On the existence of completely saturated packings and completely reduced coverings

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Abstract

We prove the following conjecture of G. Fejes Toth, G. Kuperberg, and W. Kuperberg: every body K in either n-dimensional Euclidean or n-dimensional hyperbolic space admits a completely saturated packing and a completely reduced covering. Also we prove the following counterintuitive result: for every $\epsilon > 0$, there is a body K in hyperbolic n-space which admits a completely saturated packing with density less than ϵ but which also admits a tiling.

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1 Introduction

Two of the most basic problems in the theory of packings and coverings is to find the most efficient packing and covering by replicas of a given set K in either n-dimensional Euclidean space or n-dimensional hyperbolic space ([FeK]). The term 'most efficient' is intentionally ambiguous as different circumstances have led to different interpretations. Generally, the density of a packing of Euclidean space is defined as the limit of the relative fraction of volume occupied by the bodies in the packing in a ball of radius r centered at some point p as r tends to infinity. It can be shown that if this limit exists for some point p, it exists for every point p and has the same value. A densest packing is then one which maximizes this density. It is easy to prove the existence of a densest packing in Euclidean space. The density of a covering can be defined similarly and a thinnest covering is one that minimizes the density.

One disadvantage of this definition is that is does not depend on local structure. For example, if a finite number of bodies in a densest packing are removed, it remains a densest packing. Another disadvantage is that it is not clear that this definition carries over well to hyperbolic space. For instance, the above limit may exist for every point in hyperbolic space yet take on different values for different points. For these and other reasons, the authors of [FKK] introduced the concept of a completely saturated packing. For such a packing, it is not possible to replace a finite number of bodies of the packing with a greater number of bodies and still remain a packing. Similarly a completely reduced covering is one in which it is not possible to replace a finite number of bodies of the covering with a smaller number of bodies and still remain a covering.

In [FKK], it was proven that any convex body of Euclidean space admits a completely saturated packing (and more generally any body with the strict nested similarity property) (see also [Kup]).

The authors conjectured that completely saturated packings and completely reduced coverings exist for any body K in either n-dimensional Euclidean or hyperbolic space. In section 4, we prove this conjecture. The proof requires few properties of either Euclidean or hyperbolic space and easily extends to a more general setting (see Remark after Theorem 2.2).

Our techniques are based on those developed in [BoR]. In that paper, a new notion of efficiency was introduced which intuitively applies to a class of measures on a space of packings rather than to individual packings. Such measures are often easier to analyze but at the same time they carry information about the 'average' properties of some packings. In particular, Birkhoff's ergodic theorem (in the Euclidean case) and recent results due to Nevo and Stein (in the hyperbolic case) show that this new notion (to be explained in section 3) coincides (in a measure-theoretic way) to the usual notion of density given by limits of volumes in expanding balls. This will enable us to construct a peculiar family of bodies K that admit tilings of hyperbolic space but such that they also admit completely saturated packings with arbitrarily low density.

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2 Notation and Results

Let S denote either n-dimensional hyperbolic or n-dimensional Euclidean space with a distinguished point $\mathcal{O}$ which we call the origin. Let G be the group of orientation-preserving isometries of S.

A **body** K is a nonempty connected compact subset of S which equals the closure of its interior. Let $\Sigma_K = \{g \in G | gK = K\}$.

A **packing** by a body K is a collection $\mathcal{P} = \{g_1K, g_2K, ...\}$ of congruent copies of K that pairwise have disjoint interiors. Note that if $g_1K \in \mathcal{P}$ for some $g_1 \in G$, then $g_1k \in \mathcal{P}$ for all $k \in \Sigma_K$ since $kK = K$ implies $g_1kK = g_1K$. Conversely, if $g_1K = g_2K$ then $g_2^{-1}g_1 \in \Sigma_K$. So we let $\hat{\mathcal{P}}$ be the subset of the space G/Σ_K of left cosets defined by $g\Sigma_K \in \hat{\mathcal{P}}$ if any only $gK \in \mathcal{P}$. We note that $\hat{\mathcal{P}}$ uniquely determines $\mathcal{P}$ (given K) so if $\mathcal{F} \subset G/\Sigma_K$, we will say that $\mathcal{F}$ determines the packing $\mathcal{P}$ if $\hat{\mathcal{P}} = \mathcal{F}$. Later, we will use this correspondence to define a metric on the space of packings.

A **covering** by a body K is a collection $\mathcal{C} = \{g_1K, g_2K, ...\}$ of congruent copies of K such that for every point $p \in S$, $p \in g_jK$ for some $g_jK \in \mathcal{C}$. As above, we let $\hat{\mathcal{C}}$ be the subset of the space G/Σ_K of left cosets defined by $g\Sigma_K \in \hat{\mathcal{C}}$ if any only $gK \in \mathcal{C}$. A **tiling** by K is a packing that is also a covering.

A **completely saturated** packing $\mathcal{P}$ is such that for no packing $\mathcal{F}_1 \subset \mathcal{P}$ does there exist another finite packing $\mathcal{F}_2$ with more bodies that $\mathcal{F}_1$ such that $(\mathcal{P} - \mathcal{F}_1) \cup \mathcal{F}_2$ is a packing.

A **completely reduced** covering $\mathcal{C}$ is such that for no finite subset $\mathcal{F}_1 \subset \hat{\mathcal{C}}$ does there exist another finite set $\mathcal{F}_2 \subset G/\Sigma_K$ with fewer elements that $\mathcal{F}_1$ such that $(\hat{\mathcal{C}} - \mathcal{F}_1) \cup \mathcal{F}_2$ determines a covering. Note that this is the equivalent to the definition given earlier.

Our main result is the following.

Theorem 2.1. *For any body K in S, there exists a completely saturated packing by K and a completely reduced covering by K.*

Remark: The theorem extends with almost no changes to the case in which $S = G/H$ where G is a unimodular Lie group, H is a compact subgroup of H and G satisfies the following property. For every compact subset C of G there exists a discrete cocompact subgroup $G' < G$ such that there exists a fundamental domain F for G' and $C \subset F$. In this setting, 'congruent copies of a body $K \subset S$' means subsets of the form gK where $g \in G$ and G acts on subsets of S in the natural way.

