ABSOLUTELY CONTINUOUS SPECTRUM OF JACOBI MATRICES

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ABSTRACT. We study the discrete Schrödinger operator perturbed by a three diagonal matrix of a certain class. We follow the approach of [3] and prove that the absolutely continuous part of the spectrum is essentially supported by [-2,2].

0. INTRODUCTION

Let S be the shift operator in l^2 and let

$$H = S + S^* + Q$$

be the discrete Schrödinger operator, perturbed by a three diagonal matrix $Q = S\Lambda + \Lambda S^* + \Lambda_0$. We find conditions on Q which guarantee that the Lebesgue measure vanishes on subsets of the interval $[-2, 2]$ whose spectral measure is zero. In the case $\Lambda = 0$ and $\Lambda_0 \in \mathfrak{S}_2$ such a result has been proved by P.Deift and R.Killip [3].

The sharpness of the result obtained in [3] is confirmed by examples constructed by B.Simon [11] where the author shows that for each $\varepsilon > 0$ there is a potential Λ_0 satisfying $|\Lambda_0(n)| = O(n^{-1/2+\varepsilon})$ and for which the operator $H = S + S^* + \Lambda_0$ has a pure point spectrum. Previously, perturbations of discrete operators by a potential Λ_0 were also studied by A.Kiselev in [7]. Perturbations by discrete "second order differential" operators $2\text{Re}(S\Lambda)$ were considered in [6] by J.Janas and S.N.Naboko. However, the conditions which were imposed on Λ in [6] were much stronger then the ones obtained in this article.

Notice also that [3] was a culmination of a long sequence of papers concerning a.c. spectral properties of Schrödinger operators in $L^2(0, \infty)$ (see, for example, [1], [2] and [10]). It was proved in [3] that for the operator $-d^2/dx^2 + V$, the condition $V \in L^2$ suffices for the a.c. spectrum to be essentially supported by $(0, \infty)$. In both discrete and continuous cases P.Deift and R.Killip used well known trace formulas for Schrödinger operators involving L^2-norms of the corresponding potentials. This result has been recently generalized by S.Molchanov, M.Novitskii and B.Vainberg,

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1991 *Mathematics Subject Classification.* Primary 35P15; Secondary 35L15, 47A75, 35J10.

[9], where the authors used higher order trace formulas involving first KDV integrals.

Our result is somewhat surprising. By applying high order trace formulas for Jacobi matrices (using the approach [3], [9]) we obtain, for example, that for a discrete Schrödinger operator the $\mathfrak{S}_2$-condition on Λ_0 can be replaced by the $\mathfrak{S}_2$-condition on the *mean values* of Λ_0 and $\mathfrak{S}_4$-condition on Λ_0, see Theorem 1.2.

It is interesting to try to understand whether a version of such a theorem could be true for a Schrödinger operator in $L^2(0,\infty)$. In this case the two conditions from Theorem 1.2 would be replaced by $V\in L^4(0,\infty)$ and $\int_x^{x+a} V(t)\,dt \in L^2(0,\infty)$ for some $a>0$.

Note that there is a possibility of investigating the a.c. spectrum with the help of the Gilbert-Pearson [5] theory. It was established in [5] that the a.c. spectrum is related to the absence of subordinate solutions (see also [8]).

1. DESCRIPTION OF THE OPERATOR

Let S be the shift operator in $l^2(\mathbb{N})$ or in $l^2(\mathbb{Z})$ whose action on the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ (or $\{e_n\}_{n=-\infty}^{\infty}$) is given by $Se_n = e_{n+1}$. Let Λ, Λ_0 be selfadjoint diagonal operators: $\Lambda e_n = \lambda_n e_n$, $\Lambda_0 e_n = \lambda_n^0 e_n$, $\lambda_n > -1$. We study the spectrum of the operator

$$H = S + S^* + Q, \quad \text{where } Q = 2\mathrm{Re}(S\Lambda) + \Lambda_0.$$

We need the following classes of operators:

