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PLANE-LIKE MINIMIZERS IN PERIODIC MEDIA: JET FLOWS AND GINZBURG-LANDAU

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PLANE-LIKE MINIMIZERS IN PERIODIC MEDIA: JET FLOWS AND GINZBURG-LANDAU

by

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A Dada.

...conosco le leggi del mondo e te ne farò dono...

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This dissertation will consider the functional

$$\mathcal{G}(u) = \int a_{i,j}(x)\, \partial_i u\, \partial_j u \,+\, F(x,u)\, dx\,,$$

where the coefficients $a_{i,j}$ are Lipschitz, elliptic and periodic under integer translations and F is a "double-well potential", periodic under integer translations in x.

We will show that the functional $\mathcal{G}$ has "plane-like" minimizers: More precisely, we prove that, for any $\theta \in [0,1)$, there exists a constant M_0, depending only on θ, n, the bounds on $a_{i,j}$ and F stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{G}$ for which the set $\{|u| \leq \theta\}$ is contained in the strip $\{x \text{ s.t. } x\cdot\omega \in [0, M_0|\omega|]\}$. Also, u enjoys the following property of "quasi-periodicity": if $\omega \in \mathbf{Q}^n$, then u is periodic (with respect to the identification induced by ω); if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.

In particular, the results apply to the potentials $F(x) = Q(x)\,(1-u^2)$, and $F = Q(x)\,(1-u^2)^2$, which have application in the Ginzburg-Landau model.

A particular case is given by the functional

$$\mathcal{J}(u) = \int a_{i,j}(x)\,\partial_i u\,\partial_j u \;+\; Q(x)\,\chi_{(-1,1)}(u)\,dx\,,$$

where Q is bounded and bounded away from zero. Such functional $\mathcal{J}$ has application in models for fluid jets and capillarity.

In this case, the result stated above can be made uniform in θ, i.e.: there exists a universal constant M_0, depending only on n, the bounds on $a_{i,j}$ and Q, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{J}$ for which the set $\{|u| < 1\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.

As a limiting case, we re-obtain the recent result in [CL]: if we consider a Lipschitz Riemannian metric g on $\mathbf{R}^n$, given any $\omega \in \mathbf{R}^n$, there exists a class A, codimension 1, $g-$minimal surface contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.

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Chapter 1

Set up and notation

Many of the following pages will be devoted to the detailed study of the following functional:

$$\mathcal{J}_\Omega(u) = \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j u(x) \;+\; Q(x)\,\chi_{(-1,1)}(u)\,dx\,, \tag{1.1}$$

We assume $a_{i,j}$ to be Lipschitz and periodic under integer translations (i.e. $a_{i,j}(x+e) = a_{i,j}(x)$ for any $e \in \mathbf{Z}^n$). We also assume the matrix $a_{i,j}$ to be symmetric and positive definite: $\lambda\mathbf{1} \leq a_{i,j}(x) \leq \Lambda\mathbf{1}$, with $0 < \lambda \leq \Lambda < +\infty$. We assume that Q is periodic under integer translations, bounded and positive, say $0 < Q_{\min} \leq Q \leq Q_{\max} < +\infty$. In two dimensions, the functional above is a model for jets of incompressible, irrotational, inviscid fluids. In this setting, the function u has the physical meaning of a stream function; the first term in the functional takes into account the Equation of continuity, while the second term expresses Bernoulli's law on the free surface: this is shown in Lemma 3.2.2 here and in [AC]. The particular case $a_{i,j} = \delta_{i,j}$ corresponds to a homogeneous medium. We will give a sketchy physical motivation in Section 1.3. For a more accurate reading, the interested reader may see, for instance, [BZ] and [Se] for a classical introduction, and also [ACF1] and [ACF3] for recent developments.

An extension of the "fluid jet" functional (1.1) above is the following Ginzburg-Landau functional

$$\mathcal{G}_\Omega(u) = \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j u(x) \;+\; F(x,u)\,dx\,, \tag{1.2}$$

We assume the same hypotheses on $a_{i,j}$ as above; F is supposed to be non-negative and bounded, periodic under integer translations of the $x-$variable, and $F(x,1) = F(x,-1) = 0$. We also assume that $\inf_{|t|\le\theta} F(x,t) \ge \gamma(\theta)$, where $\gamma(\theta)$ is a decreasing, strictly positive function in the interval $[0,1)$, and that there exist $d \in [0,2]$ and $\ell \in (0,1)$ so that:

- $F(x,t) \ge \mathrm{const}\,(1-|t|)^d$, if $|t| \in (\ell,1)$;
- F is continuous in u, for $|u|<1$; F_u is defined and in $\mathrm{L}^\infty_{\mathrm{loc}}$ for $|u|<1$;
- if $d>0$, $F_u(x,u)$ is continuous for $|u|<1$, and

  $$F_u(x,-1+s) \ge \mathrm{const}\,s^{d-1}\,, \qquad F_u(x,1-s) \ge -\mathrm{const}\,s^{d-1}\,,$$

  if $s<\ell$;
- if $d=2$, $F_u(x,u)$ is decreasing for u near 1 and increasing for u near -1.

Such assumptions are made in order to make use of the results in [CC]. Important examples are given by the the potentials $F(x) = Q(x)\,(1-u^2)$ and $F = Q(x)\,|1-u^2|^2$, with Q non-negative and periodic: indeed, functionals similar to the latter were introduced in [GL] for the study of phase transition problems occurring in superconductivity and later applied in superfluid models (see [GP]). A sketch of the physical motivation for the phase transition will be given in Section 1.3. For a (much deeper) discussion about phase transition models, see, for instance, [LL], p. 206. For a related "gradient penalization method" in the theory of phase transitions, see the discussion in [Gu], p. 188. See also [R], which is a translation of van der Waals' pioneering work[1] and presents an interesting historical review.

The functionals in (1.1) and (1.2) were also dealt with in the theory of minimal surfaces: see, for instance, [Md] and [CC]. Indeed, one can see that this functionals provide an interface which is converging (uniformly, up to a subsequence) to a minimal surface. This relation also provides the bridge between the results in [CL] and the ones presented here. We discuss this in Section 1.2 and we will come back to this point in Chapter 7.

In the following, for a vector $\omega \in \mathbf{R}^n$ and $M \in \mathbf{R}$, we will consider the strip of size M by

$$S_M^\omega := \{x \in \mathbf{R}^n \text{ s.t. } x \cdot \omega \in [0, M]\} .$$

In order to avoid degeneracies caused by by "too thin" strips, we will assume $M \geq 10|\omega|$.

Often, we will consider a rational frequency $\omega \in \mathbf{Q}^n$ and the equivalence relation $\sim$ induced by ω:

$$x \sim y \qquad \text{iff} \qquad x - y = k \in \mathbf{Z}^n \quad \text{with} \quad \omega \cdot k = 0 . \tag{1.3}$$

We will denote by $\tilde{S}_M^\omega = S_M^\omega / \sim$ the quotient space induced by such relation (which is topologically equivalent to the product of an $(n-1)-$dimensional

[1] The Dutch original appeared in Verhand. Konink. Akad. Wetensch. Amsterdam (Sect. 1) 1 (1892), no. 8, 1–56; Jbuch 24, 967.

torus and a real interval). We also denote by $\mathcal{L}^n$ the $n-$dimensional Lebesgue measure.

1.1 Main results and scheme of the thesis

The main result addressed in this work is the following: We prove that, for any $\theta \in [0,1)$, there exists a constant M_0, depending only on θ, n, the bounds on $a_{i,j}$ and F stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{G}$ in (1.2) for which the set $\{|u| \le \theta\}$ is contained in the strip $\{x \text{ s.t. } x\cdot\omega \in [0, M_0|\omega|]\}$. Also, u enjoys the following property of "quasi-periodicity": if $\omega \in \mathbf{Q}^n$, then u is periodic (with respect to the identification induced by ω); if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.

A particular case of interest is the functional

$$\mathcal{J}(u) = \int a_{i,j}(x)\,\partial_i u\,\partial_j u \,+\, Q(x)\,\chi_{(-1,1)}(u)\,dx\,,$$

where Q is bounded and bounded away from zero.

In this case, the result stated above can be made uniform[2] in θ, i.e.: there exists a universal constant M_0, depending only on n, the bounds on $a_{i,j}$ and Q, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{J}$ in (1.1) for which the set $\{|u| < 1\}$ is contained in the strip $\{x \text{ s.t. } x\cdot\omega \in [0, M_0|\omega|]\}$.

[2]It is easily seen that the results can *not* be made uniform in θ for general potentials F. For instance, in one dimension, the solutions of Ginzburg-Landau are heteroclinic connections between -1 and $+1$.

