On dissipative Schrödinger-type operators in one dimension, their dilations and eigenfuction expansions

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Abstract

We study in detail Schrödinger-type operators on a bounded interval of $\mathbb{R}$ with dissipative boundary conditions. The characteristic function of this operator is computed, its minimal self-adjoint dilation is constructed and the generalized eigenfunction expansion for the dilation is developed. The problem is motivated by semiconductor physics.

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1 Introduction

Quasi stationary quantum mechanical models in semiconductor device simulation mostly rely on self-adjoint Schrödinger operators for the description of particle densities in connection with some fixed equilibrium distribution function, cf. e.g. [15],[16], [22],[25]-[28]. The self-adjointness assumption for Schrödinger's operator is natural since only for self-adjoint observables quantum mechanics is well developed. Unfortunately, such an approach has the disadvantage that only closed systems can be well described. In particular, considering the self-adjoint Schrödinger operator in some bounded region the current density [21] on the boundary of this region is necessarily zero due to the fact that the complex conjugation commutes with the Schrödinger operator. However, one approach in semiconductor device simulation is to describe the semiconductor by different models in different parts of the simulation domain, cf. e.g. [17]. In particular one is interested in the embedding of a quantum mechanical described structure, e.g. by some Schrödinger-Poisson system, into a potential flow governed by the well-known drift diffusion model [13],[14],[23],[24]. A natural boundary condition for models adjacent to each other is the continuity of the normal component of the current, which is impossible if the Schrödinger operator in the quantum mechanical model is self-adjoint.

To overcome this difficulty there are different proposals. In [8],[9],[10] Schrödinger-type operators are used whose boundary conditions depend on the spectral parameter. The approach goes back to [12]. However, the problem is that instead of one operator which describes the physical system one has to do with a family of Schrödinger-type operators which implies different conceptual difficulties.

In [18] we regarded a Schrödinger-type operator with non-selfadjoint boundary conditions. The corresponding open quantum system is driven by an adjacent potential flow acting on the boundary. This approach has the advantage that exactly one Schrödinger operator describes the physical system. However, the price which one has to pay is the non-selfadjointness of the Hamiltonian.

Following this line of investigation, we consider a Schrödinger-type operator H on a bounded interval $[a,b]$ of the real axis $\mathbb{R}$ with dissipative boundary conditions, effective mass $m > 0$ satisfying $m + \frac{1}{m} \in L^\infty([a,b])$ and real valued potential $V \in L^2([a,b])$ defined by

$$\mathrm{dom}(H) = \left\{ \begin{array}{l} g \in W^{1,2}([a,b]) : \frac{1}{m(x)} g'(x) \in W^{1,2}([a,b]), \\ \frac{1}{2m(a)} g'(a) = -\kappa_a g(a), \ \frac{1}{2m(b)} g'(b) = \kappa_b g(b) \end{array} \right\} \tag{1.1}$$

and

$$(Hg)(x) = (l(g))(x) \quad g \in \mathrm{dom}(H). \tag{1.2}$$

where

$$(l(g))(x) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}g(x) + V(x)g(x), \tag{1.3}$$

and $\kappa_a, \kappa_b \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Im\mathrm{m}(z) > 0\}$. In a forthcoming paper we intend to include such operators in Schrödinger-Poisson systems. To overcome conceptual difficulties arising from the non-selfadjointness of H we use the well-known fact from harmonic analysis of operators in Hilbert spaces [11] that each maximal dissipative operator admits a minimal self-adjoint dilation. In the future, we adopt this self-adjoint dilation as the Hamiltonian of a closed larger system in which the original system described by H is embedded. This has the advantage that one can use the usual quantum mechanical formalism. For the physical background of this approach the reader is referred to [33].

From this point of view it is important to have at hand an explicit construction of the self-adjoint dilation of H. Self-adjoint dilations of dissipative Schrödinger operators ($m(x) \equiv 1$) on $\mathbb{R}_+$ were constructed in [2],[3],[5],[7], [30]-[33], for the whole $\mathbb{R}^n$, $n \geq 1$, in [29]. Dissipative Schrödinger operators with vector-valued potentials on $\mathbb{R}_+$ were considered in [4] and [6]. Although the operator H seems to be not so far from cases considered above an explicit construction of the self-adjoint dilation for H is not available in the literature.

Finally, we are interested in physical quantities such as carrier and current densities related to Schrödinger operators. For self-adjoint Schrödinger operators these quantities can be expressed in terms of eigenfunctions of the Schrödinger operator. Considering dissipative Schrödinger operators one has to replace these eigenfunctions by generalized eigenfunctions of the self-adjoint dilation. Thus one has naturally to develop an eigenfunction expansion for the dilation.

The paper is organized as follows. In section 2 we introduce the dissipative Schrödinger-type operator and summarize its properties. Section 3 is devoted to the characteristic function of H and its properties. In section 4 we explicitly indicate the minimal self-adjoint dilation of H. After that in section 5 we calculate the generalized eigenfunctions of the self-adjoint dilation. In the last section we briefly discuss possible generalizations of the model proposed in [15].

2 Dissipative Schrödinger-type operators

Let $V \in L^2([a,b])$ and let $m, m^{-1} \in L^\infty([a,b])$, $m > 0$. In accordance with [18] we define the sesquilinear form

$$\begin{aligned} \mathfrak{t}[g,f] \quad := \quad & -\kappa_a g(a)\overline{f(a)} - \kappa_b g(b)\overline{f(b)} + \\ & \int_a^b dx\, \frac{1}{2\,m(x)} g'(x)\overline{f'(x)} + V(x)g(x)\overline{f(x)}, \end{aligned} \tag{2.1}$$

for $f, g \in \mathrm{dom}(\mathfrak{t}) = W^{1,2}([a,b])$ and $\kappa_a, \kappa_b \in \mathbb{C}$. By Theorem 2.20 of [18] the form $\mathfrak{t}$ is closed on $\mathfrak{H} = L^2([a,b])$ and sectorial. Hence, one can associate with $\mathfrak{t}[\cdot,\cdot]$ a maximal sectorial operator H. If either $\kappa_a \in \mathbb{C}_+$ or $\kappa_b \in \mathbb{C}_+$, then the operator H is dissipative, i.e., $\Im\mathrm{m}(Hg,g) \leq 0$ for $g \in \mathrm{dom}(H)$. In [18] such an operator is called anti-dissipative. Indeed, from (2.1) we get

$$(Hg,g) = -\kappa_a|g(a)|^2 - \kappa_b|g(b)|^2 + \int_a^b dx\, \frac{1}{2\,m(x)}|g'(x)|^2 + V(x)|g(x)|^2 \tag{2.2}$$

which yields $\Im\mathrm{m}(Hg,g) \leq 0$ for $g \in \mathrm{dom}(H)$. In our one dimensional situation this dissipative operator H admits an explicit description which coincides with that of (1.1)-(1.3). A dissipative operator is called maximal dissipative if it does not admit any proper dissipative extension. Since H is maximal sectorial the operator is also maximal dissipative.

The spectrum of the operator H is discrete and the only accumulation point is infinity. Furthermore, the operator H possesses a Riesz basis. For a detailed analysis of the spectral properties of H the reader is referred to [18].

Each maximal dissipative operator L admits a unique orthogonal decomposition into a self-adjoint operator L_s and a completely non-selfadjoint operator $L_{c.n.s}$, i.e.

$$L = L_s \oplus L_{c.n.s.}. \tag{2.3}$$

The operator L is called purely maximal dissipative operator if L_s is absent. In this sense the operator H is purely maximal dissipative, cf. [18]. In particular, this yields that H has no real eigenvalues.