We will only prove the existence of completely saturated packings for K as the other assertion can be proven similarly.

For any point $p \in S$, we let $B_n(p)$ denote the closed ball of radius n centered at p. Also we let λ_S denote the usual measure on S. Finally for any packing $\mathcal{P}$ we let $c(\mathcal{P}) = \cup_{gK \in \mathcal{P}} gK$ be the closed subset associated to $\mathcal{P}$.

We obtain the following counterintuitive result:

Theorem 2.2. *Let S be n-dimensional hyperbolic space for some $n > 1$. Let $\epsilon > 0$. Then there is a body K in S that admits a tiling $\mathcal{T}$ and a completely saturated packing $\mathcal{P}$ with the following properties. For every point $p \in S$, the limit*

$$\lim_{r \to \infty} \frac{\lambda_S(c(\mathcal{P}) \cap B_r(p))}{\lambda_S(B_r(p))} \tag{1}$$

exists and is independent of p. Furthermore this common limit is less than ϵ.

The paper is organized as follows. In section 3, we introduce the main tools of our study. In section 4, we prove Theorem 2.1. In section 5, we prove Theorem 2.2.

3 Measures on the Space of Packings

Let $\Sigma_{\mathcal{O}}$ be the stabilizer $\{g \in G | \, g\mathcal{O} = \mathcal{O}\}$ of the origin $\mathcal{O}$. It is isomorphic to $SO(n-1)$. Let $\pi : G \to S$ be the canonical projection identifying $G/\Sigma_{\mathcal{O}}$ with S such that $\pi(g) = g\mathcal{O}$ for every $g \in G$. Let λ_G be the normalization of Haar measure on G such that λ_S, the usual measure on S, has the following relationship with λ_G (see [Rat] for the hyperbolic case). For any Borel set $E \subset S$, $\lambda_S(E) = \lambda_G(\pi^{-1}(E))$. We note the general fact (see [Lan]) that the Haar measure on any locally compact group G_0 is determined uniquely up to scalar multiplication by the property that it is a nonzero Borel measure that is invariant under left multiplication by every element $g \in G_0$ (i.e. $\lambda_{G_0}(E) = \lambda_{G_0}(gE)$ for every $g \in G$ and Borel set $E \subset G$ where λ_{G_0} is a Haar measure of G_0).

Let d_S be the usual distance function on S. Let d_G be any left-invariant distance function on G satisfying $d_S(p_1, p_2) = \inf\{d_G(\tilde{p}_1, \tilde{p}_2) | \, \tilde{p}_1 \in \pi^{-1}(p_1) \text{ and } \tilde{p}_2 \in \pi^{-1}(p_2)\}$. The left-invariant of d_G means that for any $g_0, g_1, g_2 \in G$, $d_G(g_1, g_2) = d_G(g_0 g_1, g_0 g_2)$. For any body K in S, we define a distance function d^K on G/Σ_K by $d^K(g_1\Sigma_K, g_2\Sigma_K) = \inf\{d_G(g_1 k_1, g_2 k_2) | \, k_1, k_2 \in \Sigma_K\}$. In other words, d^K is the pushforward of d_G.

Let B_n, B_n^K, and $\tilde{B}_n$ denote the closed balls of radius n centered at $\mathcal{O}$ in S, Σ_K in G/Σ_K and the identity in G respectively. If X is a subset of G/Σ_K and $\epsilon > 0$ we let $N_\epsilon(X)$ denote the open ϵ-neighborhood of X in G/Σ_K.

Let C_K denote the set of packings of S by K. First, for every $n > 0$, define a pseudometric d_n on C_K by

$$d_n(\mathcal{P}_1, \mathcal{P}_2) = \inf\{\epsilon > 0 | \, B_n^K \cap \hat{\mathcal{P}}_1 \subset N_\epsilon(\hat{\mathcal{P}}_2) \text{ and } B_n^K \cap \hat{\mathcal{P}}_2 \subset N_\epsilon(\hat{\mathcal{P}}_1)\}. \tag{2}$$

Next we define a metric d_K on C_K by:

$$d_K(\mathcal{P}_1, \mathcal{P}_2) = \sup_{n>0} \min\{\frac{d_n(\mathcal{P}_1, \mathcal{P}_2)}{n}, \frac{1}{n}\}. \tag{3}$$

It can be checked that C_K is a compact metric space on which G acts jointly continuously in the obvious way, i.e. for $\mathcal{P} \in C_K$ and $h \in G$, $h\mathcal{P} = \{hgK |\, gK \in \mathcal{P}\}$ (cf [RaW]). We note that in Euclidean space, if h is translation by x, then $h\mathcal{P}$ is commonly denoted by $\mathcal{P} + x$. Although the metric d_K will be convenient in what is to follow, we are really only concerned with the topology that it induces. The idea is that two packings are close in C_K if in a large ball about the origin, the bodies of one packing can by small rigid motions be made to coincide with the bodies of the other packing.

Let $M = M(K)$ be the set of Borel probability measures on C_K. We let $M_I = M_I(K)$ be the subset of M consisting of all those measures that are G-invariant. In other words, $\mu \in M$ is in M_I if and only if $\mu(gE) = \mu(E)$ for every $g \in G$ and Borel set $E \subset C_K$.

