# Plane-like minimizers in periodic media: jet flows and Ginzburg-Landau-type functionals

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**Abstract**

This paper is divided in three parts. In the first part, we consider the functional

$$\mathcal{J}(u) = \int a_{i,j}(x)\,\partial_i u\,\partial_j u \;+\; Q(x)\,\chi_{(-1,1)}(u)\,dx\,,$$

for $a_{i,j}$ and $Q$ periodic under integer translations. We assume that $Q$ is bounded and bounded away from zero, and $a_{i,j}$ is a bounded elliptic matrix. We prove that there exists a universal constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $Q$ stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer $u$ for the functional $\mathcal{J}$ for which the set $\{|u| < 1\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.

Furthermore, such $u$ enjoys the following property of "quasi-periodicity": if $\omega \in \mathbf{Q}^n$, then $u$ is periodic (with respect to the identification induced by $\omega$); if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then $u$ can be approximated uniformly on compact sets by periodic class A minimizers.

The functional $\mathcal{J}$ introduced above has application in models for fluid jets and capillarity.

In the second part of the paper, we extend the previous results to the functional

$$\mathcal{G}(u) = \int a_{i,j}(x)\,\partial_i u\,\partial_j u \;+\; F(x,u)\,dx\,,$$

where $a_{i,j}$ is Lipschitz and satisfies the same assumption mentioned above and $F$ fulfills suitable conditions (roughly, $F$ is a "double-well potential"). We prove that, for any $\theta \in [0,1)$, there exists a constant $M_0$, depending only on $\theta$, $n$, the bounds on $a_{i,j}$ and $F$ stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer $u$ for the functional $\mathcal{G}$ for which the set $\{|u| \le \theta\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$. Also, $u$ enjoys the property of "quasi-periodicity" stated above.

In particular, the results apply to the potentials $F(x) = Q(x)\,(1-u^2)$, and $F = Q(x)\,|1-u^2|^2$, which have application in the Ginzburg-Landau model.

In the third part, we show how the results above contain, as a limit case, the recent Theorem of Caffarelli and de la Llave.

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**PART I — Plane-like minimizers for jets of fluid**

# 1 Introduction

In the first part of the present paper, we consider the functional

$$\mathcal{J}_\Omega(u) = \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j u(x) \;+\; Q(x)\,\chi_{(-1,1)}(u)\,dx\,, \tag{1.1}$$

We assume $a_{i,j}$ to be bounded and periodic under integer translations (i.e. $a_{i,j}(x+e) = a_{i,j}(x)$ for any $e \in \mathbf{Z}^n$). We also assume the matrix $a_{i,j}$ to be symmetric and positive definite: $\lambda\mathbf{1} \le a_{i,j}(x) \le \Lambda\mathbf{1}$, with $0 < \lambda \le \Lambda < +\infty$. We assume that $Q$ is periodic under integer translations, bounded and positive, say $0 < Q_{\min} \le Q \le Q_{\max} < +\infty$. In two dimensions, the functional above is a model for jets of incompressible, irrotational, inviscid fluids: the interested reader may see, for instance, [BZ] and [Se] for a classical introduction, and also [ACF1] and [ACF3] for recent developments. In particular, the first term in the functional (as shown, for instance, in Lemma 2.4 below) takes into account the Equation of continuity, while the second term (as shown in [AC]) expresses Bernoulli's law on the free surface. The function $u$ has the physical meaning of a stream function. The particular case $a_{i,j} = \delta_{i,j}$ corresponds to a homogeneous medium. The functional in (1.1) was also dealt with in the theory of minimal surfaces: see, for instance, [Md] and [CC].

The main result of the first part of the present paper is the following:

**Theorem 1.1.** *There exists a universal constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $Q$ stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer $u$ for the functional $\mathcal{J}$ for which the set $\{|u| < 1\}$ is contained in the strip $\{x \text{ s.t. } x\cdot\omega \in [0, M_0|\omega|]\}$.*

*Furthermore, such $u$ enjoys the following property of "quasi-periodicity":*

- *if $\omega \in \mathbf{Q}^n$, then $u$ is periodic (with respect to the identification induced by $\omega$);*
- *if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then $u$ can be approximated uniformly on compact sets by periodic class A minimizers.*

We say that a minimizer is "class A" minimizer if the functional does not increase under any compact (but not necessarily small) modification (see Definition 7.5 below).

The proof of Theorem 1.1 will make use of the results of *existence and rigidity* of minimizers of [AC], the *regularity result* of [GG], several *density estimates* of [CC] and a *geometric construction* of [CL].

In the second part of this paper, we will extend the results of Theorem 1.1 to more general Ginzburg-Landau-type functionals (see below Section 8).

We remark that the results in the second part of the paper also contain the ones of the first part as a particular case. However, we preferred to treat the case of the "fluid jets" separately both because of its physical relevance and because some of the arguments are easier or more explicit in this case. Moreover, the main result of the first part (Theorem 1.1) is slightly stronger than the one proved in the second part (Theorem 8.1).

The detailed study of the functional $\mathcal{J}$ is also going to show some of the properties of the fluid jets interesting in themselves, as the growth from the free boundary and the regularity issues.

The scheme of the first part of the paper is the following: first of all, we consider the "compact" case in which $\omega \in \mathbf{Q}^n$. In Section 2, we recall some results of existence and regularity of constrained minimizers. In particular, minimizers are Hölder continuous, with a uniform control on the Hölder constant. We also point out that minimizers are indeed Lipschitz.

Also, minimizers satisfy the following rigidity property: if a minimizer takes value $\theta \in (-1, 1)$ at some point $x$, it must take both the values 1 and $-1$ at some point $y$, whose distance from $x$ can be uniformly bounded.

In Section 3, we recall and derive some density estimates, stating, roughly, that, in some sense, both the sets where a minimizer attains value 1 and where it attains value $-1$ have "full dimension", while the set where the minimizer is different from $\pm 1$ behaves like a set "of codimension one".

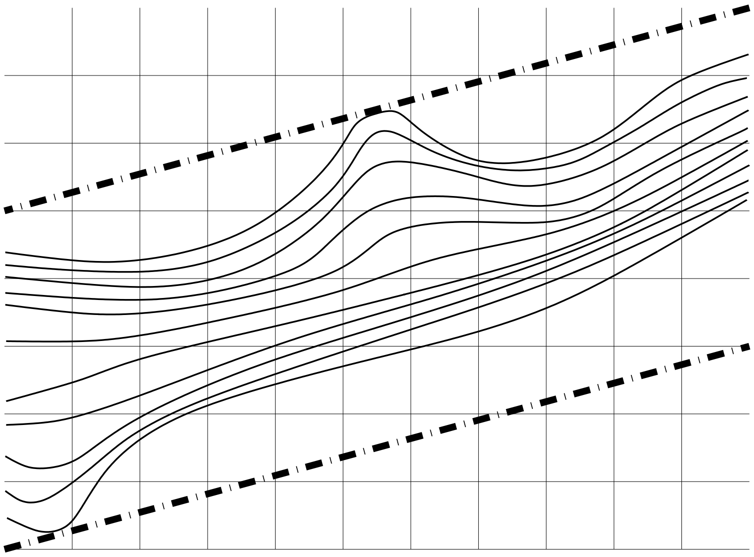
In Section 4, we discuss the notion of "minimal minimizer". In Section 5, we prove that the minimal minimizer satisfies the so called Birkhoff property. Roughly, the Birkhoff property states that the set where the minimal minimizer is in absolute value less than one cannot cross its integer translates. In Section 6, we state and proof an "abstract" Geometric Lemma. The density property will allow us to fit this Lemma into our purposes. In Section 7 we prove that, for rational frequencies, the minimal minimizer of strips wide enough is indeed unconstrained and hence class A. Finally, we extend the result to irrational frequencies.

Before starting with the technical statements and proofs, we would like to discuss a heuristic (absolutely non-rigorous!) "proof" of the existence of plane-like minimizers. For simplicity, we will consider now a two-dimensional case; given any vector $v \in \mathbf{R}^2$, let us construct two "walls" in direction $v$ and at distance $M$ from each other. Let us consider a fluid flowing through the "pipe" given by these two walls (see the picture below). This fluid can be thought as being "constrained" by the walls.

In some part of the space, the fluid is probably not touching the constraining walls, while, somewhere else, it might stick to them.

Let us now consider bigger and bigger $M$'s, i.e. let us enlarge the pipe. Our "hope" is that, for a suitable (universal!) $M$, the fluid will not stick to the walls anymore. At this point, the fluid is actually "unconstrained": we can remove the pipe and obtain a free "plane-like" fluid jet.

Obviously, some work needs to be done to make this argument rigorous, especially to show that $M$ is indeed a uniform quantity, independent on the given direction $v$. This will be precisely the target of the following pages.



*A jet of fluid "constrained between two walls"*

## 2 Existence and regularity of minimizers constrained in a strip

In this section, we recall some known properties of existence and regularity of a minimizer for the functional $\mathcal{J}$ when we prescribe boundary values for $u$ in a strip. The results of this section are an offspring of [AC] and [GG]. In the following, for a vector $\omega \in \mathbf{R}^n$ and $M \in \mathbf{R}$, we denote the strip of size $M$ by

$$S_M^\omega := \{x \in \mathbf{R}^n \text{ s.t. } x \cdot \omega \in [0, M]\}.$$

When $\omega \in \mathbf{Q}^n$, we can recover compactness by an identification of the space. Namely, we consider the following relation $\sim$ induced by $\omega$:

$$x \sim y \qquad \text{iff} \qquad x - y = k \in \mathbf{Z}^n \quad \text{with} \quad \omega \cdot k = 0. \tag{2.1}$$

We will denote by $\tilde{S}_M^\omega = S_M^\omega / \sim$ the quotient space induced by such relation (which is topologically equivalent to the product of an $(n-1)$–dimensional torus and a real interval).

We also set

$$\mathcal{X}_M^\omega := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}_M^\omega) \text{ s.t. } \nabla u \in \mathrm{L}^2(\tilde{S}_M^\omega)$$
$$\text{with } u(x) = 1 \text{ for any } x \text{ s.t. } \omega \cdot x = 0 \text{ and}$$
$$u(x) = -1 \text{ for any } x \text{ s.t. } \omega \cdot x = M\}\,.$$

We also denote by $\mathcal{L}^n$ the $n-$dimensional Lebesgue measure. In the sequel, we will assume $M \geq 10|\omega|$, in order to avoid a degeneracy caused by "too thin" strips.

We start by discussing the existence of such constrained minimizers:

**Theorem 2.1 (Alt and Caffarelli).** *For any $\omega \in \mathbf{Q}^n$ and for any $M > 0$, there exist an absolute minimum of $\mathcal{J}_{S_M^\omega}$ in the space $\mathcal{X}_M^\omega$. Such minimizer is periodic (with respect to the identification $\sim$).*

*Proof.* The argument of the Existence Theorem 1.3 of [AC] goes through with obvious minor modifications. The assumption $\omega \in \mathbf{Q}^n$ has been made in order to restrict ourselves to a compact domain, and have that $\mathcal{J}$ is finite when evaluated, for instance, on the admissible linear function. $\square$

With a little abuse of notation, we sometimes identify the functions in $\mathcal{X}_M^\omega$ with their constant extension outside the strip $S_M^\omega$. We also denote by $\mathcal{M}_M^\omega$ the set of the minima found by Theorem 2.1.

Comparing $\mathcal{J}_{S_M^\omega}(u)$, $\mathcal{J}_{S_M^\omega}(\min\{u,1\})$ and $\mathcal{J}_{S_M^\omega}(\max\{u,-1\})$, the reader may easily convince herself that we can assume the minimizers in $\mathcal{M}_M^\omega$ to take value in $[-1,1]$.

From now on, in order to apply Theorem 2.1, we will restrict our attention to rational frequencies, i.e., if not differently stated, we will assume $\omega \in \mathbf{Q}^n$. This restriction will be dropped at the end of Section 7, by an approximation argument.

It is a standard consequence of Poincaré inequality that, indeed, $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^n)$ (see, for instance [S]). Further regularity is addressed by the following

**Theorem 2.2 (Giaquinta and Giusti).** *If $u \in \mathcal{M}_M^\omega$, then $u$ is $\mathrm{C}^\alpha$ for any $\alpha \in [0,1)$. Furthermore, given any $K \subset\subset S_M^\omega$, the $\mathrm{C}^\alpha$ norm of $u$ in $K$ is bounded by a constant depending only on $\alpha$, $n$, the bounds on $a_{i,j}$ and $Q$ and on $\mathrm{dist}\,(K, \partial S_M^\omega)$.*

Theorem 2.2 follows, for instance, from [GG] or Theorem 7.6 of [G2]. The interested reader may also see Chapter 5 of [G] and, for a different approach, [DEF].

We now point out that, if the coefficients are Lipschitz, the minimizer is actually Lipschitz.

**Theorem 2.3.** *Assume that the coefficients $a_{i,j}$ are Lipschitz. If $u \in \mathcal{M}_M^\omega$, then $u$ is Lipschitz. Furthermore, given any $K \subset\subset S_M^\omega$, the Lipschitz norm of $u$ in $K$ is bounded by a constant depending only on $n$, the bounds on $a_{i,j}$ and $Q$ and on $\mathrm{dist}\,(K, \partial S_M^\omega)$.*

Theorem 2.3 can be seen as an extension of the results by Alt, Caffarelli and Friedman to the functional in (1.1) under a more general assumption on the coefficients. The proof of Theorem 2.3 is deferred to the Appendix, for the reader's convenience. We would like to remark that Theorem 2.2 is valid even if the coefficients $a_{i,j}$ are only bounded, while for the proof of Theorem 2.3 we need the coefficients to be Lipschitz.

The proof of Theorem 1.1 will not use explicitly Theorem 2.3, but only Theorem 2.2. In particular, Theorem 1.1 does not require any regularity assumption on the coefficients.

We now point out the PDE-properties of the minimizers:

**Lemma 2.4.** *Let $u \in \mathcal{M}^{\omega}_{M}$. Then: $\partial_i(a_{i,j}\partial_j u) = 0$ weakly in $\{|u| < 1\}$, $\partial_i(a_{i,j}\partial_j u) \geq 0$ weakly in $\{u < 1\}$, and $\partial_i(a_{i,j}\partial_j u) \leq 0$ weakly in $\{u > -1\}$.*

The proof of this Lemma is elementary.