The proof of this result will make use of the techniques of Alt, Caffarelli and Friedman about the *existence and rigidity* of minimizers, the *regularity result* by Giaquinta and Giusti, several *density estimates* of Caffarelli and Córdoba. We will also borrow several *geometric constructions* from the recent paper by Caffarelli and de la Llave, in which similar results were proven for minimal surfaces.

This thesis will go on as follows: we conclude this Chapter with some physical motivation and some comments on how this work relates with the analogous minimal surface problem. Then, to warm up, in Chapter 2, we will introduce a couple of elementary tools, that will be crucial during the proof. One is the so-called Birkhoff property (actually introduced in the literature by Mather). Intuitively, the Birkhoff property states that the set where the minimal minimizer is in absolute value less than one cannot cross its integer translates (which is evident, for instance, in the case of the geodesics). Then, we state and proof an "abstract" Geometric Lemma. The density property will allow us to fit this Lemma into our purposes.

In Chapter 3, we turn our attention to the regularity of the minimizers. We first recall the Theorem of Giaquinta and Giusti that provides Hölder regularity. Then, using a technique borrowed from Alt, Caffarelli and Friedman, we obtain Lipschitz regularity (for this we need that the coefficients are Lipschitz).

In Chapter 4 we recall the density estimates of Caffarelli and Córdoba and we adapt them to our case.

Chapters 5 and 6 deal with the proof of the main results in this thesis:

there we construct the plane-like minimizers for the fluid jet model and for the Ginzburg-Landau functional.

In Chapter 7 we recall the Theorem of Modica and we point out that the minimal surfaces can be seen as a limiting case of fluid jets or Ginzburg-Landau. In particular, the recent results by de la Llave and Caffarelli can be re-obtained from the ones here passing to the limit.

1.2 Plane-like minimal surfaces: the works of Morse, Hedlund, Caffarelli and de la Llave

Morse and Hedlund (see [Mo] and [H]) first proved the existence of "line-like" class A minimal geodesic on the two-dimensional torus. Namely, if we consider a periodic metric on $\mathbf{R}^2$, they prove that there exists a positive constant M so that, given any vector $\omega \in \mathbf{R}^2$, there exists a class A minimal geodesic whose support is contained in a strip of size M in direction ω. Their proofs are purely geometric and rely on a study on how the class A minimal geodesic may intersect and how they behave on periods which are multiple of a given one.

The natural extension of this problem was whether there are line-like class A minimal geodesics on the three-dimensional torus. With a very elegant example, Hedlund proved that, in general, this is not the case. Indeed, consider the unit cube and put inside three non-intersecting tubes in the three fundamental directions. Put on this cube a metric for which one pays almost zero to travel inside this cubes. Repeating periodically this fundamental cube, one obtains a metric on $\mathbf{T}^3$. Say that, with this metric, we want to travel from the origin to a point on the diagonal. If this point is far away, it is much better to stay on the "highways" given by the tubes rather than to go in the traffic outside the tubes[3]. Then, to reach a faraway point on the diagonal, it is better not to stay close to the diagonal, and hence there are no line-like geodesics in the diagonal direction in this example.

But this counterexample is not the end of the story: after many years, the question was reconsidered by Moser in [M], asking whether there are *codimension 1* plane-like minimal surfaces for a given metric on $\mathbf{T}^n$ (obviously, $n = 2$ gives exactly the case of Moser and Hedlund). Bangert addressed the case $n = 3$ in [Ba]. Caffarelli and de la Llave have given a positive answer to this question for any dimension in [CL]. They used De Giorgi's idea of considering codimension 1 surfaces that are the perimeter of a set: in this way, several densities estimates (as the Isoperimetric formula) were immediately available. The minimizer they found is constructed as a minimal minimizer, intersecting all the minimizing sets. Such minimizer indeed enjoys the so-called Birkhoff property. Using the density estimates and the Birkhoff property they can localize the minimizer into a wide enough strip. Their strategy will be largely used here in what follows.

1.3 (More or less) physical motivations

1.3.1 Fluid jets in $\mathbf{R}^2$

Consider a (homogeneous, incompressible, irrotational, inviscid) fluid jet in the plane (subject to no gravity). Let v be its velocity and p the pressure.

[3] If the highway were I-35 this might not be the case:)...

The Equation of continuity simply reads here $\operatorname{div} v = 0$. Bernoulli's law (that can be easily derived from energy balance, or from the Euler equation) states that the motion of the particles of fluid preserves the quantity $|v|^2/2 + p$.

Since v is irrotational, it admits a potential ψ, i.e. $v = \nabla\psi$. From the Equation of continuity, $\Delta\psi = 0$. Hence, there exist a function u (usually called *stream function*) so that

$$\partial_1 u = -\partial_2 \psi \qquad \text{and} \qquad \partial_2 u = \partial_1 \psi \, .$$

By construction, $\nabla u \cdot v = 0$, therefore the particles of fluid move along the level sets of u. Also, using the Equation of continuity and Bernoulli's law, one sees that

$$\Delta u = 0 \tag{1.4}$$

in the region occupied by the fluid jet, and that

$$|\nabla u|^2/2 + p = |u_\nu|^2/2 + p = \text{ const} \tag{1.5}$$

on the boundary of the jet. One can prove (see [AC] and Lemma 3.2.2 below) that the minimizers of the functional

$$\int |\nabla u|^2 + Q\, \chi_{(-1,1)}(u)$$

satisfy (1.4) and (1.5) above, for a suitable choice of Q.

1.3.2 Phase transition models

Let us consider some kind of material that has two phases, that we label by $+1$ and -1. One may expect that the equilibria in the coexistence of such phases are given by the minimization of some energy $\int W(u(x))\,dx$, where W is a "double-well potential" with minima[4] in -1 and $+1$. However, it is easy to see that this problem would admit infinitely many solutions, with no restriction on the shape of the interface between the sets $\{u = 1\}$ and $\{u = -1\}$.

On the other hand, a physically reasonable criterion would be that the interface has minimal area. To force the minimizer to have minimal interface area, it seems convenient to add a small gradient term, that would penalize too wide interfaces. Namely, one consider the minimization of

$$\int \varepsilon|\nabla u|^2 + W(u)\,dx\,.$$

This problem is now well posed, and it has been proven (see [Md]) that the minimizers u_ε converges (in L^1 up to subsequence) to a function u_0 that takes only values 1 and -1 and so that the interface between the sets $\{u_0 = 1\}$ and $\{u_0 = -1\}$ has minimal area. Moreover (see [CC]), the level sets of u_ε converge uniformly to the interface.

[4]This speculation can be made in a more rigorous way talking about the Gibbs free energy per unit volume W, which is a function of the density distribution u: see, for instance, [LL] and [Gu].

Chapter 2

Elementary (but useful!) tools

2.1 The Birkhoff property

In this section, we define a global geometric property: this property can be seen as a generalization of the so called Birkhoff property in Aubry-Mather theory (see [M], [Ma] and [MF]). A similar property has been pointed out in Proposition 8.1 of [CL] in the minimal surface setting.

If $k \in \mathbf{R}^n$, we denote by $\mathcal{T}_k$ the translation by the vector k.

Definition 2.1.1. Let $E \subseteq \mathbf{R}^n$, $\varpi \in \mathbf{R}^n$. We say that E satisfies the Birkhoff property with respect to the vector ϖ if:

- for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \leq 0$, $\mathcal{T}_k E \subseteq E$;
- for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \geq 0$, $\mathcal{T}_k E \supseteq E$.

A straightforward consequence of the definition is the following

Proposition 2.1.1. *Let $E \subseteq \mathbf{R}^n$. Assume that E satisfies the Birkhoff property with respect to ω. Then, there exists a constant ϱ, depending only on n, such that: if $\mathcal{C}(E)$ contains a ball of radius ϱ, then $\mathcal{C}(E)$ contains a strip of width 1 that intersects the ball.*

2.2 The Geometric Lemma

This section is devoted to the statement and the proof of a Geometric Lemma. The application of this "abstract" Lemma to our case will be given in Section 4.4.

Lemma 2.2.1 (Geometric Lemma). *Let $\beta \subset E \subseteq \mathbf{R}^n$, μ and ν be Radon measures on $\mathbf{R}^n$. Take $x_0 \in \beta$. Fix $r > 0$. Assume that there exist $\delta' \in (0, 1/10]$ such that:*

(i) $\forall x \in B_r(x_0)$ *and* $\forall \rho \in [\delta' r, r/10]$,

$$\mu(B_\rho(x)) \leq \mathrm{const}\, \rho^n\,;$$

(ii) $\forall x \in \beta \cap B_r(x_0)$ *and* $\forall \rho \in [\delta' r, r/10]$

$$\nu(B_\rho(x)) \geq \mathrm{const}\, \rho^{n-1}\,;$$

(iii) $\mu(E \cap B_r(x_0)) \geq \mathrm{const}\, r^n$*;*

(iv) $\nu(B_{2r}(x_0)) \leq \mathrm{const}\, r^{n-1}$.

