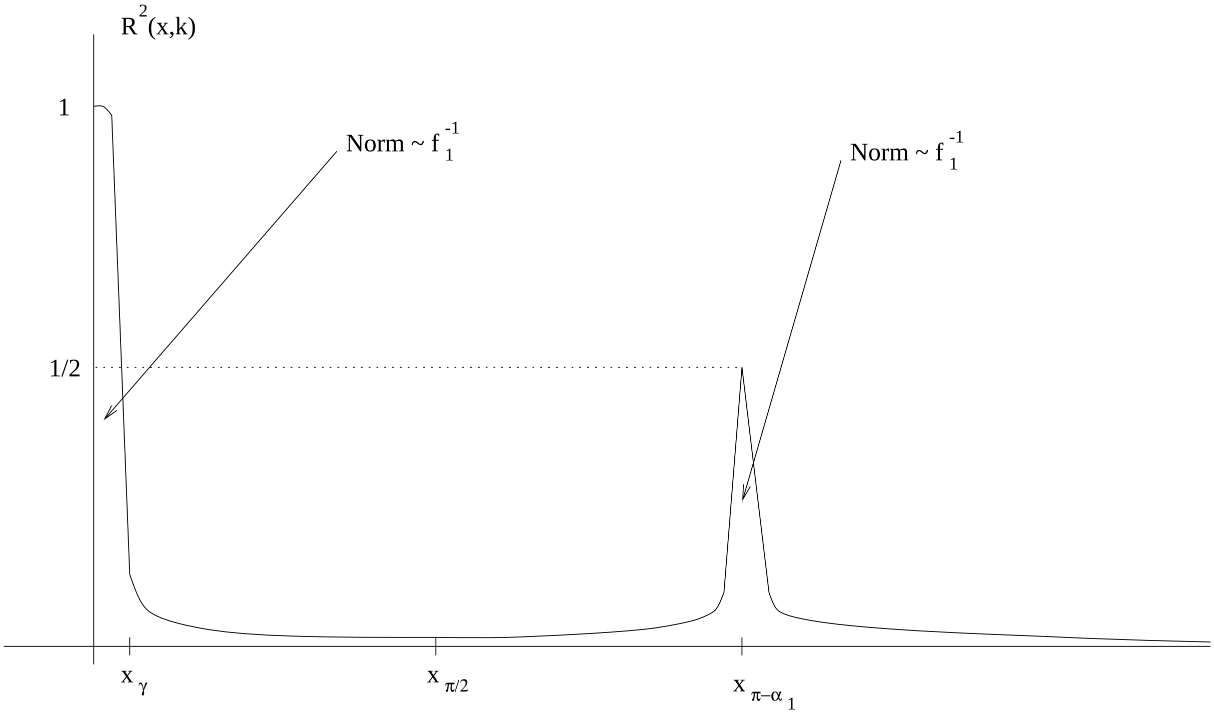

FIGURE 1. The structure of the eigenfunction at $k = k_{1,2}^{(n)}$. The error terms are not pictured and lead to small oscillations around the main value.

Notice that for $x_{\pi-\alpha_1} \le y \le x \le (f_1)^{-1} g$,

$$\int_y^x \zeta(s)\,ds = f_1 \int_y^x \left(\frac{\sin \delta\overline{\theta}_1}{\delta\overline{\theta}_1} \cos \alpha_1 + \frac{\cos \delta\overline{\theta}_1 - 1}{\delta\overline{\theta}_1} \sin \alpha_1 \right) ds + O(\mathcal{E}_n) \tag{2.45}$$

by the same argument as in the proof of Lemma 2.4. Therefore, since $\alpha_1 \sim \tilde{g}^{-1}$,

$$e^{\int_y^x \zeta(s)\,ds} \le e^{f_1(x-y)} (1 + O(\tilde{g}^{-1} g, \mathcal{E}_n)) \tag{2.46}$$

for $x \le (f_1)^{-1} g$. All oscillatory terms in (2.44) (that is, all terms except the $O(|V|)$ term) are estimated by integration by parts similarly to Lemma 2.4 and Lemma 2.7. For example, for any of the $2^{n-1} - 1$ terms entering into $W(y)$, we have for $x_{\pi-\alpha_1} \le x \le f_1^{-1} g$

$$\left| \int_{x_{\pi-\alpha_1}}^x e^{\int_y^x \zeta(s)\,ds} V_j(y) \sin(\theta_1 + \theta_2)\,dy \right| \le \tag{2.47}$$

$$Ca^{-1} e^g \left(f_j + \int_{x_{\pi-\alpha_1}}^{\infty} \left(|\tilde{f}_j'(y)| + |\zeta(y)| |\tilde{f}_j(y)| + |\tilde{f}_j(y)| O(|V(y)| + |V'(y)|) a^{-1} \right) dy \right),$$

where $\tilde{f}_j$ is given by (2.34) as before. Similarly to the computation in Lemma 2.7, the total bound in (2.47) is

$$Ce^g \left(f_j a^{-1} + a^{-2} \int\limits_{x_{\pi-\alpha_1}}^{\infty} \tilde{f}_j(y) \sum_l \tilde{f}_l(y)\, dy \right).$$

The term $\delta k_1 \cos 2(\theta_1 + \theta_2)$ is estimated similary (but gives smaller error), while the term $\delta k_1 O(|V(y)|)$ integrated over $[x_{\pi-\alpha_1}, f_1^{-1}g]$ with the exponential gives at most $(f_1)^{-1}\delta k_1 e^g \sum_j f_j$. Summing together all errors we obtain

$$|\delta\overline{\theta}(x)| \leq C 2^n \tilde{g}^{-1} g e^g a^{-2} \sum_j f_j. \tag{2.48}$$

Now notice that

$$(\log R(x, k_1^{(n)})^2)' = \frac{k_1^{(n-1)}}{k_1^{(n)}} f_1(-\cos(\delta\overline{\theta}_1 + \alpha_1) - \cos(\theta_2 + 3\theta_1)) + \frac{W(x)}{k_1^{(n)}} \sin 2\theta(x, k_1^{(n)}).$$

Using (2.42) and estimates parallel to the ones considered in Lemma 2.4 we find for $x_{\pi-\alpha_1} \leq x \leq (f_1)^{-1}g$ that

$$R^2(x, k_1^{(n)}) = \frac{1}{2} e^{-f_1(x - x_{\pi-\alpha_1})} \left(1 + O(\tilde{\mathcal{E}}_n, g\tilde{g}^{-1})\right),$$

where as before $\tilde{\mathcal{E}}_n = 2^n e^g a^{-2} \sum_j f_j$ (we are being generous here, replacing $g^2 \tilde{g}^{-1} \tilde{\mathcal{E}}_n + \mathcal{E}_n$ by $\tilde{\mathcal{E}}_n$). We also absorbed the error $\mathcal{E}_n$ coming from estimates of oscillatory integrals parallel to Lemma 2.4 into $\tilde{\mathcal{E}}_n$. In particular,

$$\|R(x, k_1^{(n)})\|^2_{L^2[x_{\pi-\alpha_1}, (f_1)^{-1}g]} = \frac{1}{2} f_1^{-1} \left(1 + O(\tilde{\mathcal{E}}_n, g\tilde{g}^{-1})\right) \tag{2.49}$$

and, given the upper bound on $x_{\pi-\alpha_1}$ of Lemma 2.5,

$$R^2(f_1^{-1}g, k_1^{(n)}) \leq e^{-g/2}(1 + C(\tilde{\mathcal{E}}_n + \tilde{g}^{-1}g)). \tag{2.50}$$