Although we will not use it, we show here how to construct a natural class of measures in M_I. If $\mathcal{P}$ is any packing such that the symmetry group $\Sigma_{\mathcal{P}} = \{g \in G |\, g\mathcal{P} = \mathcal{P}\}$ of $\mathcal{P}$ is such that G/Σ_K is compact then there is a canonical measure $\mu_{\mathcal{P}} \in M_I$ associated to $\mathcal{P}$. There is a homeomorphism q from the orbit $O(\mathcal{P}) = \{g\mathcal{P} |\, g \in G\}$ to $G/\Sigma_{\mathcal{P}}$ defined by $q(g\mathcal{P}) = g\Sigma_{\mathcal{P}}$. This homeomorphism commutes with the action by G where G acts on $G/\Sigma_{\mathcal{P}}$ by left multiplication. There is a natural measure on $G/\Sigma_{\mathcal{P}}$ induced from λ_G since the quotient map from G to $G/\Sigma_{\mathcal{P}}$ is a covering map in which the covering transformations are all measure-preserving (in fact they are isometries of G). This measure is invariant under the action of G since λ_G is invariant. Now a measure $\hat{\mu}_{\mathcal{P}}$ on $O(\mathcal{P})$ can be induced (via q) from the measure on $G/\Sigma_{\mathcal{P}}$. This measure can then be extended to a measure $\mu_{\mathcal{P}}$ on all of C_K by $\mu_{\mathcal{P}}(E) = \hat{\mu}_{\mathcal{P}}(E \cap O(\mathcal{P}))$. Thus for every packing whose symmetry group is cocompact, there is a natural measure in M_I associated to that packing.

For any packing $\mathcal{P}$ by a body K, we let $c(\mathcal{P}) = \cup_{gK \in \mathcal{P}}\, gK \subset S$ be the closed subset associated to $\mathcal{P}$. The density of a measure $\mu \in M_I$, $D(\mu)$ is defined by

$$D(\mu) := \mu(\{\mathcal{P} \in C_K |\, \mathcal{O} \in c(\mathcal{P})\}). \tag{4}$$

In other words, the density of μ is the probability (with respect to μ) that the origin lies in a body of a random packing. We define the (measure-theoretic) **optimal density** of a body K to be $D(K) := \sup_{\mu \in M_I} D(\mu)$. A measure $\mu \in M_I$ is said to be **optimally dense** (for K) if $D(\mu) = D(K)$ and μ is ergodic. By ergodic we mean that for any $E \subset C_K$ such that $gE = E$ for all $g \in G$, $\mu(E) \in \{0, 1\}$.

We will sketch a proof that when S is Euclidean, the optimal density of K is equal to the usual notion. To be precise, for a packing $\mathcal{P}$ and any point $p \in S$, let $D(\mathcal{P}, p)$ be defined by

$$\delta(\mathcal{P}, p) = \lim_{r \to \infty} \frac{\lambda_S(c(\mathcal{P}) \cap B_r(p)}{\lambda_S(B_r)} \tag{5}$$

whenever this limit exists. Let $\delta(K) = \sup_{\mathcal{P} \in C_K} \delta(\mathcal{P}, p)$. Note that $\delta(K)$ does not depend on the point p chosen since $\delta(\mathcal{P}, p) = \delta(g\mathcal{P}, \mathcal{O})$ where $g \in G$ such that $gp = \mathcal{O}$. We will show that $\delta(K) = D(K)$.

Groemer [Gro] proved that, when S is Euclidean, for any body $K \subset S$ there exists a packing $\mathcal{P}$ by K such that $\delta(\mathcal{P}, p) = \delta(K)$ for all points $p \in S$ and such that this limit converges uniformly in p. By standard ergodic theory, there exists a measure $\mu \in M_I$ whose support is contained in the closure of the orbit $O(\mathcal{P}) = \{g\mathcal{P} | \, g \in G\}$. Since convergence is uniform in p, $\delta(\mathcal{P}', \mathcal{O}) = \delta(\mathcal{P}, \mathcal{O})$ for all $\mathcal{P}'$ in the closure of $O(\mathcal{P})$. By Birkhoff's ergodic theorem, $\delta(\mathcal{P}', \mathcal{O}) = D(\mu)$ for μ-almost every packing $\mathcal{P}'$. Thus $D(\mu) = \delta(K)$ and so $D(K) \geq \delta(K)$.

For the other way, let μ be an optimally dense measure (the existence of which is proven below). Then Birkhoff's ergodic theorem implies that for μ-almost every packing $\mathcal{P}$, $\delta(\mathcal{P}, \mathcal{O}) = D(\mu)$. Thus $\delta(K) \geq D(K)$. So the two notions are equal. In the hyperbolic case, we can only show that $\delta(K) \geq D(K)$ (see Corollary 5.2). In fact, for any given ϵ we will construct a body K such that there exists a tiling by K (and hence $\delta(K) = 1$) but $D(K) < \epsilon$. This immediately implies that the subspace of C_K consisting of all tilings by K has measure zero with respect to any measure in M_I. So, by restricting our attention to invariant measures, many 'pathological' packings (or tilings) are ignored. It may seem odd to some readers that we call $D(K)$ the optimal density in spite of the fact $\delta(K)$ may be greater than $D(K)$. So we emphasize that $D(K)$ is just the *measure theoretic* optimal density.

The main theorem of this paper is:

Theorem 3.1. *Let $U \subset C_K$ be the set of all packings by K that are not completely saturated. Then for any optimally dense measure μ for K, $\mu(U) = 0$.*

Theorem 2.1 follows immediately from this one and the following existence theorem.

Theorem 3.2. *For any body $K \subset S$, there exists an optimally dense measure μ for K.*

A proof of this theorem appears in [BoR, Theorem 3.2]. We include the same proof here for completeness.

Proof. We let M_I have the weak* topology. Equivalently $\mu_n \to \mu \in M_I$ if and only if for every continuous function $f : C_K \to \mathbb{R}$, $\int_{C_K} f \, d\mu_n \to \int_{C_K} f \, d\mu$. By standard functional analysis, since C_K is a compact metric space, M_I is compact.