Then, there exists δ, depending only on n and on the constants involved in $(i) - (iv)$ above, so that, if $\delta' \leq \delta$, we can find $\bar{x} \in B_r(x_0)$ such that

$$B_{\delta r}(\bar{x}) \subset (E - \beta) \cap B_{2r}(x_0)\,. \tag{2.1}$$

A similar Lemma can be stated when assumption (ii) above holds when β is a dense subset of ∂E, for instance, the"reduced boundary", as in the case of the minimal surfaces: see [CL] (and, for instance, [G1] or [MM] for the theory of minimal surfaces). In the case studied in [CL], μ is the Lebesgue measure and ν is the Haussdorff measure on the reduced boundary. In our application, μ will be the Lebesgue measure and $\nu(B) := \mathcal{J}_B(u_M^\omega)$. Namely, we will apply the Geometric Lemma to the sets $E := \{u > -1\}$, or $E := \{u < 1\}$, and $\beta := \{|u| < 1\}$. The hypotheses of the Geometric Lemma will be satisfied using the density estimates of Chapter 4.

The rough idea of the proof of the Geometric Lemma is the following. We first cover β with balls of radius δr: since β behaves like an $(n-1)$−dimensional set, we need $O(1/\delta^{n-1})$ of such balls to cover β. We then extend this cover to the whole set E: since E behaves like an n−dimensional set, we need $O(1/\delta^n)$ of such balls to cover E. Therefore, if δ is small enough, there is at least one ball in the cover of E that does not touch β.

In order to make the argument above rigorous, we need the following covering argument:

Lemma 2.2.2. *Assume the hypotheses (ii) and (iv) of Lemma 2.2.1. Fix $\delta \in [5\delta', 1/2]$. Then, there exists a family of points $y_k \in \beta$, $k = 1, \dots, N$, such that:*

- $B_{\delta r/5}(y_k) \cap B_{\delta r/5}(y_h) = \emptyset$ *if* $k \neq h$;
- $\beta \cap B_r(x_0) \subseteq \bigcup_{k=1}^N B_{\delta r}(y_k)$;
- $N \leq \mathrm{const}\, \delta^{-(n-1)}$.

Proof. Take any $y_1 \in \beta \cap B_r(x_0)$. If $\beta \cap B_r(x_0) \subseteq B_{\delta r}(y_1)$, we can stop. Otherwise, take $y_2 \in \beta \cap B_r(x_0)$ not belonging to $B_{2\delta r/5}(y_1)$. Iterating this

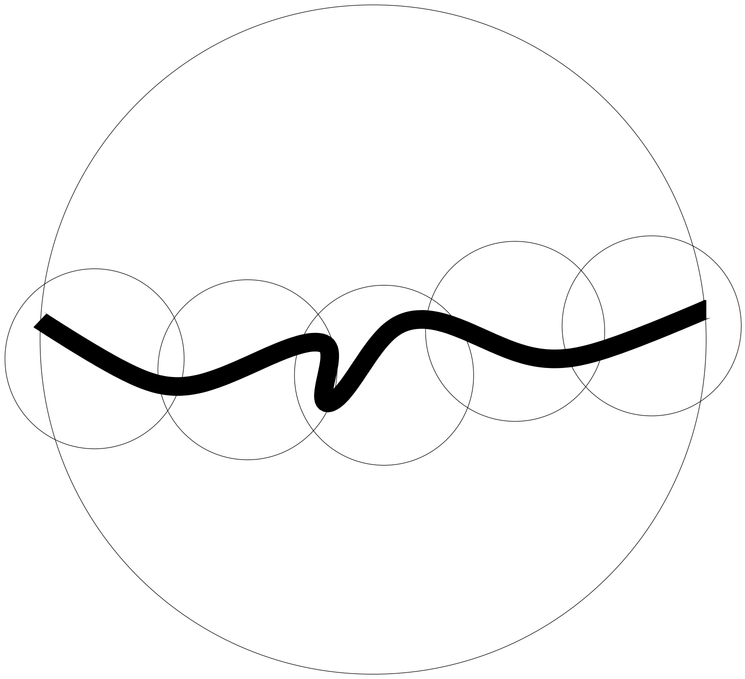

Figure 2.1: The Geometric Lemma

procedure, assume to have a family of y_k's such that $y_k \in \beta \cap B_r(x_0)$ for $1 \leq k \leq N$, verifying $B_{\delta r/5}(y_k) \cap B_{\delta r/5}(y_h) = \emptyset$ if $k \neq h$. If $\beta \cap B_r(x_0) \subseteq \bigcup_{k=1}^N B_{\delta r}(y_k)$, we stop; otherwise select $y_{N+1} \in \beta \cap B_r(x_0)$ outside $\bigcup_{k=1}^N B_{2\delta r/5}(y_k)$.

We now estimate the "good" N for which we stop the procedure: in light of hypothesis (ii) of Lemma 2.2.1,

$$\nu\Big(\bigcup_{k=1}^N B_{\delta r/5}(y_k)\Big) = \sum_{k=1}^N \nu\Big(B_{\delta r/5}(y_k)\Big) \geq \text{const}\, N\, (\delta r)^{n-1}\,.$$

Also, by construction, $B_{\delta r/5}(y_k) \subseteq B_{2r}(x_0)$. Hence, hypothesis (iv) of Lemma 2.2.1 implies that

$$\nu\Big(\bigcup_{k=1}^N B_{\delta r/5}(y_k)\Big) \leq \text{const}\, r^{n-1}\,.$$

These two inequalities show that $N \leq \text{const}\, \delta^{1-n}$. $\square$

We now address the proof of Lemma 2.2.1:

Proof. Consider a finite overlapping cover $\mathcal{B}$ of $B_r(x_0)$, consisting in balls radius δr, with $\#\mathcal{B} \leq \mathrm{const}\,\delta^{-n}$.

Since $\mathcal{B}$ is finite overlapping, any of the balls $\Big\{B_{\delta r}(y_k)\Big\}_{1\leq k\leq N}$, constructed in Lemma 2.2.2 cannot intersect more than $\tilde{C}$ balls belonging to $\mathcal{B}$, where $\tilde{C}$ is a universal constant.

Then, out of the collection $\mathcal{B}$, only $\mathrm{const}\,\delta^{-(n-1)}$ (at most) may intersect $\beta \cap B_r(x_0)$, since

$$\begin{aligned} & \#\{B \in \mathcal{B} \text{ s.t. } B \cap \beta \cap B_r(x_0) \neq \emptyset\} \leq \\ \leq\ & \#\{B \in \mathcal{B} \text{ s.t. } B \cap B_{\delta r}(y_k) \neq \emptyset \text{ for some } k\} \leq \\ \leq\ & \tilde{C} N \leq \mathrm{const}\,\delta^{-(n-1)}\,. \end{aligned} \tag{2.2}$$

We now prove that, if δ is chosen suitably small, at least one ball B^* of the collection $\mathcal{B}$ must be contained in $E - \beta$. If not,

$$\bigcup_{B\in\mathcal{B}} B = \Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\subseteq \mathcal{C}(E)}} B\Big) \cup \Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big),$$

so, by (2.2) and hypotheses (i) and (iii),

$$\begin{aligned} \mathrm{const}\,r^n\ &\leq\ \mu(E\cap B_r(x_0)) \leq \mu\Big(E \cap \Big(\bigcup_{B\in\mathcal{B}} B\Big)\Big) \leq \\ &\leq\ \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\subseteq\mathcal{C}(E)}} B\Big)\Big) + \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big)\Big) = \\ &=\ 0 + \mu\Big(E\cap\Big(\bigcup_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} B\Big)\Big) \leq \\ &\leq\ \sum_{\substack{B\in\mathcal{B}\\ B\cap\beta\neq\emptyset}} \mu(B) \leq \mathrm{const}\,\delta^{-(n-1)}\,(\delta r)^n\,, \end{aligned}$$

which is a contradiction if δ is small enough. Such ball B^* satisfy the desired property (2.1). $\square$

Chapter 3

Regularity of the minimizers

This chapter will deal with the problem of the regularity of minimizers for the functional related with the fluid jets:

$$\mathcal{J}_\Omega(u) = \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j u(x)\;+\;Q(x)\,\chi_{(-1,1)}(u)\,dx\,.$$

In the following, for a vector $\omega \in \mathbf{R}^n$ and $M \in \mathbf{R}$, we denote the strip of size M by

$$S_M^\omega := \{x \in \mathbf{R}^n \text{ s.t. } x\cdot\omega \in [0,M]\}\,.$$

We will consider a rational frequency $\omega \in \mathbf{Q}^n$ and we will denote by $\tilde{S}_M^\omega = S_M^\omega/\sim$ the quotient space induced by such relation (which is topologically equivalent to the product of an $(n-1)-$dimensional torus and a real interval). We also set

$$\mathcal{X}_M^\omega := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}_M^\omega) \text{ s.t. } \nabla u \in \mathrm{L}^2(\tilde{S}_M^\omega)$$

$$\text{with } u(x) = 1 \text{ for any } x \text{ s.t. } \omega\cdot x = 0 \text{ and}$$

$$u(x) = -1 \text{ for any } x \text{ s.t. } \omega\cdot x = M\}\,.$$

We will consider the set $\mathcal{M}_M^\omega$ of minimizers for the functional $\mathcal{J}_{S_M^\omega}$ in the space $\mathcal{X}_M^\omega$.