4. Finally, we consider the solution on $[f_1^{-1}g, \infty)$. Here,

$$V_1(x) = -\min\left(\delta k_1, \frac{g}{2x}\right) \frac{k_1^{(n-1)}}{4} (\sin 2\theta_1 + \sin 2\theta_2).$$

Equations for $\delta\theta_1$ and $R(x, k_1^{(n)})$ can be written in the following form:

$$\begin{aligned} (\log R^2(x, k_1^{(n)})' = {} & -\frac{k_1^{(n-1)}}{8k_1^{(n)}} \min\left(\delta k_1, \frac{g}{2x}\right) (1 + \cos 2\delta\theta_1 - \cos 4\theta_1 - \cos 2(\theta_1 + \theta_2)) + \\ & + \frac{W(x)}{k_1^{(n)}} \sin 2\theta_1 \end{aligned} \tag{2.51}$$

$$\begin{aligned} (\delta\theta_1)' = {} & \delta k_1 (1 + O(\delta k_1)) + \frac{k_1^{(n-1)}}{8k_1^{(n)}} \min\left(\delta k_1, \frac{g}{2x}\right) \sin 2\delta\theta_1 (1 - \cos 2(\theta_1 + \theta_2)) + \\ & + \frac{W(x)}{2k_1^{(n)}} (\cos 2\theta_2 - \cos 2\theta_1). \end{aligned} \tag{2.52}$$

First of all, any contribution to the solution R of the equation (2.51) coming from the last term involving W or two last summands in brackets in the first term is of the order $O(\mathcal{E}_n)$ by the same estimates as before (in Lemma 2.4). Therefore,

$$\frac{R(x,k_1^{(n)})^2}{R(f_1^{-1}g,k_1^{(n)})^2} = e^{-\frac{1}{8}\min(\delta k_1, \frac{g}{2x})\int_{f_1^{-1}g}^{x}(1+\cos 2\delta\theta_1)\,dy(1+O(\delta k_1))}(1+O(\mathcal{E}_n)). \tag{2.53}$$

Furthermore, same estimates show that for any $x > f_1^{-1}g$,

$$\int\limits_{f_1^{-1}g}^{x} W(y)(\cos 2\theta_2 - \cos 2\theta_1)\,dy = O(\mathcal{E}_n) \tag{2.54}$$

as well. Consider a sequence of points y_n, where $y_1 = f_1^{-1}g$, and

$$y_n = \min\{y : \delta\theta_1(y) = \pi n + \delta\theta_1(y_1)\}.$$

According to (2.52), (2.54) and (2.6), we have $y_n - y_{n-1} \le (2\pi + O(\mathcal{E}_n))(\delta k_1)^{-1}$, and in each (y_{n-1}, y_n) there is an interval I_n of length at least $(\pi/4 + O(\mathcal{E}_n))(\delta k_1)^{-1}$ where $\cos 2\delta\theta_1 > 0$. Let us assume for simplicity that the error terms in the above estimates do not consume more than half of the principal terms. Then from (2.51) we see that on (y_n, y_{n+1})

$$R(y,k_1^{(n)})^2 \le CR(y_1,k_1^{(n)})^2 e^{-\sum_{j=1}^{n-1} b_j}, \tag{2.55}$$

where

$$b_n = \frac{1}{16}\int\limits_{I_n} \min(\delta k_1, \frac{g}{2x})\,dx \ge C\min\left(1, \frac{g}{g\tilde{g}^{-1} + 3\pi n}\right). \tag{2.56}$$

Taking into account (2.50) and the estimates on b_n, y_n we have

$$\|R(y,k_1^{(n)})\|^2_{L^2[f_1^{-1}g,\infty)} \le Ce^{-g/2}\sum_{n=0}^{\infty} e^{-\sum_{j=1}^{n} b_j}(\delta k_1)^{-1}. \tag{2.57}$$

Recall that $\delta k_1 = f_1^{-1}e^{g^{3/4}}$. Provided that g is large enough so that the sum in (2.57) is finite, we can write

$$\|R(y,k_1^{(n)})\|^2_{L^2[f_1^{-1}g,\infty)} = f_1^{-1}O(e^{-g/6}). \tag{2.58}$$

Combining the estimates (2.37), (2.41), (2.49) and (2.58), we get the Splitting Lemma. □

3. A Continuity Lemma

In this section, we prove an auxiliary technical lemma that we are going to need in the construction. The estimates here are fairly straightforward, if technical. Moreover, we choose simplicity over sophistication and prove a version of the result which is sufficient for the application we have in mind. With more extensive technical effort, the estimates in this section can be significantly improved.

Assume that the potential $V^{(n)}(x)$ is given on an interval (x_n, ∞) by $V^{(n)}(x) = \sum_{j=1}^{2^{n-1}} V_j^{(n)}(x)$, where similarly to (2.7) we have

$$V_j^{(n)}(x) = \begin{cases} -2f_j^{(n)}k_j^{(n-1)}\sin(\theta_{2j-1}^{(n)} + \theta_{2j}^{(n)}), & x_n \le x \le x^{(n)}_{j,\pi-\alpha_j^{(n)}} \\ 2f_j^{(n)}k_j^{(n-1)}\sin(\theta_{2j-1}^{(n)} + \theta_{2j}^{(n)}), & x^{(n)}_{j,\pi-\alpha_j^{(n)}} \le x \le x_n + (f_j^{(n)})^{-1}g_n \\ -\min\left(\delta k_j^{(n)}, \frac{g_n}{2x}\right)\frac{k_j^{(n-1)}}{4}(\sin 2\theta_{2j-1}^{(n)} + \sin 2\theta_{2j}^{(n)}), & x > x_n + (f_j^{(n)})^{-1}g_n. \end{cases} \tag{3.1}$$

It is assumed that $\delta\theta_j^{(n)}(x_n) < \pi/2$; $x^{(n)}_{j,\pi-\alpha_j^{(n)}}$ is then defined as a minimal value of $x > x_n$ where $\delta\theta_j^{(n)}(x)$ reaches $\pi - \alpha_j^{(n)}$. We adopt this definition in order to avoid making too many assumptions on what happens for $x \le x_n$, although in the construction process that appears in the following sections it is straightforward to check that $\delta\theta_j^{(n)}(x)$ does not reach $\pi - \alpha_j^{(n)}$ for $x \le x_n$. Similarly, in the actual construction we will always have $x_n << (f_j^{(n)})^{-1}g_n$. As usual, $\delta k_j^{(n)} = \tilde{g}_n^{-1}f_j^{(n)}$. For simplicity, we are also going to assume that

$$\|V^{(n)}\|_{L^\infty(\mathbb{R}^+)} \le \frac{1}{2}\mathrm{min}_j\{(k_j^{(n)})^2\}. \tag{3.2}$$