# 3 Density estimates

We will now recall some *density estimates* on the level sets of the minimizers. Similar density estimates play a crucial role in several results related to ours: see, for instance [CC] and [CL]. Here, the density estimates will be the tool to apply the Geometric Lemma we will present in Section 6. Several results of this section will be based on [CC].

**Theorem 3.1.** *Assume $u \in \mathcal{M}^{\omega}_{M}$. Then:*

*(i) there exist positive constants $c$, $r_0$, $r_1$ (depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$) such that*

$$\mathcal{J}_{B_r(x)}(u) \leq c\, r^{n-1}\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$;*

*(ii) for any $\theta_0 \in [0,1)$, for any $\theta \in [-\theta_0, \theta_0]$ and for any $\mu_0 > 0$, if $\mathcal{L}^n(B_K(x) \cap \{u > \theta\}) \geq \mu_0$, there exist positive constants $c^\star$, $r_0$, $r_1$ (depending on $n$, $K$, $\mu_0$, $\theta_0$ and the bounds on $a_{i,j}$ and $Q$) such that*

$$\mathcal{L}^n\Big(\{u > \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$. Analogously, if $\mathcal{L}^n(B_K(x) \cap \{u < \theta\}) \geq \mu_0$,*

$$\mathcal{L}^n\Big(\{u < \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$.*

We include the proof of this result in the Appendix (see in particular page 35): our proof is a variation of the one in [CC], where the result was proved in the case $a_{i,j} = \delta_{i,j}$ (we indeed simplify a bit the proof and correct some flawns). From $(i)$ of Theorem 3.1, the following bound on the "width" of $\{|u| < 1\}$ immediately follows:

**Lemma 3.2.** *Let $u \in \mathcal{M}^{\omega}_{M}$. Assume that $u(\bar{x}) \in (-1,1)$. Then, there exist $r_0$ and $r_1$ (depending only on $n$, the bounds on $a_{i,j}$ and $Q$) such that $u$ takes either the value $1$ or $-1$ in $B_{r_0}(\bar{x})$, provided that $B_{r_0+r_1}(\bar{x}) \subset S^{\omega}_{M}$.*

We would like now to deduce some other density estimates, that will be needed in the application of the Geometric Lemma in Section 6. To do this, we need some better control on the "width" of $\{|u| < 1\}$ than the one obtained in Lemma 3.2: indeed, in Lemma 3.6 we will show that *both* the values 1 and $-1$ are attained in a universal neighborhood of any point in $\{|u| < 1\}$. This result will be a consequence of some properties of "rigidity" of minimizers, related to the "linear growth from the free boundary" proved by Alt, Caffarelli and Friedman. The following results will lead us to this target.

In order to compare the functional in (1.1) with the one dealt with, for instance, in [AC], we observe the following:

**Lemma 3.3.** *Let $u \in \mathcal{M}^\omega_M$ and let $\Omega$ be an open connected set contained in $S^\omega_M$, with a Lipschitz boundary. If $u$ does not take the value 1 in $\Omega$, then $u$ minimizes*

$$\int_\Omega a_{i,j}\, \partial_i v\, \partial_j v \, + \, Q(x)\chi_{\{v>-1\}}\,,$$

*among all $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in \mathrm{L}^2(\Omega)$ and $v = u$ on $\partial\Omega$.*

*Analogously, if $u$ does not take the value $-1$ in $\Omega$, then $u$ minimizes*

$$\int_\Omega a_{i,j}\, \partial_i v\, \partial_j v \, + \, Q(x)\chi_{\{v<1\}}\,,$$

*among all $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in \mathrm{L}^2(\Omega)$ and $v = u$ on $\partial\Omega$.*

The proof is straightforward.

**Lemma 3.4.** *Let $\Omega$ be a Lipschitz domain. Let $u$ be a minimizer of*

$$\tilde{\mathcal{J}}_\Omega(u) := \int_\Omega a_{i,j}\partial_i u \partial_j u + Q(x)\chi_{\{u>0\}}\,, \tag{3.1}$$

*among all the functions $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that $\nabla v \in \mathrm{L}^2(\Omega)$ so that $v = u$ on $\partial\Omega$. Assume that $\tilde{\mathcal{J}}_\Omega(u) < \infty$ and that $a_{i,j}$ and $Q$ satisfy the hypotheses stated at page 2.*

*Then, for any $x \in \{u > 0\}$,*

$$\mathrm{dist}(x, \{u = 0\}) \le C u(x)\,,$$

*provided that $B_{\mathrm{dist}(x,\{u=0\})}(x) \subseteq \Omega$, where $C$ is a positive constant, depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$.*

*Proof.* By scaling, we may assume $x = 0$ and $\mathrm{dist}\,(x, \{u = 0\}) = 1$. Set $\varepsilon := u(0)$. Since $B_1 \subseteq \{u > 0\}$, the first variation of the functional shows that $\partial_i(a_{i,j}\partial_j u) = 0$ on $B_1$. Thus, by Harnack inequality,

$$\sup_{B_{1/2}} u \le c^*\varepsilon\,,$$

for a universal constant $c^*$.

Let $\varphi$ be the radially symmetric function defined as follows: $\varphi = 0$ in $B_{1/4}$, $\varphi = 2c^*\varepsilon$ on $\partial B_{1/2}$ and growing linearly on the rays of $B_{1/2} - B_{1/4}$. Set $v := \min\{u, \varphi\}$. Then $v$ is an admissible function, so

$$\begin{aligned}\int_{B_{1/2}} a_{i,j}\partial_i u \partial_j u + Q\chi_{\{u>0\}} &\le \int_{B_{1/2}} a_{i,j}\partial_i v \partial_j v + Q\chi_{\{v>0\}} \le \\ &\le \int_{B_{1/2}} a_{i,j}\partial_i u \partial_j u + \int_{B_{1/2}} a_{i,j}\partial_i \varphi \partial_j \varphi + \int_{B_{1/2}-B_{1/4}} Q\chi_{\{v>0\}} .\end{aligned}$$

Since $|\nabla\varphi| \le \operatorname{const}\varepsilon$ and $u > 0$ on $B_{1/4}$, the previous inequality implies

$$\varepsilon^2 \ge \int_{B_{1/2}} Q\chi_{\{u>0\}} - \int_{B_{1/2}-B_{1/4}} Q\chi_{\{v>0\}} \ge Q_{\min}|B_{1/4}| .$$

$\square$

**Lemma 3.5.** *Let $\Omega$, $u$ and $\tilde{\mathcal{J}}$ be as in Lemma 3.4. Let $x \in \partial\{u > 0\}$. Then, for any $r \ge 0$,*

$$\sup_{B_r(x)} u \ge Cr ,$$

*where $C$ is a positive constant (depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$), provided $B_r(x) \subseteq \Omega$.*

*Proof.* By scaling, we may assume $x = 0$ and $r = 1$. Since $0 \in \partial\{u > 0\}$, for any $\varepsilon > 0$ there exists a point $x_1 \in B_\varepsilon$, so that $u(x_1) > 0$. We now construct a polygonal, starting as close as we wish to zero, along which $u$ grows up at least linearly. Set

$$d_1 := \operatorname{dist}(x_1, \partial\{u > 0\}) \le |x_1| \le \varepsilon .$$

By Lemma 3.4,

$$u(x_1) \ge c^* d_1 , \tag{3.2}$$

for a suitable constant $c^*$. Let $z_1 \in \partial B_{d_1}(x_1) \cap \{u = 0\}$. Define

$$\rho_1 := \sup\{\rho \text{ s.t. } B_\rho(z_1) \subseteq B_1 \text{ and } \sup_{B_\rho(z_1)} u \ge c^*\rho\} .$$

By (3.2), $\rho_1 \ge d_1$. Also, if $B_{\rho_1}(z_1)$ touches $\partial B_1$, then

$$\rho_1 = 1 - |z_1| \ge 1 - d_1 - \varepsilon \ge 1 - 2\varepsilon \ge \frac{1}{2} ,$$

if $\varepsilon \le 1/4$, so

$$\sup_{B_1} u \ge \sup_{B_{\rho_1}(z_1)} u \ge \frac{c^*}{2} ,$$

and the claim would be proven. Thus, we may restrict ourselves to the case in which

$$\sup_{B_{\rho_1}(z_1)} u = c^*\rho_1 . \tag{3.3}$$

Take $\tilde{x}_1 \in \bar{B}_{\rho_1}(z_1)$ attaining the sup in (3.3). In fact, by Maximum Principle, $\tilde{x}_1 \in \partial B_{\rho_1}(z_1)$. We want now to show that there exist some universal $\delta > 0$ such that

$$\sup_{B_{2\rho_1}(z_1)} u \geq (1+\delta)\, u(\tilde{x}_1)\,, \tag{3.4}$$

provided $B_{2\rho_1}(z_1) \subseteq B_1$. To prove this, assume

$$\sup_{B_{2\rho_1}(z_1)} u < (1+\delta)\, u(\tilde{x}_1)$$

and define

$$v(x) := \frac{1}{\rho_1} u(z_1 + \rho_1 x)\,, \qquad \forall x \in B_2\,.$$

Then

$$||v||_{\mathrm{L}^\infty(B_2)} \leq \frac{(1+\delta)\, u(\tilde{x}_1)}{\rho_1} = (1+\delta)c^*\,, \tag{3.5}$$

having used (3.3). Assuming $\delta \leq 1$, we get a uniform bound on $||v||_{\mathrm{L}^\infty(B_2)}$. So, by the Theorem of Giaquinta and Giusti (see [GG] or [G2]), $v \in \mathrm{C}^\alpha(B_{3/2})$ and

$$||v||_{\mathrm{C}^\alpha(B_{3/2})} \leq \tilde{c}\,, \tag{3.6}$$

for a universal $\tilde{c}$. Set

$$w := \frac{2v}{c^*(1+\delta)} - 1\,,$$

so that $\partial_i(a_{i,j}\partial_j w) \geq 0$ in $B_{3/2}$ and $w \leq 1$ because of (3.5). Also, by (3.6) and the fact that $w(0) = -1$, we deduce that $\mathcal{L}^n(\{w \leq 0\} \cap B_{3/2}) \geq \text{const}$. Then, by De Giorgi Oscillation Lemma (see, for instance the Appendix of [C] or Theorem 4.9 in [HL] and references therein), we infer $\sup_{B_1} w \leq \lambda$, for a universal $\lambda \in (0,1)$. In particular, since $\tilde{x}_1 \in \bar{B}_{\rho_1}(z_1)$,

$$\lambda \geq w\left(\frac{\tilde{x}_1 - z_1}{\rho_1}\right) = \frac{2}{1+\delta} - 1\,,$$

which is a contradiction if $\delta$ is small enough. This proves (3.4). By construction, $u(\tilde{x}_1) \geq u(x_1)$. Hence, there exists a point $x_2 \in \bar{B}_{2\rho_1}(z_1)$ (and in fact $x_2 \in \partial B_{2\rho_1}(z_1)$ by Maximum Principle) such that

$$u(x_2) \geq u(\tilde{x}_1) + \delta c^* \rho_1 \geq u(x_1) + \theta|x_1 - x_2|\,,$$

for universal $\delta$, $\theta > 0$, provided that $B_{3\rho_1}(x_1) \subseteq B_1$.

Iterating, we obtain a polygonal of vertexes $x_1, x_2, \ldots, x_k$ such that: for $2 \leq j \leq k$, $x_j \in \partial B_{3\rho_{j-1}}(x_{j-1})$, $|x_j - x_{j-1}| \geq \rho_j$ and

$$\begin{aligned} u(x_j) &\geq (1+\delta)u(x_{j-1}) \geq \qquad (3.7)\\ &\geq u(x_{j-1}) + \theta|x_j - x_{j-1}|\,, \qquad (3.8) \end{aligned}$$

provided $B_{3\rho_{j-1}}(x_{j-1}) \subseteq B_1$. In particular, iterating (3.8),

$$u(x_j) \geq \theta \ell_j$$

where $\ell_j$ is the length of the polygonal joining $x_1, \ldots, x_j$. We want to prove that we can take $j$ big enough, and make $\ell_j$ bigger than some universal quantity. This would finish the proof of the lemma.

Indeed, we must be able to iterate the scheme and find a suitable $j^*$ such that $|x_{j^*}| \geq 1/20$. Otherwise, if $|x_j| < 1/20$ for any $j$, then $\rho_j \leq |x_j - x_{j-1}| < 1/10$ for any $j$; and so $B_{3\rho_j}(x_j) \subset B_1$ for any $j$, and iterating (3.7) we would get $\sup_{B_1} u = +\infty$.