Comparing $\mathcal{J}_{S_M^\omega}(u)$, $\mathcal{J}_{S_M^\omega}(\min\{u,1\})$ and $\mathcal{J}_{S_M^\omega}(\max\{u,-1\})$, the reader may easily convince herself that we can assume the minimizers in $\mathcal{M}_M^\omega$ to take value in $[-1,1]$.

In Section 3.1, we will recall the celebrated Theorem of Giaquinta and Giusti [GG], stating that the minimizers are Hölder continuous.

In Section 3.2, using some techniques developed by Alt, Caffarelli and Friedman, we will point out that, under the assumption that the coefficients are Lipschitz, then the minimizers are Lipschitz[1].

It is easy to see that Lipschitz regularity is optimal even for smooth coefficients (due to the growth from the free boundary). However, I do not know whether the assumption on the coefficients to be Lipschitz can be removed or not.

3.1 Hölder regularity: the Theorem of Giaquinta and Giusti

In this Section we recall one of the main results of [GG]. Let $\mathcal{F}$ be a functional of the type

$$\mathcal{F}_\Omega(u) = \int_\Omega f(x,u,\nabla u)\,dx\,.$$

We assume a very mild regularity on f and a suitable growth condition: for instance, it is enough to ask $f(x,u,p)$ to be measurable in x for each $(u,p) \in$

[1]Hölder continuity of the minimizers can actually be shown for bounded measurable coefficients, while the proofs of the Lipschitz regularity and of the density estimates given here rely on the Lipschitz regularity of the coefficients.

$\mathbf{R} \times \mathbf{R}^n$, to be continuous in (u, p) for almost every $x \in \Omega$, and to satisfy

$$\frac{1}{a}|p|^m - k \leq f(x, u, p) \leq a|p|^m + k\,,$$

for suitable constants $a > 1$, $k > 0$.

We say that $u \in \mathrm{W}^{1,m}_{\mathrm{loc}}(\Omega)$ is a *local minimum* for $\mathcal{F}$ if

$$\mathcal{F}_{\mathrm{supp}\varphi}(u) \leq \mathcal{F}_{\mathrm{supp}\varphi}(u + \varphi)\,,$$

for any test function φ whose support is well contained in Ω.

Theorem 3.1.1 (Giaquinta and Giusti). *Let $\mathcal{F}$ be a functional satisfying the assumptions above, and let u be a local minimum. Then u is Hölder continuous.*

The result can actually be stated for quasi minima (see [G2]). With other techniques it is also possible to prove a similar result for $\omega-$minima (see [DEF]). For completeness, we now sketch the proof of Theorem 3.1.1. The idea of the proof is that local minima satisfy a suitable Caccioppoli Inequality. From this, one can apply (a suitable modification of) De Giorgi's iteration method.

For the sake of simplicity, we will sketch the proof for the functional (1.1): the extension to the general case does not provide any conceptual difficulty.

We will set, for $k \in \mathbf{R}$ and $R > 0$,

$$A(k, R) = \{u > k\} \cap Q_R \qquad \text{and} \qquad B(k, R) = \{u < k\}\,.$$

Notice that, a.e. k,

$$\mathcal{L}^n(A(k,R)) + \mathcal{L}^n(B(k,R)) = \mathcal{L}^n(Q_R)\,.$$

Here is the Caccioppoli Inequality we need:

Proposition 3.1.2. *With the above setting, for any $k \in \mathbf{R}$, for any $0 < \rho < R <$ dist $(0, \partial\Omega)$,*

$$\frac{1}{C}\int_{A(k,\rho)} |\nabla u|^2 \leq \frac{1}{(R-\rho)^2}\int_{A(k,R)} |u-k|^2 \,+\, \mathcal{L}^n(A(k,R))\,,$$

where C is a constant, depending only on the assumptions on $a_{i,j}$ and Q.

Proof. Let η be a smooth function supported inside Q_R so that $Q\big|_{Q_\rho} = 1$. Set $w = (u-k)^+$, $v = u - \eta w$. Since $\mathfrak{J}(u) \leq \mathfrak{J}(v)$ and $u = v$ outside $A(k,R)$,

$$\int_{A(k,R)} a_{i,j}\partial_i u \partial_j u \leq \int_{A(k,R)} a_{i,j}\partial_i v \partial_j v + Q(x)\left(\chi_{(-1,1)}(v) - \chi_{(-1,1)}(u)\right).$$

Since in $A(k,R)$ we have that $v = u(1-\eta) + k\eta$, we infer that $a_{i,j}\partial_i v\partial_j v$ is controlled by

$$(1-\eta)^2\, a_{i,j}\partial_i u \partial_j u + \frac{|u-k|^2}{(R-\rho)^2}\,.$$

$\square$

In the same way, one can prove that

$$\frac{1}{C}\int_{B(k,\rho)} |\nabla u|^2 \leq \frac{1}{(R-\rho)^2}\int_{B(k,R)} |u-k|^2 \,+\, \mathcal{L}^n(B(k,R))\,.$$

Proposition 3.1.3. *If $|k_0| \leq \sup_{Q_R} |u|$, then*

$$\frac{1}{C}\sup_{Q_{R/2}} u \leq \left(\frac{1}{R^n}\int_{A(k_0,R)} (u-k_0)^2\right)^{1/2} \cdot \left(\frac{\mathcal{L}^n(A(k_0,R))}{R^n}\right)^{\alpha/2} + k_0 + R\,,$$

with $\alpha^2 + \alpha = 2/n$.

Proof. Without loss of generality, $k_0 = 0$. Let $\sigma < \tau$ and let η be a smooth function supported in $Q_{(\sigma+\tau)/2}$, with $\eta\Big|_{Q_\sigma} = 1$. Set $\zeta = \eta(u-k)^+$. Then, by Hölder and Poincaré Inequalities,

$$\int_{A(k,\sigma)} (u-k)^2 \leq \int_{Q_\sigma} \zeta^2 \leq C \int |\nabla\zeta|^2 \, \mathcal{L}^n(A(k,\tau))^{2/n} \,.$$

Since $\nabla\zeta = (\nabla u(u-k)^+ + \eta\nabla u) \cdot \chi_{A(k,(\sigma+\tau)/2)}$, the last term is controlled by

$$\mathcal{L}^n(A(k,\tau))^{2/n} \Big(\int_{A(k,(\sigma+\tau)/2)} |\nabla u|^2 + \frac{1}{(\tau-\sigma)^2} \int_{A(k,(\sigma+\tau)/2)} (u-k)^2 \Big) \,.$$

Using Proposition 3.1.2, one bounds the above quantity with

$$\mathcal{L}^n(A(k,\tau))^{2/n} \Big(\frac{1}{(\tau-\sigma)^2} \int_{A(k,\tau)} (u-k)^2 + \mathcal{L}^n(A(k,\tau)) \Big) \,.$$

Thus, collecting the estimates above, one has that, for any $\rho < r \leq R$,

$$\frac{1}{C} \int_{A(k,\rho)} (u-k)^2 \leq \frac{\mathcal{L}^n(A(k,r))^{\frac{2}{n}}}{(r-\rho)^2} \int_{A(k,r)} (u-k)^2 + \mathcal{L}^n(A(k,r))^{\frac{2+n}{n}} \,. \quad (3.1)$$