In order to compute the resolvent of H let us introduce elementary solutions $v_a(x,z)$ and $v_b(x,z)$ which are defined by

$$l(v_a(x,z)) - zv_a(x,z) = 0, \quad v_a(a,z) = 1, \quad \frac{1}{2m(a)}v_a'(a,z) = -\kappa_a \tag{2.4}$$

$$l(v_b(x,z)) - zv_b(x,z) = 0, \quad v_b(b,z) = 1, \quad \frac{1}{2m(b)}v_b'(b,z) = \kappa_b. \tag{2.5}$$

The existence of these solutions for each $z \in \mathbb{C}$ can be proved by writing (2.4) and (2.5) in integral form

$$v_a(x,z) = 1 - 2\kappa_a \int_a^x dt\, m(t) + 2\int_a^x dt\, m(t) \int_a^t ds\, (V(s) - z)v_a(s,z) \tag{2.6}$$

and

$$v_b(x,z) = 1 - 2\kappa_b \int_x^b dt\, m(t) + 2\int_x^b dt\, m(t) \int_t^b ds\, (V(s) - z)v_b(s,z). \tag{2.7}$$

Since (2.6) and (2.7) are Volterra-type equations they are always soluble. Moreover, one gets that v_a and v_b as well as $\frac{1}{2\,m}v_a'$ and $\frac{1}{2\,m}v_b'$ are absolutely continuous. Further, let $W(z) := W(v_a(x,z), v_b(x,z))$ be the Wronskian of these solutions, i.e.

$$W(z) := v_a(x,z)\frac{1}{2m(x)}v_b'(x,z) - v_b(x,z)\frac{1}{2m(x)}v_a'(x,z). \tag{2.8}$$

We note that $W(z)$ depends only on z and is independent from x. In particular, for $x = a$ and $x = b$ one gets

$$W(z) = \frac{1}{2m(a)}v_b'(a,z) + \kappa_a v_b(a,z) = \kappa_b v_a(b,z) - \frac{1}{2m(b)}v_a'(b,z). \tag{2.9}$$

Furthermore, the functions

$$v_{*a}(x,z) := \overline{v_a(x,\overline{z})} \quad \text{and} \quad v_{*b}(x,z) := \overline{v_b(x,\overline{z})}, \tag{2.10}$$

$x \in [a,b]$ and $z \in \mathbb{C}$ are solutions of

$$l(v_{*a}(x,z)) - zv_{*a}(x,z) = 0 \quad v_{*a}(a,z) = 1 \quad \frac{1}{2m(a)}v'_{*a}(a,z) = -\overline{\kappa_a}, \tag{2.11}$$

$$l(v_{*b}(x,z)) - zv_{*b}(x,z) = 0 \quad v_{*b}(b,z) = 1 \quad \frac{1}{2m(b)}v'_{*b}(b,z) = \overline{\kappa_b}. \tag{2.12}$$

Obviously, for the Wronskian $W_*(z)$ one gets that

$$W_*(z) := v_{*a}(x,z)\frac{1}{2m(x)}v'_{*b}(x,z) - v_{*b}(x,z)\frac{1}{2m(x)}v'_{*a}(x,z) = \overline{W(\overline{z})}. \tag{2.13}$$

Similarly to (2.9) we obtain

$$W_*(z) = \frac{1}{2m(a)}v'_{*b}(a,z) + \overline{\kappa_a}v_{*b}(a,z) = \overline{\kappa_b}v_{*a}(b,z) - \frac{1}{2m(b)}v'_{*a}(b,z). \tag{2.14}$$

Let us define the following kernels

$$k(x,y;z) = -\frac{1}{W(z)}\begin{cases} v_b(x,z)v_a(y,z) & : \quad y \le x \\ v_a(x,z)v_b(y,z) & : \quad x < y \end{cases}, \text{ if } \quad W(z) \neq 0, \tag{2.15}$$

and

$$k_*(x,y;z) = -\frac{1}{W_*(z)}\begin{cases} v_{*b}(x,z)v_{*a}(y,z) & : \quad y \le x \\ v_{*a}(x,z)v_{*b}(y,z) & : \quad x < y \end{cases}, \text{ if } \quad W_*(z) \neq 0. \tag{2.16}$$

Theorem 2.1 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then the resolvent of the maximal dissipative operator H admits the representation*

$$\begin{aligned}((H-z)^{-1}f)(x) = \int_a^b dy\, k(x,y;z)f(y) = \\ -\frac{v_b(x,z)}{W(z)}\int_a^x dy\, v_a(y,z)f(y) - \frac{v_a(x,z)}{W(z)}\int_x^b dy\, v_b(y,z)f(y),\end{aligned} \tag{2.17}$$

$f \in L^2([a,b])$ and $z \in \varrho(H)$. For the resolvent of the adjoint operator H^* one has the representation

$$((H^* - z)^{-1} f)(x) = \int_a^b dy\, k_*(x,y;z) f(y) = \tag{2.18}$$
$$-\frac{v_{*b}(x,z)}{W_*(z)} \int_a^x dy\, v_{*a}(y,z) f(y) - \frac{v_{*a}(x,z)}{W_*(z)} \int_x^b dy\, v_{*b}(y,z) f(y),$$

$f \in L^2([a,b])$ and $z \in \varrho(H^*)$.

We omit the proof. Note that $z \in \sigma(H) \Leftrightarrow W(z) = 0$ and $z \in \sigma(H^*) \Leftrightarrow W(z) = 0$.

3 The characteristic function

In the following we consider the case that both complex numbers κ_a and κ_b,

$$\kappa_a = q_a + \frac{i}{2}\alpha_a^2 \quad \text{and} \quad \kappa_b = q_b + \frac{i}{2}\alpha_b^2, \tag{3.1}$$

$q_a, q_b \in \mathbb{R}$ belong to $\mathbb{C}_+$, i.e. $\alpha_a, \alpha_b > 0$. The other case that only κ_a or κ_b belongs to $\mathbb{C}_+$ can be handled mutatis mutandis setting formally either $\alpha_a = 0$ or $\alpha_b = 0$ in the formulas below. Let us introduce the operator-valued function $T(z) : \mathcal{H} \longrightarrow \mathbb{C}^2$,

$$T(z)f := \begin{pmatrix} \alpha_b((H-z)^{-1}f)(b) \\ -\alpha_a((H-z)^{-1}f)(a) \end{pmatrix} \tag{3.2}$$

for $z \in \varrho(H)$ and $f \in L^2([a,b])$. Using (2.17) we find

$$T(z)f = \frac{1}{W(z)} \begin{pmatrix} -\alpha_b \int_a^b dy\ v_a(y,z) f(y) \\ \alpha_a \int_a^b dy\ v_b(y,z) f(y) \end{pmatrix} \tag{3.3}$$

for $f \in L^2([a,b])$. The adjoint operator is given by

$$\begin{aligned} \left(T(z)^*\xi\right)(x) &= \frac{1}{W_*(\overline{z})} \left(-\alpha_b v_{*a}(x,\overline{z}), \alpha_a v_{*b}(x,\overline{z})\right) \xi \\ &= \frac{1}{W_*(\overline{z})} \left(-\alpha_b v_{*a}(x,\overline{z}) \xi^b + \alpha_a v_{*b}(x,\overline{z}) \xi^a\right) \end{aligned} \tag{3.4}$$

$x \in [a,b]$, where

$$\xi = \begin{pmatrix} \xi^b \\ \xi^a \end{pmatrix} \in \mathbb{C}^2. \tag{3.5}$$

Similarly, we set

$$T_*(z)f := \begin{pmatrix} \alpha_b((H^* - z)^{-1}f)(b) \\ -\alpha_a((H^* - z)^{-1}f)(a) \end{pmatrix} \tag{3.6}$$

for $z \in \varrho(H^*)$ and $f \in L^2([a,b])$. Using (2.18) we find

$$T_*(z)f = \frac{1}{W_*(z)} \begin{pmatrix} -\alpha_b \int_a^b dy\ v_{*a}(y,z)f(y) \\ \alpha_b \int_a^b dy\ v_{*b}(y,z)f(y) \end{pmatrix}. \tag{3.7}$$

The adjoint operator has the representation

$$\begin{aligned} (T_*(z)^*\xi)(x) &= \frac{1}{W(\overline{z})} (-\alpha_b v_a(x,\overline{z}), \alpha_a v_b(x,\overline{z}))\,\xi \\ &= \frac{1}{W(\overline{z})} \left(-\alpha_b v_a(x,\overline{z})\xi^b + \alpha_a v_b(x,\overline{z})\xi^a\right) \end{aligned} \tag{3.8}$$

$x \in [a,b]$, $\xi \in \mathbb{C}^2$.

Lemma 3.1 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then one has*

$$(H^* - z)^{-1} - (H - z)^{-1} = -iT_*(\overline{z})^* T_*(z) = -iT(\overline{z})^* T(z) \tag{3.9}$$

for $z \in \varrho(H) \cap \varrho(H^*)$.