Therefore, $\ell_{j^*} \geq |x_{j^*}| - |x_1| \geq \frac{1}{20} - \varepsilon$. □

We can now sharpen the result of Lemma 3.2:

**Lemma 3.6.** *Let $u \in \mathcal{M}_M^\omega$. Assume that $u(\bar{x}) \in (-1, 1)$. Then, there exist $r_0$ and $r_1$ (depending only on $n$, the bounds on $a_{i,j}$ and $Q$) such that $u$ takes both the values $1$ and $-1$ in $B_{r_0}(\bar{x})$, provided that $B_{r_0+r_1}(\bar{x}) \subset S_M^\omega$.*

*Proof.* By Lemma 3.2, $u$ takes, say, the value $-1$ in $B_{r_0}(\bar{x})$. Without loss of generality, we assume $r_0 \geq 3/C$, where $C$ is the constant in Lemma 3.5. If $u$ did not assume the value 1 too, using Lemmas 3.3 and 3.5, we get

$$2 \geq \sup_{B_{r_0}(x_0)} u + 1 \geq C r_0 \geq 3\,,$$

which is a contradiction. □

We are now in position to improve the previous density estimates:

**Lemma 3.7.** *Let $u \in \mathcal{M}_M^\omega$. Assume that $x \in \{|u| < 1\}$. Then, there exist positive constants $r_0$, $r_1$ and $c$, depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$, such that*

$$\mathcal{J}_{B_r(x)}(u) \geq c r^{n-1}\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S_M^\omega$.*

*Proof.* By Lemma 3.6, Theorem 2.2 and Theorem 3.1

$$\mathcal{L}^n\Big(B_r(x) \cap \{u > 9/10\}\Big) \geq \text{const}\, r^n$$

and

$$\mathcal{L}^n\Big(B_r(x) \cap \{u < -9/10\}\Big) \geq \text{const}\, r^n\,.$$

Let $a$ be the average of $u$ in $B_r(x)$. If $a \leq 0$,

$$\begin{aligned} \int_{B_r(x)} |u - a| \;&\geq\; \int_{B_r(x)\cap\{u>9/10\}} (u - a) \geq \frac{9}{10}\mathcal{L}^n\Big(B_r(x) \cap \{u > 9/10\}\Big) \geq \\ &\geq\; \text{const}\, r^n\,. \end{aligned}$$

Analogously, if $a \geq 0$,

$$\int_{B_r(x)} |u - a| \geq \frac{9}{10}\mathcal{L}^n\Big(B_r(x) \cap \{u < -9/10\}\Big) \geq \text{const}\, r^n\,.$$

So, in any case, using Poincaré and Cauchy Inequalities:

$$\begin{aligned}
\mathrm{const}\, r^{n-1} &\leq \frac{1}{r}\int_{B_r(x)} |u-a| \leq \mathrm{const}\int_{B_r(x)} |\nabla u| \leq \\
&\leq \mathrm{const}\sqrt{\int_{B_r(x)} |\nabla u|^2}\sqrt{\mathcal{L}^n\Big(B_r(x)\cap\{|u|<1\}\Big)} \leq \\
&\leq \mathrm{const}\Big(\int_{B_r(x)} a_{i,j}\partial_i u\partial_j u + Q\chi_{(-1,1)}(u)\Big)\,.
\end{aligned}$$

□

Let us summarize some of the density estimates obtained in this section:

**Proposition 3.8.** *Let $u \in \mathcal{M}^{\omega}_{M}$. Assume that $x \in \{|u| < 1\}$. Then, there exist positive constants $r_0$, $r_1$, $c$ and $C$, depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$, such that:*

$$\mathcal{J}_{B_r(x)}(u) \geq cr^{n-1}\,,$$

$$\mathcal{L}^n\Big(B_r(x)\cap\{u>-1\}\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(B_r(x)\cap\{u<1\}\Big) \geq cr^n\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$. Moreover, for any $y \in S^{\omega}_{M}$,*

$$\mathcal{J}_{B_r(y)} \leq Cr^{n-1}\,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(y) \subset S^{\omega}_{M}$.*

# 4 The minimal minimizer

Following the ideas developed in [CL], we now turn our attention to a special minimizer of $\mathcal{J}$, namely the *minimal minimizer*. The minimal minimizer is defined by

$$u^{\omega}_{M}(x) := \inf_{u\in\mathcal{M}^{\omega}_{M}} u(x)\,.$$

By definition, for any $u \in \mathcal{M}^{\omega}_{M}$ and $\theta \in \mathbf{R}$,

$$\{u^{\omega}_{M} > \theta\} \subseteq \{u > \theta\}\,. \tag{4.1}$$

We now want to prove that $u^{\omega}_{M}$ is still a minimizer: this will be accomplished in the following Proposition 4.3. For this, we need some easy auxiliary observation.

**Lemma 4.1.** *For any $u$, $v \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}^{\omega}_{M})$ with $\nabla u$, $\nabla v \in \mathrm{L}^2(\tilde{S}^{\omega}_{M})$,*

$$\mathcal{J}_{S^{\omega}_{M}}(u) + \mathcal{J}_{S^{\omega}_{M}}(v) \geq \mathcal{J}_{S^{\omega}_{M}}(\min\{u,v\}) + \mathcal{J}_{S^{\omega}_{M}}(\max\{u,v\})\,.$$

The proof is a straightforward computation.

**Corollary 4.2.** *If $u$, $v \in \mathcal{M}^{\omega}_{M}$, then $\min\{u,v\}$, $\max\{u,v\} \in \mathcal{M}^{\omega}_{M}$.*

*Proof.* Since $\mathcal{J}_{S^{\omega}_{M}}(u) \leq \mathcal{J}_{S^{\omega}_{M}}(\min\{u,v\})$ and $\mathcal{J}_{S^{\omega}_{M}}(v) \leq \mathcal{J}_{S^{\omega}_{M}}(\max\{u,v\})$, by Lemma 4.1 we deduce $\mathcal{J}_{S^{\omega}_{M}}(u) = \mathcal{J}_{S^{\omega}_{M}}(\min\{u,v\})$ and $\mathcal{J}_{S^{\omega}_{M}}(v) = \mathcal{J}_{S^{\omega}_{M}}(\max\{u,v\})$. □

**Proposition 4.3.** $u^{\omega}_{M} \in \mathcal{M}^{\omega}_{M}$.

The proof of Proposition 4.3 is now a standard application of the Theorem of Ascoli and it is omitted.

## 5 Birkhoff property for the minimal minimizer

In this section, we prove a geometric property about the global behavior of the minimal minimizer. This property can be seen as a generalization of the so called Birkhoff property in Aubry-Mather theory (see [M], [Ma] and [MF]). A similar property has been pointed out in Proposition 8.1 of [CL] in the minimal surface setting.

If $k \in \mathbf{R}^n$, we denote by $\mathcal{T}_k$ the translation by the vector $k$.

**Definition 5.1.** *Let $E \subseteq \mathbf{R}^n$, $\varpi \in \mathbf{R}^n$. We say that $E$ satisfies the Birkhoff property with respect to the vector $\varpi$ if:*

- *for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \leq 0$, $\mathcal{T}_k E \subseteq E$;*
- *for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \geq 0$, $\mathcal{T}_k E \supseteq E$.*

**Proposition 5.2.** *Let $E \subseteq \mathbf{R}^n$. Assume that $E$ satisfies the Birkhoff property with respect to $\omega$. Then, there exists a constant $\varrho$, depending only on $n$, such that: if $\mathcal{C}(E)$ contains a ball of radius $\varrho$, then $\mathcal{C}(E)$ contains a strip of width $1$ that intersects the ball.*

The proof is elementary.

**Proposition 5.3.** *Let $\theta \in \mathbf{R}$. Then, the set $\{u^\omega_M > \theta\}$ satisfies the Birkhoff property with respect to $\omega$. Namely:*

- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \leq 0$, then $\{u^\omega_M > \theta\} \supseteq \{\mathcal{T}_k u^\omega_M > \theta\}$;*
- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \geq 0$, then $\{u^\omega_M > \theta\} \subseteq \{\mathcal{T}_k u^\omega_M > \theta\}$.*

*Proof.* Set $v := \min\{u^\omega_M, \mathcal{T}_k u^\omega_M\}$. If $k \cdot \omega \leq 0$, by Corollary 4.2, $v$ is a minimizer in the strip $\mathcal{T}_k S^\omega_M$. By construction, $\mathcal{T}_k u^\omega_M$ is the minimal minimizer in $\mathcal{T}_k S^\omega_M$. Thus, from (4.1),

$$\{\mathcal{T}_k u^\omega_M > \theta\} \subseteq \{v > \theta\} \subseteq \{u^\omega_M > \theta\}.$$

Analogously, if $k \cdot \omega \geq 0$, $v$ is a minimizer in $S^\omega_M$, so

$$\{u^\omega_M > \theta\} \subseteq \{v > \theta\} \subseteq \{\mathcal{T}_k u^\omega_M\}.$$

□

## 6 The Geometric Lemma

This section is devoted to the statement and the proof of a Geometric Lemma. The application of this "abstract" Lemma to our case will be given in Proposition 6.3.

**Lemma 6.1 (Geometric Lemma).** *Let $\beta \subset E \subseteq \mathbf{R}^n$, $\mu$ and $\nu$ be Radon measures on $\mathbf{R}^n$. Take $x_0 \in \beta$. Fix $r > 0$. Assume that there exist $\delta' \in (0, 1/10]$ such that:*

*(i)* $\forall x \in B_r(x_0)$ *and* $\forall \rho \in [\delta' r, r/10]$,

$$\mu(B_\rho(x)) \leq \mathrm{const}\, \rho^n\,;$$

*(ii)* $\forall x \in \beta \cap B_r(x_0)$ *and* $\forall \rho \in [\delta' r, r/10]$

$$\nu(B_\rho(x)) \geq \text{const}\,\rho^{n-1}\,;$$

*(iii)* $\mu(E \cap B_r(x_0)) \geq \text{const}\,r^n$*;*

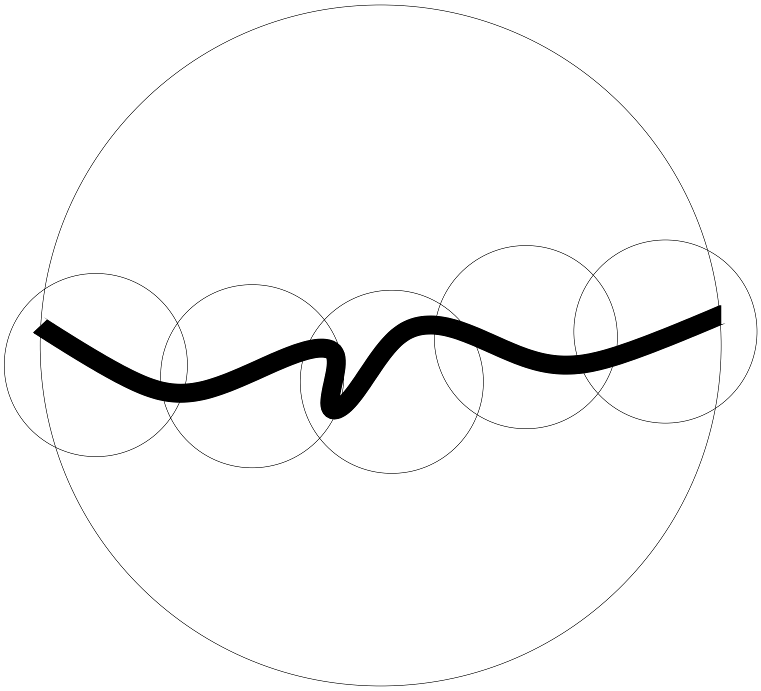
*(iv)* $\nu(B_{2r}(x_0)) \leq \text{const}\,r^{n-1}$.

*Then, there exists $\delta$, depending only on $n$ and on the constants involved in $(i)-(iv)$ above, so that, if $\delta' \leq \delta$, we can find $\bar{x} \in B_r(x_0)$ such that*

$$B_{\delta r}(\bar{x}) \subset (E - \beta) \cap B_{2r}(x_0)\,. \tag{6.1}$$

A similar Lemma can be stated when assumption $(ii)$ above holds when $\beta$ is a dense subset of $\partial E$ (for instance, the"reduced boundary", as in the case of the minimal surfaces: see [CL] and, for instance, [G1] or [MM] for the theory of minimal surfaces). In the case studied in [CL], $\mu$ is the Lebesgue measure and $\nu$ is the Haussdorff measure on the reduced boundary. In our application, $\mu$ will be the Lebesgue measure and $\nu(B) := \mathcal{J}_B(u^\omega_M)$. Namely, we will apply the Geometric Lemma to the sets $E := \{u > -1\}$, or $E := \{u < 1\}$, and $\beta := \{|u| < 1\}$. The hypotheses of the Geometric Lemma will be satisfied using the density estimates of Section 3.

The rough idea of the proof of the Geometric Lemma is the following. We first cover $\beta$ with balls of radius $\delta r$: since $\beta$ behaves like an $(n-1)-$dimensional set, we need $O(1/\delta^{n-1})$ of such balls to cover $\beta$. We then extend this cover to the whole set $E$: since $E$ behaves like an $n-$dimensional set, we need $O(1/\delta^n)$ of such balls to cover $E$. Therefore, if $\delta$ is small enough, there is at least one ball in the cover of $E$ that does not touch $\beta$.



In order to make the argument above rigorous, we need the following auxiliary covering argument:

**Lemma 6.2.** *Assume the hypotheses (ii) and (iv) of Lemma 6.1. Fix* $\delta \in [5\delta', 1/2]$. *Then, there exists a family of points* $y_k \in \beta$, $k = 1, \dots, N$, *such that:*

- $B_{\delta r/5}(y_k) \cap B_{\delta r/5}(y_h) = \emptyset$ *if* $k \neq h$;
- $\beta \cap B_r(x_0) \subseteq \bigcup_{k=1}^N B_{\delta r}(y_k)$;
- $N \leq \operatorname{const} \delta^{-(n-1)}$.

*Proof.* Take any $y_1 \in \beta \cap B_r(x_0)$. If $\beta \cap B_r(x_0) \subseteq B_{\delta r}(y_1)$, we can stop. Otherwise, take $y_2 \in \beta \cap B_r(x_0)$ not belonging to $B_{2\delta r/5}(y_1)$. Iterating this procedure, assume to have a family of $y_k$'s such that $y_k \in \beta \cap B_r(x_0)$ for $1 \leq k \leq N$, verifying $B_{\delta r/5}(y_k) \cap B_{\delta r/5}(y_h) = \emptyset$ if $k \neq h$. If $\beta \cap B_r(x_0) \subseteq \bigcup_{k=1}^N B_{\delta r}(y_k)$, we stop; otherwise select $y_{N+1} \in \beta \cap B_r(x_0)$ outside $\bigcup_{k=1}^N B_{2\delta r/5}(y_k)$.

We now estimate the "good" $N$ for which we stop the procedure: in light of hypothesis (*ii*) of Lemma 6.1,

$$\nu\Big(\bigcup_{k=1}^N B_{\delta r/5}(y_k)\Big) = \sum_{k=1}^N \nu\Big(B_{\delta r/5}(y_k)\Big) \geq \operatorname{const} N \, (\delta r)^{n-1} \, .$$

Also, by construction, $B_{\delta r/5}(y_k) \subseteq B_{2r}(x_0)$. Hence, hypothesis (*iv*) of Lemma 6.1 implies that

$$\nu\Big(\bigcup_{k=1}^N B_{\delta r/5}(y_k)\Big) \leq \operatorname{const} r^{n-1} \, .$$

These two inequalities show that $N \leq \operatorname{const} \delta^{1-n}$. □

We now address the proof of Lemma 6.1:

*Proof.* Consider a finite overlapping cover $\mathcal{B}$ of $B_r(x_0)$, consisting in balls radius $\delta r$, with $\#\mathcal{B} \leq \operatorname{const} \delta^{-n}$.

Since $\mathcal{B}$ is finite overlapping, any of the balls $\Big\{B_{\delta r}(y_k)\Big\}_{1 \leq k \leq N}$, constructed in Lemma 6.2 cannot intersect more than $\tilde{C}$ balls belonging to $\mathcal{B}$, where $\tilde{C}$ is a universal constant.