$$\mathfrak{S}_p = \{T:\ \mathrm{tr}\,(T^*T)^{p/2} < \infty\,\}.$$

Theorem 1.1. *Let $m>0$ be an integer number. Assume that $Q = 2\mathrm{Re}(S\Lambda)+\Lambda_0$, where the operators $\sum_{n=0}^m S^n\Lambda S^{-n}$ and $\sum_{n=0}^m S^n\Lambda_0 S^{-n}$ are of the Hilbert-Schmidt class and $\Lambda,\ \Lambda_0\in\mathfrak{S}_3$. Then the spectral measure of the operator H does not vanish on subsets $K\subset\mathbb{R}_+=(-2,2)$ whose Lebesgue measure is positive.*

When $\Lambda = 0$ we prove the following

Theorem 1.2. *Let $m>0$ be an integer number. Assume that $Q=\Lambda_0$, where the operator $\sum_{n=0}^m S^n\Lambda_0 S^{-n}$ is of the Hilbert-Schmidt class and $\Lambda_0\in\mathfrak{S}_4$. Then the spectral measure of the operator H does not vanish on subsets $K\subset\mathbb{R}_+=(-2,2)$ whose Lebesgue measure is positive.*

2. TRACE FORMULAE

For every $k\in\mathbb{C}\setminus\{0\}$ there exists a solution to

(2.1)
$$(1+\lambda_n)\psi_{n+1} + (1+\lambda_{n-1})\psi_{n-1} + \lambda_n^0\psi_n = (k+1/k)\psi_n,\ \ \forall n\in\mathbb{Z},$$

such that $\psi_n = k^{-n}$ on the right of the "support" of Q. To the left of the support of Q this solution takes the form

$$\psi_n = a(k)k^{-n} + b(k)k^n,$$

where a and b depend analytically on k. We substitute ψ_n by $\psi_n = f_n\phi_n$ where

$$f_n = \begin{cases} \prod_1^{n-1}(1+\lambda_k) & \text{if } n \geq 2, \\ 1 & \text{if } n \leq 1. \end{cases}$$

Then

$$(1+\lambda_n)^2\phi_{n+1} + \phi_{n-1} + \lambda_n^0\phi_n = (k+1/k)\phi_n,$$

We may rewrite this equation as

$$v_n = \tau - \frac{k}{k^2-1}\sum_{m=n}^{\infty}(1-k^{2(n-m)})\Big(\lambda_m^0 v_m + \frac{2\lambda_m + \lambda_m^2}{k}v_{m+1}\Big), \tag{2.2}$$

where $v_n = k^n\phi_n$ and $\tau = \prod_1^\infty(1+\lambda_k)^{-1}$.

For $|k| > 1$ this equation can be solved by repeated substitution (or Neumann series), from which we conclude that

$$\begin{gathered} a = \lim_{n\to-\infty} v_n = \tau\Big(1 - \frac{\mathrm{tr}\Lambda_0}{k} + \frac{1}{k^2}\{\mathrm{tr}(I-(I+\Lambda)^2) + \\ \sum\sum\frac{(1-k^{2|n-m|})}{2}\lambda_m^0\lambda_n^0\}\Big) + O\Big(\frac{1}{k^3}\Big), \quad \text{as } k\to\infty. \end{gathered} \tag{2.3}$$

Therefore

$$\tau^{-1}a = 1 - \frac{\mathrm{tr}\Lambda_0}{k} - \frac{1}{k^2}\{\mathrm{tr}((I+\Lambda)^2 - I) + \frac{(\mathrm{tr}\Lambda_0^2 - (\mathrm{tr}\Lambda_0)^2)}{2}\} + O(\frac{1}{k^3}), \tag{2.4}$$

as $k\to\infty$. The crucual circumstance is that

$$\tau^{-1}a = \det(I + Q(H_0 - z)^{-1}), \quad H_0 = S + S^*,\ z = k + k^{-1}.$$

Here, as in the previous section, S and Q are operators in $l^2(\mathbb{Z})$. It follows from this relation that

$$\log(a(k)) = \log(\tau) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{tr}(Q(H_0-z)^{-1})^n, \ \ z\to\infty.$$