Let $Z = \{\mathcal{P} \in C_K | \, \mathcal{O} \in c(\mathcal{P})\}$. Let χ_Z be the characteristic function of Z. Because χ_Z is upper semicontinuous there exists a decreasing sequence f_j of continuous real valued functions on C_K which converge pointwise to χ_Z. Choose a sequence $\mu_k \in \mathcal{M}_I$ such that $D(\mu_k) = \int_{C_K} F_{\mathcal{O}} \, d\mu_k \to D(K)$ as $k \to \infty$, and, using the compactness of $\mathcal{M}_I$, assume without loss of generality that μ_k converges to some $\mu_\infty \in \mathcal{M}_I$. Then $\int_{C_K} f_j \, d\mu_k \to \int_{C_K} f_j \, d\mu_\infty$ as $j \to \infty$, and $\int_{C_K} f_j \, d\mu_\infty \searrow D(\mu_\infty)$ as $k \to \infty$. Since $\int_{C_K} f_j \, d\mu_k \geq D(\mu_k)$ and $D(\mu_k) \to D(K)$ as $k \to \infty$, $D(\mu_\infty) \geq D(K)$. From the Krein-Milman theorem there exists an ergodic measure $\tilde{\mu} \in \mathcal{M}_I$ for which $D(\tilde{\mu}) = \int_{C_K} \chi_Z \, d\tilde{\mu} \geq D(\mu_\infty)$, and thus $D(\tilde{\mu}) \geq D(K)$. But then from the definition of $D(K)$, $D(\tilde{\mu}) = D(K)$. $\square$

4 A Family of saturating maps

Before proving the theorem, we outline the case in which S is the Euclidean plane. Let T_j be the usual square tiling by squares of side length j. We assume that the origin is in the center of a tile of T_j for all j. Let G_j be the group of translations which fix T_j. Let U_j be the set of packings which are not saturated in the square of T_j that contains the origin. To be precise, $\mathcal{P} \in U_j$ if and only if there is a finite packing $\mathcal{F}_1 \subset \mathcal{P}$ and a finite packing $\mathcal{F}_2$ such that $\mathcal{F}_2$ has more elements than $\mathcal{F}_1$, for all $gK \in \mathcal{F}_1 \cup \mathcal{F}_2$, gK is contained in the square of T_j that contains the origin and $(\mathcal{P} - \mathcal{F}_1) \cup \mathcal{F}_2$ is a packing.

For each j, there exists a Borel map $\Phi_j : C_K \to C_K$ satisfying:

1. $gK \in \mathcal{P}$ and gK intersects an edge of T_j if and only if $gK \in \Phi_j(\mathcal{P})$ and gK intersects an edge of T_j;
2. For any given square in T_j, the number of bodies of $\Phi_j(\mathcal{P})$ that are contained in that square is as large as possible given the above constraint;
3. Φ_j commutes with G_j.

The idea is that Φ_j replaces $\mathcal{P} \in C_K$ with a $\Phi_j(\mathcal{P})$ that is saturated with respect to the squares of T_j. Because Φ_j commutes with G_j, it induces a map $\Phi_{j*} : M \to M$ such that if $\mu \in M_I$ then $\Phi_{j*}(\mu)$ is invariant under G_j. Since G_j is cocompact and discrete we can average $\Phi_{j*}(\mu)$ over a fundamental domain for G_j in G to obtain a measure that is invariant under all of G. This new measure will have a density greater than μ unless $\mu(U_j) = 0$. Since $\cup_j U_j = U$ is the set of all non-completely saturated packings, this shows that an optimally dense measure must satisfy $\mu(U) = 0$ which proves the theorem.

By a fundamental domain F of a subgroup $H < G$, we shall mean a connected set in S equal to the closure of its interior such that $\{hF | \, h \in H\}$ is a tiling by F.

For the general case, let $\{G_j\}_{j=0}^{\infty}$ be a sequence of discrete cocompact subgroups of G such that there exist fundamental domains F_j (in S) for G_j with $B_j \subset F_j$. We will also assume that F_j and G_j have been chosen so that for all $g \in \pi^{-1}(F_j)$, $g^{-1} \in \pi^{-1}(F_j)$. For the Euclidean case, we could, for example, let G_j be the cubic lattice generated by the translations $\tau_{i,j} := (x_1, ..., x_n) \to (x_1, ..., x_i + j, x_{i+1}, ..., x_n)$ for $1 \le i \le n$. Then we could choose F_j to be the cube of side length j whose center is the origin and whose faces are parallel to the coordinate planes. For the hyperbolic case, we refer to [FKK, Theorem 4.1] for the existence of $\{G_j\}_{j=0}^{\infty}$.

For $\mathcal{P} \in C_K$ and $F \subset S$, let $\mathcal{P} * F$ be the packing consisting of all elements $gK \in \mathcal{P}$ such that the interior of gK intersects F nontrivially. Also let ∂F denote the boundary of F, i.e. the intersection of F with the closure of its complement. Let $X_j = \{x \in C_K | x * \partial F_j = x\}$ with the subspace topology. For any $x \in X_j$, a filling f for x is a packing by K such that $f * \partial F_j = x$, $f * F_j = f$ and the number of elements of f is as large as possible given these constraints.

Lemma 4.1. *For each $j \ge 0$, there exists a Borel map $\phi_j : X_j \to C_K$ such that for each $x \in X_j$, $\phi_j(x)$ is a filling for x.*

Proof. Let $j \ge 0$ be fixed and let $X = X_j$. Given $x \in X$, f a filling for x and $m > 0$, let $Y(f, m) = \{x' \in X | \text{ there exists a filling } f' \text{ for } x' \text{ with } D_K(f', f) < 2^{-m}\}$. From the compactness of F_j it follows that given $m > 0$, there exists a finite number of packings $x_1, ..., x_p \in X$ and fillings f_k for x_k such that $X = \bigcup_{k=1}^{p} Y(f_k, m)$.

Thus for a given integer $m > 0$, there exists a finite partition $\{A_{m,k}\}_{k=1}^{r_m}$ of X such that each $A_{m,k}$ is Borel and contained in some $Y(f, m)$. We will assume that $\{A_{m+1,k}\}_{k=1}^{r_{m+1}}$ refines $\{A_{m,k'}\}_{k'=1}^{r_m}$ (i.e. for every $A_{m+1,k}$, there exists an $A_{m,k'}$ such that $A_{m+1,k} \subset A_{m,k'}$). For each $m > 0$ and $1 \le k \le r_m$, choose $a(m,k) \in A_{m,k}$ and $f(m,k)$ a filling for $a(m,k)$ so that the following are satisfied:

1. $\{a(m,k)\}_{k=1}^{r_m} \subset \{a(m+1,k')\}_{k'=1}^{r_{m+1}}$ for all $m > 0$.
2. For every $m > 0$, $1 \le k \le r_m$, $a \in A_{m,k}$ there is a filling f for a such that $d_K(f, f(m,k)) < 2^{-m+1}$. This is possible since for each $A_{m,k}$ there exists $x \in X$ and a filling f for x with $A_{m,k} \subset Y(f,m)$.
3. If for some m and k, k', $A_{m+1,k'} \subset A_{m,k}$, then $d_K(f(m+1,k'), f(m,k)) < 2^{-m+1}$.