Then, out of the collection $\mathcal{B}$, only $\operatorname{const} \delta^{-(n-1)}$ (at most) may intersect $\beta \cap B_r(x_0)$, since

$$\begin{aligned} & \#\{B \in \mathcal{B} \text{ s.t. } B \cap \beta \cap B_r(x_0) \neq \emptyset\} \leq \\ \leq \; & \#\{B \in \mathcal{B} \text{ s.t. } B \cap B_{\delta r}(y_k) \neq \emptyset \text{ for some } k\} \leq \tilde{C} N \leq \operatorname{const} \delta^{-(n-1)} \, . \end{aligned} \tag{6.2}$$

We now prove that, if $\delta$ is chosen suitably small, at least one ball $B^*$ of the collection $\mathcal{B}$ must be contained in $E - \beta$. If not,

$$\bigcup_{B\in\mathcal{B}} B = \Big( \bigcup_{\substack{B\in\mathcal{B}\\ B\subseteq\mathcal{C}(E)}} B\Big) \cup \Big( \bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big),$$

so, by (6.2) and hypotheses $(i)$ and $(iii)$,

$$\begin{aligned}
\mathrm{const}\, r^n &\le \mu(E\cap B_r(x_0)) \le \mu\Big(E\cap\Big(\bigcup_{B\in\mathcal{B}} B\Big)\Big) \le \\
&\le \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\subseteq\mathcal{C}(E)}} B\Big)\Big) + \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big)\Big) = \\
&= 0 + \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big)\Big) \le \\
&\le \sum_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} \mu(B) \le \mathrm{const}\, \delta^{-(n-1)}\, (\delta r)^n\,,
\end{aligned}$$

which is a contradiction if $\delta$ is small enough. Such ball $B^*$ satisfy the desired property (6.1). ◻

Following is the application of the Geometric Lemma to our case:

**Proposition 6.3.** *Let $u \in \mathcal{M}^\omega_M$. There exist positive constants $\kappa$, $r_0$ and $r_1$, depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$, such that: for any $x \in \{|u| < 1\}$ and any $r \ge r_0$, there exist $z_1$ and $z_2$, such that*

$$B_{\kappa r}(z_1) \subseteq \{u = 1\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u = -1\} \cap B_r(x)\,,$$

*provided $B_{2r+r_1}(x) \subset S^\omega_M$.*

*Proof.* Apply the Geometric Lemma to the sets $E := \{u > -1\}$, or $E := \{u < 1\}$, and $\beta := \{|u| < 1\}$. Take $\mu := \mathcal{L}^n$ and $\nu(B) := \mathcal{J}_B(u^\omega_M)$. The hypotheses of the Geometric Lemma are fulfilled in light of Proposition 3.8. ◻

# 7 The minimal minimizer is unconstrained and class A

**Proposition 7.1.** *There exists $M_0 > 0$, depending only on $n$ and on the bounds on $a_{i,j}$ and $Q$, such that if $M \ge M_0|\omega|$, then the set $\{|u^\omega_M| < 1\}$ does not touch the upper constraint $\{\omega \cdot x = M\}$. Also, $\forall\alpha \ge 0$, $u^\omega_M = u^\omega_{M+\alpha}$.*

*Proof.* Let $S$ be a strip of width 3 around the hyperplane $L := \{x \cdot \omega = M/2\}$. First of all, we will prove that there exists a strip $H$ of width 1 parallel to the constraints and at distance universally bounded from $L$, on which

$$\text{either } u^\omega_M = -1 \text{ or } u^\omega_M = 1\,. \tag{7.1}$$

For that, we consider two cases:

**Case 1:** there exists a point $x^* \in S$ so that $u^\omega_M(x^*) \in (-1, 1)$. Then, from Proposition 6.3 there exists a ball (whose radius is big if $M$ is big), contained in $\{u^\omega_M = -1\} = \mathcal{C}\{u^\omega_M > -1\}$. By the results in Section 5, we obtain a strip of width 1 intersecting such ball and contained in $\{u^\omega_M = -1\}$.

**Case 2:** $u^\omega_M(S) \subseteq (-\infty, -1] \cup [1, +\infty)$. By the continuity of $u^\omega_M$, either $u^\omega_M(S) \subseteq (-\infty, -1]$ or $u^\omega_M(S) \subseteq [1, +\infty)$.

In any case, (7.1) is proved. With no loss of generality, assume also $\omega_1 > 0$. Now we infer that it is not possible that $u^\omega_M\Big|_H = 1$: Otherwise, since the set $\{u^\omega_M = 1\}$ satisfies the Birkhoff property with respect to $\omega$ (see Section 5), we obtain that

$$\bigcup_{j\in\mathbf{N}} \mathcal{T}_{(-j,0,\dots,0)} H \subseteq \bigcup_{j\in\mathbf{N}} \mathcal{T}_{(-j,0,\dots,0)} \{u^\omega_M = 1\} \subseteq \{u^\omega_M = 1\}\,,$$

thus $u^\omega_M = 1$ in the whole portion of space trapped between the lower constraint and $H$. But then we can consider the integer translation $v := \mathcal{T}_{(-1,0,\dots,0)} u^\omega_M$, which has the same energy as $u^\omega_M$, contradicting the fact that $u^\omega_M$ is the minimal minimizer.

Therefore, $u^\omega_M\Big|_H = -1$. Then, again, since the set $\{u^\omega_M > -1\}$ satisfies the Birkhoff property with respect to $\omega$, for any $j \in \mathbf{N}$,

$$\mathcal{T}_{(j,0,\dots,0)} H \cap \{u^\omega_M > -1\} = \emptyset\,,$$

so $u^\omega_M = -1$ in the whole portion of space trapped between the upper constraint and $H$. $\square$

In order to clearly state the next result, we introduce the following notation. If $a > 0$, we define

$$S^\omega_{M,a} := \{x \in \mathbf{R}^n \text{ s.t. } x \cdot \omega \in [-a, M + a]\}$$

and

$$\begin{aligned} \mathcal{X}^\omega_{M,a} := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}^\omega_M) \text{ s.t. } \nabla u \in \mathrm{L}^2(\tilde{S}^\omega_M) \\ \text{with } u(x) = 1 \text{ for any } x \text{ s.t. } \omega \cdot x = -a \text{ and} \\ u(x) = -1 \text{ for any } x \text{ s.t. } \omega \cdot x = M + a\}\,. \end{aligned}$$

**Theorem 7.2.** *There exists a constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $Q$, so that if $M \geq M_0|\omega|$, then the minimal minimizer $u^\omega_M$ is an unconstrained minimizer, i.e. for any $a > 0$ and any $u \in \mathcal{X}^\omega_{M,a}$, $\mathcal{J}_{S^\omega_{M,a}}(u^\omega_M) \leq \mathcal{J}_{S^\omega_{M,a}}(u)$.*

*Proof.* Proposition 7.1 above establishes that the upper constraint $\{x \cdot \omega = M\}$ is irrelevant. Hence, we have only to show that the lower constraint $L_0 := \{x \cdot \omega = 0\}$ is not necessary either. Given $a > 0$, take $k \in \mathbf{Z}^n$ so that $k \cdot \omega > a$: for instance, if $\omega_1 \neq 0$, take $k := A \cdot \mathrm{sign}(\omega_1) e_1$ with $A \in \mathbf{N}$ sufficiently big. And consider translations $\mathcal{T}_k u$. $\square$

**Remark 7.3.** Following [CL], we would like to sketch an alternative geometric argument for proving that the minimal minimizer is unconstrained. Let $\varrho$ be the constant in Proposition 5.2. We assume, without loss of generality, that $\varrho$, $M/4 \in \mathbf{N}$ and that all the components of $\omega$ are multiple of $4\varrho$. Let us consider a non-overlapping covering of the strip $S^\omega_M$. If $K_\omega$ is the number of cubes of side $4\varrho$ intersecting the plane $\{\omega \cdot x = 0\}$, we have that the number of cubes in the covering is $MK_\omega/4$. On the other hand, let $\nu$ be the number of cubes in the covering whose dialation of a factor $1/2$ intersects the set $\{|u^\omega_M| < 1\}$. We want to prove that $\nu < MK_\omega/4$: from this, we would obtain the existence of a cube of side $2\varrho$ outside $\{|u^\omega_M| < 1\}$. In the same way as above, this and the Birkhoff property would imply that the minimizer is unconstrained.

To show that $\nu < MK_\omega/4$, we first notice that, comparing with a one-directional linear function interpolating $-1$ and $1$, one sees that $\mathcal{J}_{S^\omega_M}(u^\omega_M) \le \text{const}K_\omega$. And, by Lemma 3.7, if a cube $Q$ of size $2\varrho$ intersects $\{|u| < 1\}$, then $\mathcal{J}_{2Q}(u^\omega_M) \ge \text{const}$. From these two estimates, it follows that $\nu \le \text{const}K_\omega$ and so $\nu < MK_\omega/4$, provided that $M$ is big enough.

A similar geometric construction has been used in [CL] for dealing with manifoolds with negative curvature.

**Remark 7.4.** From Theorem 7.2 and Theorem 2.2, it follows that, if $M \ge M_0|\omega|$, then, for any $\alpha \in [0,1)$, $||u^\omega_M||_{C^\alpha(\mathbf{R}^n)} \le C^\star$, for a suitable $C^\star$, depending only on $\alpha$, $n$ and on the bounds on $a_{i,j}$ and $Q$. More precisely, by Theorem 2.3, $||u^\omega_M||_{C^{0,1}(\mathbf{R}^n)} \le C^\star$.

**Definition 7.5.** *We say that $u$ is a* **class A** *minimizer for the functional $\mathcal{J}$ if for any admissible function $v$, verifying $v = u$ in $\mathcal{C}(K)$, we have $\mathcal{J}_K(u) \le \mathcal{J}_K(v)$.*

Notice that the definition of class A requires the functional not to increase under any compact (but not necessarily small) modification. This property was introduced for geodesics in [Mo], and it appeared also in Aubry-Mather theory.

**Theorem 7.6.** *There exists a constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $Q$, so that if $M \ge M_0|\omega|$, then the minimal minimizer $u^\omega_M$ is a class A minimizer.*

*Proof.* Given a compact perturbation, we can always consider $u^\omega_M$ as periodic with a period larger than the diameter of the perturbation and (in light of Theorem 7.2) as a minimizer in a strip of width bigger than the size of the perturbation. □

In order to prove Theorem 1.1, it suffices now to consider the case of an irrational frequency $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, since the rational frequency case was dealt with in the previous pages. An irrational frequency $\omega$ will now be handled passing to the limit rational frequencies $\omega_n$ approximating $\omega$.

Namely, consider minimal minimizers $u^{\omega_n}_M$. By Remark 7.4 and the Theorem of Ascoli, $u^{\omega_n}_M$ converges to some $u_\infty$ uniformly on compact sets. Accordingly, $u_\infty$ is a class A minimizer. This completes the proof of Theorem 1.1.

## PART II — Plane-like minimizers for Ginzburg-Landau-type models

## 8 The Ginzburg-Landau-type functional

In the second part of this paper, we will deal with the functional

$$\mathcal{G}_\Omega(u) = \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j u(x) \;+\; F(x,u)\,dx\,, \tag{8.1}$$

The assumptions on the coefficients $a_{i,j}$ are analogous to the one in Part I. Moreover, we assume in this Part that the coefficients $a_{i,j}$ are Lipschitz (indeed, the constants in the density estimates may depend, in this case, on the Lipschitz norm of the coefficients); $F$ is supposed to be non-negative and bounded, periodic under integer translations of the $x-$variable, and $F(x,1) = F(x,-1) = 0$. We also assume that $\inf_{|t|\le\theta} F(x,t) \ge \gamma(\theta)$, where $\gamma(\theta)$ is a decreasing, strictly positive function in the interval $[0,1)$, and that there exist $d \in [0,2]$ and $\ell \in (0,1)$ so that:

- $F(x,t) \ge \mathrm{const}\,(1-|t|)^d$, if $|t| \in (\ell,1)$;
- $F$ is continuous in $u$, for $|u| < 1$; $F_u$ is defined and in $\mathrm{L}^\infty_{\mathrm{loc}}$ for $|u| < 1$;
- if $d > 0$, $F_u(x,u)$ is continuous for $|u| < 1$, and

  $$F_u(x,-1+s) \ge \mathrm{const}\,s^{d-1}\,, \qquad F_u(x,1-s) \ge -\mathrm{const}\,s^{d-1}\,,$$

  if $s < \ell$;
- if $d = 2$, $F_u(x,u)$ is decreasing for $u$ near 1 and increasing for $u$ near -1.

Important examples are given by the the potentials $F(x) = Q(x)\,(1-u^2)$ and $F = Q(x)\,|1-u^2|^2$, with $Q$ non-negative and periodic: indeed, functionals similar to the latter were introduced in [GL] for the study of phase transition problems occurring in superconductivity and later applied in superfluid models (see [GP]). For a discussion about phase transition models, see, for instance, [LL], p. 206. For a related "gradient penalization method" in the theory of phase transitions, see the discussion in [Gu], p. 188. See also [R], which is a translation of van der Waals' pioneering work[1] and presents an interesting historical review.

The main result of the second part of the present paper is the following:

**Theorem 8.1.** *There exists a universal constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $F$ stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer $u$ for the functional $\mathcal{G}$ for which the set $\{|u| \le 99/100\}$ is contained in the strip $\{x$ s.t. $x\cdot\omega \in [0, M_0|\omega|]\}$.*

*Furthermore, such $u$ enjoys the following property of "quasi-periodicity":*

- *if $\omega \in \mathbf{Q}^n$, then $u$ is periodic (with respect to the identification induced by $\omega$);*
- *if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then $u$ can be approximated uniformly on compact sets by periodic class A minimizers.*

[1] The Dutch original appeared in Verhand. Konink. Akad. Wetensch. Amsterdam (Sect. 1) 1 (1892), no. 8, 1–56; Jbuch 24, 967.

The reader may compare Theorems 8.1 and 1.1: notice, in particular, that in Theorem 1.1 it was possible to control the set $\{|u| < 1\}$ in a uniform way and no regularity of the coefficients was required there.