We also suppose that the condition (2.8) of the Splitting Lemma holds:

$$4\sum_j f_j^{(n)}k_j^{(n-1)} < \frac{1}{3}a_n, \quad g_n >> 1, \;\; 2^n g_n a_n^{-2}\sum_{j=1}^{2^{n-1}} f_j^{(n)} << 1, \quad \delta k_j^{(n)} < \frac{1}{12}a_n \tag{3.3}$$

for all j. The potential $V^{(n)}(x)$ on $(0, x_n)$ comes from the previous steps, but for now we make no more specific assumptions about it. Recall our notation $a_{n+1} = \mathrm{min}_{j,j'}|k_j^{(n)} - k_{j'}^{(n)}|$. Let $\tilde{x}_n$ be such that for any $x \ge \tilde{x}_n$,

$$|V^{(n)}(x)| \le \frac{1}{8}a_{n+1}\mathrm{min}_j\{(k_j^{(n)})\}. \tag{3.4}$$

Also, take $\tilde{x}_n > g_n\mathrm{max}_j\{(f_j^{(n)})^{-1}\}$ for convenience.

Lemma 3.1. *Assume that $V^{(n)}(x)$ is given by (3.1) for $x > x_n$. Then*

$$\frac{\partial\theta}{\partial k}(x, k_j^{(n)}) \ge P_n\left(\frac{x}{\tilde{x}_n}\right)^{\beta(k_j^{(n)})} \tag{3.5}$$

for any $x \ge 2\tilde{x}_n$. Here $\beta(k) = \frac{k_j^{(n-1)}g_n}{16k}$. Moreover, if k is such that

$$|k - k_j^{(n)}| \le \frac{1}{4}a_{n+1}, \tag{3.6}$$

then

$$\frac{\partial\theta}{\partial k}(x, k) \le D_n\left(1 + \left(\frac{x}{\tilde{x}_n}\right)^{\beta(k)}\right) \tag{3.7}$$

for any x*. Here* D_n*,* P_n *are positive constants defined only in terms of* $V^{(n)}(x)$ *for* $x \leq x_n$*,* x_n*,* $\{f_j^{(n)}\}$*,* a_{n+1} *and* g_n*.*

Proof. It is clear that the value of $\tilde{x}_n$ depends only on x_n, $f_j^{(n)}$, a_{n+1} and g_n, and can be easily estimated in terms of these quantities:

$$\tilde{x}_n \leq C\max\left(2^n g_n a_{n+1}^{-1}, x_n + g_n \max_j\{(f_j^{(n)})^{-1})\}\right).$$

Notice that from (2.3) it follows that

$$\left(\frac{\partial\theta}{\partial k}(x,k)\right)' = 1 + \frac{1}{k^2}V(x)(\sin\theta(x,k))^2 - \frac{1}{k}V(x)\sin 2\theta(x,k)\left(\frac{\partial\theta}{\partial k}(x,k)\right).$$

Along with the boundary condition $\frac{\partial\theta}{\partial k}(0,k) = 0$ (which follows from the fact that we consider solutions satisfying fixed boundary condition for all k) this implies in our setting

$$\frac{\partial\theta}{\partial k}(x,k) = \int\limits_0^x e^{-\frac{1}{k}\int_y^x V^{(n)}(s)\sin(2\theta(s,k))\,ds}\left(1 + \frac{1}{k^2}V^{(n)}(y)(\sin\theta)^2\right)dy. \tag{3.8}$$

For any k satisfying the assumption (3.6) of the lemma we have for $\tilde{x}_n \leq y \leq x$ by an estimate parallel to that in Lemma 2.4 and using (3.4) and (3.3)

$$\int\limits_y^x V^{(n)}(y)\sin 2\theta(y,k)\,dy = -\frac{k_j^{(n-1)}g_n}{16}\int\limits_y^x \frac{\cos(2\theta(s,k) - 2\theta_j^{(n)}(s))}{s}\,ds + O(2^n a_{n+1}^{-2} g_n \sum_j f_j^{(n)}). \tag{3.9}$$

Equations (3.8), (3.9) together imply a very rough estimate

$$\frac{\partial\theta}{\partial k}(x,k) \leq C\tilde{x}_n e^{\frac{1}{k}\int\limits_0^{\tilde{x}_n}|V^{(n)}(y)|\,dy} + Ce^{C2^n a_{n+1}^{-2} g_n \sum_j f_j^{(n)}}\int\limits_{\tilde{x}_n}^x e^{\beta(k)\log(x/y)}\,dy,$$

where C is a universal constant. Similarly,

$$\frac{\partial\theta}{\partial k}(x,k_j^{(n)}) \geq C_1 e^{-C2^n a_{n+1}^{-2} g_n \sum_j f_j^{(n)}}\int\limits_{\tilde{x}_n}^x e^{\beta(k_j^{(n)})\log(x/y)}\,dy.$$

Notice that

$$\int\limits_{\tilde{x}_n}^x e^{\beta(k)\log(x/y)}\,dy = \frac{\tilde{x}_n}{\beta(k)-1}\left(\left(\frac{x}{\tilde{x}_n}\right)^{\beta(k)} - \frac{x}{\tilde{x}_n}\right).$$

The parameter $\beta(k)$ is large because of g_n, and we can assume for simplicity $\beta(k) > 2$ for all k of interest (lying in a compact set where all $k_j^{(n)}$ lie). Then we get the result of the lemma. We can take, for example,

$$D_n = C\tilde{x}_n^2 e^{\frac{1}{k}\int\limits_0^{\tilde{x}_n}|V^{(n)}(y)|\,dy + C2^n a_{n+1}^{-2} g_n \sum_j f_j^{(n)}}, \tag{3.10}$$

$$P_n = C_1\frac{\tilde{x}_n}{\beta(k_j^{(n)})-1}e^{-C2^n a_{n+1}^{-2} g_n \sum_j f_j^{(n)}}. \tag{3.11}$$

□

Now we prove the principal result of this section.

Lemma 3.2 (Continuity Lemma). *Choose $\tilde{g}_{n+1}$ so that $\tilde{g}_{n+1}^{1/4} \geq 2^n D_n P_n^{-1}$. Then there exists x_n' such that for any $x_{n+1} > x_n'$ the following holds. If*

$$|k - k_j^{(n)}| \leq 2\tilde{g}_{n+1}^{-1/2} \left(\frac{\partial \theta}{\partial k}(x_{n+1}, k_j^{(n)}) \right)^{-1}, \tag{3.12}$$

then

$$R(x,k)^2 = R(x, k_j^{(n)})^2 (1 + O(\tilde{g}_{n+1}^{-1/4})) \tag{3.13}$$

and

$$\frac{\partial \theta}{\partial k}(x,k) = \frac{\partial \theta}{\partial k}(x, k_j^{(n)})(1 + O(\tilde{g}_{n+1}^{-1/4})) \tag{3.14}$$

for all $x \leq x_{n+1}$.