Define $U(k,r) = \int_{A(k,r)} (u-k)^2$. Notice that, if $h < k$, $A(k,r) \subseteq A(h,r)$ and $U(k,r) \leq U(h,r)$. Hence,

$$\frac{U(h,r)}{(k-h)^2} \geq \frac{1}{(k-h)^2} \int_{A(k,r)} (u-h)^2 \geq \mathcal{L}^n(A(k,r)) \,. \quad (3.2)$$

From (3.1) and (3.2), one gets that

$$\frac{1}{C} U(k,\rho) \leq U(h,r) \cdot \mathcal{L}^n(A(h,r))^{\frac{2}{n}} \cdot \Big(\frac{1}{(r-\rho)^2} + \frac{1}{(k-h)^2} \Big) \quad (3.3)$$

From this, using again (3.2) and the definition of α, we infer that

$$\frac{1}{C} \mathcal{L}^n(A(k,\rho))^\alpha U(k,\rho) \leq \frac{U(h,r)^{1+\alpha} \, \mathcal{L}^n(A(h,r))^{\frac{2}{n}}}{(k-h)^{2\alpha}} \Big(\frac{1}{(r-\rho)^2} + \frac{1}{(k-h)^2} \Big) \,.$$

Set now $\Phi(k,r) = U(k,r) \, \mathcal{L}^n(A(k,r))^\alpha$. The above formula then reads

$$\Phi(k,\rho) \leq \frac{C}{(k-h)^{2\alpha}} \Big(\frac{1}{(r-\rho)^2} + \frac{1}{(k-h)^2} \Big) \Phi(h,r)^{1+\alpha} \,, \quad (3.4)$$

where we also used that $\alpha(\alpha+1) = 2/n$. Let $d \geq R$ be a suitable constant, to be chosen later. Set

$$k_i = d(1 - 2^{-i}) \qquad \text{and} \qquad r_i = \frac{R}{2}(1 + 2^{-i})\,.$$

We now apply (3.4), with $\rho = r_{i+1}$, $r = r_i$, $k = k_{i+1}$, $h = k_i$ and $\Phi_i = \Phi(k_i, r_i)$: from this inductive relation it is standard (see, for instance, Lemma 7.1 of [G2]) to deduce that Φ_i goes to zero for i going to infinity, if one chooses d of the order of $R + R^{-1/\alpha}\Phi_0^{1/2}$. Thus, $\Phi(d, R/2) = 0$ and so $\mathcal{L}^n(A(d, R/2)) = 0$. Therefore, $\sup_{Q_{R/2}} u \leq Cd$. $\square$

We can now state the Oscillation Lemma we need here: we define $M(R) = \sup_{Q_R} u$, $m(R) = \inf_{Q_R} u$, $o(R) = \operatorname{osc}_{Q_R} u$.

Lemma 3.1.4. *Let $k_0 = (M(2R) + m(2R))/2$ and assume that*

- $\mathcal{L}^n(A(k_0, R)) \leq \gamma \mathcal{L}^n(Q_R)$, *for some $\gamma \in (0, 1)$;*
- $o(2R) \geq 2^{\nu+1} R$, *for some $\nu \in \mathbf{N}$.*

Set $k_\nu = M(2R) - 2^{-\nu-1} o(2R)$. Then,

$$\mathcal{L}^n(A(k_\nu, R)) \leq \nu^{-\frac{n}{2(n-1)}} \mathcal{L}^n(Q_R)\,.$$

Proof. Fix $k_0 < h < k$. Let v be the continuous function equal to $u - h$ in the set $\{h < u < k\}$ and constant outside. Obviously, $v = 0$ outside $A(k_0, R)$ and

$$\mathcal{L}^n(Q_R - A(k_0, R)) \geq (1 - \gamma)\,\mathcal{L}^n(Q_R)\,.$$

Thus, by Sobolev Inequality,

$$\Big(\int_{Q_R} v^{\frac{n}{n-1}}\Big)^{1-\frac{1}{n}} \leq \int_{A(h,R)-A(k,R)} |\nabla v| =$$
$$= \int_{A(h,R)-A(k,R)} |\nabla u| \leq D\sqrt{\int_{A(h,R)} |\nabla u|^2}\,,$$

where we set $D = \sqrt{\mathcal{L}^n(A(h,R)-A(k,R))}$. Also,

$$\Big(\int_{A(k,R)} |\nabla v|^{\frac{n}{n-1}}\Big)^{1-\frac{1}{n}} = (k-h)\,\mathcal{L}^n(A(k,R))^{1-\frac{1}{n}}\,.$$

Therefore,

$$(k-h)\,\mathcal{L}^n(A(k,R))^{1-\frac{1}{n}} \leq D\sqrt{\int_{A(h,R)} |\nabla u|^2}\,. \tag{3.5}$$

From Proposition 3.1.2,

$$\int_{A(h,R)} |\nabla u|^2 \leq C\Big(R^{n-2}(M(2R)-h)^2 + R^n\Big)\,. \tag{3.6}$$

If $h \leq k_\nu$,

$$M(2R)-h \geq M(2R)-k_\nu = 2^{\nu-1}o(2R) \geq R\,,$$

so, by means of (3.5) and (3.6),

$$(k-h)\,\mathcal{L}^n(A(k,R))^{1-\frac{1}{n}} \leq CD\,R^{\frac{n-2}{2}}\,(M(2R)-h)\,.$$

Hence, taking

$$k = k_i = M(2R) - 2^{-i-1}o(2R) \qquad \text{and} \qquad h = k_{i-1}$$

we obtain

$$\mathcal{L}^n(A(k_i,R))^{\frac{n-1}{n}} \leq CD_iR^{\frac{n-2}{2}}\,,$$

where $D_i = \sqrt{\mathcal{L}^n(A(k_i,R)) - \mathcal{L}^n(A(k_{i-1},R))}$. Hence,

$$\nu\,\mathcal{L}^n(A(k_\nu,R))^{\frac{2(n-1)}{n}} \le \sum_{i=1}^{\nu} \mathcal{L}^n(A(k_i,R))^{\frac{2(n-1)}{n}} \le$$

$$\le \sum_{i=1}^{\nu} C D_i^2 R^{n-2} \le C R^{n-2}\,\mathcal{L}^n(A(k_\nu,R) - A(k_0,R)) \le C R^{2(n-1)}\,.$$

□

We are now in position to finish the proof of Theorem 3.1.1: following the notation in Lemma 3.1.4 above, without loss of generality, we may assume that $\mathcal{L}^n(A(k_0,R)) \le \frac{1}{2}\mathcal{L}^n(Q_R)$. We may also assume that $\nu^{-\frac{n}{2(n-1)}}$ is small, say less than some ε_0. From Lemma 3.1.4, if $o(2R) \ge 2^{\nu+1}R$, then

$$\mathcal{L}^n(A(k_\nu,R)) \le \varepsilon_0\,\mathcal{L}^n(Q_R)\,.$$

From this and Proposition 3.1.3, we get that

$$\begin{aligned} M(R/2) - k_\nu &\le C\Big[\Big(M(2R) - k_\nu\Big)\cdot\Big(\frac{\mathcal{L}^n(A(k_\nu,R))}{R^n}\Big)^{\frac{\alpha}{2}} + R\Big] \le \\ &\le C[\varepsilon_0^{\alpha/2}(M(2R) - k_\nu) + R]\,; \end{aligned}$$

therefore, if ε_0 is small enough,

$$M(R/2) - k_\nu \le \frac{1}{2}(M(2R) - k_\nu) + CR\,.$$

Subtracting $m(2R)$ to both sides, we get

$$o(R/2) \le \Big(1 - \frac{1}{2^{\nu+1}}\Big)o(2R) + CR\,. \tag{3.7}$$

From these considerations, we have that either (3.7) holds or $o(2R) \le 2^{\nu+1}R$. In any case,

$$o(R/2) \le \Big(1 - \frac{1}{2^{\nu+1}}\Big)o(2R) + 2^\nu CR\,. \tag{3.8}$$

Iterating (3.8), a standard recursive argument (see, for instance, Lemma 7.3 of [G2] at page 248) proves that u is Hölder continuous. This finishes the proof of Theorem 3.1.1.

3.2 Lipschitz regularity

We will now prove the following

Theorem 3.2.1. *If $u \in \mathcal{M}^{\omega}_{M}$, and $a_{i,j} \in \mathrm{C}^{0,1}(\mathbf{R}^n)$, then u is $\mathrm{C}^{0,1}$. Furthermore, given any $K \subset\subset S^{\omega}_{M}$, the Lipschitz norm of u in K is bounded by a constant depending only on n, the bounds on $a_{i,j}$ and Q and on* $\mathrm{dist}\,(K, \partial S^{\omega}_{M})$.