The proof of Theorem 8.1 will use some of the machinery already developed in the first part of the paper. The main obstacle in shadowing the proof presented above consists in the fact that we do not expect in general to have the rigidity and non-degeneracy properties stated in Lemmas 3.4 and 3.5. Namely, for smooth potentials, the minimizers do not reach the values 1 and $-1$, so, in this case, we are not in presence of a free boundary. However, the arguments involving the minimal minimizer and the Birkhoff property still hold. Therefore, to extend the previous proof we have to obtain the density estimates that fit our purposes by another argument. The proof will go as follows: In Section 9 we discuss the existence and regularity of constrained minimizers. The arguments will be in analogy with the ones in Section 2; the main difference is that, in this case, we can not exactly prescribe the values on the boundary of the strip to be $\pm 1$: this will be replaced by the weaker constraint of requiring the function to be bigger than, say, $99/100$ if $\omega \cdot x \leq 0$ and smaller that $-99/100$ if $\omega \cdot x \geq M$. Obviously, the explicit choice of the values $\pm 99/100$ is by no means important.

In Section 10 we discuss the density estimates that we need. From this, it follows that the minimal minimizer is unconstrained and class A, provided that the strip is wide enough.

# 9 Existence and regularity of minimizers constrained in a strip

As in the first part, we first turn our attention to a rational frequency $\omega$. The irrational frequency case will then be handled by a limit argument.

We recall now the equivalence relation "$\sim$" defined in (2.1), and we define $\tilde{\mathbf{R}}^n := \mathbf{R}^n / \sim$ (topologically, $\tilde{\mathbf{R}}^n$ is the product between an $(n-1)-$dimensional torus and a real line). We consider the space of admissible functions

$$\mathcal{Y}_M^\omega := \{u \in \mathrm{H}^1_{\mathrm{loc}}(\tilde{\mathbf{R}}^n)$$

$$\text{with } u(x) \geq 99/100 \text{ for any } x \text{ s.t. } \omega \cdot x \leq 0 \text{ and}$$

$$u(x) \leq -99/100 \text{ for any } x \text{ s.t. } \omega \cdot x \geq M\}\,.$$

Following is the extension of the existence Theorem 2.1 we need:

**Theorem 9.1.** *For any $\omega \in \mathbf{Q}^n$ and for any $M > 0$, there exist an absolute minimum in $\mathcal{Y}_M^\omega$ of the functional*

$$\mathcal{G}_{\tilde{\mathbf{R}}^n}(u) := \int_{\tilde{\mathbf{R}}^n} a_{i,j}\partial_i u \partial_j u + F\,.$$

*Such minimizer is periodic (with respect to the identification $\sim$).*

*Proof.* Set $u_0 := -\frac{2\omega\cdot x}{M} + 1$ in $\{\omega \cdot x \in [0, M]\}$, and continuously extend $u_0$ as a constant to $\tilde{\mathbf{R}}^n$. For any $R > 0$, define $D_R := \{\omega \cdot x \in [-R, R]\}$ and $\tilde{D}_R := D_R / \sim$ (topologically, $\tilde{D}_R$ is the product between an $(n-1)-$dimensional torus and an interval).

Consider a minimizing sequence $u_k$: by cutting off the values above 1 and below $-1$, we may assume $-1 \le u_k \le 1$. We may also assume $\mathcal{G}_{\tilde{\mathbf{R}}^n}(u_k) \le \mathcal{G}_{\tilde{\mathbf{R}}^n}(u_0)$, hence $||\nabla u_k||_{\mathrm{L}^2(\tilde{\mathbf{R}}^n)}$ is bounded. From these facts and Poincaré inequality, it follows that $||u_k||_{\mathrm{L}^2(\tilde{D}_R)}$ is bounded.

Therefore, we may assume that $u_k$ converges to a suitable $u$ in $\mathrm{L}^2(\tilde{D}_R)$ (and a.e.) and that $\nabla u_k$ converges to $\nabla u$ weakly in $\mathrm{L}^2(\tilde{D}_R)$, so that

$$\liminf_{k\longrightarrow\infty} \int_{\tilde{\mathbf{R}}^n} a_{i,j}\partial_i u_k \partial_j u_k \ge \liminf_{k\longrightarrow\infty} \int_{\tilde{D}_R} a_{i,j}\partial_i u_k \partial_j u_k \ge \int_{\tilde{D}_R} a_{i,j}\partial_i u \partial_j u \,,$$

and $\int_{\tilde{D}_R} |F(x,u_k) - F(x,u)|$ goes to zero by the Dominated Convergence Theorem. □

The set of minimizers obtained via Theorem 9.1 will be denoted by $\mathcal{M}^{\omega}_{M}(\mathcal{G})$. The minimizers whose existence has just been proven are $\mathrm{C}^{\alpha}$, for a suitable $\alpha \in [0,1)$, with a uniform bound on the $\mathrm{C}^{\alpha}$ norm in the interior: this follows from Giusti and Giaquinta's Theorem (see [GG]). Also, in the case of smooth coefficients $a_{i,j}$ and a smooth potential $F$, further regularity can be obtained by applying standard elliptic estimates to the Euler-Lagrange equation of the minimizers. Indeed, in this case, by inspecting the first variational of the functional, it is easy to see that minimizers verify

$$2\partial_i(a_{i,j}\partial_j u) = F_u \,.$$

## 10 Density estimates

**Theorem 10.1.** *Assume $u \in \mathcal{M}^{\omega}_{M}(\mathcal{G})$. Then:*

*(i) there exist positive constants $c$, $r_0$, $r_1$ (depending only on $n$ and on the bounds on $a_{i,j}$ and $F$) such that*

$$\mathcal{G}_{B_r(x)}(u) \le c\, r^{n-1} \,,$$

*for any $r \ge r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$;*

*(ii) for any $\theta_0 \in [0,1)$, for any $\theta \in [-\theta_0, \theta_0]$ and for any $\mu_0 > 0$, if $\mathcal{L}^n(B_K(x) \cap \{u > \theta\}) \ge \mu_0$, there exist positive constants $c^\star$, $r_0$, $r_1$ (depending on $n$, $K$, $\mu_0$, $\theta_0$ and the bounds on $a_{i,j}$ and $F$) such that*

$$\mathcal{L}^n\Big(\{u > \theta\} \cap B_r(x)\Big) \ge c^\star r^n \,,$$

*for any $r \ge r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$. Analogously, if $\mathcal{L}^n(B_K(x) \cap \{u < \theta\}) \ge \mu_0$,*

$$\mathcal{L}^n\Big(\{u < \theta\} \cap B_r(x)\Big) \ge c^\star r^n \,,$$

*for any $r \ge r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$.*

The proof of this result, which is deferred to the Appendix, is a modification of the one in [CC], where the result was proved in the case $a_{i,j} = \delta_{i,j}$. We now derive some easy consequences from the previous result:

**Corollary 10.2.** *Fix $\theta_0 \in [0,1)$. Assume $u \in \mathcal{M}^{\omega}_M(\mathcal{G})$ and $u(x) \in [-\theta_0, \theta_0]$. Then, there exist positive constants $c$, $r_0$, $r_1$ (depending only on $\theta_0$, $n$ and on the bounds on $a_{i,j}$ and $F$) such that*

$$\mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(\{u < -\theta_0\} \cap B_r(x)\Big) \geq cr^n \,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_M$.*

*Proof.* Define $\theta^* := (1+\theta_0)/2$. From the fact that $u$ is Hölder continuous and $(ii)$ of Theorem 10.1,

$$\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(\{u < \theta^*\} \cap B_r(x)\Big) \geq cr^n \,, \tag{10.1}$$

so that

$$\begin{aligned} &\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) - \mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \leq \\ \leq \quad &\mathcal{L}^n\Big(\{-\theta^* < u \leq \theta_0\} \cap B_r(x)\Big) \leq \frac{1}{\gamma(\theta^*)} \int_{\{-\theta^* < u \leq \theta_0\} \cap B_r(x)} F \,, \end{aligned}$$

thus, by $(i)$ of Theorem 10.1,

$$\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) - \mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \leq cr^{n-1} \,.$$

From this and (10.1),

$$\mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \geq \; \mathrm{const}\, r^n - \; \mathrm{const}\, r^{n-1} \geq \; \mathrm{const}\, r^n \,,$$

if $r$ big enough. □

In the sequel, following a standard notation in Geometric Measure Theory (see, for instance [G1]), we denote the perimeter of a Borel set $E$ in an open set $\Omega$ as

$$P(E,\Omega) := \sup\{\int_E \operatorname{div} \varphi \;\text{ s.t. }\; \varphi \in \mathrm{C}^1_0(\Omega, \mathbf{R}^n) \,,\; |\varphi(x)| \leq 1\} \,.$$

The following result complete the density estimates we need in the second part of the paper:

**Lemma 10.3.** *Fix $\theta_0 \in [0,1)$. Assume $u \in \mathcal{M}^{\omega}_M(\mathcal{G})$ and $u(x) \in [-\theta_0, \theta_0]$. Then, there exist positive constants $c$, $r_0$, $r_1$ (depending only on $\theta_0$, $n$ and on the bounds on $a_{i,j}$ and $F$) such that*

$$\mathcal{L}^n\Big(\{|u| < \theta_0\} \cap B_r(x)\Big) \geq cr^{n-1} \,,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_M$.*

*Proof.* Define

$$u^*(x) := \begin{cases} u(x) & \text{if} \quad |u(x)| < \theta_0 \\ \theta_0 & \text{if} \quad u(x) \geq \theta_0 \\ -\theta_0 & \text{if} \quad u(x) \leq -\theta_0 \end{cases} .$$

Using the Coarea Formula and the Isoperimetric Inequality (see, for instance, [G1], p. 20 and p. 25), we infer from Corollary 10.2 that

$$\begin{aligned} & \int_{B_r(x)\cap\{|u|<\theta_0\}} |\nabla u| = \int_{B_r(x)} |\nabla u^*| \geq \\ \geq \quad & \int_{-\theta_0}^{\theta_0} \Big( P(\{u^* < t\}, B_r(x)) \Big)\, dt \geq \\ \geq \quad & \text{const} \int_{-\theta_0}^{\theta_0} \left( \min \left\{ \mathcal{L}^n\Big(\{u^* < t\} \cap B_r(x)\Big) \, , \, \mathcal{L}^n\Big(\{u^* \geq t\} \cap B_r(x)\Big) \right\}^{(n-1)/n} \right) dt = \\ = \quad & \text{const} \int_{-\theta_0}^{\theta_0} \left( \min \left\{ \mathcal{L}^n\Big(\{u < t\} \cap B_r(x)\Big) \, , \, \mathcal{L}^n\Big(\{u \geq t\} \cap B_r(x)\Big) \right\}^{(n-1)/n} \right) dt \geq \\ \geq \quad & \text{const } r^{n-1} . \end{aligned}$$

Therefore, by Cauchy Formula and $(i)$ of Theorem 10.1,

$$\begin{aligned} & \text{const}\, r^{n-1} \leq \frac{1}{A} \int_{B_r(x)} |\nabla u|^2 + A \cdot \mathcal{L}^n\Big(\{|u| < \theta_0\} \cap B_r(x)\Big) \leq \\ \leq \quad & \frac{\text{const}}{A} r^{n-1} + A \cdot \mathcal{L}^n\Big(\{|u| < \theta_0\} \cap B_r(x)\Big) . \end{aligned}$$

Choosing $A$ suitably big, this proves the claim. □

Let us summarize the density estimates we developed towards the application of the Geometric Lemma:

**Proposition 10.4.** *Fix $\theta_0 \in [0, 1)$. Let $u \in \mathcal{M}^\omega_M$. Assume that $x \in \{|u| < \theta_0\}$. Then, there exist positive constants $r_0$, $r_1$, $c$ and $C$, depending only on $\theta_0$, $n$ and on the bounds on $a_{i,j}$ and $F$, such that:*

$$\begin{aligned} & \mathcal{L}^n\Big(B_r(x) \cap \{|u| < \theta_0\}\Big) \geq c r^{n-1} , \\ & \mathcal{L}^n\Big(B_r(x) \cap \{u > -\theta_0\}\Big) \geq c r^n \qquad \text{and} \qquad \mathcal{L}^n\Big(B_r(x) \cap \{u < \theta_0\}\Big) \geq c r^n , \end{aligned}$$

*for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^\omega_M$. Moreover, for any $y \in S^\omega_M$,*

$$\mathcal{L}^n\Big(B_r(y) \cap \{|u| < \theta_0\}\Big) \leq C r^{n-1} ,$$

*for any $r \geq r_0$, provided $B_{r+r_1}(y) \subset S^\omega_M$.*

Notice in particular that with the estimates developed above, we are in position to apply the Geometric Lemma discussed in Section 6. Hence, we obtain the following

**Proposition 10.5.** *Fix $\theta_0 \in [0,1)$. Let $u \in \mathcal{M}^\omega_M(\mathcal{G})$. There exist positive constants $\kappa$, $r_0$ and $r_1$, depending only on $\theta_0$, $n$ and on the bounds on $a_{i,j}$ and $F$, such that: for any $x \in \{|u| \leq \theta_0\}$ and any $r \geq r_0$, there exist $z_1$ and $z_2$, such that*

$$B_{\kappa r}(z_1) \subseteq \{u \geq \theta_0\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u \leq -\theta_0\} \cap B_r(x)\,,$$

*provided $B_{2r+r_1}(x) \subset S^\omega_M$.*

The reader can compare the result above with Proposition 6.3, and the results in Proposition 10.4 with the ones in Proposition 3.8. Note that all through this Section the constants in the estimates depend on $\theta_0$ (i.e., roughly, on the value attained at the center). The results in Section 3 and Proposition 6.3 give slightly sharper results than the ones here, in the case of functional (1.1).

Notice also that the sets where we applied the Geometric Lemma here are $E := \{u > -\theta_0\}$ or $E := \{u < \theta_0\}$ and $\beta := \{|u| < \theta_0\}$; also, the measure $\mu$ and $\nu$ appearing in the "abstract" Geometric Lemma will be taken here to be the Lebesgue measure and the restriction of the Lebesgue measure to the set $\{|u| < \theta_0\}$, respectively.

In the following Proposition 10.6, we will apply Proposition 10.5 with $\theta_0 = 99/100$, to get uniform bounds.

**Proposition 10.6.** *There exists $M_0 > 0$, depending only on $n$ and on the bounds on $a_{i,j}$ and $F$, such that if $M \geq M_0|\omega|$, then the set $\{u^\omega_M > -99/100\}$ does not touch the upper constraint $\{\omega \cdot x = M\}$. Also, $\forall \alpha \geq 0$, $u^\omega_M = u^\omega_{M+\alpha}$.*

The proof of Proposition 10.6 is analogous to the one of Proposition 7.1.