2. Let us introduce the operator

$$M = (I+\Lambda)^2 - I.$$

Then

$$\log(a(k)) = -\sum_{n=0}^{\infty}A_n k^{-n}, \qquad k\to\infty,$$

where (compare with [4], p. 155) $A_n = A_n(\Lambda, \Lambda_0)$,

$$A_0 = \log(\tau), \quad A_1 = \mathrm{tr}\Lambda_0,$$

$$A_2 = \frac{1}{2}\mathrm{tr}\Lambda_0^2 + \mathrm{tr}(M),$$

$$A_3 = \mathrm{tr}(\frac{1}{3}\Lambda_0^3 + (\Lambda_0 + S^*\Lambda_0 S)(\frac{1}{2}I + M)),$$

$$A_4 = \frac{1}{4}\mathrm{tr}(\{(\Lambda_0^2 + M + SMS^*)^2 + 2(\Lambda_0 + S^*\Lambda_0 S)^2(M+1)\}) +$$

$$+\frac{1}{4}\mathrm{tr}\{(M + SMS^*)^2 - 2(M^2 - 2M)\}.$$

The function $a(k)$ vanishes when $k + 1/k$ is an eigenvalue of (2.1). Let $\{\beta_n\}$ be the zeros of $a(k)$ lying in the domain $|k| > 1$. We introduce the Blaschke product

$$B = \prod_n \frac{\beta_n - k}{1 - \beta_n k}\frac{\beta_n}{|\beta_n|}, \qquad (\overline{\beta_n} = \beta_n).$$

On the unit circle, when $k = e^{i\theta}$, we have $|B| = 1$ and Im$[\log(B/a)]$ is an odd function of θ and so, by (2.4)

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \log|a| \sin^2(m\theta)d\theta = \frac{1}{2\pi i}\int_{|k|=1} \log(B/a)\frac{(k^{2m}-1)^2 dk}{k^{2m+1}} =$$

$$2\log(\tau) + A_{2m} - \sum_n f(\beta_n^{2m}),$$

where $f(t) = 1/2(t - t^{-1}) - \log(|t|) > 0$ for $t > 1$. Finally, we obtain

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \log|a| \sin^2(m\theta)d\theta + \sum_n f(\beta_n^{2m}) = 2\log(\tau) + A_{2m}.$$

Which leads to the inequality:

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \log|a| \sin^2(m\theta)d\theta \leq 2\log(\tau) + A_{2m}. \tag{2.5}$$

In particular for $m = 4$ we can estimate the right hand side by

$$\begin{aligned} 2\log(\tau) + A_4 \leq \frac{1}{4}\mathrm{tr}(\{(\Lambda_0^2 + M + SMS^*)^2 + \\ 2(\Lambda_0 + S^*\Lambda_0 S)^2(M+1)\}) + \\ +\frac{1}{4}\mathrm{tr}(M + SMS^*)^2 + R_3(\Lambda). \end{aligned} \tag{2.6}$$

where $R_3(\Lambda)$ is finite if $\Lambda \in \mathfrak{S}_3$.

3. Assume for a moment that

$$Q = \Lambda_0, \ \ \Lambda_0 = \Lambda_0^* = \mathrm{diag}\{\lambda_n^0\}.$$

Since

$$(H_0 - z)^{-1} = -\frac{1}{k}(I - S/k)^{-1}(I - S^*/k)^{-1} = -\frac{1}{k}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{S^{j-m}}{k^{j+m}} =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{k^{n+1}}\sum_{m=0}^{n} S^{n-2m},$$

we obtain

$$\mathrm{tr} Q(H_0 - z)^{-1} Q(H_0 - z)^{-1} =$$

$$\sum_{p=0}^{\infty}\frac{1}{k^{p+2}}\mathrm{tr}\sum_{n=0}^{p} S^{-n}\Lambda S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda\sum_{j=0}^{p-n} S^{p-2j} = \sum_{p=0}^{\infty}\frac{J_p}{k^{p+2}},$$

where

$$J_{2(p+1)} = \mathrm{tr}\sum_{n=0}^{p} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda_0\sum_{j=n-p}^{p} S^{p-2j} =$$

$$\mathrm{tr}\sum_{n=0}^{p} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda_0 S^{2m} + \mathrm{tr}\sum_{n=p+1}^{2p} S^{-n}\Lambda_0 S^{n}\sum_{m=n-p}^{p} S^{-2m}\Lambda_0 S^{2m}.$$