Proof. From (3.5) it follows that $\frac{\partial \theta}{\partial k}(x, k_j^{(n)}) \to \infty$ as $x \to \infty$, and thus we can choose x_n' so that for any $x_{n+1} \geq x_n'$ the values of k satisfying (3.12) also satisfy (3.6). Then Lemma 3.1 applies, and thus we have for any x

$$\frac{\frac{\partial \theta}{\partial k}(x,k)}{\frac{\partial \theta}{\partial k}(x_{n+1}, k_j^{(n)})} \leq D_n P_n^{-1} \left(1 + \left(\frac{x}{\tilde{x}_n} \right)^{\beta(k)} \right) \left(\frac{x_{n+1}}{\tilde{x}_n} \right)^{-\beta(k_j^{(n)})}.$$

Hence, by definition of $\tilde{g}_{n+1}$, we have

$$|\theta(x,k) - \theta(x, k_j^{(n)})| \leq 2^{-n} \tilde{g}_{n+1}^{-1/4} \left(1 + \left(\frac{x}{\tilde{x}_n} \right)^{\beta(k)} \right) \left(\frac{x_{n+1}}{\tilde{x}_n} \right)^{-\beta(k_j^{(n)})}. \tag{3.15}$$

Recall from (2.2) that

$$R(x,k)^2 / R(x, k_j^{(n)})^2 = e^{\frac{1}{k}\int_0^x V^{(n)}(y) \sin 2\theta(y,k)\, dy - \frac{1}{k_j^{(n)}} \int_0^x V^{(n)}(y) \sin 2\theta(y, k_j^{(n)})\, dy}.$$

First of all, we can replace $1/k$ with $1/k_j^{(n)}$ in the first term in the exponent at the expense of creating an error factor of size at most

$$e^{C|k - k_j^{(n)}| \int_0^{x_{n+1}} |V^{(n)}(y)|\, dy}$$

for $x \leq x_{n+1}$. From (3.12) and a bound (3.5) of Lemma 3.1 it is clear that we can choose x_n' so that this factor does not exceed $1 + O(\tilde{g}_{n+1}^{-1/4})$. Next, we need to bound

$$\frac{1}{k_j^{(n)}} \int_0^x V^{(n)}(y) |\theta(y,k) - \theta(y, k_j^{(n)})|\, dy. \tag{3.16}$$

Using (3.15), we can estimate the expression (3.16) for $x \leq x_{n+1}$ by

$$C 2^{-n} \tilde{g}_{n+1}^{-1/4} x_{n+1}^{-\beta(k_j^{(n)})} \tilde{x}_n^{\beta(k_j^{(n)}) - \beta(k)} \int_0^{x_{n+1}} y^{\beta(k)} |V^{(n)}(y)|\, dy. \tag{3.17}$$

The contribution to (3.17) coming from the integral over $[0, \tilde{x}_n]$ can be made $O(\tilde{g}_{n+1}^{-1/4})$ by the choice of x'_n. Using an inequality $|V^{(n)}(y)| \leq C2^n\beta(k)y^{-1}$ for $y > \tilde{x}_n$, we estimate the remaining part of (3.17) by $C\tilde{g}_{n+1}^{-1/4}(x_{n+1}/\tilde{x}_n)^{\beta(k)-\beta(k_j^{(n)})}$. Notice that from the definition of $\beta(k)$, we have

$$|\beta(k) - \beta(k_j^{(n)})| \leq Cg_n|k - k_j^{(n)}| \leq Cg_n\tilde{g}_{n+1}^{-1/2}\left(\frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n)})\right)^{-1}. \tag{3.18}$$

By (3.5) of Lemma 3.1 and the definition of $\tilde{g}_{n+1}$, the right hand side in (3.18) does not exceed $C(x_{n+1}/\tilde{x}_n)^{-\beta(k_j^{(n)})}$. Thus,

$$\left(\frac{x_{n+1}}{\tilde{x}_n}\right)^{\beta(k)-\beta(k_j^{(n)})} \leq e^{C(\log(x_{n+1}/\tilde{x}_n))(x_{n+1}/\tilde{x}_n)^{-\beta(k_j^{(n)})}}, \tag{3.19}$$

which is bounded by some fixed universal constant. Combining the estimates, we see that if we choose x'_n to satisfy all of the above requirements, then (3.13) holds for any $x \leq x_{n+1}$, provided that $x_{n+1} \geq x'_n$.

Notice also that from (3.15) and (3.19), it follows that for $x \in (0, x_{n+1})$ we have $|\theta(x,k) - \theta(x, k_j^{(n)})| \leq C\tilde{g}_{n+1}^{-1/4}$. Thus (3.14) follows from (3.13) and (3.8), which can also be written as

$$\frac{\partial\theta}{\partial k}(x,k) = R(x,k)^{-2}\int_0^x R(y,k)^2\left(1 + \frac{1}{k^2}V^{(n)}(y)(\sin\theta)^2\right)dy. \tag{3.20}$$

We used (3.2) in the last estimate. □

4. A Brick of Construction

Our goal is to organize a Cantor-like process, splitting each eigenvalue we have on the nth step into two on the $(n+1)$th step. To do this we need to see how norm splitting works in a more general setting than Splitting Lemma, which starts at the origin. The needed estimates are provided by the following lemma, which helps to connect different steps.

Assume that for some j, we have for k satisfying

$$|k - k_j^{(n)}| \leq 2\tilde{g}_{n+1}^{-1/2}\left(\frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n)})\right)^{-1}$$

the bounds of Continuity Lemma:

$$R(x,k)^2 = R(x, k_j^{(n)})^2(1 + O(\tilde{g}_{n+1}^{-1/4})) \tag{4.1}$$

and

$$\frac{\partial\theta}{\partial k}(x,k) = \frac{\partial\theta}{\partial k}(x, k_j^{(n)})(1 + O(\tilde{g}_{n+1}^{-1/4})) \tag{4.2}$$

for all $x \le x_{n+1}$. Set

$$f_j^{(n+1)} = \tilde{g}_{n+1}^{1/2} \left(\frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n)}) \right)^{-1} \tag{4.3}$$

$$\delta k_j^{(n+1)} = \tilde{g}_{n+1}^{-1/2} \left(\frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n)}) \right)^{-1}, \tag{4.4}$$

and let $k_{2j}^{(n+1)}$, $k_{2j-1}^{(n+1)}$ be the ends of the interval of size $\delta k_j^{(n+1)}$ centered at $k_j^{(n)}$. For $x > x_{n+1}$, define potential $V^{(n+1)}(x)$ to be given by (3.1) with $n+1$ replacing n. We are going to assume also that the conditions (3.3) of the Splitting Lemma are satisfied (with n replaced by $n+1$ in our current situation). Define a parameter $A_j^{(n)}$ by

$$A_j^{(n)} = \frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n)}) R(x_{n+1}, k_j^{(n)})^2. \tag{4.5}$$

We have

Lemma 4.1 (Connection Lemma)**.** *Under the above assumptions, $k_{2j-1}^{(n+1)}$ and $k_{2j}^{(n+1)}$ are eigenvalues of $H_{V^{(n+1)}}$, and the norms of the corresponding eigenfunctions on (x_{n+1}, ∞) satisfy*

$$\|R(x, k_{2j,2j-1}^{(n+1)})\|^2_{L^2(x_{n+1},\infty)} = A_j^{(n)}(1 + O(\tilde{\mathcal{E}}_{n+1}, \tilde{g}_{n+1}^{-1/4})). \tag{4.6}$$