The rest of the proof of Theorem 8.1 follows exactly as in Section 7.

**PART III — Back to the minimal surfaces**

# 11 The result of [CL] as a limit case

Now we point out that the results of [CL] about plane-like minimal surfaces can be recovered as a limiting case of jet flows or Ginzburg-Landau minimizers.

Namely, we now turn our attention to the $\varepsilon-$dependent functional

$$\mathcal{G}^\varepsilon_\Omega(u) := \int_\Omega \varepsilon a_{i,j}\partial_i u \partial_j u + \frac{1}{\varepsilon}F(u)\,dx\,. \tag{11.2}$$

The assumptions on $a_{i,j}$ and $F$ will be the same as in Section 8 and $\varepsilon > 0$ is a small parameter[2].

It is known (see [Md], [B], [OS] and [CC]) that the minimizers $u_\varepsilon$ of (11.2) converge (when $\varepsilon \longrightarrow 0$, up to subsequences) to a step function, and that the level sets of $u_\varepsilon$ converge

[2]For simplicity, we take here $F$ independent of $x$: a more general situation can be dealt with using the same techniques, following [B]

uniformly to an interface, which minimizes $\int_{\partial^\star U} \sqrt{a_{i,j}\nu_i\nu_j}\, d\mathcal{H}^{n-1}$. Here, as customary, $\partial^\star U$ denotes the reduced boundary of a Caccioppoli set $U$, and $\nu$ is its normal. This has the following geometric interpretation (see [BPV]): if we consider a Riemannian metric $g$ on $\mathbf{R}^n$, with a suitable choice of $a_{i,j}$, minimizers of $\int_{\partial^\star U} \sqrt{a_{i,j}\nu_i\nu_j}\, d\mathcal{H}^{n-1}$ are simply minimal surfaces with respect to the notion of perimeter induced by $g$. Also, the non-degeneracy of $g$ translates into the uniform ellipticity of $a_{i,j}$ and, if $g$ is periodic, the coefficients $a_{i,j}$ are periodic as well.

Thus, the result we now address is the following:

**Theorem 11.1.** *There exists a universal constant $M_0$, depending only on $n$, the bounds on $a_{i,j}$ and $F$, but independent of $\varepsilon$, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer $u^\varepsilon$ for the functional $\mathcal{G}^\varepsilon$, for which the set $\{|u^\varepsilon| \le 99/100\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.*

*Furthermore, such $u^\varepsilon$ enjoys the following property of "quasi-periodicity":*

- *if $\omega \in \mathbf{Q}^n$, then $u^\varepsilon$ is periodic (with respect to the identification induced by $\omega$);*
- *fixed $\varepsilon > 0$, if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then $u^\varepsilon$ can be approximated uniformly on compact sets by periodic class A minimizers of $\mathcal{G}^\varepsilon$.*

In the case of the jet flows, the result is actually uniform in $\theta$ and no regularity of the coefficients is needed. In particular, sending $\varepsilon$ to zero, we re-obtain the main result[3] of [CL]:

**Theorem 11.2.** *Let $g$ be a bounded metric on $\mathbf{T}^n$. There exists a universal constant $M$ (depending only on $n$ and on the nondegeneracy of $g$), such that, given any $\omega \in \mathbf{R}^n$, there exists a codimension 1 surface contained in the strip $\{\omega \in [0, M|\omega|]\}$, which is class A minimal with respect to the perimeter induced by $g$. Also, if $\omega$ is rational, the surface obtained is periodic (with respect to the identification induced by $\omega$); if $\omega$ is irrational, the surface obtained can be approximated locally uniformly by periodic class A minimal surfaces.*

Following a notation introduced in the previous sections, we will denote by $\mathcal{M}^\omega_M(\mathcal{G}^\varepsilon)$ the set of minimizers among the admissible functions in the strip $S^\omega_M$: see page 4 and 19 for details.

The proof of Theorem 11.1 readily follows from the work done in the first and second part of the paper, using the following rescaling: for any function $u$ we define

$$\mathcal{R}^\varepsilon u(x) := u(\varepsilon x)\,. \tag{11.3}$$

Notice that the renormalization $\mathcal{R}^\varepsilon$ preserves the double-well potential and the ellipticity constants, but does *not* preserve integer periodicity. Therefore we can not apply the arguments involving the Birkhoff property to $\mathcal{R}^\varepsilon u$. Thus, we only uses the rescaling for extending the Geometric Lemma to $\mathcal{R}^\varepsilon u$, since this result did *not* use the integer periodicity. Then, apply the Birkhoff property arguments directly to $u$.

**Remark 11.3.** A natural question arising in this context is whether the minimal surfaces that are plane-like minimizers are always graphs.

[3]Theorem 11.2 was also proved by [Mo] and [H] in the case $n = 2$ and by [Ba] in the case $n = 3$.

The reader may easily convince herself that the answer to such question, in general, is no, just looking at a periodic metric on $\mathbf{R}^2$ which is very small on a set with an "S" shape and very big elsewhere.

Analogously, one can construct examples of metrics whose plane-like minimizers for the functional (11.2) are not monotone in any direction.

**APPENDIX**

# 12 Proof of Theorem 2.3

The proof will be accomplished in several steps.

**Lemma 12.1.** *Fix $\mu > 0$. Let $\beta \in (0,1)$. Assume that $v \in \mathrm{H}^1(B_r)$, $v \geq 0$, $\partial_i(a_{i,j}\partial_j v) \leq 0$, and*

$$v \geq \mu \quad \text{on} \quad \partial B_r \cap \{x_n \leq (-1+\beta)r\}\,.$$

*Then, there exists a constant $\theta \in (0,\beta)$, depending only on $n$, $\beta$ and the bounds on $a_{i,j}$, such that: $v(0,\ldots,(-1+\theta)r) \geq \mu\beta/2$.*

*Proof.* By scaling, it is enough to prove the claim for $r = 1$. Set $c^\star := 2n||a_{i,j}||_{\mathrm{W}^{1,\infty}}$. Let $\psi$ be the solution of $\partial_i(a_{i,j}\partial_j\psi) = c^\star\mu$ in $B_1$, with $\psi = 0$ on $\partial B_1$. By elliptic estimates (see, for instance [GT], Theorem 8.16 and Theorem 8.27)

$$|\psi(0,\ldots,0,(-1+\theta))| \leq \mathrm{osc}_{B_1\cap B_{2\theta}(0,\ldots,0,-1)}\psi \leq c\theta^\alpha\mu\,, \tag{12.4}$$

for suitable constants $c$ and $\alpha$, depending only on $n$ and the bounds on $a_{i,j}$. Define

$$\phi := \mu[-(x_n+1)+\beta] \quad \text{and} \quad \zeta := \phi + \psi\,. \tag{12.5}$$

We have $\partial_i(a_{i,j}\partial_j\zeta) = -\partial_i a_{i,n}\mu + c^\star\mu > 0$. We now check that $v \geq \zeta$ on $\partial B_1$. Indeed, if $x \in \partial B_1 \cap \{x_n \leq -1+\beta\}$,

$$\zeta(x) = \phi(x) \leq \mu\beta \leq v(x)\,;$$

on the other hand, if $x \in \partial B_1 \cap \{x_n > -1+\beta\}$, $\zeta(x) \leq 0 \leq v(x)$.

Hence, by Maximum Principle, $v \geq \zeta$ in $B_1$. In particular, by (12.4) and (12.5), $v(0,\ldots,0,-1+\theta) \geq \mu(\beta - c\theta^\alpha - \theta) \geq \mu\beta/2$, if $\theta$ is small enough with respect to $\beta$. □

We would like to remark that an alternative proof of the Lemma above can be accomplished by using the boundary Harnack inequality.

**Lemma 12.2.** *Let $u \in \mathcal{M}^\omega_M$. Suppose that $x_0 \in S^\omega_M$ verifies*

$$\mathrm{dist}\,(x_0, \partial S^\omega_M) > 2\,\mathrm{dist}\,(x_0, \{|u| = 1\})\,.$$

*Then,*

$$\min\{1-u(x_0),\, 1+u(x_0)\} \leq C\,\mathrm{dist}\,(x_0, \{|u|=1\})\,,$$

*where $C$ is a suitable constant, depending only on $n$, the bounds on $a_{i,j}$ and $F$.*

*Proof.* If $u(x_0) = \pm 1$, there is nothing to prove. Set $U_\pm = 1 \pm u$, and $d_0 := \operatorname{dist}(x_0, \{|u| = 1\})$.

We assume that $\min\{U_+(x_0), U_-(x_0)\} \geq M d_0$, and derive an upper bound for $M$. Notice that $B_{9d_0/10}(x_0) \subset \{|u| < 1\}$, so by Lemma 2.4 and Harnack inequality

$$\inf_{B_{3d_0/4(x_0)}} U_\pm \geq \text{const} \cdot \sup_{B_{3d_0/4(x_0)}} U_\pm \geq \text{const}\, M d_0\,. \tag{12.6}$$

Let $y \in B_{d_0}(x_0) \cap \{|u| = 1\}$. Without loss of generality, we may assume $y = 0$. We now denote, for short, $B := B_{d_0}$ and $U := U_\pm$.

Let $v := v_\pm$ be the minimum of

$$\int_B a_{i,j}\partial_i\zeta\partial_j\zeta$$

among the functions $\zeta \in \mathrm{H}^1(B)$ with $\zeta = U_\pm$ on $\partial B$. Hence,

$$\begin{aligned} \text{const} \int_B |\nabla(U-v)|^2 &\leq \int_B a_{i,j}\partial_i(U-v)\partial_j(U-v) = \\ &= \int_B a_{i,j}\partial_i u\partial_j u - \int_B a_{i,j}\partial_i v\partial_j v\,. \end{aligned} \tag{12.7}$$

By construction, $V = V_\pm := \pm(v-1) = u$ on $\partial B$: hence by (12.7) and the fact that $u \in \mathcal{M}^\omega_M$,

$$\int_B |\nabla(U-v)^2| \leq \text{const} \int_B Q\chi_{(-1,1)}(V) - \chi_{(-1,1)}(u)\,. \tag{12.8}$$

By Maximum Principle and the fact that $0 \leq U \leq 2$, we deduce that either $0 < v < 2$ or $v$ is constant on $B$.

In the latter case, since $u \in \mathcal{M}^\omega_M$, $u$ must be constant in $B$: in particular $u(x_0) = u(0) = \pm 1$, and the claim is proven.

Thus, we can now restrict ourselves to the case $0 < v < 2$ in $B$, i.e. $-1 < V < 1$. From (12.8),

$$\int_B |\nabla(U-v)|^2 \leq \text{const}\, \mathcal{L}^n(\{|u| = 1\} \cap B)\,. \tag{12.9}$$

From (12.6) and Lemma 12.1, there is a suitable constant $\kappa \in (0,1)$ and a suitable point $x^\star \in B_{\kappa d_0}$ so that $v(x^\star) \geq \text{const}\, M d_0$. Harnack inequality then implies

$$\inf_{B_{\kappa d_0}} v \geq \text{const}\, M d_0\,. \tag{12.10}$$

Set now $v^*(x) := \frac{1}{d_0} v(d_0 x)$, for $x \in B_1$. Let us now introduce the following barrier:

$$w(x) = \bar{c}M(e^{-\mu|x|^2} - e^{-\mu})\,,$$

with $\bar{c}$ and $\mu$ suitable constants, properly chosen to satisfy

$$\begin{aligned} &\partial_i(a_{i,j}\partial_j w) > 0 \qquad \text{in } B_1 - B_\kappa\,, \\ &w = 0 \qquad \text{on } \partial B_1 \qquad \text{and} \qquad w \leq v^* \qquad \text{on } \partial B_\kappa\,, \end{aligned}$$

where the last relation can be fulfilled in light of (12.10). So, by Maximum Principle,

$$v^*(x) \geq w(x) \geq \text{const}\, M(1-|x|) \qquad \text{in } B_1 - B_\kappa\,.$$

Recalling (12.10), we get

$$v^*(x) \geq \text{const}\, M(1-|x|) \qquad \text{in } B_1\,.$$

Rescaling back:

$$v(x) \geq \text{const}\, M d_0 (1-\frac{|x|}{d_0}) \qquad \text{in } B_{d_0}\,. \tag{12.11}$$

We can now follow [AC], p. 112: Fix any $y \in B_{d_0/2}$ and consider the following transformation of $B_{d_0}$ such that $y$ becomes the new origin:

$$T_y(x) := \frac{d_0 - |x|}{d_0} y + x\,,$$

and set, for any function $f$, $f_y(x) := f(T_y(x))$.