On the other side,

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{-n}\Lambda_0 S^{n}\sum_{m=n-p}^{p} S^{-2m}\Lambda_0 S^{2m} =$$

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{2p-n} S^{-2(m+n-p)}\Lambda_0 S^{2(m+n-p)} =$$

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{n-2p}\Lambda_0 S^{2p-n}\sum_{m=0}^{2p-n} S^{-2m}\Lambda_0 S^{2m} = \mathrm{tr}\sum_{n=0}^{p-1} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda_0 S^{2m}.$$

Therefore

$$J_{2(p+1)} = \mathrm{tr}\sum_{n=0}^{p} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda_0 S^{2m} + \mathrm{tr}\sum_{n=0}^{p-1} S^{-n}\Lambda_0 S^{n}\sum_{m=0}^{n} S^{-2m}\Lambda_0 S^{2m}.$$

So we observe, that

$$J_{2(p+1)} = J_{2p} + \mathrm{tr} S^{-p}\Lambda_0 S^{p}\sum_{m=0}^{p} S^{-2m}\Lambda_0 S^{2m} + \mathrm{tr} S^{-p+1}\Lambda_0 S^{p-1}\sum_{m=0}^{p-1} S^{-2m}\Lambda_0 S^{2m}$$

$$= J_{2p} + \mathrm{tr} S^{-p}\Lambda_0 S^{p}\sum_{m=0}^{2p} S^{-m}\Lambda_0 S^{m} =$$

$$= J_{2p} + 2\mathrm{tr}\Lambda \sum_{m=1}^{p} S^{-m}\Lambda S^{m} + \mathrm{tr}\Lambda^{2}.$$

Since

$$J_2 = \mathrm{tr}\Lambda_0^2,$$

by induction we obtain that

$$J_{2(p+1)} = \mathrm{tr}(\sum_{n=0}^{p} S^{-n}\Lambda_0 S^{n})^2.$$

As a result we have

$$\mathrm{tr}Q(H_0-z)^{-1}Q(H_0-z)^{-1} = \sum_{p=2}^{\infty} \frac{J_p}{k^p},$$

where

$$J_{2(p+1)} = 2\mathrm{tr}(\sum_{n=0}^{p} S^{-n}\Lambda S^{n})^2.$$

4. Let us make another temporary assumption, that

$$Q = S\Lambda + \Lambda S^*, \quad \Lambda = \Lambda^*.$$

Then

$$\mathrm{tr}Q(H_0-z)^{-1}Q(H_0-z)^{-1} = 2\mathrm{tr}\Lambda(H_0-z)^{-1}\Lambda(H_0-z)^{-1}+$$

$$+\mathrm{tr}S\Lambda(H_0-z)^{-1}S\Lambda(H_0-z)^{-1} + \mathrm{tr}\Lambda S^*(H_0-z)^{-1}\Lambda S^*(H_0-z)^{-1}.$$

Consider the last term in the sum

$$\mathrm{tr}\Lambda S^*(H_0-z)^{-1}\Lambda S^*(H_0-z)^{-1} = \mathrm{tr}S\Lambda S^*(H_0-z)^{-1}\Lambda S^*(H_0-z)^{-1}S^* =$$

$$=: \sum_{p=2}^{\infty} \frac{D_p}{k^p}, \quad \text{where} \quad D_{2(p+1)} = \mathrm{tr}\sum_{n=0}^{2p} S^{-n+1}\Lambda S^{n-1}\sum_{m=0}^{n} S^{-2m}\Lambda \sum_{j=n-p}^{p} S^{2j-2} =$$

$$\mathrm{tr}\sum_{n=0}^{2p} S^{-n+1}\Lambda S^{n-1}\sum_{m=0}^{n} S^{-2m}\Lambda \sum_{j=n-p-1}^{p-1} S^{2j} =$$