Proof. To simplify notation, let us assume without loss of generality that $j = 1$. According to our assumptions, $\delta\theta_1^{(n+1)}(x_{n+1}) = \tilde{g}_{n+1}^{-1/2}(1 + O(\tilde{g}_{n+1}^{-1/4}))$. If $\delta\theta_1^{(n+1)}(x_{n+1})$ were zero, we would be exactly in a situation of the Splitting Lemma construction, and would have two eigenfunctions of norm $\sim 2(f_j^{(n+1)})^{-1}R(x_{n+1}, k_j^{(n)})^2$. We have to take into account the fact that on the interval where $\delta\theta_1^{(n+1)}(x) \in [0, \delta\theta_1^{(n+1)}(x_{n+1})]$ a significant decay takes place in the Splitting Lemma, and we lose this decay in our situation.

Consider an auxiliary problem for $\overline{R}$, $\delta\overline{\theta}$ satisfying same equations (2.10), (2.11) as $R(x, k_1^{(n+1)})$, $\delta\theta_1^{(n+1)}$ but with $\overline{V}^{(n+1)}$ given by (2.7) with $n+1$ instead of n. In other words, consider the case of $(n+1)$st step potential starting right away at the origin. The Small Angle and Splitting Lemmas imply that

$$\|\overline{R}(x, k_1^{(n+1)})\|^2_{L^2(x_{\delta\theta_1(x_{n+1})},\infty)} = (f_1^{(n+1)})^{-1}(1 + O(\tilde{\mathcal{E}}_{n+1}, \tilde{g}_{n+1}^{-1/2} g_{n+1})). \tag{4.7}$$

But the same estimates as in these Lemmas apply to $R(x, k_1^{(n+1)})$, $\delta\theta_1$ on (x_{n+1}, ∞), by identical arguments. The only adjustment we need to make is by a factor

$$R(x_{n+1}, k_1^{(n+1)})^2 / \overline{R}(x_{\delta\theta_1(x_{n+1})}, k_1^{(n+1)})^2.$$

By Lemma 2.6 (Small Angle Lemma),

$$\overline{R}(x_{\delta\theta_1(x_{n+1})}, k_1^{(n+1)})^2 = \left(\frac{\delta k_1^{(n+1)}}{\delta\theta_1^{(n+1)}(x_{n+1}) f_1^{(n+1)} + \delta k_1^{(n+1)}} \right) \left(1 + O(\mathcal{E}_{n+1}, \tilde{g}_{n+1}^{-1} g_{n+1})\right). \tag{4.8}$$

According to (4.7), (4.8) we have
(4.9)

$$\|R(x,k_{1,2}^{(n+1)})\|^2_{L^2(x_{n+1},\infty)} = \frac{R(x_{n+1},k_1^{(n)})^2}{f_1^{(n+1)}}\left(\frac{\delta\theta_1^{(n+1)}(x_{n+1})f_1^{(n+1)}}{\delta k_1^{(n+1)}}+1\right)\left(1+O(\tilde{\mathcal{E}}_{n+1},\tilde{g}_{n+1}^{-1/4})\right).$$

We utilized assumption (4.1) in the error estimate, and also used $\tilde{g}_{n+1}^{-1}g_{n+1} < C\tilde{g}_{n+1}^{-1/4}$. Because of the assumption (4.2), we have

$$\frac{\delta\theta_1^{(n+1)}(x_{n+1})}{\delta k_1^{(n+1)}} = \frac{\partial\theta}{\partial k}(x_{n+1},k_1^{(n)})(1+O(\tilde{g}_{n+1}^{-1/4})). \tag{4.10}$$

The term $(f_1^{(n+1)})^{-1}R(x_{n+1},k_1^{(n)})^2 = A_j^{(n)}\tilde{g}_{n+1}^{-1/2}$, and hence contributes a correction of higher order. Thus by definition of $A_j^{(n)}$ and (4.10), (4.9) leads exactly to (4.6). $\square$

The usefulness of Lemma 4.1 becomes clear if one looks at (3.20), which in particular implies

$$\frac{\partial\theta}{\partial k}(x,k) = R(x,k)^{-2}\|R(y,k)\|^2_{L^2(0,x)}(1+O(\|V\|_{L^\infty})).$$

Thus in the asymptotic regime where V is small, one can expect, informally,

$$\frac{\partial\theta}{\partial k}(x,k) \sim R(x,k)^{-2}\|R(y,k)\|^2_{L^2(0,x)},$$

and the role of A is played by $\|R(x,k)\|^2_{L^2(0,x)}$. Lemma 4.1 allows to get two eigenvalues with the norm on (x,∞) same as $A = \|R(x,k)\|^2_{L^2(0,x)}$. This will provide the mechanism for norm doubling in the asymptotic regime which is crucial for the construction.

5. The Construction

Now we are ready to provide a precise description of the construction of our potential $V(x)$. In the beginning, we take a potential given by (2.7), like in Splitting Lemma with $n=1$. Start by choosing $\delta k_1^{(1)}$, $k_1^{(0)}$, g_1 and $f_1^{(1)}$. The precise choice of the parameters on the first step of construction is not so important, but we are going to keep it similar to the subsequent steps. To satisfy (2.6) we choose $k_1^{(0)}$ sufficiently large and $\delta k_1^{(1)}$ small compared to $k_1^{(0)}$. We take g_1 large, impose a relationship $\delta k_1^{(1)} = \tilde{g}_1^{-1}f_1^{(1)}$ and take $e^{g_1}f_1^{(1)}$ small. The smallness of the latter number and $\delta k_1^{(1)}$ can be achieved by taking $f_1^{(1)}$ sufficiently small while keeping g_1 fixed. In particular, we take g_1 large enough and $f_1^{(1)}$ small enough for the conditions (3.3) of the Splitting to be satisfied. The only additional restriction we need to impose on the parameters comes from the condition $|V(x)| \le h(x)/(1+x)$. By the monotonicity of h and the definition of potential, the inequality

$$2f_1^{(1)}k_1^{(0)} \le \frac{h((f_1^{(1)})^{-1}g_1)}{1+(f_1^{(1)})^{-1}g_1}, \tag{5.1}$$

implies the needed bound for $x \geq (f_1^{(1)})^{-1} g_1$. It can be rewritten as $h((f_1^{(1)})^{-1}g_1) \geq 2(g_1 + f_1^{(1)})k_1^{(0)}$ which holds for sufficiently small $f_1^{(1)}$. To simplify consideration for smaller x, we are going to assume without loss of generality that $h(x)/(1+x)$ is decreasing. Then (5.1) also implies the bound for $x < (f_1^{(1)})^{-1}g_1$. The Splitting Lemma then gives us two eigenvalues at $k_{1,2}^{(1)}$ with corresponding eigenfunctions having norms $(f_1^{(1)})^{-1}(1 + O(e^{g_1}f_1^{(1)}, \tilde{g}_1^{-1/2}g_1))$.