For any $\zeta \in \partial B_1$, define

$$r_\zeta := \inf\{r \in [d_0/8, d_0] \text{ s.t. } U(r\zeta) = 0\}\,,$$

if this set is non-empty. For almost all such $\zeta$ we have

$$v_y(r_\zeta \zeta) = \int_{r_\zeta}^{d_0} \nabla(U_y - v_y)(\rho\zeta)\cdot\zeta\, d\rho \leq \sqrt{\int_{r_\zeta}^{d_0} |\nabla(U_y - v_y)(\rho\zeta)|^2\, d\rho} \cdot \sqrt{d_0 - r_\zeta}\,. \tag{12.12}$$

By (12.11), for any $r \in [0, d_0]$,

$$\begin{aligned} v_y(r\zeta) &\geq \text{const}\, M\Big(d_0 - \Big|\frac{d_0 - r}{d_0} y + r\zeta\Big|\Big) \geq \\ &\geq \text{const}\, M\Big(d_0 - \frac{d_0 - r}{d_0}|y| - r\Big) \geq \text{const}\, M\,\frac{d_0 - r}{2}\,, \end{aligned}$$

since $|y| \leq d_0/2$. From this and (12.12)

$$M^2(d_0 - r_\zeta) \leq \text{const} \int_{r_\zeta}^{d_0} |\nabla(U_y - v_y)(\rho\zeta)|^2\, d\rho\,. \tag{12.13}$$

Also, using polar coordinates and the definition of $r_\zeta$,

$$\begin{aligned} &\int_{B_{d_0} - B_{d_0/4}(y)} \chi_{\{U=0\}} \leq \text{const} \int_{B_{d_0} - B_{d_0/8}} \chi_{\{U_y=0\}} \leq \\ &\leq \text{const} \int_{\partial B_1} \int_{d_0/8}^{d_0} \chi_{\{U_y=0\}}(\rho\zeta)\, d\rho\, d\zeta = \text{const} \int_{\partial B_1} \int_{r_\zeta}^{d_0} \chi_{\{U_y=0\}}(\rho\zeta)\, d\rho\, d\zeta \leq \\ &\leq \text{const} \int_{\partial B_1} (d_0 - r_\zeta)\, d\zeta\,. \end{aligned}$$

From this, integrating both sides of (12.13), it follows that

$$\begin{aligned} & \text{const}\, M^2 \int_{B_{d_0}-B_{d_0/4}(y)} \chi_{\{U=0\}} \le \int_{\partial B_1} \int_{d_0/8}^{d_0} |\nabla (U_y - v_y)(\rho\zeta)|^2 \, d\rho \, d\zeta \le \\ \le \;\; & \text{const} \int_{B_{d_0}} |\nabla (U-v)|^2 \,. \end{aligned}$$

Applying this result to two points $y_1$ and $y_2$ in $B_{d_0/2}$ so that $B_{d_0/4}(y_1)$ and $B_{d_0/4}(y_2)$ do not overlap, and summing up the two estimates, we obtain

$$M^2 \int_{B_{d_0}} \chi_{\{U=0\}} \le \text{ const} \int_{B_{d_0}} |\nabla (U-v)|^2 \,.$$

Thus, recalling (12.9), and writing $U = U_\pm$ explicitly:

$$M^2 \mathcal{L}^n(\{U_\pm = 0\} \cap B_{d_0}) \le \text{ const}\, \mathcal{L}^n(\{|u|=1\} \cap B_{d_0}) \,.$$

Since $\{|u|=1\} \subseteq \{U_+ = 0\} \cup \{U_- = 0\}$, we deduce that

$$M^2 \mathcal{L}^n(\{|u|=1\} \cap B_{d_0}) \le 2 \,\text{const}\, \mathcal{L}^n(\{|u|=1\} \cap B_{d_0}) \,.$$

Accordingly, recalling also (12.9), if $M$ were too big, we get $|u| < 1$ in $B_{d_0}$, which is a contradiction. □

**Theorem 12.3.** *If $u \in \mathcal{M}_M^\omega$, then $u \in \mathrm{C}^{0,1}(S_M^\omega)$. Furthermore, for any compact domain $D$ well contained in $S_M^\omega$ with*

$$D \cap \{|u| = 1\} \neq \emptyset \,, \tag{12.14}$$

*there exists a suitable positive constant $C_D$, depending only on* $\mathrm{dist}\,(D, \partial S_M^\omega)$, *$n$ and the bounds on $a_{i,j}$ and $F$ such that*

$$||u||_{\mathrm{C}^{0,1}} \le C_D \,. \tag{12.15}$$

*Proof.* First of all, we show that $\nabla u$ is bounded in $D \cap \mathcal{N}$, where $\mathcal{N}$ is a suitable neighborhood of $\partial\{|u|=1\}$.

Indeed: if $\mathrm{dist}\,(x, \{|u|=1\}) \le \frac{1}{10} \mathrm{dist}\,(x, \partial S_M^\omega)$, we are under the hypotheses of Lemma 12.2, so that (setting $B := B_{\mathrm{dist}\,(x,\{|u|=1\})/10}(x)$ for short)

$$\sup_{x_0 \in B} \min\{1 - u(x_0),\, 1 + u(x_0)\} \le \text{ const dist}\,(x, \{|u|=1\}) \,. \tag{12.16}$$

Also, by Lemma 2.4, $\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0$ in $B$; therefore, by elliptic estimates,

$$|\nabla (1 \pm u)(x)| \le \text{ const} \frac{\sup_B |1 \pm u|}{\mathrm{dist}\,(x, \{|u|=1\})} \,.$$

From this and (12.16), we deduce a bound on $\nabla u$ in $D \cap \mathcal{N}$.

This and standard elliptic estimates imply that $u \in \mathrm{C}^{0,1}(S_M^\omega)$.

We now prove (12.15): Take $D'$ such that $D \subset\subset D' \subset\subset S_M^\omega$, with

$$\mathrm{dist}\,(D', \partial S_M^\omega) \geq \frac{9}{10}\,\mathrm{dist}\,(D, \partial S_M^\omega)\,.$$

With no loss of generality, we may assume $D$ and $D'$ to be connected. Fix any $x \in D'$ and set $r_0 := \frac{1}{5}\,\mathrm{dist}\,(D, \partial S_M^\omega)$. Consider a covering of $D'$ with balls $B_{r_0}(x_j)$, centered in $D'$, with $x_0 := x$ and $x_j \in B_{r_0/2}(x_{j-1})$. Let $k$ be the number of such balls: obviously, $k$ depends only on $n$, $D'$ and $r_0$, hence we may think $k$ to depend only on $n$, $D$ and $\mathrm{dist}\,(D, S_M^\omega)$.

By (12.14), there exists $k^* \leq k$ so that

$$B_{r_0}(x_{k^*}) \cap \{|u| = 1\} \neq \emptyset\,,$$

while

$$B_{r_0}(x_j) \cap \{|u| = 1\} = \emptyset\,, \quad \forall j < k^*\,.$$

It follows that

$$\mathrm{dist}\,(x_{k^*}, \{|u| = 1\}) \leq \frac{2}{9}\,\mathrm{dist}\,(x_{k^*}, \partial S_M^\omega)\,,$$

so we can apply Lemma 12.2 and infer that

$$\min\{1 - u(x_{k^*}),\, 1 + u(x_{k^*})\} \leq \mathrm{const}\, r_0\,. \tag{12.17}$$

Also, $\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0$ in $B_{r_0}(x_j)$ for $j < k^*$ because of Lemma 2.4. So, we deduce from Harnack inequality that

$$\mathrm{const}\,(1 \pm u(x_{j+1})) \geq 1 \pm u(x_j)\,.$$

Iterating, using (12.17) and recalling that $x_0 = x$, we get

$$\sup_{x \in D'} \min\{1 - u(x),\, 1 + u(x)\} \leq \mathrm{const}\, r_0\,. \tag{12.18}$$

Take now $\zeta \in D$, with $|u(\zeta)| < 1$.

Two cases are possible: If

$$\mathrm{dist}\,(\zeta, \{|u| = 1\}) \geq \frac{1}{3}\,\mathrm{dist}\,(D, \partial D')\,,$$

then

$$\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0 \qquad \text{on } B_{\mathrm{dist}\,(D,\partial D')/3}(\zeta)\,;$$

thus, elliptic estimates imply that

$$|\nabla(1 \pm u)(\zeta)| \leq \frac{\mathrm{const}}{\mathrm{dist}\,(D, \partial D')} \sup_{B_{\mathrm{dist}\,(D,\partial D')/3}(\zeta)} |1 \pm u|\,.$$

From this and (12.18), $|\nabla u(\zeta)| \leq C_D$.

On the other hand: If

$$\mathrm{dist}\,(\zeta, \{|u| = 1\}) < \frac{1}{3}\,\mathrm{dist}\,(D, \partial D')\,,$$

as done before, we can apply again Lemma 12.2 and elliptic estimates (via Lemma 2.4) to get (12.15). □

We are now in position to complete the proof of Theorem 2.3.

If $\mathrm{Int}(S^\omega_M) \cap \{|u| = 1\} \neq \emptyset$, the claim in Theorem 2.3 follows by Theorem 12.3.

If, viceversa, $\mathrm{Int}(S^\omega_M) \cap \{|u| = 1\} = \emptyset$, by Lemma 2.4 $\partial_i(a_{i,j}\partial_j u = 0$ in $\mathrm{Int}(S^\omega_M)$, so the claim follows from standard elliptic estimates (for instance, Theorem 8.33 and Corollary 8.35 of [GT]).

## 13 Proof of Theorems 3.1 and 10.1

This proof is a variation of the one in [CC], where the result was proved in the case $a_{i,j} = \delta_{i,j}$. Here, we simplify a bit the proof and correct some minor flawns of [CC]. Let $g$ be a radial smooth function that is identically equal to $-1$ in $B_{r-1}$ and takes value 1 on $\partial B_r$. Define $u^\star = \min\{u, g\}$. Since $u$ is a minimizer,

$$\begin{aligned} \mathcal{J}_{B_r}(u) &\leq \mathcal{J}_{B_r}(u^\star) \leq \mathrm{const}\,(\int_{B_r - B_{r-1}} |\nabla u^\star|^2 + r^{n-1}) \leq \\ &\leq \mathrm{const}\,(\int_{B_r - B_{r-1}} (|\nabla u|^2 + |\nabla g|^2) + r^{n-1}) \leq \\ &\leq \mathrm{const} \int_{B_r - B_{r-1}} |\nabla u|^2 + r^{n-1}\,. \end{aligned}$$

Let us now cover $B_r - B_{r-1}$ with balls $\mathcal{B}_1, \dots \mathcal{B}_K$ of radius 2; obviously, one can take $K \leq \mathrm{const}\, r^{n-1}$. Let $\mathcal{B}'_i$ be the ball concentric to $\mathcal{B}_i$ with radius 3. Then, from the estimate above and Caccioppoli Inequality,

$$\begin{aligned} \mathcal{J}_{B_r}(u) &\leq \mathrm{const}\,(\sum_{i=1}^K \int_{\mathcal{B}_i} |\nabla u|^2 + r^{n-1}) \leq \\ &\leq \mathrm{const}\,(\sum_{i=1}^K \int_{\mathcal{B}'_i} |u|^2 + r^{n-1})\,. \end{aligned}$$

Since $|u| \leq 1$, this finishes the proof of $(i)$.

We now prove $(ii)$. Notice that it is enough to prove it for any $\theta$ close to $-1$. Indeed: assume the result to be true for $\theta^\star$ close to $-1$, and let $\theta \in [-\theta_0, \theta_0]$, with $\theta^\star \leq -\theta_0$. Then

$$\mu_0 \leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_K\Big) \leq \mathcal{L}^n\Big(\{u > \theta^\star\} \cap B_K\Big)\,,$$

therefore, using the result for $\theta^\star$, our hypotheses on $F$ and $(i)$, we infer that

$$\begin{aligned} Cr^n &\leq \mathcal{L}^n\Big(\{u > \theta^\star\} \cap B_r\Big) \leq \\ &\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + \mathcal{L}^n\Big(\{\theta^\star < u \leq \theta\} \cap B_r\Big) \leq \\ &\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + \frac{1}{\inf_{u \in [\theta^\star, \theta_0]} F} \int_{B_r} F \leq \\ &\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + Cr^{n-1}\,, \end{aligned}$$

and so $\mathcal{L}^n\Big(\{u>\theta\}\cap B_r\Big)\geq Cr^n$. From these consideration, in the rest of the proof, we may and do assume that $\theta$ is close to $-1$.

We distinguish three cases:

**Case 1:** if $0<d<2$. We define

$$\begin{aligned}\mathcal{V}(r) &= \mathcal{L}^n(B_r\cap\{u>\theta\})\,,\\ \mathcal{A}(r) &= \int_{B_r\cap\{u\leq\theta\}} F(x,u)\,.\end{aligned}$$

We want to prove that

$$\begin{aligned}&C\Big((\mathcal{V}(r-1))^{\frac{n-1}{n}}+\mathcal{A}(r-1)\Big)\leq\\ \leq\;& \mathcal{V}(r)-\mathcal{V}(r-1)+\mathcal{A}(r)-\mathcal{A}(r-1)\,.\end{aligned}\tag{13.19}$$

Indeed, if (13.19) holds, then, by induction, one sees that $\mathcal{V}(k)+\mathcal{A}(k)\geq Ck^n$; recalling $(i)$, we get $\mathcal{V}(k)\geq Ck^n$, for $k$ big enough, proving $(ii)$ – notice that we used here the hypothesis $\mathcal{V}(K)\geq\mu_0$ to start the induction.

We now prove (13.19). Let $h$ be an auxiliary function to be chosen later in such a way that $-1\leq h\leq 1$, $h=1$ on $\partial B_r$ and $h=-1$ in $B_{r-1}$. We will denote by $C$ suitable constants and by $A$ a free parameter, to be chosen in the sequel. Let $\sigma=\min\{u,h\}$ and $\beta=\min\{u-\sigma,1+\theta\}$ By Sobolev and Cauchy Inequalities, and the fact that $\nabla\beta$ vanishes outside $\{u-\sigma\leq 1+\theta\}$,

$$\begin{aligned}&\Big(\int_{B_r}\beta^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}}\leq\\ \leq\;& C\int_{B_r\cap\{u-\sigma\leq1+\theta\}}|\beta|\,|\nabla\beta|\leq\\ \leq\;& CA\Big(\int_{B_r\cap\{u-\sigma\leq1+\theta\}}(a_{i,j}\partial_i u\partial_j u-a_{i,j}\partial_i\sigma\partial_j\sigma-2a_{i,j}\partial_i(u-\sigma)\partial_j\sigma)\Big)+\\ &+\frac{C}{A}\int_{B_r\cap\{u-\sigma\leq1+\theta\}}(u-\sigma)^2\,.\end{aligned}\tag{13.20}$$

Integration by parts, together with the minimality property of $u$, implies

$$\begin{aligned}&\Big(\int_{B_r}\beta^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}}\leq\\ \leq\;& CA\Big[\int_{B_r\cap\{u-\sigma\leq1+\theta\}}(F(x,\sigma)-F(x,u))+\\ &+2\int_{B_r\cap\{u-\sigma\leq1+\theta\}}\Big(a_{i,j}\partial^2_{i,j}\sigma\,(u-\sigma)+(\partial_i a_{i,j})\partial_j\sigma\,(u-\sigma)\Big)\Big]+\\ &+\frac{C}{A}\int_{B_r\cap\{u-\sigma\leq1+\theta\}}(u-\sigma)^2\,.\end{aligned}\tag{13.21}$$

Since $\sigma = h = -1$ in $B_{r-1}$, the left hand side of (13.21) is bigger than

$$C\left(\mathcal{L}^n(B_{r-1} \cap \{u > \theta\})\right)^{\frac{n-1}{n}}.$$

We now estimate the right hand side of (13.21) in $B_{r-1}$, in which $\sigma = -1$: Notice now that, if $\theta$ is close to $-1$ and $u \le \theta$, we have, by our assumptions on $F$, that

$$F(x, u) \ge C\,(1 + u)^d \ge \frac{C}{2^{2-d}}(1 + u)^2 \tag{13.22}$$

so that

$$\frac{C}{A}\int_{B_{r-1}\cap\{u\le\theta\}}(1+u)^2 - \frac{CA}{2}\int_{B_{r-1}\cap\{u\le\theta\}}F(x,u) \le 0$$

if we choose $A$ suitably big. Therefore, the right hand side of (13.21) in $B_{r-1}$ is negative and less than $-C\,\mathcal{A}(r-1)$.