$$\mathrm{tr}\sum_{n=0}^{p-1} S^{-n+1}\Lambda S^{n-1}\sum_{m=0}^{n} S^{-2m}\Lambda S^{2m} + \mathrm{tr}\sum_{n=p+1}^{2p} S^{-n+1}\Lambda S^{n-1}\sum_{m=n-p-1}^{p-1} S^{-2m}\Lambda S^{2m}+$$

$$S^{-p+1}\Lambda S^{p-1}\sum_{m=0}^{p-1} S^{-2m}\Lambda S^{2m}.$$

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{-n+1}\Lambda S^{n-1}\sum_{m=n-p-1}^{p-1} S^{-2m}\Lambda S^{2m} =$$

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{n-2p-1}\Lambda S^{2p-n+1}\sum_{m=0}^{2p-n} S^{-2m}\Lambda S^{2m} =$$

$$\mathrm{tr}\sum_{n=0}^{p-1} S^{-n-1}\Lambda S^{n+1}\sum_{m=0}^{n} S^{-2m}\Lambda S^{2m}.$$

Thus,

$$D_{2(p+1)} = D_{2p}+\mathrm{tr}S^{-p+2}\Lambda S^{p-2}\sum_{m=0}^{p-1} S^{-2m}\Lambda S^{2m}-\mathrm{tr}S^{-p+2}\Lambda S^{p-2}\sum_{m=0}^{p-2} S^{-2m}\Lambda S^{2m}+$$

$$+\mathrm{tr}S^{-p}\Lambda S^{p}\sum_{m=0}^{p-1} S^{-2m}\Lambda S^{2m} + \mathrm{tr}S^{-p+1}\Lambda S^{p-1}\sum_{m=0}^{p-1} S^{-2m}\Lambda S^{2m} =$$

$$D_{2p} + \mathrm{tr}S^{-p}\Lambda S^{p}\sum_{m=0}^{p-1} S^{-2m}\Lambda S^{2m} + \mathrm{tr}S^{-p}\Lambda S^{p}\sum_{m=0}^{p-1} S^{-2m-1}\Lambda S^{2m+1}+$$

$$+\mathrm{tr}S^{p}\Lambda S^{-p} = D_{2p} + \mathrm{tr}S^{-p}\Lambda S^{p}\sum_{m=0}^{2p} S^{-m}\Lambda S^{m} =$$

$$= D_{2p} + 2\mathrm{tr}\Lambda\sum_{m=1}^{p} S^{-m}\Lambda S^{m} + \mathrm{tr}\Lambda^2.$$

Since

$$D_2 = 0 = \mathrm{tr}\Lambda^2 - \mathrm{tr}\Lambda^2,$$

we obtain

$$D_{2(p+1)} = \mathrm{tr}(\sum_{n=0}^{p} S^{-n}\Lambda S^{n})^2 - \mathrm{tr}\Lambda^2.$$

As a result we have

$$\mathrm{tr}Q(H_0-z)^{-1}Q(H_0-z)^{-1} = 2\sum_{p=2}^{\infty}\frac{I_p}{k^p},$$

where

$$I_{2(p+1)} = 2\mathrm{tr}(\sum_{n=0}^{p} S^{-n}\Lambda S^{n})^2 - \mathrm{tr}\Lambda^2.$$

5. Finally we consider the general case, when $\Lambda \neq 0,\ \Lambda_0 \neq 0$. Since

$$\mathrm{tr}Q(H_0-z)^{-1} = -2\mathrm{tr}\Lambda\sum_{m=0}^{\infty}\frac{1}{k^{2m}} - \mathrm{tr}\Lambda_0\sum_{m=0}^{\infty}\frac{1}{k^{2m+1}},$$

we obtain:

$$A_{2m} = 2\mathrm{tr}\Lambda + J_{2m}/2 + I_{2m} + F_{2m}(Q), \quad m \geq 3, \tag{2.7}$$

where $F_{2m}(Q)$ is finite if one of the two following conditions hold

$$\begin{aligned} &i)\ \Lambda_0 \in \mathfrak{S}_3, \ \ \Lambda \in \mathfrak{S}_3 \\ &\qquad\qquad \text{or} \\ &ii)\ \Lambda_0 \in \mathfrak{S}_4, \ \ \Lambda = 0. \end{aligned} \tag{2.8}$$