Proof: Taking into account the boundary conditions (1.1) one gets

$$(H^*f,g) - (f,H^*g) = 2i\Im\mathrm{m}(\kappa_b)f(b)\overline{g(b)} + 2i\Im\mathrm{m}(\kappa_a)f(a)\overline{g(a)} \tag{3.10}$$

for $f,g \in \mathrm{dom}(H^*)$. By (3.1) we find

$$(H^*f,g) - (f,H^*g) = i\alpha_b^2 f(b)\overline{g(b)} + i\alpha_b^2 f(a)\overline{g(a)} \tag{3.11}$$

Setting $f = (H^* - z)^{-1}h$ and $g = (H^* - \overline{z})^{-1}k$ with $h,k \in L^2([a,b])$ and $z, \overline{z} \in \varrho(H^*)$ we obtain

$$((H^* - z)^{-1}h, k) - (h, (H^* - \overline{z})^{-1}k) = -i\,\langle T_*(z)h, T_*(\overline{z})k\rangle \tag{3.12}$$

where $< \cdot, \cdot >$ is the scalar product in $\mathbb{C}^2$. However, the relation (3.12) implies (3.9). Similarly, we prove the second relation. $\triangle$

Lemma 3.2 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then one has*

$$(H^* - \overline{z})^{-1} - (H-z)^{-1} + 2i\Im\mathrm{m}(z)(H^* - \overline{z})^{-1}(H-z)^{-1} = -iT(z)^*T(z) \quad (3.13)$$

for $z \in \varrho(H)$ and

$$(H^* - z)^{-1} - (H - \overline{z})^{-1} - 2i\Im\mathrm{m}(z)(H - \overline{z})^{-1}(H^* - z)^{-1} = -iT_*(z)^*T_*(z) \quad (3.14)$$

for $z \in \varrho(H^*)$.

Proof: Since

$$(Hf, g) - (f, Hg) = -2i\Im\mathrm{m}(\kappa_b)f(b)\overline{g(b)} - 2i\Im\mathrm{m}(\kappa_a)f(a)\overline{g(a)} \quad (3.15)$$

we find

$$\begin{aligned} &((H-z)f, g) - (f, (H-z)g) = \\ &\qquad -i\alpha_b^2 f(b)\overline{g(b)} - i\alpha_a^2 f(a)\overline{g(a)} - 2i\Im\mathrm{m}(z)(f,g). \end{aligned} \quad (3.16)$$

Setting $h = (H-z)^{-1}f$ and $k = (H-z)^{-1}g$ we find

$$\begin{aligned} &(h, (H-z)^{-1}k) - ((H-z)^{-1}h, k) = \\ &\qquad -i\langle T(z)h, T(z)k\rangle - 2i\Im\mathrm{m}(z)((H-z)^{-1}h, (H-z)^{-1}k) \end{aligned} \quad (3.17)$$

which immediately implies (3.13). Similarly, we prove (3.14). $\triangle$

By (3.13) and (3.14) we find

$$\begin{aligned} &T(z)^*T(z) - T_*(z)^*T_*(z) = \\ &\qquad 2\Im\mathrm{m}(z)[(H^* - \overline{z})^{-1} - (H - \overline{z})^{-1}][(H^* - z)^{-1} - (H-z)^{-1}] \end{aligned} \quad (3.18)$$

for $z \in \varrho(H) \cap \varrho(H^*)$. Taking into account (3.9) we finally get

$$T(z)^*T(z) - T_*(z)^*T_*(z) = -2\Im\mathrm{m}(z)T(z)^*T(\overline{z})T(\overline{z})^*T(z) \quad (3.19)$$

for $z \in \varrho(H) \cap \varrho(H^*)$. The characteristic function $\Theta_H(\cdot)$ of the maximal dissipative operator H is a two-by-two matrix-valued function which satisfies the relation

$$\Theta_H(z)T(z)f = T_*(z)f, \quad z \in \varrho(H) \cap \varrho(H^*), \quad f \in L^2([a,b]). \quad (3.20)$$

The characteristic function $\Theta_H(\cdot)$ depends meromorphically on $z \in \varrho(H) \cap \varrho(H^*)$ and is contractive in $\mathbb{C}_-$, i.e.

$$\|\Theta_H(z)\| \le 1, \quad z \in \overline{\mathbb{C}_-}, \tag{3.21}$$

$\mathbb{C}_- := \{z \in \mathbb{C} : \Im\mathrm{m}(z) < 0\}$. The last property is a consequence of (3.19). Indeed, one has to verify that

$$\|T_*(z)f\|^2 \le \|T(z)f\|^2, \quad z \in \overline{\mathbb{C}_-}, \quad f \in L^2([a,b]). \tag{3.22}$$

The inequality (3.22) is equivalent to

$$T_*(z)^*T(z) \le T(z)^*T(z), \quad z \in \overline{\mathbb{C}_-}, \tag{3.23}$$

which follows immediately from (3.19). If z is real, i.e. $z = \lambda \in \mathbb{R}$, then from (3.19) we obtain

$$T(\lambda)^*T(\lambda) = T_*(\lambda)^*T_*(\lambda) \tag{3.24}$$

which yields that $\Theta_H(\lambda)$ is unitary for each $\lambda \in \mathbb{R}$.

Similarly, one can introduce the characteristic function of H^* defined by

$$\Theta_{H^*}(z)T_*(z)f = T(z)f, \quad z \in \varrho(H) \cap \varrho(H^*), \quad f \in L^2([a,b]). \tag{3.25}$$

As above one can show that $\Theta_{H^*}(\cdot)$ is a contractive analytic function in the upper half plane. On the real axis both characteristic functions are related by

$$\Theta_{H^*}(\lambda) = \Theta_H(\lambda)^*, \quad \lambda \in \mathbb{R}, \tag{3.26}$$

which shows that the characteristic function $\Theta_{H^*}(\cdot)$ is unitary on the real axis, too. Moreover, a straightforward computation shows that

$$\Theta_{H^*}(z) = \Theta_H(\overline{z})^*, \quad z \in \varrho(H) \cap \varrho(H^*). \tag{3.27}$$

holds. Furthermore, from (3.20) and (3.25) we find

$$\Theta_{H^*}(z) = \Theta_H(z)^{-1}, \quad z \in \varrho(H) \cap \varrho(H^*). \tag{3.28}$$

Let us now compute the characteristic function $\Theta_H(z)$ of H:

Lemma 3.3 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then the characteristic function $\Theta_H(z)$ is given by*

$$\Theta_H(z) = \frac{1}{W_*(z)} \begin{pmatrix} W(z) - i\alpha_a^2 v_b(a,z) & -i\alpha_a\alpha_b \\ -i\alpha_a\alpha_b & W(z) - i\alpha_b^2 v_a(b,z) \end{pmatrix}. \tag{3.29}$$

for $z \in \varrho(H) \cap \varrho(H^*)$.

Proof: Using (3.20) we have to find a two-by-two matrix-valued function

$$\Theta_H(z) = \begin{pmatrix} \Theta_{bb}(z) & \Theta_{ba}(z) \\ \Theta_{ab}(z) & \Theta_{aa}(z) \end{pmatrix} \tag{3.30}$$

such that

$$\Theta_H(z) \begin{pmatrix} -\frac{\alpha_b}{W(z)} \int_a^b dy\, v_a(y,z) f(y) \\ \frac{\alpha_a}{W(z)} \int_a^b dy\, v_b(y,z) f(y) \end{pmatrix} = \begin{pmatrix} -\frac{\alpha_b}{W_*(z)} \int_a^b dy\, v_{*a}(y,z) f(y) \\ \frac{\alpha_a}{W_*(z)} \int_a^b dy\, v_{*b}(y,z) f(y) \end{pmatrix}. \tag{3.31}$$

Since $v_a(x,z)$ and $v_b(x,z)$ as well as $v_{*a}(x,z)$ and $v_{*b}(x,z)$ are solutions of the same second order differential equation the solutions $v_{*a}(x,z)$ and $v_{*b}(x,z)$ are linear combinations of the solutions $v_a(x,z)$ and $v_b(x,z)$. A straightforward computation proves that

$$v_{*a}(x,z) = \frac{1}{W(z)} \left\{ (W(z) - i\alpha_a^2 v_b(a,z)) v_a(x,z) + i\alpha_a^2 v_b(x,z) \right\} \tag{3.32}$$

and

$$v_{*b}(x,z) = \frac{1}{W(z)} \left\{ i\alpha_b^2 v_a(x,z) + (W(z) - i\alpha_b^2 v_a(b,z)) v_b(x,z) \right\}. \tag{3.33}$$