Theorem 3.2.1 can be seen as an extension of the results by Alt, Caffarelli and Friedman to the functional in (1.1) under a more general assumption on the coefficients. The proof will be accomplished in several steps.

First, we point out the PDE-properties of the minimizers:

Lemma 3.2.2. *Let $u \in \mathcal{M}^{\omega}_{M}$. Then $\partial_i(a_{i,j}\partial_j u) = 0$ weakly in $\{|u| < 1\}$..*

Proof. Take $\varphi \in \mathrm{C}^{\infty}_{0}(\{|u| < 1\})$ and compute the first variation of the functional. □

Lemma 3.2.3. *Let $u \in \mathcal{M}^{\omega}_{M}$. Then $\partial_i(a_{i,j}\partial_j u) \geq 0$ weakly in $\{u < 1\}$. and $\partial_i(a_{i,j}\partial_j u) \leq 0$ weakly in $\{u > -1\}$.*

Proof. Taking, a non-negative $\varphi \in \mathrm{C}^{\infty}_{0}(\{u < 1\})$ and computing the first variation, we obtain

$$0 \leq \limsup_{\varepsilon \longrightarrow 0^+} \frac{1}{2\varepsilon}\Big(\mathcal{J}_{S^{\omega}_{M}}(u - \varepsilon\varphi) - \mathcal{J}_{S^{\omega}_{M}}(u)\Big) \leq -\int_{S^{\omega}_{M}} a_{i,j}\partial_i\varphi\partial_j u\,,$$

which proves the first claim. The other claim can be proved in a similar way. □

Now, a barrier argument:

Lemma 3.2.4. *Fix $\mu > 0$. Let $\beta \in (0,1)$. Assume that $v \in \mathrm{H}^1(B_r)$, $v \geq 0$, $\partial_i(a_{i,j}\partial_j v) \leq 0$, and*

$$v \geq \mu \quad \text{on} \quad \partial B_r \cap \{x_n \leq (-1+\beta)r\}\,.$$

Then, there exists a constant $\theta \in (0,\beta)$, depending only on n, β and the bounds on $a_{i,j}$, such that: $v(0,\ldots,(-1+\theta)r) \geq \mu\beta/2$.

Proof. By scaling, it is enough to prove the claim for $r = 1$. Set $c^\star := 2n||a_{i,j}||_{\mathrm{W}^{1,\infty}}$. Let ψ be the solution of $\partial_i(a_{i,j}\partial_j\psi) = c^\star\mu$ in B_1, with $\psi = 0$ on ∂B_1. By elliptic estimates (see, for instance [GT], Theorem 8.16 and Theorem 8.27)

$$|\psi(0,\ldots,0,(-1+\theta))| \leq \mathrm{osc}_{B_1\cap B_{2\theta}(0,\ldots,0,-1)}\psi \leq c\theta^\alpha\mu\,, \tag{3.9}$$

for suitable constants c and α, depending only on n and the bounds on $a_{i,j}$. Define

$$\phi := \mu[-(x_n+1)+\beta] \quad \text{and} \quad \zeta := \phi + \psi\,. \tag{3.10}$$

We have $\partial_i(a_{i,j}\partial_j\zeta) = -\partial_i a_{i,n}\mu + c^\star\mu > 0$. We now check that $v \geq \zeta$ on ∂B_1. Indeed, if $x \in \partial B_1 \cap \{x_n \leq -1+\beta\}$,

$$\zeta(x) = \phi(x) \leq \mu\beta \leq v(x)\,;$$

on the other hand, if $x \in \partial B_1 \cap \{x_n > -1+\beta\}$, $\zeta(x) \leq 0 \leq v(x)$.

Hence, by Maximum Principle, $v \geq \zeta$ in B_1. In particular, by (3.9) and (3.10), $v(0, \dots, 0, -1+\theta) \geq \mu(\beta - c\theta^\alpha - \theta) \geq \mu\beta/2$, if θ is small enough with respect to β. $\square$

We would like to remark that an alternative proof of the Lemma above can be accomplished by using the boundary Harnack inequality.

Lemma 3.2.5. *Let $u \in \mathcal{M}_M^\omega$. Suppose that $x_0 \in S_M^\omega$ verifies*

$$\operatorname{dist}(x_0, \partial S_M^\omega) > 2 \operatorname{dist}(x_0, \{|u| = 1\}) .$$

Then,

$$\min\{1 - u(x_0),\, 1 + u(x_0)\} \leq C \operatorname{dist}(x_0, \{|u| = 1\}) ,$$

where C is a suitable constant, depending only on n, the bounds on $a_{i,j}$ and F.

Proof. If $u(x_0) = \pm 1$, there is nothing to prove. Set $U_\pm = 1 \pm u$, and $d_0 := \operatorname{dist}(x_0, \{|u| = 1\})$.

We assume that $\min\{U_+(x_0),\, U_-(x_0)\} \geq M d_0$, and derive an upper bound for M. Notice that $B_{9d_0/10}(x_0) \subset \{|u| < 1\}$, so by Lemma 3.2.2 and Harnack inequality

$$\inf_{B_{3d_0/4(x_0)}} U_\pm \geq \text{const} \cdot \sup_{B_{3d_0/4(x_0)}} U_\pm \geq \text{const}\, M d_0 . \tag{3.11}$$

Let $y \in B_{d_0}(x_0) \cap \{|u| = 1\}$.

Without loss of generality, we may assume $y = 0$. We now denote, for short, $B := B_{d_0}$ and $U := U_\pm$.

Let $v := v_\pm$ be the minimum of

$$\int_B a_{i,j} \partial_i \zeta \partial_j \zeta$$

among the functions $\zeta \in \mathrm{H}^1(B)$ with $\zeta = U_\pm$ on ∂B. Hence,

$$\begin{aligned} \text{const} \int_B |\nabla(U-v)|^2 \leq \int_B a_{i,j}\partial_i(U-v)\partial_j(U-v) = \\ = \int_B a_{i,j}\partial_i u \partial_j u - \int_B a_{i,j}\partial_i v \partial_j v \,. \end{aligned} \tag{3.12}$$

By construction, $V = V_\pm := \pm(v-1) = u$ on ∂B: hence by (3.12) and the fact that $u \in \mathcal{M}^\omega_M$,

$$\int_B |\nabla(U-v)^2| \leq \text{ const} \int_B Q\chi_{(-1,1)}(V) - \chi_{(-1,1)}(u)\,. \tag{3.13}$$

By Maximum Principle and the fact that $0 \leq U \leq 2$, we deduce that either $0 < v < 2$ or v is constant on B.

In the latter case, since $u \in \mathcal{M}^\omega_M$, u must be constant in B: in particular $u(x_0) = u(0) = \pm 1$, and the claim is proven.

Thus, we can now restrict ourselves to the case $0 < v < 2$ in B, i.e. $-1 < V < 1$. From (3.13),

$$\int_B |\nabla(U-v)|^2 \leq \text{ const}\,\mathcal{L}^n(\{|u| = 1\} \cap B)\,. \tag{3.14}$$

From (3.11) and Lemma 3.2.4, there is a suitable constant $\kappa \in (0,1)$ and a suitable point $x^\star \in B_{\kappa d_0}$ so that $v(x^\star) \geq \text{ const}\, M d_0$. Harnack inequality then implies

$$\inf_{B_{\kappa d_0}} v \geq \text{ const}\, M d_0\,. \tag{3.15}$$

Set now $v^*(x) := \frac{1}{d_0} v(d_0 x)$, for $x \in B_1$. Let us now introduce the following barrier:

$$w(x) = \bar{c}M(e^{-\mu|x|^2} - e^{-\mu})\,,$$

with $\bar{c}$ and μ suitable constants, properly chosen to satisfy

$$\partial_i(a_{i,j}\partial_j w) > 0 \qquad \text{in } B_1 - B_\kappa\,,$$
$$w = 0 \qquad \text{on } \partial B_1 \qquad \text{and} \qquad w \le v^* \qquad \text{on } \partial B_\kappa\,,$$

where the last relation can be fulfilled in light of (3.15). So, by Maximum Principle,

$$v^*(x) \ge w(x) \ge \text{ const}\, M(1-|x|) \qquad \text{in } B_1 - B_\kappa\,.$$

Recalling (3.15), we get

$$v^*(x) \ge \text{ const}\, M(1-|x|) \qquad \text{in } B_1\,.$$

Rescaling back:

$$v(x) \ge \text{ const}\, M d_0 (1 - \frac{|x|}{d_0}) \qquad \text{in } B_{d_0}\,. \tag{3.16}$$

We can now follow [AC], p. 112: Fix any $y \in B_{d_0/2}$ and consider the following transformation of B_{d_0} such that y becomes the new origin:

$$T_y(x) := \frac{d_0 - |x|}{d_0} y + x\,,$$

and set, for any function f, $f_y(x) := f(T_y(x))$.