On the integration over $B_r - B_{r-1}$, we split the domain of integration. First of all, since $\theta$ is close to $-1$ and $\sigma \le u$, our assumptions on $F$ imply that $F(x, \sigma) \le F(x, u)$ if $u \le \theta$. Thus,

$$CA\int_{B_r - B_{r-1}} F(x,\sigma) - F(x,u) \le CA\,\mathcal{L}^n(\{u > \theta\} \cap (B_r - B_{r-1})).$$

Also, recalling (13.22),

$$\begin{aligned} &\frac{C}{A}\int_{B_r - B_{r-1}}(u-\sigma)^2 \le \\ \le\ &\frac{C}{A}\int_{(B_r - B_{r-1})\cap\{u\le\theta\}}(u+1)^2 + \frac{C}{A}\mathcal{L}^n(\{u>\theta\}\cap(B_r - B_{r-1})) \le \\ \le\ &\frac{C}{A}\int_{(B_r - B_{r-1})\cap\{u\le\theta\}}F(x,u) + \frac{C}{A}\mathcal{L}^n(\{u>\theta\}\cap(B_r - B_{r-1})). \end{aligned}$$

Finally, we define, $d' = \max\{1, d\}$ and, for $\rho \in [r-1, r]$,

$$h(\rho) = 2(\rho - r + 1)_+^{\frac{2}{2-d'}} - 1. \tag{13.23}$$

Notice that this setting is possible for our assumption on $d$. With this definition, we can complete the estimate on the right hand side of (13.21): Indeed, from a direct calculation one sees that, in $B_r - B_{r-1}$,

$$|\nabla h| + |D^2 h| \le C(\rho - r + 1)^{\frac{2(d'-1)}{2-d'}} \le C(h+1)^{d'-1}.$$

Also, in the set $\{u > \sigma\}$ we have $\sigma = h$ and then the last formula implies

$$\begin{aligned} (|\nabla h| + |D^2 h|)\cdot(u-\sigma) &\le C(h+1)^{d'-1}\cdot(u-h) \le \\ &\le C\,(u+1)^{d'}, \end{aligned} \tag{13.24}$$

in $(B_r - B_{r-1}) \cap \{u > \sigma\}$. Therefore, recalling that $\sigma = -1$ in $B_{r-1}$, splitting the domain of integration into the sets $\{u \leq \theta\}$ and $\{u > \theta\}$, using (13.24) and our hypotheses on $F$, we get

$$\begin{aligned}
&\int_{B_r} \Big( a_{i,j} \partial^2_{i,j} \sigma \, (u - \sigma) + (\partial_i a_{i,j}) \partial_j \sigma \, (u - \sigma) \Big) = \\
&= \int_{(B_r - B_{r-1}) \cap \{u > \sigma\}} \Big( a_{i,j} \partial^2_{i,j} \sigma \, (u - \sigma) + (\partial_i a_{i,j}) \partial_j \sigma \, (u - \sigma) \Big) \leq \\
&\leq C \mathcal{L}^n \Big( (B_r - B_{r-1}) \cap \{u > \theta\} \Big) + C \int_{(B_r - B_{r-1}) \cap \{u \leq \theta\}} (u + 1)^{d'} \leq \\
&\leq C \mathcal{L}^n \Big( (B_r - B_{r-1}) \cap \{u > \theta\} \Big) + C \int_{(B_r - B_{r-1}) \cap \{u \leq \theta\}} F(x, u) \, .
\end{aligned}$$

Collecting all these estimates, the proof of (13.19) follows. This finishes the proof of (ii) in the first case.

**Case 2:** if $d = 2$. We use suitable positive parameters $\Theta$ and $T$: we will fix $\Theta$ small enough and then choose $T$ suitably large.

Let $k \in \mathbf{N}$. We use a barrier function $h = h_k \in \mathrm{C}^2(B_{(k+1)T})$ so that $-1 \leq h \leq 1$, $h = 1$ on $\partial B_{(k+1)T}$,

$$(h + 1) + |\nabla h| + |D^2 h| \leq C(h + 1) \leq C e^{-\Theta T} \tag{13.25}$$

in $B_{kT}$, and

$$|\nabla h| + |D^2 h| \leq C \Theta (h + 1) \tag{13.26}$$

in $B_{(k+1)T}$. From the last formula and our assumptions on $F$, we deduce, if $\Theta$ is suitably small,

$$|\nabla h| + |D^2 h| \leq \sqrt{\Theta} F_u(x, h) \tag{13.27}$$

in $B_{(k+1)T}$. We postpone the explicit construction of such $h$ to the end of the proof. Define $\theta' = \theta - C e^{-\Theta T}$; if $T$ is large enough, $\theta' > -1$. Define also

$$\sigma = \min\{u, h\} \quad \text{and} \quad \beta = \min\{u - \sigma, 1 + \theta'\} \, .$$

As in (13.20)–(13.21), we have that

$$\begin{aligned}
&\Big( \int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \Big)^{\frac{n-1}{n}} \leq \\
\leq \; & CA \Big[ \int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} \Big( F(x, \sigma) - F(x, u) + a_{i,j} \partial^2_{i,j} \sigma (u - \sigma) + \\
&+ \partial_i a_{i,j} \partial_j \sigma (u - \sigma) \Big) \Big] + \\
&+ \frac{C}{A} \int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} (u - \sigma)^2 \, ,
\end{aligned} \tag{13.28}$$

where $A$ is a free parameter, to be chosen suitably large in what follows. We now estimate the left hand side of (13.28). From (13.25), if $T$ is big enough, we get $\theta - h \geq (1 - \theta_0)/2$ in $B_{kT}$. Thus, the left hand side of (13.28) is bigger than $C \mathcal{V}(kT)^{\frac{n-1}{n}}$.

Let us now estimate the right hand side of (13.28). First of all, we consider the contribution in $\{u \le \theta\}$. Let us observe that, since $-1 \le \sigma \le u \le 1$,

$$
\begin{aligned}
& (u+1)^2 - (\sigma+1)^2 - \frac{1}{2}(u-\sigma)^2 = \\
= \;& (u+\sigma)\,(u-\sigma) + 2(u-\sigma) - \frac{1}{2}(u-\sigma)^2 = \\
= \;& (u-\sigma)\,(\frac{1}{2}u + \frac{3}{2}\sigma + 2) \ge 0\,;
\end{aligned}
$$

accordingly, in $\{\sigma < u \le \theta\}$,

$$
\begin{aligned}
& F(x,u) - F(x,\sigma) = \int_h^u F_u(x,\zeta)\, d\zeta \ge \\
& \ge C \int_h^u (\zeta+1)\, d\zeta = C\Big[(u+1)^2 - (h+1)^2\Big] \ge \\
& \ge C(u-h)^2\,. \qquad (13.29)
\end{aligned}
$$

Consequently, choosing $A$ suitably large and recalling (13.27), the contribution of the right hand side of (13.28) in $\{u \le \theta\}$ is controlled by

$$
\int_{B_{(k+1)T}\cap\{\sigma<u\le\theta\}} \Big( F(x,\sigma) - F(x,u) + C\sqrt{\Theta} F_u(x,\sigma)(u-\sigma) \Big)\,. \qquad (13.30)
$$

We now show that this quantity is indeed negative. Since we ruled out $\theta$ to be close to $-1$, we have that $F$ and $F_u$ are monotone in $(-1,\theta)$. Hence, in $\{\sigma < u \le \theta\}$, $F(x,\sigma) - F(x,u)$ is negative and, furthermore,

$$
|F_u(x,\sigma)(u-\sigma)| \le |F(x,\sigma) - F(x,u)|\,.
$$

Since we assumed $\Theta$ to be small, we have shown that the quantity in (13.30) is negative.

Let us now bound the right hand side of (13.28) in $\{u > \theta\}$. First, we notice that this term has no contribution in $B_{kT}$: indeed, from (13.25),

$$
B_{kT} \cap \{\sigma < u \le \sigma + 1 + \theta'\} \subseteq B_{kT} \cap \{\sigma < u \le \theta\}\,.
$$

Thus, the right hand side of (13.28) is bounded by

$$
\int_{(B_{(k+1)T}-B_{kT})\cap\{u>\theta\}} \Big( F(x,\sigma) - F(x,u) + (\sigma+1)(u-\sigma) + (u-\sigma)^2 \Big)\,, \qquad (13.31)
$$

and this term can be bounded by

$$
\mathcal{L}^n\Big( \{u > \theta\} \cap (B_{(k+1)T} - B_{kT}) \Big)\,.
$$

Collecting all the estimates, we get

$$
C(\mathcal{V}(kT))^{\frac{n-1}{n}} \le \mathcal{V}((k+1)T) - \mathcal{V}(kT)\,.
$$

Let us now define $\alpha_k = \mathcal{V}(kT) - \mathcal{V}((k-1)T)$. Notice that $\sum_{1\le j\le k} \alpha_j = \mathcal{V}(kT)$ and therefore

$$C\Big(\sum_{1\le j\le k} \alpha_j\Big)^{\frac{n-1}{n}} \le \alpha_{k+1}\,.$$

By induction over $k$, one easily gets that $\alpha_k \ge ck^{n-1}$ and thus $\mathcal{V}(kT) \ge ck^n$, which is the desired result for the case $d = 2$.

We now construct explicitly the barrier introduced at page 33. Define the following functions $\Phi : [0,1] \longrightarrow \mathbf{R}$, $\Psi : [1,(k+1)T] \longrightarrow \mathbf{R}$:

$$\begin{aligned} \Phi(\rho) &= 2e^{\Theta[\frac{3}{8}\rho^6 - \frac{10}{8}\rho^4 + \frac{15}{8}\rho^2 - (k+1)T]} - 1 \quad \textit{and} \\ \Psi(\rho) &= 2e^{\Theta[\rho-(k+1)T]} - 1\,. \end{aligned}$$

By explicit computations,

$$\Phi(1) = \Psi(1)\,, \quad \Phi'(1) = \Psi'(1) \quad \text{and} \quad \Phi''(1) = \Psi''(1)\,.$$

Thus, the function $\bar h$ agreeing with $\Phi$ in $[0,1]$ and with $\Psi$ in $[1,(k+1)T]$ belongs to $\mathrm{C}^2([0,(k+1)T],\mathbf{R})$. Define $h(x) = \bar h(|x|)$. Notice that

$$|\Phi'(\rho)| \le C\Theta\rho(\Phi+1)\,, \qquad |\Phi''(\rho)| \le C\Theta(\Phi+1) \tag{13.32}$$

in $[0,1]$ and

$$|\Psi'(\rho)| + |\Psi''(\rho)| \le C\Theta(\Psi+1) \tag{13.33}$$

in $[1,(k+1)T]$. Moreover,

$$\begin{aligned} &(h+1) + |\nabla h| + |D^2 h| \le \\ \le\ &(\bar h + 1) + \Big(1 + \frac{2}{|x|}\Big)|\bar h'| + |\bar h''|\,. \end{aligned} \tag{13.34}$$

By means of (13.32), we bound the right hand side of (13.34) in $B_1$ by $C(\Phi+1) \le Ce^{\Theta(C-(k+1)T)}$. Similarly, in $B_{(k+1)T} - B_1$, using (13.33), we bound (13.34) by $C(\Psi+1) \le Ce^{-\Theta T}$. This proves (13.25). In a similar way, one can prove (13.26). This finishes the proof of the case $d = 2$.

**Case 3:** if $d = 0$. This will be treated similarly to Case 1, but we will be able not to take derivatives of the coefficients (so that the results in Part I do not depend on any regularity of $a_{i,j}$). Defining $\sigma$ and $\beta$ as in Case 1, by Sobolev and Cauchy Inequalities, the ellipticity of the coefficients and the minimality of $u$, we deduce that

$$\begin{aligned} &\Big(\int_{B_r} \beta^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}} \le \\ \le\ &C\int_{B_r\cap\{u-\sigma\le 1+\theta\}} |\beta|\,|\nabla\beta| \le \\ \le\ &CA\Big(\int_{B_r\cap\{0<u-\sigma\le 1+\theta\}} (|\nabla u|^2 - |\nabla\sigma|^2 - 2\nabla(u-\sigma)\cdot\nabla\sigma)\Big) + \end{aligned}$$

$$
\begin{aligned}
& +\frac{C}{A}\int_{B_r\cap\{u-\sigma\le 1+\theta\}}(u-\sigma)^2 \le \\
\le\;& CA\Big(\int_{B_r\cap\{0<u-\sigma\le 1+\theta\}}(Ca_{i,j}\partial_i u\partial_j u-|\nabla\sigma|^2+2(u-\sigma)\,\Delta\sigma)\Big)+ \\
& +\frac{C}{A}\int_{B_r\cap\{u-\sigma\le 1+\theta\}}(u-\sigma)^2 \le \\
\le\;& CA\Big[\int_{B_r\cap\{0<u-\sigma\le 1+\theta\}}|\nabla\sigma|^2+(F(x,\sigma)-F(x,u))+ \\
& +2\int_{B_r\cap\{u-\sigma\le 1+\theta\}}(u-\sigma)\,\Delta\sigma\Big]+\frac{C}{A}\int_{B_r\cap\{u-\sigma\le 1+\theta\}}(u-\sigma)^2\,.
\end{aligned}
$$

The reader may compare this with (13.20) and (13.21). Defining $h$ as in (13.23), we immediately get

$$
\begin{aligned}
& \int_{B_r\cap\{u>\sigma\}}|\nabla h|^2 \le C\,\mathcal{L}^n\Big((B_r-B_{r-1})\cap\{u>\sigma\}\Big)\le \\
\le\;& C\,\mathcal{L}^n\Big((B_r-B_{r-1})\cap\{u>\theta\}\Big)+C\int_{(B_r-B_{r-1})\cap\{u\le\theta\}}\chi_{(-1,1)}(u)\,.
\end{aligned}
$$

From this and the estimates developed for Case 1, the recursive relation (13.19) readily follows. This finishes the proof of the case $d=0$.

The proof of Theorems 3.1 and 10.1 is completed.

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