Let us now describe the induction process. We are going to assume that on the nth step we have the potential $V^{(n)}(x)$, in particular given by (3.1) on (x_n, ∞). We assume that this potential leads to eigenvalues $k_j^{(n)}$, $j = 1, \dots 2^n$ and the following holds:

1. The norms of the corresponding eigenfunctions satisfy for every j

$$\|R(x, k_j^{(n)})\|^2_{L^2(0,\infty)} \leq C_1 2^n \prod_{l=1}^{n} (1 + Er_l), \tag{5.2}$$

where $|Er_l| \leq C_2 2^{-l}$, and $C_{1,2}$ are fixed constants which do not depend on n, l.

2. The angle derivative in energy satisfies for every j

$$\frac{\partial \theta}{\partial k}(x, k_j^{(n)}) \leq C_1 2^n R(x, k_j^{(n)})^{-2} \prod_{l=1}^{n}(1 + Er_l) \tag{5.3}$$

for any x.

3. The conditions (3.3) of the Splitting Lemma are satisfied, and $|V^{(n)}(x)| \leq h(x)/(1+x)$.

4. We assume that (2.6) holds; this will be nearly automatic because of the rapid decrease of $\delta k_j^{(n)}$.

Clearly for $n = 1$ these assumptions are satisfied with $x_n = 0$ and sufficiently large $C_{1,2}$. C_1, in particular, should be taken $\sim (f_1^{(1)})^{-1}$ or larger. The fact that (5.3) is satisfied for $n = 1$ follows from (5.2) and (3.20). The structure of the eigenfunctions that we are constructing is illustrated by Figure 2.

Now let us make an induction step. Choose $\tilde{g}_{n+1}$ so that $\tilde{g}_{n+1}^{1/4} > 2^n D_n P_n^{-1}$, where D_n, P_n are given by (3.10), (3.11). We should also make sure that g_{n+1} is large enough so that the second condition of (3.3) needed for Splitting and Connection Lemmas holds; but this is basically automatic after the first step, since g_n is virtually forced by construction to be an increasing sequence. We require in addition that $\tilde{g}_{n+1}^{-1/4} \leq C2^{-n-1}$ (which is again basically automatic). For each $j = 1, \dots, 2^n$, choose

$$f_j^{(n+1)} = \tilde{g}_{n+1}^{1/2} \left(\frac{\partial \theta}{\partial k}(x_{n+1}, k_j^{(n)}) \right)^{-1}, \quad \delta k_j^{(n+1)} = \tilde{g}_{n+1}^{-1/2} \left(\frac{\partial \theta}{\partial k}(x_{n+1}, k_j^{(n)}) \right)^{-1}, \tag{5.4}$$

where the splitting point x_{n+1} remains the only parameter of the $(n+1)$st step to be chosen. Define

$$\tilde{f}_{n+1} = C a_{n+1}^2 2^{-3(n+1)} e^{-g_{n+1}}. \tag{5.5}$$

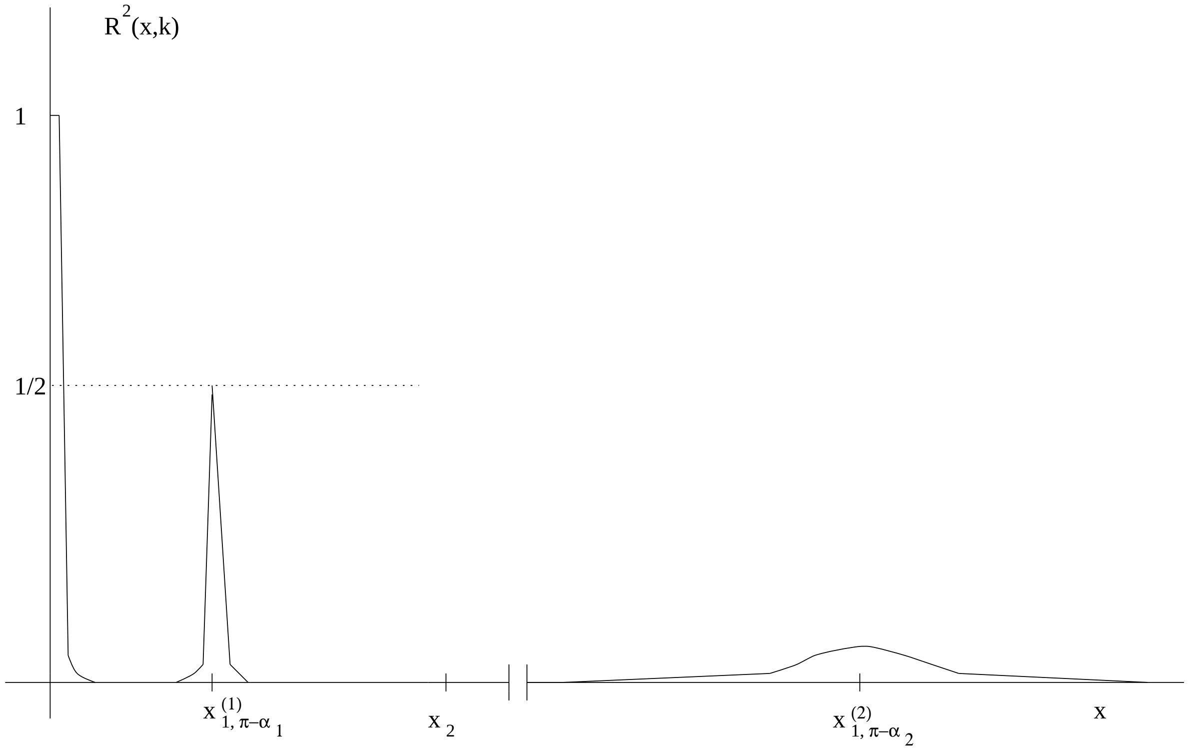

FIGURE 2. The structure of the eigenfunction at $k = k_1^{(n)}$. Only the first two stages are shown. The factor by which the eigenfunction grows from the splitting point x_n to its top value at $x^{(n)}_{1,\pi-\alpha_1^{(n)}}$ is $\sim (1/2)\tilde{g}_n^{1/2}$. This can be seen from the argument in Connection Lemma. Each bump contains roughly as much norm as everything before it.

One condition that we impose on x_{n+1} is that $f_j^{(n+1)}$ defined by (5.4) are smaller than $\tilde{f}_{n+1}$ for every j. This can always be achieved by taking x_{n+1} large enough because of (3.5). Notice that such choice ensures that $\tilde{\mathcal{E}}_{n+1} \leq C2^{-n-1}$. Also, the constant C is (5.5) is chosen so that the first, third and fourth conditions of (3.3) needed for Splitting and Connection Lemmas on the $(n+1)$st step hold.

The second condition imposed on x_{n+1} is that Lemma 3.2 (Continuity Lemma) applies, that is, $x_{n+1} > x_n'$, x_n' is defined in Lemma 3.2, and $\delta k_j^{(n)} \leq \frac{1}{12} a_{n+1}$ for all j.