For any $\zeta \in \partial B_1$, define

$$r_\zeta := \inf\{r \in [d_0/8, d_0] \text{ s.t. } U(r\zeta) = 0\}\,,$$

if this set is non-empty. For almost all such ζ we have

$$\begin{aligned} v_y(r_\zeta \zeta) &= \int_{r_\zeta}^{d_0} \nabla(U_y - v_y)(\rho\zeta)\cdot\zeta\, d\rho \le \\ &\le \sqrt{\int_{r_\zeta}^{d_0} |\nabla(U_y - v_y)(\rho\zeta)|^2\, d\rho} \cdot \sqrt{d_0 - r_\zeta}\,. \end{aligned} \tag{3.17}$$

By (3.16), for any $r \in [0, d_0]$,

$$\begin{aligned} v_y(r\zeta) &\geq \text{const}\, M\Big(d_0 - \Big|\frac{d_0 - r}{d_0} y + r\zeta\Big|\Big) \geq \\ &\geq \text{const}\, M\Big(d_0 - \frac{d_0 - r}{d_0}|y| - r\Big) \geq \text{const}\, M\,\frac{d_0 - r}{2}\,, \end{aligned}$$

since $|y| \leq d_0/2$. From this and (3.17)

$$M^2(d_0 - r_\zeta) \leq \text{const} \int_{r_\zeta}^{d_0} |\nabla(U_y - v_y)(\rho\zeta)|^2 \, d\rho\,. \tag{3.18}$$

Also, using polar coordinates and the definition of r_ζ,

$$\begin{aligned} &\int_{B_{d_0} - B_{d_0/4}(y)} \chi_{\{U=0\}} \leq \text{const} \int_{B_{d_0} - B_{d_0/8}} \chi_{\{U_y=0\}} \leq \\ &\leq \text{const} \int_{\partial B_1} \int_{d_0/8}^{d_0} \chi_{\{U_y=0\}}(\rho\zeta)\, d\rho\, d\zeta = \\ &= \text{const} \int_{\partial B_1} \int_{r_\zeta}^{d_0} \chi_{\{U_y=0\}}(\rho\zeta)\, d\rho\, d\zeta \leq \\ &\leq \text{const} \int_{\partial B_1} (d_0 - r_\zeta)\, d\zeta\,. \end{aligned}$$

From this, integrating both sides of (3.18), it follows that

$$\begin{aligned} &\text{const}\, M^2 \int_{B_{d_0} - B_{d_0/4}(y)} \chi_{\{U=0\}} \leq \int_{\partial B_1} \int_{d_0/8}^{d_0} |\nabla(U_y - v_y)(\rho\zeta)|^2 \, d\rho\, d\zeta \leq \\ \leq \quad &\text{const} \int_{B_{d_0}} |\nabla(U - v)|^2\,. \end{aligned}$$

Applying this result to two points y_1 and y_2 in $B_{d_0/2}$ so that $B_{d_0/4}(y_1)$ and $B_{d_0/4}(y_2)$ do not overlap, and summing up the two estimates, we obtain

$$M^2 \int_{B_{d_0}} \chi_{\{U=0\}} \leq \text{const} \int_{B_{d_0}} |\nabla(U - v)|^2\,.$$

Thus, recalling (3.14), and writing $U = U_\pm$ explicitly:

$$M^2 \mathcal{L}^n(\{U_\pm = 0\} \cap B_{d_0}) \leq \text{const}\, \mathcal{L}^n(\{|u| = 1\} \cap B_{d_0})\,.$$

Since $\{|u| = 1\} \subseteq \{U_+ = 0\} \cup \{U_- = 0\}$, we deduce that

$$M^2 \mathcal{L}^n(\{|u| = 1\} \cap B_{d_0}) \leq 2 \,\mathrm{const}\, \mathcal{L}^n(\{|u| = 1\} \cap B_{d_0}) \,.$$

If M were too big, recalling (3.14), we get $|u| < 1$ in B_{d_0}, which is a contradiction. □

Theorem 3.2.6. *If $u \in \mathcal{M}_M^\omega$, then $u \in \mathrm{C}^{0,1}(S_M^\omega)$. Furthermore, for any compact domain D well contained in S_M^ω with*

$$D \cap \{|u| = 1\} \neq \emptyset \,, \tag{3.19}$$

there exists a suitable positive constant C_D, depending only on $\mathrm{dist}\,(D, \partial S_M^\omega)$, *$n$ and the bounds on $a_{i,j}$ and F such that*

$$||u||_{\mathrm{C}^{0,1}} \leq C_D \,. \tag{3.20}$$

Proof. First of all, we show that ∇u is bounded in $D \cap \mathcal{N}$, where $\mathcal{N}$ is a suitable neighborhood of $\partial\{|u| = 1\}$.

Indeed: if $\mathrm{dist}\,(x, \{|u| = 1\}) \leq \frac{1}{10} \mathrm{dist}\,(x, \partial S_M^\omega)$, we are under the hypotheses of Lemma 3.2.5, so that (setting $B := B_{\mathrm{dist}\,(x,\{|u|=1\})/10}(x)$ for short)

$$\sup_{x_0 \in B} \min\{1 - u(x_0),\, 1 + u(x_0)\} \leq \,\mathrm{const}\, \mathrm{dist}\,(x, \{|u| = 1\}) \,. \tag{3.21}$$

Also, by Lemma 3.2.2, $\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0$ in B; therefore, by elliptic estimates,

$$|\nabla(1 \pm u)(x)| \leq \,\mathrm{const}\, \frac{\sup_B |1 \pm u|}{\mathrm{dist}\,(x, \{|u| = 1\})} \,.$$

From this and (3.21), we deduce a bound on ∇u in $D \cap \mathcal{N}$.

This and standard elliptic estimates imply that $u \in \mathrm{C}^{0,1}(S_M^\omega)$.

We now prove (3.20): Take D' such that $D \subset\subset D' \subset\subset S_M^\omega$, with

$$\operatorname{dist}(D', \partial S_M^\omega) \geq \frac{9}{10} \operatorname{dist}(D, \partial S_M^\omega)\,.$$

With no loss of generality, we may assume D and D' to be connected. Fix any $x \in D'$ and set $r_0 := \frac{1}{5} \operatorname{dist}(D, \partial S_M^\omega)$. Consider a covering of D' with balls $B_{r_0}(x_j)$, centered in D', with $x_0 := x$ and $x_j \in B_{r_0/2}(x_{j-1})$. Let k be the number of such balls: obviously, k depends only on n, D' and r_0, hence we may think k to depend only on n, D and $\operatorname{dist}(D, S_M^\omega)$.

By (3.19), there exists $k^* \leq k$ so that

$$B_{r_0}(x_{k^*}) \cap \{|u| = 1\} \neq \emptyset\,,$$

while

$$B_{r_0}(x_j) \cap \{|u| = 1\} = \emptyset\,, \quad \forall j < k^*\,.$$

It follows that

$$\operatorname{dist}(x_{k^*}, \{|u| = 1\}) \leq \frac{2}{9} \operatorname{dist}(x_{k^*}, \partial S_M^\omega)\,,$$

so we can apply Lemma 3.2.5 and infer that

$$\min\{1 - u(x_{k^*}),\, 1 + u(x_{k^*})\} \leq \operatorname{const} r_0\,. \tag{3.22}$$

Also, $\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0$ in $B_{r_0}(x_j)$ for $j < k^*$ because of Lemma 3.2.2. So, we deduce from Harnack inequality that

$$\operatorname{const}(1 \pm u(x_{j+1})) \geq 1 \pm u(x_j)\,.$$

Iterating, using (3.22) and recalling that $x_0 = x$, we get

$$\sup_{x \in D'} \min\{1 - u(x),\, 1 + u(x)\} \leq \operatorname{const} r_0\,. \tag{3.23}$$

Take now $\zeta \in D$, with $|u(\zeta)| < 1$.

Two cases are possible: If

$$\operatorname{dist}(\zeta, \{|u| = 1\}) \geq \frac{1}{3} \operatorname{dist}(D, \partial D'),$$

then

$$\partial_i(a_{i,j}\partial_j(1 \pm u)) = 0 \qquad \text{on } B_{\operatorname{dist}(D,\partial D')/3}(\zeta);$$

thus, elliptic estimates imply that

$$|\nabla(1 \pm u)(\zeta)| \leq \frac{\text{const}}{\operatorname{dist}(D, \partial D')} \sup_{B_{\operatorname{dist}(D,\partial D')/3}(\zeta)} |1 \pm u|.$$

From this and (3.23), $|\nabla u(\zeta)| \leq C_D$.