Inserting (3.32) and (3.33) into (3.31) we obtain

$$T_*(z) f = \begin{pmatrix} -\frac{\alpha_b}{W_*(z)} \int_a^b dy\, v_{*a}(y,z) f(y) \\ \frac{\alpha_a}{W_*(z)} \int_a^b dy\, v_{*b}(y,z) f(y) \end{pmatrix} = \tag{3.34}$$

$$\frac{1}{W_*(z)} \begin{pmatrix} W(z) - i\alpha_a^2 v_b(a,z) & -i\alpha_b\alpha_a \\ -i\alpha_a\alpha_b & W(z) - i\alpha_b^2 v_a(b,z) \end{pmatrix} T(z) f$$

which verifies (3.29). $\triangle$

Using (2.9) and (3.32) we find that

$$W_*(z) + i\alpha_b^2 v_{*a}(b,z) = W(z) - i\alpha_a^2 v_b(a,z). \tag{3.35}$$

Similarly, from (2.9) and (3.33) we get

$$W_*(z) + i\alpha_a^2 v_{*b}(a,z) = W(z) - i\alpha_b^2 v_a(b,z). \tag{3.36}$$

Inserting (3.35) and (3.36) into (3.29) we obtain

$$\Theta_H(z) = I_{\mathbb{C}^2} + i\frac{1}{W_*(z)}\begin{pmatrix} \alpha_b^2 v_{*a}(b,z) & -\alpha_b\alpha_a \\ -\alpha_b\alpha_a & \alpha_a^2 v_{*b}(a,z) \end{pmatrix}. \tag{3.37}$$

On the formal level the characteristic function can be expressed as follows. We introduce the unclosed operator $\alpha : \mathcal{H} \longrightarrow \mathbb{C}^2$.

$$\alpha f = \begin{pmatrix} \alpha_b f(b) \\ -\alpha_a f(a) \end{pmatrix}, \qquad f \in \mathrm{dom}(\alpha) = C([a,b]), \tag{3.38}$$

one gets the representation

$$\Theta_H(z) = I_{\mathbb{C}^2} - i\alpha T(\overline{z})^*. \tag{3.39}$$

If we assume for a moment that the operator α^* makes sense then we get the formula

$$\Theta_H(z) = I_{\mathbb{C}^2} - i\alpha(H^* - z)^{-1}\alpha^*, \quad z \in \varrho(H)^*, \tag{3.40}$$

which is expected. Indeed, if the imaginary part of the dissipative operator H is a bounded operator, then formula (3.40) is well-known, e.g. [1].

4 Dilations

Since H is a maximal dissipative operator there is a larger Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and a self-adjoint operator K on $\mathfrak{K}$ such that one has

$$P^{\mathfrak{K}}_{\mathfrak{H}}(K - z)^{-1}|\mathfrak{H} = (H - z)^{-1}, \quad z \in \mathbb{C}_+. \tag{4.1}$$

The operator K is called a self-adjoint dilation of the maximal dissipative operator H. Obviously, from the condition (4.1) one gets

$$P^{\mathfrak{K}}_{\mathfrak{H}}(K - z)^{-1}|\mathfrak{H} = (H^* - z)^{-1}, \quad z \in \mathbb{C}_-. \tag{4.2}$$

If the condition

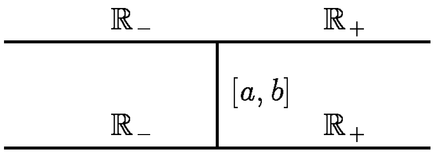
$$\bigvee_{z\in\mathbb{C}\setminus\mathbb{R}} (K-z)^{-1}\mathfrak{H} = \mathfrak{K} \tag{4.3}$$

is satisfied, then K is called a minimal self-adjoint dilation of H. Minimal self-adjoint dilations of maximal dissipative operators are determined up to a certain isomorphism, in particular, all minimal self-adjoint dilations are unitarily equivalent. The minimal self-adjoint dilation of a self-adjoint operator is the self-adjoint operator itself, thus of real interest is only the minimal self-adjoint dilation of the completely non-selfadjoint part of a maximal dissipative operator. Since H is purely maxim dissipative we are interested in the minimal self-adjoint dilation of H only.

Our next aim is to obtain an explicit description of the self-adjoint dilation of H. To this end we introduce the Hilbert space $\mathfrak{K}$ given by

$$\mathfrak{K} = \mathcal{D}_- \oplus \mathfrak{H} \oplus \mathcal{D}_+ \tag{4.4}$$

where $\mathcal{D}_\pm := L^2(\mathbb{R}_\pm, \mathbb{C}^2)$. Introducing the domain $\hat{\Omega}$



one can write the Hilbert space $\mathfrak{K}$ as $L^2(\hat{\Omega}, dx)$. Further, we define

$$\vec{g} := g_- \oplus g \oplus g_+ \tag{4.5}$$

where

$$g_-(x) := \begin{pmatrix} g_-^b(x) \\ g_-^a(x) \end{pmatrix} \qquad \text{and} \quad g_+(x) := \begin{pmatrix} g_+^b(x) \\ g_+^a(x) \end{pmatrix} \tag{4.6}$$

for $x \in \mathbb{R}_-$ and $x \in \mathbb{R}_+$, respectively. Let us introduce the matrices $K_\pm^a$ and $K_\pm^b$ which are defined by

$$K_-^a := \frac{1}{\alpha_a}\begin{pmatrix} 0 & 0 \\ 1 & \kappa_a \end{pmatrix} \qquad \text{and} \qquad K_+^a := \frac{1}{\alpha_a}\begin{pmatrix} 0 & 0 \\ 1 & \overline{\kappa_a} \end{pmatrix} \tag{4.7}$$

as well as

$$K_-^b := \frac{1}{\alpha_b}\begin{pmatrix} 1 & -\kappa_b \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad K_+^b := \frac{1}{\alpha_b}\begin{pmatrix} 1 & -\overline{\kappa_b} \\ 0 & 0 \end{pmatrix}. \tag{4.8}$$

We note that

$$K_-^{a*}K_-^a - K_+^{a*}K_+^a = iE \quad \text{and} \quad K_-^{b*}K_-^b - K_+^{b*}K_+^b = -iE \tag{4.9}$$

as well as

$$K_-^{b*}K_-^a = K_-^{a*}K_-^b = 0 \quad \text{and} \quad K_+^{b*}K_+^a = K_+^{a*}K_+^b = 0, \tag{4.10}$$

where

$$E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.11}$$

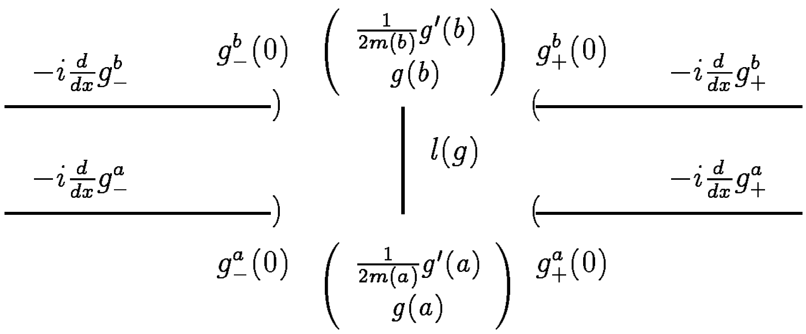
Further we need the relation

$$(l(g), f) - (g, l(f)) = \langle Eg_b, f_b \rangle - \langle Eg_a, f_a \rangle\,, \tag{4.12}$$

which holds for $f, \frac{1}{m}f, g, \frac{1}{m}g \in W^{1,2}([a,b])$ where

$$g_a = \begin{pmatrix} \frac{1}{2m(a)}g'(a) \\ g(a) \end{pmatrix} \quad \text{and} \quad g_b = \begin{pmatrix} \frac{1}{2m(b)}g'(b) \\ g(b) \end{pmatrix}. \tag{4.13}$$

Similarly, one defines f_a and f_b. On the Hilbert space $L^2(\hat{\Omega}, dx)$ we define a self-adjoint operator K which should play the role of a self-adjoint dilation of the maximal dissipative operator H. We choose the operator K in the form



where the problem is to find suitable boundary conditions such that the arising operator is self-adjoint. This is the content of the following theorem.