The third condition comes from the constraint $|V^{(n+1)}(x)| \leq h(x)/(1+x)$. The potential $V^{(n+1)}$ on $[x_{n+1}, \infty)$ is going to be defined by (3.1) with $n+1$ instead of n, so the latter requirement is going to be satisfied if for every j,

$$f_j^{(n+1)} \leq 2^{-n-1} \frac{h(x_{n+1} + (f_j^{(n+1)})^{-1} g_{n+1})}{x_{n+1} + (f_j^{(n+1)})^{-1} g_{n+1}}. \tag{5.6}$$

By (3.5) we have

$$\left(\frac{\partial \theta}{\partial k}(x, k_j^{(n)})/x \right) \overset{x\to\infty}{\longrightarrow} \infty$$

(since we assume that g_n is large enough). Thus, for large enough x_{n+1} we have $x_{n+1} \leq (f_j^{(n+1)})^{-1} g_{n+1}$ (recall that $f_j^{(n+1)}$ is defined by (5.4)). Then the condition (5.6) reduces to

$$2^{n+2} g_{n+1} \leq h(x_{n+1} + (f_j^{(n+1)})^{-1} g_{n+1}).$$

This can always be achieved by taking x_{n+1} sufficiently large while g_{n+1} is fixed.

By the choice of $\delta k_j^{(n+1)}$, x_{n+1} and $\tilde{g}_{n+1}$, the result of Lemma 3.2 (Continuity Lemma) applies to $k_j^{(n+1)}$, and therefore we can use Lemma 4.1 (Connection Lemma). Together, Continuity and Connection Lemmas show that $k_j^{(n+1)}$ are eigenvalues with eigenfunctions satisfying

$$\|R(x, k_j^{(n+1)})\|_{L^2}^2 \leq C_1 2^n \prod_{l=1}^{n} (1 + Er_l) \left((1 + C\tilde{g}_{n+1}^{-1/4}) + (1 + C(\tilde{g}_{n+1}^{-1/4} + \tilde{\mathcal{E}}_{n+1})) \right).$$

Here the first summand in the brackets comes from $(0, x_{n+1})$ and was estimated by the induction assumption (5.2) and Continuity Lemma, while the second summand comes from (x_{n+1}, ∞) and was estimated using (5.3), Continuity and Connection Lemmas. Altogether, we obtain

$$\|R(x, k_j^{(n+1)})\|_{L^2}^2 \leq C_1 2^{n+1} \prod_{l=1}^{n+1} (1 + Er_l), \tag{5.7}$$

where $Er_{n+1} \leq C_2 2^{-n-1}$ and thus satisfies the required bound. Next,

$$\frac{\partial\theta}{\partial k}(x, k_j^{(n+1)}) = \frac{\partial\theta}{\partial k}(x_{n+1}, k_j^{(n+1)}) \frac{R(x_{n+1}, k_j^{(n+1)})^2}{R(x, k_j^{(n+1)})^2} + \int_{x_{n+1}}^{x} \frac{R(y, k_j^{(n+1)})^2}{R(x, k_j^{(n+1)})^2} (1 + O(|V(y)|))\, dy.$$

By induction assumption and Continuity Lemma, the first summand does not exceed

$$R(x, k_j^{(n+1)})^{-2} C_1 2^n \prod_{l=1}^{n} (1 + Er_l)(1 + C\tilde{g}_{n+1}^{-1/4}).$$

The second summand, by Connection Lemma, does not exceed

$$R(x, k_j^{(n+1)})^{-2} C_1 2^n \prod_{l=1}^{n} (1 + Er_l)(1 + C(\tilde{\mathcal{E}}_{n+1} + \tilde{g}_{n+1}^{-1/4}))(1 + C2^n \tilde{f}_{n+1}).$$

Therefore, taking into account the above estimates on $\tilde{\mathcal{E}}_{n+1}$, $\tilde{g}_{n+1}^{-1/4}$ and the choice of $\tilde{f}_{n+1}$, we get that

$$\frac{\partial\theta}{\partial k}(x, k_j^{(n+1)}) = R(x, k_j^{(n+1)})^{-2} C_1 2^{n+1} \prod_{l=1}^{n+1} (1 + Er_l),$$

with $E_{n+1} \leq C_2 2^{-n-1}$. The third component of the induction step was already handled by the choice of x_{n+1}. The fourth component is easily satisfied, since by inspection of the argument it is clear that $\delta k_j^{(n)}$ decay rapidly in n. In fact we can assume freely that $g_n > 2^n$ (there is no restriction on how large g_n we can choose on each step, only a bound from below). Since $f_j^{(n)} \le C2^{-3n}e^{-g_n}$ for any j and $\delta k_j^{(n)}$ is even smaller, we can clearly arrange for (2.6) to be valid. This concludes the induction step.

Notice that by construction, $\max_j \delta k_j^{(n+1)} \le \tilde{f}_{n+1}\tilde{g}_{n+1}^{-1/2}$, and by definition of a_{n+1}, $\tilde{f}_{n+1} \le 2^{-3n}e^{-g_n}(\min_j \delta k_j^{(n)})^2$. It is easy to see from these relations that for all sufficiently large n,

$$\sum_{m=n+1}^{\infty} \max_j(\delta k_j^{(m)}) \le \frac{1}{3}\min_j(\delta k_j^{(n)}). \tag{5.8}$$

Denote $I_j^{(n)}$ the intervals with centers at $k_j^{(n)}$ of size $\frac{1}{3}\delta k_j^{(n)}$. Then by (5.8), for sufficiently large n, $2I_j^{(n)}$ are disjoint for different j (where by $2I_j^{(n)}$ we mean the intervals centered at $k_j^{(n)}$ twice the size of $I_j^{(n)}$). At the same time, all eigenvalues $k_l^{(m)}$ that are generated from $k_j^{(n)}$ at a later steps and their intervals $I_l^{(m)}$ lie in $I_j^{(n)}$ (these are the eigenvalues $k_l^{(m)}$, $l = (j-1)2^{m-n}+1, \dots j2^{m-n}$ for any $m > n$). For the size of $I_j^{(n)}$ we have an estimate $|I_j^{(n)}| \le Ce^{-2^n}$ provided that we choose $g_n > 2^n$ (the estimate is even stronger, but this will suffice).

Let us summarize our findings.

Theorem 5.1. *Applying the construction described above we obtain a sequence of potentials $V^{(n)}(x)$ with the following properties. Each $V^{(n)}(x)$ satisfies $|V^{(n)}(x)| \le h(x)/(1+x)$. If $m > n$, then $V^{(m)}(x) = V^{(n)}(x)$ for $x \le x_n$, where $\{x_n\}$ is a strictly increasing sequence tending to infinity as n grows. For each n, the operator $H_{V^{(n)}}$ has 2^n eigenvalues $k_j^{(n)}$, $j = 1, \dots 2^n$ such that for any j*

$$\int_0^{\infty} R(x, k_j^{(n)})^2\, dx \le B^{-1}2^n, \tag{5.9}$$

where $B > 0$ is a universal constant independent of n. Moreover, for each $k_j^{(n)}$ we can define an interval $I_j^{(n)}$ centered at $k_j^{(n)}$ such that $|I_j^{(n)}| \le Ce^{-2^n}$ and for any $m > n$, all $k_l^{(m)}$ with $l = (j-1)2^{m-n}+1, \dots j2^{m-n}$ belong to $I_j^{(n)}$. At the same time, intervals $2I_j^{(n)}$, $j = 1, \dots, 2^n$ for fixed n are disjoint from each other.