We define a map $\alpha_m : X \to C_K$ by $\alpha_m(x) = f(m,k)$ if $x \in A_{m,k}$. Each α_m is Borel (since each $A_{m,k}$ is Borel). We claim that $\{\alpha_m\}_{m=1}^{\infty}$ converges pointwise as to a map ϕ. Let $x \in X$. Then by the third condition above, if $m < m'$ then

$$D_K(\alpha_m(x), \alpha_{m'}(x)) \le \Sigma_{k=m}^{m'-1} 2^{-k+1} \le \Sigma_{k=m}^{\infty} 2^{-k+1} \to 0 \tag{6}$$

as $m \to \infty$. So the sequence $\{\alpha_m(x)\}_{m=1}^{\infty}$ is Cauchy and therefore converges in C_K to an element $\phi(x)$. Since all the maps α_m are Borel, ϕ must be Borel, too.

We claim that $\phi(x)$ is a filling for x for each $x \in X$. So let $x \in X$ and for each $m > 0$ choose a filling f_m for x such that $d_K(f_m, \alpha_m(x)) < 2^{-m+1}$. By the definition of $A_{m,k}$ and the second condition of $f(m,k)$ listed above, such an f_m exists for all $m > 0$. By the triangle inequality, $d_K(f_m, \phi(x)) \le 2^{-m+1} + \Sigma_{k=m}^{\infty} 2^{-k+1}$ which goes to zero as m goes to infinity. Hence the sequence $\{f_m\}_{m>0}$ converges to $\phi(x)$. Since f_m is a filling for x for each m, it follows that $\phi(x)$ is also a filling for x. Now we are done.

□

We choose maps Φ_j satisfying the conclusion of the above lemma. Define $\Phi_j : C_K \to C_K$ by the following properties:

1. for any $\mathcal{P} \in C_K$, $\Phi_j(\mathcal{P}) * F_j = \phi_j(\mathcal{P} \cap \partial F_j) * F_j$;
2. for any $g \in G_j$ and $\mathcal{P} \in C_K$, $\Phi_j(g\mathcal{P}) = g\Phi_j(\mathcal{P})$.

It should be clear from this definition that for each j, Φ_j is Borel. Let $\mu \in M_I$ be given. Let $\mu'_j = \Phi_{j*}(\mu)$, i.e. for every Borel set $E \subset C_K$, $\mu'_j(E) = \mu(\Phi_j^{-1}(E))$. By the above, μ'_j is a Borel probability measure that is invariant under G_j. Let $\mu_j \in M_I$ be defined by

$$\mu_j(E) = \frac{1}{\lambda_S(F_j)} \int_{\pi^{-1}(F_j)} \mu'_j(g^{-1}E)\, d\lambda_G(g) \tag{7}$$

for any Borel $E \subset C_K$. It is easy to show that μ_j is G-invariant and therefore really is in M_I. The next two lemmas will provide tools for calculating $D(\mu)$ and $D(\mu_j)$ with respect to the relative density within F_j.

Lemma 4.2.

$$D(\mu_j) = \int_{C_K} \frac{\lambda_S(c(\mathcal{P}) \cap F_j)}{\lambda_S(F_j)}\, d\mu'_j(\mathcal{P}) \tag{8}$$

Proof. Let $Z = \{\mathcal{P} \in C_K | \mathcal{O} \in c(\mathcal{P})\}$. Let χ_Z denote the characteristic function of Z. By definition of density and of μ_j,

$$\begin{aligned}
D(\mu_j) &= \mu_j(Z) && (9)\\
&= \frac{1}{\lambda_S(F_j)} \int_{\pi^{-1}(F_j)} \mu'_j(g^{-1}Z)\, d\lambda_G(g) && (10)\\
&= \frac{1}{\lambda_S(F_j)} \int_{\pi^{-1}(F_j)} \int_{C_K} \chi_{g^{-1}Z}(\mathcal{P})\, d\mu'_j(\mathcal{P})\, d\lambda_G(g) && (11)\\
&= \int_{C_K} \frac{1}{\lambda_S(F_j)} \int_{\pi^{-1}(F_j)} \chi_{g^{-1}Z}(\mathcal{P})\, d\lambda_G(g)\, d\mu'_j(\mathcal{P}) && (12)
\end{aligned}$$

Hence we will be done once we show that for any $\mathcal{P} \in C_K$,

$$\int_{\pi^{-1}(F_j)} \chi_{g^{-1}Z}(\mathcal{P})\, d\lambda_G(g) = \lambda_S(c(\mathcal{P}) \cap F_j). \tag{13}$$

So,

$$\begin{aligned}
\int_{\pi^{-1}(F_j)} \chi_{g^{-1}Z}(\mathcal{P})\, d\lambda_G(g) &= \lambda_G(\{g \in \pi^{-1}(F_j) | \, \mathcal{P} \in g^{-1}Z\}) && (14)\\
&= \lambda_G(\{g \in \pi^{-1}(F_j) | \, g\mathcal{P} \in Z\}) && (15)\\
&= \lambda_G(\{g \in \pi^{-1}(F_j) | \, \mathcal{O} \in c(g\mathcal{P})\}) && (16)\\
&= \lambda_G(\{g \in \pi^{-1}(F_j) | \, g^{-1}\mathcal{O} \in c(\mathcal{P})\}) && (17)\\
&= \lambda_G(\{g \in \pi^{-1}(F_j) | \, g\mathcal{O} \in c(\mathcal{P})\}) && (18)\\
&= \lambda_G(\pi^{-1}(c(\mathcal{P}) \cap F_j)) && (19)\\
&= \lambda_S(c(\mathcal{P}) \cap F_j) && (20)
\end{aligned}$$

Equation (18) holds because λ_G is inversion-invariant (see [Rat] for the hyperbolic case) and because $\pi^{-1}(F_j) = \pi^{-1}(F_j)^{-1}$. The last equation holds since for any Borel set $E \subset S$, $\lambda_S(E) = \lambda_G(\pi^{-1}(E))$.