Theorem 4.1 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then the operator K defined by*

$$\mathrm{dom}(K) := \left\{ \begin{array}{c} \vec{g} \in \mathcal{K} : g_\pm \in W^{1,2}(\mathbb{R}_\pm, \mathbb{C}^2),\ g, \frac{1}{m}g' \in W^{1,2}([a,b]) \\ K^a_- g_a + K^b_- g_b = g_-(0),\ K^a_+ g_a + K^b_+ g_b = g_+(0) \end{array} \right\} \tag{4.14}$$

and

$$K\vec{g} := -i\frac{d}{dx}g_- \oplus l(g) \oplus -i\frac{d}{dx}g_+, \qquad \vec{g} \in \mathrm{dom}(K), \tag{4.15}$$

is self-adjoint.

Proof: We find

$$(K\vec{g}, \vec{f}) = \left\langle -i\frac{d}{dx}g_-, f_- \right\rangle + (l(g), f) + \left\langle -i\frac{d}{dx}g_+, f_+ \right\rangle, \tag{4.16}$$

$\vec{g}, \vec{f} \in \mathrm{dom}(K)$. One has

$$\left\langle -i\frac{d}{dx}g_-, f_- \right\rangle = \langle -ig_-(0), f_-(0) \rangle + \left\langle g_-, -i\frac{d}{dx}f_- \right\rangle \tag{4.17}$$

and

$$\left\langle -i\frac{d}{dx}g_+, f_+ \right\rangle = \langle ig_+(0), f_+(0) \rangle + \left\langle g_+, -i\frac{d}{dx}f_+ \right\rangle. \tag{4.18}$$

Inserting (4.12), (4.17) and (4.18) into (4.16) we obtain

$$\begin{aligned} &(K\vec{g}, \vec{f}) - (\vec{g}, K\vec{f}) = \\ &\quad -i\langle g_-(0), f_-(0) \rangle + \langle Eg_b, f_b \rangle - \langle Eg_a, f_a \rangle + i\langle g_+(0), f_+(0) \rangle. \end{aligned} \tag{4.19}$$

Using (4.14) we find

$$\begin{aligned} (K\vec{g}, \vec{f}) - (\vec{g}, K\vec{f}) &= -i\left\langle \{K^a_- g_a + K^b_- g_b\}, \{K^a_- f_a + K^b_- f_b\} \right\rangle + \\ &\langle Eg_b, f_b \rangle - \langle Eg_a, f_a \rangle + i\left\langle \{K^a_+ g_a + K^b_+ g_b\}, \{K^a_+ f_a + K^b_+ f_b\} \right\rangle. \end{aligned} \tag{4.20}$$

By (4.10) and (2.3) one has

$$\begin{aligned} (K\vec{g}, \vec{f}) - (\vec{g}, K\vec{f}) &= -i\left\langle \{K^{a*}_- K^a_- - K^{a*}_+ K^a_+\}g_a, f_a \right\rangle - \\ &i\left\langle \{K^{b*}_- K^b_- - K^{b*}_+ K^b_+\}g_b, f_b \right\rangle + \langle Eg_b, f_b \rangle - \langle Eg_a, f_a \rangle. \end{aligned} \tag{4.21}$$

Using (4.9) we obtain $(K\vec{g},\vec{f}) = (\vec{g}, K\vec{f})$ for $\vec{g},\vec{f} \in \mathrm{dom}(K)$ which proves the symmetry of K.

Next we are going to verify that K is self-adjoint. Let $\vec{f} \in \mathrm{dom}(K^*)$. It is not hard to see that in this case one has $f_\pm \in W^{1,2}(\mathbb{R}_\pm)$ and $f, \frac{1}{m}g' \in W^{1,2}([a,b])$. It remains to show that the boundary conditions of (4.14) are satisfied. To this end we note that $(Kg,f) = (g, K^*f)$, $g \in \mathrm{dom}(K)$, and (4.19) imply

$$0 = -i\left\langle g_-(0), f_-(0)\right\rangle + \left\langle Eg_b, f_b\right\rangle - \left\langle Eg_a, f_a\right\rangle + i\left\langle g_+(0), f_+(0)\right\rangle. \tag{4.22}$$

Using the boundary conditions (4.14) we find

$$\begin{aligned} 0 = &\left\langle K^a_- g_a + K^b_- g_b, f_-(0)\right\rangle + \\ &i\left\langle Eg_b, f_b\right\rangle - i\left\langle Eg_a, f_a\right\rangle - \left\langle K^a_+ g_a + K^b_+ g_b, f_+(0)\right\rangle. \end{aligned} \tag{4.23}$$

Hence we get

$$\begin{aligned} 0 = &\left\langle g_a, \left\{K^{a*}_- f_-(0) - K^{a*}_+ f_+(0) - iEf_a\right\}\right\rangle + \\ &\left\langle g_b, \left\{K^{b*}_- f_-(0) - K^{b*}_+ f_+(0) + iEf_b\right\}\right\rangle. \end{aligned} \tag{4.24}$$

From (4.24) we obtain

$$K^{a*}_- f_-(0) - K^{a*}_+ f_+(0) - iEf_a = 0 \tag{4.25}$$

and

$$K^{b*}_- f_-(0) - K^{b*}_+ f_+(0) + iEf_b = 0 \tag{4.26}$$

From (4.25) and (4.26) we deduce

$$f_a = iEK^{a*}_- f_-(0) - iEK^{b*}_+ f_+(0) \tag{4.27}$$

and

$$f_b = -iEK^{b*}_- f_-(0) + iEK^{b*}_+ f_+(0) \tag{4.28}$$

Now by a straightforward computation one verifies that $K^a_- f_a = K^b_- f_b = f_-(0)$ and $K^a_+ f_a + K^b_+ f_b = f_+(0)$ which shows that $f \in \mathrm{dom}(K)$. Hence $K^* = K$. $\triangle$

In the picture representation the boundary conditions can be expressed as follows:

$$\alpha_b g^b_-(0) = \frac{1}{2m(b)} g'(b) - \kappa_b g(b) \qquad \frac{1}{2m(b)} g'(b) - \overline{\kappa_b} g(b) = \alpha_b g^b_+(0)$$

$$-i\frac{d}{dx} g^b_- \qquad\qquad -i\frac{d}{dx} g^b_+$$

$$l(g)$$

$$-i\frac{d}{dx} g^a_- \qquad\qquad -i\frac{d}{dx} g^a_+$$

$$\alpha_a g^a_-(0) = \frac{1}{2m(a)} g'(a) + \kappa_a g(a) \qquad \frac{1}{2m(a)} g'(a) + \overline{\kappa_a} g(a) = \alpha_a g^a_+(0)$$

From the picture one immediately sees that the boundary conditions are local ones. In order to show that K is a self-adjoint dilation of the maximal dissipative operator let us compute the resolvent of the self-adjoint operator K.

Theorem 4.2 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then the resolvent of K admits the representation*

$$\begin{aligned}
&(K - z)^{-1}(f_- \oplus f \oplus f_+) = \\
&i\int_{-\infty}^{x} dy\; e^{i(x-y)z} f_-(y) \;\oplus\; (H - z)^{-1} f + iT_*(\overline{z})^* \int_{-\infty}^{0} dy\; e^{-iyz} f_-(y) \;\oplus \\
&i\int_{0}^{x} dy\; e^{i(x-y)z} f_+(y) + ie^{izx} T(z) f + i\Theta_H(\overline{z})^* \int_{-\infty}^{0} dy\; e^{i(x-y)z} f_-(y)
\end{aligned} \tag{4.29}$$

for $\Im\mathrm{m}(z) > 0$ and

$$\begin{aligned}
&(K - z)^{-1}(f_- \oplus f \oplus f_+) = \\
&-i\int_{x}^{0} dy\; e^{i(x-y)z} f_-(y) - ie^{izx} T_*(z) - i\Theta_H(z) \int_{0}^{\infty} dy\; e^{i(x-y)z} f_+(y) \oplus \\
&(H^* - z)^{-1} f - iT(\overline{z})^* \int_{0}^{\infty} dy\; e^{-iyz} f_+(y) \;\oplus\; -i\int_{x}^{\infty} dy\; e^{i(x-y)z} f_+(y)
\end{aligned} \tag{4.30}$$

for $\Im\mathrm{m}(z) < 0$.