3. WEYL FUNCTION

Without loss of generality we may assume that $\lambda_n = \lambda_n^0 = 0$ for $n \le 2$. Let us define the Weyl function for the operator in $l^2(\mathbb{N})$ (on the so called semi-axis)

$$m(z) := ((H - z)^{-1} e_0, e_0).$$

From the explicit representation of the vector $y = (H - z)^{-1} e_0$

$$y_n = -\psi_n / \psi_{-1}, \qquad n \ge 0,$$

we obtain an important relation which recovers m from the solution of (2.1): $m(k + 1/k) = -\psi_0/\psi_{-1}$ for $|k| > 1$. Since $\psi_n = a(k)k^{-n} + b(k)k^n$, this relation leads to

$$m(z) = -\frac{a(k) + b(k)}{a(k)k + b(k)k^{-1}}, \quad z = k + 1/k.$$

Therefore

$$r := b/a = \frac{-(1 + km)}{k(m + k)}.$$

Introducing the Wronskian

$$W_n = W_n(\psi, \overline{\psi}) = \psi_n \overline{\psi}_{n-1} - \psi_{n-1} \overline{\psi}_n$$

we observe that for $k = e^{i\theta}$, $\theta \in (0, \pi)$ the relation $\sum (W_{n+1} - W_n) = 0$ leads to

$$|a|^2 - |b|^2 = 1.$$

Therefore for values $k = e^{i\theta}$, $\theta \in (0, \pi)$, corresponding to $k + 1/k = 2\cos(\theta) + i0$, we have:

$$|a|^{-2} = 1 - |r|^2 = \frac{4\sin(\theta)\text{Im}(m)}{|m + k|^2} \le \frac{4\text{Im}(m)}{\sin(\theta)}.$$

The last inequality holds since $\text{Im}(m(\lambda + i0)) \ge 0$ for $\lambda \in \mathbb{R}$. For a compact set $K \subset [-2, 2]$ put

$$w(K) := \int_K \sqrt{4 - \lambda^2} \frac{d\lambda}{8\pi}$$

and

$$\pi\mu(K) := \int_K \text{Im} m(\lambda + i0) d\lambda,$$

then according to [3]

$$
\begin{aligned}
w(K)\log(\frac{w(K)}{\mu(K)}) &\leq \frac{-1}{2\pi}\int_{K_0}\log(\frac{4\mathrm{Im}(m)}{\sin(\theta)})\sin^2(\theta)d\theta \\
&\leq \frac{1}{\pi}\int_{K_0}\log(|a(e^{i\theta})|)\sin^2(\theta))d\theta.
\end{aligned}
\tag{3.1}
$$

where $K_0 = \{\theta \in [-\pi, \pi] : \; 2\cos(\theta) \in K\}$.

Assume now that K is a compact set with the property:

$$\inf_{K_0}\sin(m\theta) > 0. \tag{3.2}$$

Then the integral

$$\int_{-\pi}^{\pi}\log|a|\sin^2(m\theta)d\theta$$

can be estimated from below by

$$c\int_{-\pi}^{\pi}\log|a|\sin^2(\theta)d\theta.$$

with $c > 0$ independent of Λ and Λ_0. For such K we obtain:

$$w(K)\log(\frac{w(K)}{\mu(K)}) \leq C(A_{2m} + 2\log(\tau)) \tag{3.3}$$

with a constant C independent of Λ and Λ_0.