□

Lemma 4.3.

$$D(\mu) = \int_{C_K} \frac{\lambda_S(c(\mathcal{P}) \cap F_j)}{\lambda_S(F_j)}\, d\mu \tag{21}$$

Proof. By equation (13) in the previous lemma,

$$\begin{aligned}
\int_{C_K} \frac{\lambda_S(c(\mathcal{P}) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) &= \int_{C_K} \int_{\pi^{-1}(F_j)} \frac{\chi_Z(g^{-1}\mathcal{P})}{\lambda_S(F_j)}\, d\lambda_G(g)\, d\mu(\mathcal{P}) && (22)\\
&= \int_{\pi^{-1}(F_j)} \int_{C_K} \frac{\chi_Z(g^{-1}\mathcal{P})}{\lambda_S(F_j)}\, d\mu(\mathcal{P})\, d\lambda_G(g) && (23)\\
&= \int_{\pi^{-1}(F_j)} \int_{C_K} \frac{\chi_Z(\mathcal{P})}{\lambda_S(F_j)}\, d\mu(\mathcal{P})\, d\lambda_G(g) && (24)\\
&= \int_{\pi^{-1}(F_j)} \frac{D(\mu)}{\lambda_S(F_j)} d\lambda_G(g) && (25)\\
&= D(\mu) && (26)
\end{aligned}$$

The third inequality holds because μ is G-invariant. □

Proof. (of Theorem 3.1) Let U_j be the set of all packings in C_K that are unsaturated relative to F_j, i.e. for any $\mathcal{P} \in U_j$, there is a packing $\mathcal{P}'$ such that $\mathcal{P} * \partial F_j = \mathcal{P}' * \partial F_j$ but $\mathcal{P}' * F_j$ has more elements than $\mathcal{P} * F_j$.

$$\begin{aligned}
D(\mu_j) &= \int_{C_K} \frac{\lambda_S(c(\mathcal{P}) \cap F_j)}{\lambda_S(F_j)}\, d\mu_j'(\mathcal{P}) && (27)\\
&= \int_{C_K} \frac{\lambda_S(c(\Phi_j(\mathcal{P})) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) && (28)\\
&= \int_{U_j} \frac{\lambda_S(c(\Phi_j(\mathcal{P})) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) && (29)\\
&+ \int_{C_K - U_j} \frac{\lambda_S(c(\Phi_j(\mathcal{P})) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) && (30)\\
&\geq \int_{U_j} \frac{\lambda_S(c(\mathcal{P}) \cap F_j) + \lambda_S(K)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) && (31)\\
&+ \int_{C_K - U_j} \frac{\lambda_S(c(\Phi_j(\mathcal{P})) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) && (32)\\
&= \int_{C_K} \frac{\lambda_S(c(\mathcal{P}) \cap F_j)}{\lambda_S(F_j)}\, d\mu(\mathcal{P}) + \mu(U_j)\frac{\lambda_S(K)}{\lambda_S(F_j)} && (33)\\
&= D(\mu) + \mu(U_j)\frac{\lambda_S(K)}{\lambda_S(F_j)} && (34)
\end{aligned}$$

The first equality is Lemma 4.2, the second comes from the definition of μ_j' and the last equality uses Lemma 4.3.

If μ is optimally dense then by definition $D(\mu) \geq D(\mu_j)$. So, $\mu(U_j) = 0$ for all j. Hence $\mu(\bigcup_j U_j) = 0$. Since $B_j \subset F_j$ for all j, $U = \bigcup_j U_j$ is the set of all packings that are not completely saturated. □

5 Tiles with small optimal density

In this section we will prove Theorem 2.2. We assume from now on that S is n-dimensional hyperbolic space. We will need a lemma that follows from much more general ergodic theory results of Nevo and Stein. It is an immediate corollary of Theorem 3 in [NeS].

Lemma 5.1. *If G acts continuously on a compact metric space X such that there is a Borel probability measure μ on X that is invariant and ergodic under this action, then for every function $f \in L^p(X,\mu)$ $(1 < p < \infty)$ the following holds for μ-a.e. $x \in X$.*

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{\lambda_G(\tilde{B}_n)} \int_{\tilde{B}_n} f(gx)\, d\lambda_G(g). \tag{35}$$

Corollary 5.2. *Let μ be an ergodic measure in M_I. Then the following holds for μ-a.e. packing $\mathcal{P} \in C_K$.*

$$D(\mu) = \lim_{r\to\infty} \frac{\lambda_S(c(\mathcal{P}) \cap B_r)}{\lambda_S(B_r)}. \tag{36}$$

Proof. By the above lemma, (since $\chi_Z \in L^p(C_K,\mu)$ for all p) for μ-almost every packing $\mathcal{P}$ in C_K,

$$\begin{aligned} D(\mu) &= \int_{C_K} \chi_Z \, d\mu && (37)\\ &= \lim_{r\to\infty} \frac{1}{\lambda_G(\tilde{B}_r)} \int_{\tilde{B}_r} \chi_Z(g\mathcal{P})\, d\lambda_G(g) && (38)\\ &= \lim_{r\to\infty} \frac{1}{\lambda_S(B_r)} \int_{\tilde{B}_r} \chi_{g^{-1}Z}(\mathcal{P})\, d\lambda_G(g) && (39) \end{aligned}$$

If we replace $\pi^{-1}(F_j)$ with $\pi^{-1}(B_r) = \tilde{B}_r$ in the proof of equation (13), we get that

$$\int_{\tilde{B}_n} \chi_{g^{-1}Z}(\mathcal{P})\, d\lambda_G(g) = \lambda_S(c(\mathcal{P}) \cap B_r). \tag{40}$$

Equations (39) and (40) prove the corollary. □

Theorem 2.2 will follow almost immediately from this result, Theorem 3.1 and the following.