Proof: Let $\Im\mathrm{m}(z) > 0$. We set

$$g_-(x) \;:=\; i\int_{-\infty}^{x} dy\; e^{i(x-y)z} f_-(y), \tag{4.31}$$

$$g(x) := (H-z)^{-1}f + iT_*(\overline{z})^* \int_{-\infty}^{0} dy e^{-iyz} f_-(y) \tag{4.32}$$

$$g_+(x) := i\int_0^x dy\ e^{i(x-y)z} f_+(y) + ie^{izx}T(z)f + \tag{4.33}$$

$$i\Theta_H(\overline{z})^* \int_{-\infty}^{0} dy\ e^{i(x-y)z} f_-(y)$$

Obviously we have $g_- \in W^{1,2}(\mathbb{R}_-)$ and

$$\left(-i\frac{d}{dx} - z\right) g_- = f_-. \tag{4.34}$$

Setting

$$h = (H-z)^{-1}f \tag{4.35}$$

and taking into account (3.8) one gets that

$$g(x) = h(x) + \frac{1}{W(z)}\left(-\alpha_b v_a(x,z), \alpha_a v_b(x,z)\right) g_-(0) \tag{4.36}$$

which shows $g \in W^{1,2}([a,b])$. Hence

$$g'(x) = h'(x) + \frac{1}{W(z)}\left(-\alpha_b v_a'(x,z), \alpha_a v_b'(x,z)\right) g_-(0) \tag{4.37}$$

which yields $\frac{1}{m}g' \in W^{1,2}([a,b])$. Taking into account (2.4) and (2.5) we find

$$l(g) - zg = f. \tag{4.38}$$

Obviously one has $g_+ \in W^{1,2}([a,b])$. A straightforward computation shows

$$\left(-i\frac{d}{dx} - z\right) g_+ = f_+. \tag{4.39}$$

It remains to verify that $\vec{g}$ satisfies the boundary conditions (4.14). One gets

$$g_a = h_a + \frac{1}{W(z)}\begin{pmatrix} \alpha_b\kappa_a & \alpha_a\frac{1}{2m(a)}v_b'(a,z) \\ -\alpha_b & \alpha_a v_b(a,z) \end{pmatrix} g_-(0) \tag{4.40}$$

and

$$g_b = h_b + \frac{1}{W(z)}\begin{pmatrix} -\alpha_b\frac{1}{2m(b)}v_a'(b,z) & \alpha_a\kappa_b \\ -\alpha_b v_a(b,z) & \alpha_a \end{pmatrix} g_-(0). \tag{4.41}$$

Since $K_-^a h_a + K_-^b h_b = 0$ and

$$\frac{1}{W(z)} K_-^a \begin{pmatrix} \alpha_b \kappa_a & \alpha_a \frac{1}{2m(a)} v_b'(a,z) \\ -\alpha_b & \alpha_a v_b(a,z) \end{pmatrix} + \tag{4.42}$$

$$\frac{1}{W(z)} K_-^b \begin{pmatrix} -\alpha_b \frac{1}{2m(b)} v_a'(b,z) & \alpha_a \kappa_b \\ -\alpha_b v_a(b,z) & \alpha_a \end{pmatrix} = I$$

we immediately find that $K_-^a g_a + K_-^b g_b = g_-(0)$. We note that

$$g_+(0) = iT(z)f + \Theta_H(\overline{z})^* g_-(0). \tag{4.43}$$

Using (2.17) we compute that

$$K_+^a h_a = \frac{i}{W(z)} \begin{pmatrix} 0 \\ \alpha_a \int_a^b dy\ v_b(y,z) f(y) \end{pmatrix} \tag{4.44}$$

and

$$K_+^b h_b = \frac{i}{W(z)} \begin{pmatrix} -\alpha_b \int_a^b dy\ v_a(y,z) f(y) \\ 0 \end{pmatrix}. \tag{4.45}$$

By definition (3.3) we finally obtain

$$K_+^a h_a + K_+^b h_b = iT(z)f. \tag{4.46}$$

Furthermore, from (3.29), (4.40) and (4.41) we find that

$$\frac{1}{W(z)} K_+^a \begin{pmatrix} \alpha_b \kappa_a & \alpha_a \frac{1}{2m(a)} v_b'(a,z) \\ -\alpha_b & \alpha_a v_b(a,z) \end{pmatrix} g_-(0) + \tag{4.47}$$

$$\frac{1}{W(z)} K_+^b \begin{pmatrix} -\alpha_b \frac{1}{2m(b)} v_a'(b,z) & \alpha_a \kappa_b \\ -\alpha_b v_a(b,z) & \alpha_a \end{pmatrix} g_-(0) = \Theta_H(\overline{z})^* g_-(0).$$

By (4.43), (4.46) and (4.47) we finally get $K_+^a g_a + K_+^b g_b = g_+(0)$ which completes the proof. △

From (4.29) we obtain

$$P_{\mathfrak{H}}^{\mathfrak{K}} (K - z)^{-1}(0, f, 0) = (H - z)^{-1} f, \quad f \in \mathfrak{H}, \tag{4.48}$$

for $z \in \mathbb{C}_+$ and

$$P_{\mathfrak{H}}^{\mathfrak{K}} (K - z)^{-1}(0, f, 0) = (H^* - z)^{-1} f, \quad f \in \mathfrak{H}, \tag{4.49}$$

for $z \in \mathbb{C}_-$. Hence the operator K is indeed a self-adjoint dilation of the maximal dissipative operator H. It can be shown that K is minimal.

5 Eigenfunction expansion

The self-adjoint operator K is absolutely continuous and its spectrum coincides with the real axis, i.e. $\sigma(K) = \mathbb{R}$. Its multiplicity is two. Let us compute the generalized eigenfunctions $\vec{\phi}(\cdot, \lambda)$, $\lambda \in \mathbb{R}$, of K. We set

$$\vec{\phi}(x, \lambda) = \phi_-(x, \lambda) \oplus \phi(x, \lambda) \oplus \phi_+(x, \lambda) \tag{5.1}$$

for $x \in \hat{\Omega}$ where

$$\phi_-(x, \lambda) = \begin{pmatrix} \phi_-^b(x, \lambda) \\ \phi_-^a(x, \lambda) \end{pmatrix}, \qquad x \in \mathbb{R}_-, \tag{5.2}$$

and

$$\phi_+(x, \lambda) = \begin{pmatrix} \phi_+^b(x, \lambda) \\ \phi_+^a(x, \lambda), \end{pmatrix}, \qquad x \in \mathbb{R}_+. \tag{5.3}$$

From the equation

$$\begin{aligned} (K\vec{\phi})(x, \lambda) &= -i\frac{d}{dx}\phi_-(x, \lambda) \oplus l(\phi(x, \lambda)) \oplus -i\frac{d}{dx}\phi_+(x, \lambda) \\ &= \lambda \left(\phi_-(x, \lambda) \oplus \phi(x, \lambda) \oplus \phi_+(x, \lambda)\right), \end{aligned} \tag{5.4}$$

$x \in \hat{\Omega}$, we find the equations

$$-i\frac{d}{dx}\begin{pmatrix} \phi_-^b(x, \lambda) \\ \phi_-^a(x, \lambda) \end{pmatrix} = \lambda \begin{pmatrix} \phi_-^b(x, \lambda) \\ \phi_-^a(x, \lambda), \end{pmatrix}, \qquad x \in \mathbb{R}_-, \tag{5.5}$$

$$l(\phi(x, \lambda)) = \lambda\phi(x, \lambda), \qquad x \in \Omega, \tag{5.6}$$

and

$$-i\frac{d}{dx}\begin{pmatrix} \phi_+^b(x, \lambda) \\ \phi_+^a(x, \lambda) \end{pmatrix} = \lambda \begin{pmatrix} \phi_+^b(x, \lambda) \\ \phi_+^a(x, \lambda), \end{pmatrix}, \qquad x \in \mathbb{R}_+. \tag{5.7}$$