4. PERTURBATIONS OF A CLASS WIDER THAN THE HILBERT-SCHMIDT CLASS

The spectral significance of the Weyl function is represented by the relation

$$\mu(K) = (E_H(K)e_0, e_0),$$

where E_H denotes the spectral measure of H. So far we have only considered finite rank operators Λ, Λ_0. Suppose now that Λ, Λ_0 are arbitrary compact selfadjoint operators such that

$$A_{2m} + 2\log(\tau) < \infty. \tag{4.1}$$

Let $\Lambda(n), \Lambda_0(n)$ be sequences of finite rank operators not only converging to Λ and Λ_0 in $\mathfrak{S}_\infty$, but also satisfying the condition that

$$A_{2m}(\Lambda(n), \Lambda_0(n)) + 2A_0(\Lambda(n), \Lambda_0(n))$$

converges to

$$A_{2m}(\Lambda, \Lambda_0) + 2A_0(\Lambda, \Lambda_0), \quad \text{as } n \to \infty.$$

As always we assume that $I+\Lambda > cI, c>0$. Let $Q_n = 2\mathrm{Re}(S\Lambda(n)) + \Lambda_0(n)$,

$$H_n = S + S^* + Q_n, \qquad \text{and } \ \mu_n(K) = (E_{H_n}(K)e_0, e_0).$$

Then

$$\mu_n(K) \to \mu(K), \quad \text{as } \ n \to \infty,$$

since $(H_n - z)^{-1}$ converges to $(H-z)^{-1}$ uniformly on compact subsets of the upper half-plane. Indeed, the difference between resolvents is the operator

$$(H_n - z)^{-1} - (H-z)^{-1} = (H_n - z)^{-1}(Q - Q_n)(H-z)^{-1}$$

whose norm can be estimated by $C_0||Q - Q_n||$, with C_0 independent of n. Thus despite the fact that (2.5), (3.1) and (3.3) were obtained for finite rank perturbations Q, we come to the conclusion that under the restriction (3.2) the inequality

$$w(K)\log(\frac{w(K)}{\mu(K)}) \le C(A_{2m} - 2\log(I+\Lambda))$$

holds for general $\Lambda,\ \Lambda_0 \in \mathfrak{S}_\infty$ satisfying (4.1) with $I+\Lambda > 0$. This ineqality holds (with a constant independant of $\Lambda,\ \Lambda_0$) for compact sets $K \subset (-2,2)$ of positive Lebesgue measure satisfying (3.2). If (3.2) is not fulfilled then we can exclude from K small neighborhoods of zeros of $\sin(m\theta)$, such that the Lebesgue measure of the remaining set is positive. As a result we obtain that if (4.1) is fulfilled with $\Lambda,\ \Lambda_0 \in \mathfrak{S}_\infty$, then the spectral projection $E_H(K) \ne 0$ for an arbitrary compact set $K \subset [-2,2]$ of positive Lebesgue measure.

Without any difficulties we obtain from (2.6):

Theorem 4.1. *Assume that* $Q = 2\mathrm{Re}(S\Lambda) + \Lambda_0$ *where* Λ, Λ_0 *are "diagonal" compact selfadjoint operators, i.e.* $\Lambda e_n = \lambda_n e_n,\ \lambda_n \to 0$ *as* $n \to \infty$ *and* $\Lambda_0 e_n = \lambda_n^0 e_n,\ \lambda_n^0 \to 0$ *as* $n \to \infty$. *Let*

$$\sum\{(\lambda_n^0 + \lambda_{n+1}^0)^2 + (\mu_n + \mu_{n+1})^2\} < \infty, \quad \mu_n = 2\lambda_n + \lambda_n^2,$$

and

$$\Lambda \in \mathfrak{S}_3,\ \ \Lambda_0 \in \mathfrak{S}_4.$$

Then the spectral measure of the operator H *does not vanish on subsets* $K \subset \mathbb{R}_+ = (-2,2)$ *whose Lebesgue measure is positive.*

Moreover the proof of Theorem 1.1 and Theorem 1.2 is now obtained from (2.7), (2.8).

4.1. **Acknowledgements.** A.Laptev has been supported by the Swedish Natural Sciences Research Council, Grant M5105-20005157. S.Naboko wish to express his gratitude to the Mathematical Department of the Royal Institute of Technology in Stockholm for its warm hospitality. O.Safronov has been supported by the Swedish Natural Science Council M 5105-433/2000.

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