Theorem 5.3. *For every $\epsilon > 0$ there exists a body K such that K admits a tiling of S but the optimal density of K is less than ϵ.*

Remark: It is not yet known whether K can be chosen to be convex. It is known, however, that there exist convex polygons P which admit tilings of $\mathcal{H}^2$ but such that $D(P) < 1$. For example, the 5-gon with vertices at i, $2i$, $i + 1/2$, $i + 1$, and $2i + 2$ (in the upperhalf plane model) is one such.

To prove this theorem we will construct a body with some indentations and protrusions such that in any tiling by K every protrusion in any copy of K must fit into an indentation of another copy of K. There will be more indentations than protrusions implying at least, that it does not admit a tiling of a closed manifold. To show that any invariant measure in C_K has small density, we will show that the probability that the origin is in a protrusion is about equal to the probability that the origin is in any given indentation. Since there are many more indentations than protrusions, the probability that the origin is in an indentation but not in a protrusion will be high, implying that the density is low.

We will need some notation and a lemma (which is similar to a key part of the proof of proposition 2 in [BoR]). Let K be a body whose symmetry group Σ_K is trivial. For any $L \subset S$, let $C_K(L) = \{\mathcal{P} \in C_K |$ there exists $g \in \hat{\mathcal{P}}$ such that $\mathcal{O} \in gL\}$.

Lemma 5.4. *Let* K *be a body with* $\Sigma_K = \{e\}$. *Let* $\mu \in M_I(K)$ *such that* $D(\mu) \neq 0$. *If* L *is a Borel subset of* S *such that for every* $\mathcal{P} \in C_K(L)$ *there exists a unique* $g \in \hat{\mathcal{P}}$ *such that* $\mathcal{O} \in gL$ *then*

$$\mu(C_K(L)) = \mu(C_K(K))\frac{\lambda_S(L)}{\lambda_S(K)}. \tag{41}$$

Proof. Let $L_1 = L$ and $L_2 = K$. For $i = 1, 2$, and $\mathcal{P} \in C_K(L_i)$ let $\Psi_i(\mathcal{P})$ be the unique element of $\hat{\mathcal{P}}$ such that $\mathcal{O} \in \Psi_i(\mathcal{P})L_i$.

Note that if for some $i, j \in \{1, 2\}$, $\mathcal{P} \in C_K(L_i)$, $g \in G$ and $g\Psi_i(\mathcal{P})$ is in the image of Ψ_j then $\mathcal{O} \in g\Psi_i(\mathcal{P})K$. Since $\Psi_i(\mathcal{P}) \in \mathcal{P}$, $g\Psi_i(\mathcal{P}) \in g\mathcal{P}$. Hence $\Psi_j(g\mathcal{P}) = g\Psi_i(\mathcal{P})$. So if h is in the image of Ψ_i and gh is in the image of Ψ_j then $\Psi_j^{-1}(gh) = g\Psi_i^{-1}(h)$.

We define a measure λ'_G on G by the following. If E is a Borel subset of G and there exists a $g \in G$ with gE contained in the image of Ψ_i for some $i \in \{1, 2\}$, then $\lambda'_G(E) = \mu(\Psi_i^{-1}(gE))$. By the above statement and the G-invariance of μ, this is well-defined. Note that the image of Ψ_2 equals $\pi^{-1}(L_2)^{-1}$ and thus contains an open set (since $K = L_2$ does). So for any Borel subset $E \subset G$ there exists a countable partition $\{E_i\}_{i=1}^{\infty}$ of E such that for each $i > 0$, there exists $g_i \in G$ such that g_iE_i is contained in the image of Ψ_2. Then we define $\lambda'_G(E) = \Sigma_{i=1}^{\infty}\lambda'_G(E_i)$.

Clearly, λ'_G is left G-invariant. By the uniqueness of Haar measure up to scalars (see [Lan]), λ'_G must be a scalar multiple of λ_G. Since $D(\mu) = \mu(C_K(K)) \neq 0$,

$$\frac{\mu(C_K(L_1))}{\mu(C_K(L_2))} = \frac{\lambda'_G(\Psi_1(C_K(L)))}{\lambda'_G(\Psi_2(C_K(K)))} \tag{42}$$

$$= \frac{\lambda_G(\Psi_1(C_K(L)))}{\lambda_G(\Psi_2(C_K(K)))} \tag{43}$$

$$= \frac{\lambda_G(\pi^{-1}(L)^{-1})}{\lambda_G(\pi^{-1}(K)^{-1})} \tag{44}$$

$$= \frac{\lambda_G(\pi^{-1}(L))}{\lambda_G(\pi^{-1}(K))} \tag{45}$$

$$= \frac{\lambda_S(L)}{\lambda_S(K)}. \tag{46}$$

The fourth equation above holds because λ_G in inversion invariant (i.e. G is unimodular) and the last holds since for any Borel set $E \subset S$, $\lambda_S(E) = \lambda_G(\pi^{-1}(E))$.

□

Lemma 5.5. *If K, μ, and L satisfy the hypotheses of the above lemma and L is the disjoint union of L' and L'', then both L' and L'' satisfy the hypotheses of above lemma and $C_K(L)$ is the disjoint union of $C_K(L')$ and $C_K(L'')$.*

Proof. This is an easy exercise in understanding the definitions. ☐

Proof. (of Theorem 5.3) For simplicity we will only prove the dimension 2 case. We will identify S with the upperhalf plane model of $\mathcal{H}^2$ and thus regard it as a subset of the complex plane $\mathbb{C}$ (see [Rat]).

Let $\epsilon > 0$ be given. We pick a large positive integer m so that $\epsilon > 1 - \frac{m-1}{m}$. Let δ and δ' be small positive real numbers with $\delta' < \delta/2 < 1/6$.

Let R be the Euclidean rectangle with vertices i, mi, $m+mi$, and $m+i$. Let Q_0 be the Euclidean rectangle with vertices $\delta + i(1+\delta)$, $\delta + i(m-\delta)$, $1-\delta+i(1+\delta)$, and $1-\delta+i(m-\delta)$. Finally, let Q'_0 be the Euclidean rectangle with vertices $\frac{1}{2} - \delta' + i$, $\frac{1}{2} + \delta' + i$, $\frac{1}{2} - \delta' + i(1+\delta)$ and $\frac{1}{2} + \delta' + i(1+\delta)$.