The equations (5.5) - (5.7) have the solutions

$$\phi_-(x, \lambda) = \begin{pmatrix} C_-^b \\ C_-^a \end{pmatrix} e^{ix\lambda}, \qquad x \in \mathbb{R}_-, \tag{5.8}$$

$$\phi(x, \lambda) = C_a v_a(x, \lambda) + C_b v_b(x, \lambda), \qquad x \in \Omega, \tag{5.9}$$

and

$$\phi_+(x,\lambda) = \begin{pmatrix} C^b_+ \\ C^a_+ \end{pmatrix} e^{ix\lambda}, \qquad x \in \mathbb{R}_+. \tag{5.10}$$

The eigenfunctions have to satisfy the boundary conditions, cf. Theorem 4.1,

$$K^a_-\phi_a(\lambda) + K^b_-\phi_b(\lambda) = \phi_-(0,\lambda) \tag{5.11}$$

and

$$K^a_+\phi_a(\lambda) + K^b_+\phi_b(\lambda) = \phi_+(0,\lambda), \tag{5.12}$$

where

$$\phi_a(\lambda) = \begin{pmatrix} \frac{1}{2m(a)}\phi'(a,\lambda) \\ \phi(a,\lambda) \end{pmatrix} \quad \text{and} \quad \phi_b(\lambda) = \begin{pmatrix} \frac{1}{2m(b)}\phi'(b,\lambda) \\ \phi(b,\lambda) \end{pmatrix}. \tag{5.13}$$

A straightforward computation shows that

$$C_a = -\frac{\alpha_b}{W(\lambda)}C^b_- \qquad \text{and} \qquad C_b = \frac{\alpha_a}{W(\lambda)}C^a_-. \tag{5.14}$$

This yields

$$\phi(x,\lambda) = -\frac{\alpha_b}{W(\lambda)}C^b_- v_a(x,\lambda) + \frac{\alpha_a}{W(\lambda)}C^a_- v_b(x,\lambda), \qquad x \in \Omega \tag{5.15}$$

where $v_a(x,\lambda)$ and $v_b(x,\lambda)$ the elementary solutions (2.6) and (2.7). Using the adjoint operator $T_*(\lambda)^*$, cf. (3.8), we find

$$\phi(x,\lambda) = (T_*(\lambda)^* C_-)(x), \qquad x \in \Omega, \qquad \lambda \in \mathbb{R}, \tag{5.16}$$

where

$$C_- := \begin{pmatrix} C^b_- \\ C^a_- \end{pmatrix} \tag{5.17}$$

Inserting (5.16) into (5.12) we find

$$C_+ = \Theta_H(\lambda)^* C_-, \qquad \lambda \in \mathbb{R}, \tag{5.18}$$

where

$$C_+ = \begin{pmatrix} C^b_+ \\ C^a_+ \end{pmatrix}. \tag{5.19}$$

Thus, we finally have

$$\vec{\phi}^{C_-}(x,\lambda) := C_- e^{ix\lambda} \oplus (T_*(\lambda)^* C_-)(x) \oplus \Theta_H(\lambda)^* C_- e^{ix\lambda}, \tag{5.20}$$

$x \in \hat{\Omega}$, $\lambda \in \mathbb{R}$. Using the well-known formula

$$\frac{1}{x \pm i0} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x} \tag{5.21}$$

from distribution theory [34] we find

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{0} dx\ \left\langle \vec{\phi}^{C_-}(x,\lambda), \vec{\phi}^{C'_-}(x,\lambda')\right\rangle = \\ &\quad\frac{1}{2\pi}\int_{-\infty}^{0} dx\ e^{ix(\lambda-\lambda')} = \frac{1}{2}\delta(\lambda-\lambda')\left\langle C_-, C'_-\right\rangle - \frac{i}{2\pi}\mathcal{P}\frac{1}{\lambda-\lambda'}\left\langle C_-, C'_-\right\rangle\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}&\frac{1}{2\pi}\int_{0}^{+\infty} dx\ \left\langle \vec{\phi}^{C_-}(x,\lambda), \vec{\phi}^{C'_-}(x,\lambda')\right\rangle = \\ &\quad\frac{1}{2\pi}\int_{0}^{+\infty} dx\ e^{ix(\lambda-\lambda')}\left\langle \Theta_H(\lambda)^* C_-, \Theta_H(\lambda')^* C'_-\right\rangle = \\ &\quad\frac{1}{2}\delta(\lambda-\lambda')\left\langle \Theta_H(\lambda)^* C_-, \Theta_H(\lambda')^* C'_-\right\rangle + \\ &\quad\frac{i}{2\pi}\mathcal{P}\frac{1}{\lambda-\lambda'}\left\langle \Theta_H(\lambda)^* C_-, \Theta_H(\lambda')^* C'_-\right\rangle\end{aligned} \tag{5.23}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{C}^2$. Since

$$\begin{aligned}&\frac{1}{2\pi}\int_{a}^{b} dx\ \vec{\phi}^{C_-}(x,\lambda)\,\overline{\vec{\phi}^{C'_-}(x,\lambda')} = \\ &\quad\frac{1}{2\pi}\int_{a}^{b} dx (T_*(\lambda)^* C_-)(x)\,\overline{T_*(\lambda')^* C'_-)(x)} = \\ &\quad\frac{i}{2\pi}\mathcal{P}\frac{1}{\lambda-\lambda'}\left\langle C_-, C'_-\right\rangle - \frac{i}{2\pi}\mathcal{P}\frac{1}{\lambda-\lambda'}\left\langle \Theta_H(\lambda)^* C_-, \Theta_H(\lambda')^* C'_-\right\rangle\end{aligned} \tag{5.24}$$

we find

$$\left(\frac{1}{\sqrt{2\pi}}\vec{\phi}^{C_-}(\cdot,\lambda), \frac{1}{\sqrt{2\pi}}\vec{\phi}^{C'_-}(\cdot,\lambda')\right) = \delta(\lambda-\lambda')\left\langle C_-, C'_-\right\rangle. \tag{5.25}$$

Setting

$$\vec{\psi}^-(\cdot,\lambda,b) := \frac{1}{\sqrt{2\pi}}\vec{\phi}^{e_b}(\cdot,\lambda) \quad \text{where} \quad e_b := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.26}$$

and

$$\vec{\psi}^-(\cdot,\lambda,a) := \frac{1}{\sqrt{2\pi}}\vec{\phi}^{e_a}(\cdot,\lambda) \quad \text{where} \quad e_a := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.27}$$

one gets the following theorem:

Theorem 5.1 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. Then the functions $\{\psi^-(\cdot,\lambda,b), \psi^-(\cdot,\lambda,a)\}_{\lambda\in\mathbb{R}}$ perform a complete orthonormal system of generalized eigenfunctions of K, i.e*

$$\left(\vec{\psi}^-(\cdot,\lambda,\tau), \vec{\psi}^-(\cdot,\lambda',\tau')\right) = \delta(\lambda-\lambda')\delta_{\tau\tau'}, \quad \lambda,\lambda' \in \mathbb{R}, \quad \tau,\tau' = a,b, \tag{5.28}$$

and their linear span is dense in $\mathfrak{K}$.

Since the eigenfunctions of the system behaves on $\mathbb{R}_-$ like free waves one calls it the complete system of incoming eigenfunctions. Let

$$C(\lambda) := \begin{pmatrix} C^b(\lambda) \\ C^a(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{5.29}$$

In accordance with (5.20) we set

$$\vec{\phi}^{C(\lambda)}(x,\lambda) := C(\lambda)e^{ix\lambda} \oplus (T_*(\lambda)^*C(\lambda))(x) \oplus \Theta_H(\lambda)^*C(\lambda)e^{ix\lambda}, \tag{5.30}$$

$x \in \hat{\Omega}$ and $\lambda \in \mathbb{R}$. Obviously, the functions $\vec{\phi}^{C(\lambda)}(\cdot,\lambda)$ are eigenfunctions of K, i.e. $K\vec{\phi}^{C(\lambda)}(\cdot,\lambda) = \lambda\vec{\phi}^{C(\lambda)}(\cdot,\lambda)$. Moreover, one gets that

$$\frac{1}{\sqrt{2\pi}}\vec{\phi}^{C(\lambda)}(\cdot,\lambda) = C^b(\lambda)\vec{\psi}^-(\cdot,\lambda,b) + C^a(\lambda)\vec{\psi}^-(\cdot,\lambda,a), \quad \lambda \in \mathbb{R}, \tag{5.31}$$

which yields

$$\left(\frac{1}{\sqrt{2\pi}}\vec{\phi}^{C(\lambda)}(\cdot,\lambda), \frac{1}{\sqrt{2\pi}}\vec{\phi}^{C(\lambda')}(\cdot,\lambda')\right) = \delta(\lambda-\lambda')\left\langle C(\lambda), C(\lambda')\right\rangle. \tag{5.32}$$