Proof. We have already shown every aspect of the theorem except we have not directly stated (5.9). However this estimate follows directly from the first induction assumption, (5.2), since $Er_l \le C_2 2^{-l}$. □

6. Singular Continuous Spectrum

In this section we prove Theorem 1.1. We are going to define potential $V(x)$ by a condition $V(x) = V^{(n)}(x)$ for $x \le x_n$. By Theorem 5.1, this defines unambiguously $V(x)$ on all positive semi-axis. We prove the following result, which implies Theorem 1.1.

Theorem 6.1. *The singular continuous spectrum of the operator H_V is nonempty.*

Proof. Let us denote the spectral measure of H_V by μ, and spectral measures of $H_{V^{(n)}}$ by μ_n. Notice that since $V = V^{(n)}$ for $x < x_n$ and $x_n \to \infty$, the operator $H_{V^{(n)}}$ converges to H_V in a strong resolvent sense. The strong resolvent convergence implies weak convergence of μ_n to μ (see, e.g. [17], Theorem VII.20). Consider any interval $I_j^{(n)}$ with n sufficiently large. Then by Theorem 5.1 and Lemma 2.1, we have $\mu_m(I_j^{(n)}) \geq B2^{-n}$ for each $m > n$. This implies by weak convergence that $\mu(2I_j^{(n)}) \geq B2^{-n}$ for every n and $j = 1, \dots, 2^n$. Assume that μ does not have singular continuous component, so that $\mu = \mu^{\rm ac} + \mu^{\rm pp}$. Take a small number $\epsilon < B/10$, then there exists a δ_ϵ such that if $|S| < \delta_\epsilon$, $\mu^{\rm ac}(S) < \epsilon$ (we denote here by $|S|$ the Lebesgue measure of the set S). Choose n_1 so that for any $n > n_1$,

$$\sum_{j=1}^{2^n} |2I_j^{(n)}| \leq C2^{n+1}e^{-2^n} < \delta_\epsilon.$$

Then

$$\mu^{\rm ac}(\cup_{j=1}(2I_j^{(n)})) \leq \epsilon < B/10 \tag{6.1}$$

for any $n > n_1$. Let $\mu^{\rm pp} = \sum_{l=1}^\infty \rho_l \delta(x - x_l)$. Pick N_ϵ so that $\sum_{l=N_\epsilon}^\infty \rho_l < \epsilon$. Choose n_2 so that $2^{n_2} > N_\epsilon$. Then for any $n > n_2$, there are at least $2^n - 2^{n_2}$ intervals $2I_j^{(n)}$ which do not contain any of the x_l with $l \leq N_\epsilon$. Denote the set of such intervals Q_n. Then we have

$$\mu^{\rm pp}(\cup_{j\in Q_n}(2I_j^{(n)})) \leq \epsilon < B/10 \tag{6.2}$$

for any $n > n_2$. At the same time, by Theorem 5.1, we have

$$\mu(\cup_{j\in Q_n}(2I_j^{(n)})) \geq B/2$$

if $n > n_2$. For $n \geq \max(n_1, n_2)$, this gives us a contradiction with (6.1), (6.2) and our assumption on the absence of singular continuous component. □

Theorem 1.1 follows from Theorem 6.1 immediately since from Theorem 5.1 we know that $|V(x)| \leq h(x)/(1+x)$.

7. ASYMPTOTIC INCOMPLETENESS OF WAVE OPERATORS

In [4], it was proved that the Möller wave operators exist if $V \in L^p$, $p < 2$, and in addition $\lim_{x\to\infty} \int_0^x V(s)\,ds$ exists. Therefore, to prove Theorem 1.2, we only need to show that the potential $V(x)$, constructed in the previous sections, can be chosen to be conditionally integrable.

Proof of Theorem 1.2. Consider one of the components of $V^{(n)}(x)$, say $V_j^{(n)}(x)$, on (x_n, ∞). Simple integration by parts similar to Lemma 2.4 shows that for any x,

$$\left| \int_{x_n}^{x} V_j^{(n)}(y)\,dy \right| \leq C \left(f_j^{(n)} + \int_{x_n}^{\infty} \tilde{f}_j^{(n)}(y)|V^{(n)}(y)|\,dy, \right)$$

where similarly to what we had before in (2.34), $\tilde{f}_j^{(n)}$ is given by

$$\tilde{f}_j^{(n)}(y) = \begin{cases} -f_j^{(n)}, & x_n \le x \le x^{(n)}_{j,\pi-\alpha_j^{(n)}} \\ f_j^{(n)}, & x^{(n)}_{j,\pi-\alpha_j^{(n)}} \le x \le x_n + (f_j^{(n)})^{-1} g_n \\ -\min(\delta k_j^{(n)}, \frac{g_n}{2x}) \frac{k_j^{(n-1)}}{4}, & x > x_n + (f_j^{(n)})^{-1} g_n. \end{cases}$$

Since we also have $|V^{(n)}(y)| \le C \sum_j \tilde{f}_j^{(n)}(y)$, a simple computation gives that

$$\left| \int\limits_{x_n}^{x} V^{(n)}(y)\, dy \right| \le C 2^n g_n \sum_j f_j^{(n)}$$

for any $x > x_n$. According to our choice of parameters in the construction, in particular $\tilde{f}_n$, the bound on the right hand side does not exceed $C2^{-n}$. Summing up the contributions from (x_n, x_{n+1}) for different n, we get the result. □

8. Discussion

In this paper, we did not try to study fine properties of the imbedded singular continuous measure, such as Hausdorff dimension. The reasonable conjecture is that for $|V(x)| \le C(1+x)^{-\alpha}$, there could be imbedded singular continuous measures of dimension up to $1 - 2\alpha$ (see [19, 20]). One might try to use ideas of this paper to construct such measures. This, however, would have to lead to a different plan of attack and more technical effort. In particular, one should use approximations of the limiting measure by absolutely continuous, rather than by pure point, measures, cutting $V^{(n)}(x)$ off at an appropriate point. In addition, one would have to be more careful with the separation of scales on different steps of construction, which is extensively used in our argument to control the errors. Such an approach would require an improvement in control of many estimates, in particular in continuity estimates of Section 3. To keep the size of the paper in check, we did not try to pursue this direction here.

Another interesting open question is what happens sharply at the Coulomb rate of decay $1/x$. In [19], it is shown that imbedded singular spectrum cannot occur if $V(x) = o(1/x)$ or above a certain threshold if $|V(x)| \le C/(x+1)$. In [10], it was shown that any imbedded eigenvalues λ_n that can occur for $|V(x)| \le C/(x+1)$ satisfy $\sum_n \lambda_n < \infty$. This shows that a construction carried out in this paper is not possible for V decaying at a Coulomb rate. We conjecture that singular continuous spectrum cannot occur for such V.

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