For $j = 0, .., m-1$ let $Q_j = Q_0 + j$ and $Q'_j = Q'_0 + j$. Let P (for protrusion) equal $mQ_0 = \{mz |\, z \in Q\}$ and let $P' = mQ'_0$. Let $K = R \cup P \cup P' - \cup_j Q_j - \cup Q'_j$.

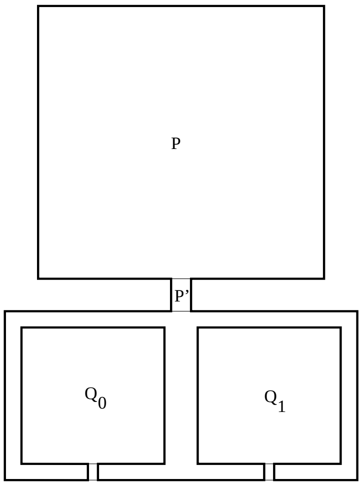

Figure 1: K

In the figure above, $m = 2$, K is in heavy outline and P, P', Q_0, Q_1, Q'_0, and Q'_1 are in light outline although Q'_0 and Q'_1 are not labeled. We let $R' = R \cup P' - \cup_j Q'_j$. Since $1 - \frac{m-1}{m} < \epsilon$ and $\frac{\lambda_S(Q_0)}{\lambda_S(R')} \to \frac{1}{m}$ as $\delta \to 0$, we can (and do) choose $\delta > 0$ so that

$$1 - (m-1)\left(\frac{\lambda_S(Q_0)}{\lambda_S(R')}\right) < \epsilon. \tag{47}$$

We let $\tau_m(z) = z + m$ and $s_m(z) = mz$ for every z in the upperhalf plane. We will assume that $\delta' > 0$ has been chosen small enough so that for any packing $\mathcal{P} \in C_K$ such that the K is in $\mathcal{P}$ and for any $gK \in \mathcal{P}$ such that $g \neq s_m^{-1}$, $gK \cap Q_0 = \emptyset$. In other words, unless the protrusion gP of gK fits into Q_0, no part of gK overlaps Q_0. This assumption implies R' is such that for every $\mathcal{P} \in C_K(R')$, there exists a unique $g \in \hat{\mathcal{P}}$ such that $\mathcal{O} \in gR'$. Thus Lemma 5.3 applies.

We now obtain a bound for the optimal density of K. Let $\mu \in M_I(K)$ such that $D(\mu) \neq 0$. By definition of density $D(\mu) = \mu(C_K(K))$. By Lemma 5.5,

$$D(\mu) = \mu(C_K(K - P)) + \mu(C_K(P)). \tag{48}$$

By Lemma 5.4, $\mu(C_K(P)) = \mu(C_K(Q_j))$ for any j. Thus

$$D(\mu) = \mu(C_K(K - P)) + \mu(C_K(Q_0)). \tag{49}$$

By Lemma 5.5,

$$\begin{aligned}
\mu(C_K(R')) &= \mu(C_K(K - P)) + \Sigma_{j=0}^{m-1}\mu(C_K(Q_j)) &(50)\\
&= \mu(C_K(K - P)) + m\mu(C_K(Q_0)). &(51)
\end{aligned}$$

Equations (49) and (51) imply that

$$D(\mu) = \mu(C_K(R')) - (m-1)\mu(C_K(Q_0)). \tag{52}$$

By Lemma 5.4,

$$\mu(C_K(Q_0)) = \mu(C_K(R'))\frac{\lambda_S(Q_0)}{\lambda_S(R')}. \tag{53}$$

This implies that

$$D(\mu) = \mu(C_K(R'))\left(1 - (m-1)\frac{\lambda_S(Q_0)}{\lambda_S(R')}\right). \tag{54}$$

Since $\mu(C_K(R')) \leq 1$,

$$D(\mu) \leq 1 - (m-1)\left(\frac{\lambda_S(Q_0)}{\lambda_S(R')}\right) < \epsilon. \tag{55}$$

Since $\mu \in M_I$ is arbitrary, $D(K) < \epsilon$. It can easily be checked that $\{s_m^i \tau_m^j K \,|\, i, j \in \mathbb{Z}\}$ is a tiling by K.

□

Remark: It follows from the above that the space of tilings by copies of K does not admit a G-invariant measure. However, this fact can be proven more directly by noting that there is an equivariant continuous map from the space of tilings by K onto the space at infinity of the hyperbolic plane. The latter admits no G-invariant Borel probability measure.

Proof. (of Theorem 2.2) Let $\epsilon > 0$. Let K be as in Theorem 5.3. Let μ be an optimally dense measure for K. Since μ is ergodic, Corollary 5.2 shows that the set X_1 of all $\mathcal{P} \in C_K$ satisfying

$$\lim_{n\to\infty} \frac{\lambda_S(c(\mathcal{P}) \cap B_n)}{\lambda_S(B_n)} = D(\mu) \tag{56}$$

has μ-measure 1. Let G' be a countable dense subset of G. Then $X_2 = \cap_{g\in G'} gX_1$ has μ-measure 1 also. Note that for any $\mathcal{P} \in X_2$, $g^{-1}\mathcal{P} \in X_1$ for all $g \in G'$. Hence

$$\lim_{n\to\infty} \frac{\lambda_S(c(\mathcal{P}) \cap B_n(g\mathcal{O}))}{\lambda_S(B_n)} = D(\mu) < \epsilon \tag{57}$$

for all $g \in G'$. Using the fact that G' is dense in G, it can be shown that the above equation holds for all $g \in G$ and for all $\mathcal{P} \in X_2$.

On the other hand, Theorem 3.1 shows that there exists a subset $X_3 \subset C_K$ of full μ-measure such that every packing in X_3 is completely saturated. Then $X_4 = X_2 \cap X_3$ has full μ-measure, in particular it is non-empty. Any $\mathcal{P} \in X_4$ satisfies the conclusion of Theorem 2.2. □

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