By a family of orthonormal bases $\{e_1(\lambda), e_2(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\mathbb{C}^2$ we mean that the components of the vectors $e_1(\lambda)$ and $e_2(\lambda)$ are Lebesgue measurable functions such that $\langle e_\tau(\lambda), e_{\tau'}(\lambda)\rangle = \delta_{\tau\tau'}$ for a.e. $\lambda \in \mathbb{R}$. Using this notion one gets following corollary:

Corollary 5.2 *Let $V \in L^2([a,b])$, $\Im\mathrm{m}(V) = 0$ and $\kappa_a, \kappa_b \in \mathbb{C}_+$. If $\{e_1(\lambda), e_2(\lambda)\}_{\lambda\in\mathbb{R}}$ is a measurable family of orthonormal bases in the Hilbert space $\mathbb{C}^2$, then the system of eigenfunctions $\{\vec{\psi}^{(1)}(\cdot,\lambda), \vec{\psi}^{(2)}(\cdot,\lambda)\}_{\lambda\in\mathbb{R}}$,*

$$\vec{\psi}^{(\tau)}(\cdot,\lambda) := \frac{1}{\sqrt{2\pi}}\vec{\phi}^{e_\tau(\lambda)}(\cdot,\lambda), \quad \lambda \in \mathbb{R}, \quad \tau = 1,2, \tag{5.33}$$

performs a complete orthonormal system of generalized eigenfunctions of K.

In particular, setting

$$\vec{\psi}^{+}(\cdot,\lambda,\tau) := \frac{1}{\sqrt{2\pi}}\vec{\phi}^{\Theta_H(\lambda)e_\tau}(\cdot,\lambda), \quad \tau = a,b, \tag{5.34}$$

where

$$\vec{\phi}^{\Theta_H(\lambda)C_+}(x,\lambda) = \Theta_H(\lambda)C_+e^{ix\lambda} \oplus (T(\lambda)^*C_+)(x) \oplus C_+e^{ix\lambda}, \tag{5.35}$$

$x \in \hat{\Omega}$, $\tau = a,b$, $C_+ \in \mathbb{C}^2$, one defines a complete orthonormal system of eigenfunctions of K. Since the eigenfunctions behaves on $\mathbb{R}_+$ like free waves one calls them the complete system of outgoing eigenfunctions. Using (5.31) one gets

$$\begin{pmatrix} \vec{\psi}^{+}(\cdot,\lambda,b) \\ \vec{\psi}^{+}(\cdot,\lambda,a) \end{pmatrix} = \Theta^t_H(\lambda)\begin{pmatrix} \vec{\psi}^{-}(\cdot,\lambda,b) \\ \vec{\psi}^{-}(\cdot,\lambda,a) \end{pmatrix} \tag{5.36}$$

where $\Theta^t_H(\lambda)$ is the transposed matrix of $\Theta_H(\lambda)$, i.e., the matrix where lines and columns are interchanged. Since $\Theta^t_H(\lambda) = \Theta_H(\lambda)$, $\lambda \in \mathbb{R}$, we find

$$\begin{pmatrix} \vec{\psi}^{+}(\cdot,\lambda,b) \\ \vec{\psi}^{+}(\cdot,\lambda,a) \end{pmatrix} = \Theta_H(\lambda)\begin{pmatrix} \vec{\psi}^{-}(\cdot,\lambda,b) \\ \vec{\psi}^{-}(\cdot,\lambda,a) \end{pmatrix}. \tag{5.37}$$

Using the incoming eigenfunctions we introduce the transformations

$$(\Phi_-\vec{g}))(\lambda) =: \hat{g}(\lambda) = \begin{pmatrix} \hat{g}^b(\lambda) \\ \hat{g}^a(\lambda) \end{pmatrix} \tag{5.38}$$

where

$$\hat{g}^\tau(\lambda) := \int_{\hat{\Omega}} dx \, \left(\vec{g}(x), \vec{\psi}^{-}(x,\lambda,\tau)\right), \quad \tau = a,b. \tag{5.39}$$

The operator $\Phi_- : \mathfrak{K} \longrightarrow \hat{\mathfrak{K}} = L^2(\mathbb{R}, \mathbb{C}^2)$ is unitary and is called the incoming Fourier transformation. The inverse incoming Fourier transformation Φ_-^{-1} is given by

$$\vec{g}(x) = \int_{\mathbb{R}} d\lambda \sum_{\tau=a,b} \vec{\psi}^-(x,\lambda,\tau)\hat{g}^\tau(\lambda), \quad \hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2). \tag{5.40}$$

We note that

$$\Phi_- K \Phi_-^{-1} = M \tag{5.41}$$

where M is the multiplication operator by the independent variable λ on $\hat{\mathfrak{K}}$, i.e.

$$\mathrm{dom}(M) := \left\{\hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) : \lambda\hat{g}(\lambda) \in L^2(\mathbb{R}, \mathbb{C}^2)\right\} \tag{5.42}$$

and

$$(M\hat{g})(\lambda) := \lambda\hat{g}(\lambda), \quad \hat{g} \in \mathrm{dom}(M). \tag{5.43}$$

Using the outgoing eigenfunctions one easily defines an outgoing Fourier transformation $\Phi_+ : \mathfrak{K} \longrightarrow \hat{\mathfrak{K}} = L^2(\mathbb{R}, \mathbb{C}^2)$.

6 Concluding remarks

In a forthcoming paper [19] we intend to use dissipative Schrödinger-type operators as ingredients for Schrödinger-Poisson systems. Such an application in mind let us make some remarks. Firstly, to include electro-magnetic radiation effects in the model one has to extend the considerations to dissipative potentials, i.e. $\Im\mathrm{m}(V(x)) \leq 0$, $x \in [a,b]$. Hence, we have to construct self-adjoint dilations K for Schrödinger-like operators with dissipative boundary conditions and dissipative potentials. This will be done in a subsequent paper. Secondly, we note that the self-adjoint dilation K of H is not semi-bounded from below. This observation has the unpleasant consequence that we can regard the operator K only very conditionally as a physical Hamiltonian since, in general, such Hamiltonians have to be semi-bounded from below. To overcome this disadvantage one can use boundary conditions which depend on the spectral parameter λ. In particular, one can consider instead of (1.2) and (1.3) a family of maximal dissipative operators $\{H(\lambda)\}_{\lambda\in\overline{\mathbb{C}_+}}$ defined by

$$\mathrm{dom}(H(\lambda)) = \left\{ \begin{array}{l} g \in W^{1,2}([a,b]): \frac{1}{m(x)}g'(x) \in W^{1,2}([a,b]), \\ \frac{1}{2m(a)}g'(a) = -\kappa_a(\lambda)g(a), \ \frac{1}{2m(b)}g'(b) = \kappa_b(\lambda)g(b) \end{array} \right\} \tag{6.1}$$

and

$$(H(\lambda)g)(x) = (l(g))(x) \quad g \in \mathrm{dom}(H). \tag{6.2}$$

If $\kappa_a(\lambda)$ and $\kappa_b(\lambda)$, $\lambda \in \mathbb{C}_+$, are suitable Nevanlinna functions, then one can prove that there is a larger Hilbert $\mathfrak{K} \supseteq \mathfrak{H}$ and a self-adjoint operator K defined on $\mathfrak{K}$, such that

$$P^{\mathfrak{K}}_{\mathfrak{H}}(K-\lambda)^{-1}|\mathfrak{H} = (H(\lambda)-\lambda)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{6.3}$$

With respect of (4.1) the operator K can be called the dilation of the family $\{H(\lambda)\}_{\lambda\in\overline{\mathbb{C}_+}}$. In contrast to the present case it is possible that this dilation K is semibounded from below.

Finally, we note that in application to Schrödinger-Poisson systems and to semiconductor physics the necessity arises to extend all considerations to dimensions two and three which generates new difficulties with respect to dilations and eigenfunction expansions.

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