On the other hand: If

$$\operatorname{dist}(\zeta, \{|u| = 1\}) < \frac{1}{3} \operatorname{dist}(D, \partial D'),$$

as done before, we can apply again Lemma 3.2.5 and elliptic estimates (via Lemma 3.2.2) to get (3.20). □

We are now in position to complete the proof of Theorem 3.2.1.

If $\operatorname{Int}(S_M^\omega) \cap \{|u| = 1\} \neq \emptyset$, the claim in Theorem 3.2.1 follows by Theorem 3.2.6.

If, viceversa, $\operatorname{Int}(S_M^\omega) \cap \{|u| = 1\} = \emptyset$, by Lemma 3.2.2, $\partial_i(a_{i,j}\partial_j u) = 0$ in $\operatorname{Int}(S_M^\omega)$, so the claim follows from standard elliptic estimates (for instance, Theorem 8.33 and Corollary 8.35 of [GT]).

Chapter 4

Density estimates

We will now recall some *density estimates* on the level sets of the minimizers. Similar density estimates play a crucial role in several results related to ours: see, for instance [CC] and [CL]. Here, the density estimates will be the tool to apply the Geometric Lemma we presented in Section 2.2.

What we are going to prove is roughly that, in some sense, both the sets where a minimizer attains value 1 and where it attains value -1 have "full dimension", while the set where the minimizer is different from ± 1 behaves like a set "of codimension one".

Several results of this chapter will be based on the techniques of [CC].

4.1 Density estimates for fluid jets

We will use in this Section the notation fixed in Chapter 1. We start by recalling the following

Theorem 4.1.1 (Caffarelli and Córdoba). *Assume $u \in \mathcal{M}^{\omega}_{M}$. Then:*

(i) there exist positive constants c, r_0, r_1 (depending only on n and on the

bounds on $a_{i,j}$ and Q) such that

$$\mathcal{J}_{B_r(x)}(u) \leq c\, r^{n-1}\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_M$;

(ii) for any $\theta_0 \in [0,1)$, $\theta \in [-\theta_0, \theta_0]$ and $\mu_0 > 0$, if $\mathcal{L}^n(B_K(x) \cap \{u > \theta\}) \geq \mu_0$, there exist positive constants $c^\star$, r_0, r_1 (depending on n, K, μ_0, θ_0 and the bounds on $a_{i,j}$ and Q) such that

$$\mathcal{L}^n\Big(\{u > \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_M$. Analogously, if $\mathcal{L}^n(B_K(x) \cap \{u < \theta\}) \geq \mu_0$,

$$\mathcal{L}^n\Big(\{u < \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_M$.

We include the proof of this Theorem in Section 4.3. Such proof is a variation of the one in [CC], where the result was proved in the case $a_{i,j} = \delta_{i,j}$ (we indeed simplify a bit the proof and correct some minor flawns). From (i) of Theorem 4.1.1, the following bound on the "width" of $\{|u| < 1\}$ immediately follows:

Lemma 4.1.2. *Let $u \in \mathcal{M}^{\omega}_M$. Assume that $u(\bar{x}) \in (-1,1)$. Then, there exist r_0 and r_1 (depending only on n, the bounds on $a_{i,j}$ and Q) such that u takes either the value 1 or -1 in $B_{r_0}(\bar{x})$, provided that $B_{r_0+r_1}(\bar{x}) \subset S^{\omega}_M$.*

We would like now to deduce some other density estimates, that will be needed for the application of the Geometric Lemma. To do this, we need some better control on the "width" of $\{|u| < 1\}$ than the one obtained in Lemma 4.1.2: indeed, in Lemma 4.1.6 we will show that *both* the values 1 and -1 are attained in a universal neighborhood of any point in $\{|u| < 1\}$. This result will be a consequence of some properties of "rigidity" of minimizers, related to the "linear growth from the free boundary" proved by Alt, Caffarelli and Friedman. The following results will lead us to this target.

In order to compare the functional in (1.1) with the one dealt with, for instance, in [AC], we observe the following:

Lemma 4.1.3. *Let $u \in \mathcal{M}^{\omega}_{M}$ and let Ω be an open connected set contained in S^{ω}_{M}, with a Lipschitz boundary. If u does not take the value 1 in Ω, then u minimizes*

$$\int_{\Omega} a_{i,j}\,\partial_i v\,\partial_j v \;+\; Q(x)\chi_{\{v>-1\}}\,,$$

among all $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in \mathrm{L}^2(\Omega)$ and $v = u$ on $\partial\Omega$.

Analogously, if u does not take the value -1 in Ω, then u minimizes

$$\int_{\Omega} a_{i,j}\,\partial_i v\,\partial_j v \;+\; Q(x)\chi_{\{v<1\}}\,,$$

among all $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in \mathrm{L}^2(\Omega)$ and $v = u$ on $\partial\Omega$.

Proof. Given an admissible function v, consider the function that agrees with u outside Ω and with v in Ω, and notice that, if u does not take the value 1 in Ω, then $\chi_{\{|u|<1\}} = \chi_{\{u>-1\}}$ in Ω. $\square$

Lemma 4.1.4. *Let Ω be a Lipschitz domain. Let u be a minimizer of*

$$\tilde{\mathfrak{J}}_{\Omega}(u) := \int_{\Omega} a_{i,j}\partial_i u \partial_j u + Q(x)\chi_{\{u>0\}}\,, \tag{4.1}$$

among all the functions $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that $\nabla v \in \mathrm{L}^2(\Omega)$ so that $v = u$ on $\partial\Omega$. Assume that $\tilde{\mathfrak{J}}_\Omega(u) < \infty$ and that $a_{i,j}$ and Q satisfy the hypotheses stated at page 1.

Then, for any $x \in \{u > 0\}$,

$$\mathrm{dist}(x, \{u = 0\}) \leq Cu(x)\,,$$

provided that $B_{\mathrm{dist}(x,\{u=0\})}(x) \subseteq \Omega$, where C is a positive constant, depending only on n and on the bounds on $a_{i,j}$ and Q.

Proof. By scaling, we may assume $x = 0$ and $\mathrm{dist}\,(x, \{u = 0\}) = 1$. Set $\varepsilon := u(0)$. Since $B_1 \subseteq \{u > 0\}$, the first variation of the functional shows that $\partial_i(a_{i,j}\partial_j u) = 0$ on B_1. Thus, by Harnack inequality,

$$\sup_{B_{1/2}} u \leq c^*\varepsilon\,,$$

for a universal constant c^*.

Let φ be the radially symmetric function defined as follows: $\varphi = 0$ in $B_{1/4}$, $\varphi = 2c^*\varepsilon$ on $\partial B_{1/2}$ and growing linearly on the rays of $B_{1/2} - B_{1/4}$. Set $v := \min\{u, \varphi\}$. Then v is an admissible function, so

$$\begin{aligned}\int_{B_{1/2}} a_{i,j}\partial_i u \partial_j u + Q\chi_{\{u>0\}} &\leq \int_{B_{1/2}} a_{i,j}\partial_i v \partial_j v + Q\chi_{\{v>0\}} \leq \\ \leq \int_{B_{1/2}} a_{i,j}\partial_i u\partial_j u &+ \int_{B_{1/2}} a_{i,j}\partial_i\varphi\partial_j\varphi + \int_{B_{1/2}-B_{1/4}} Q\chi_{\{v>0\}}\,.\end{aligned}$$

Since $|\nabla\varphi| \leq \mathrm{const}\,\varepsilon$ and $u > 0$ on $B_{1/4}$, the previous inequality implies

$$\varepsilon^2 \geq \int_{B_{1/2}} Q\chi_{\{u>0\}} - \int_{B_{1/2}-B_{1/4}} Q\chi_{\{v>0\}} \geq Q_{\min}|B_{1/4}|\,.$$

$\square$

Lemma 4.1.5. *Let Ω, u and $\tilde{\mathfrak{J}}$ be as in Lemma 4.1.4. Let $x \in \partial\{u > 0\}$. Then, for any $r \geq 0$,*

$$\sup_{B_r(x)} u \geq Cr \,,$$

where C is a positive constant (depending only on n and on the bounds on $a_{i,j}$ and Q), provided $B_r(x) \subseteq \Omega$.

Proof. By scaling, we may assume $x = 0$ and $r = 1$. Since $0 \in \partial\{u > 0\}$, for any $\varepsilon > 0$ there exists a point $x_1 \in B_\varepsilon$, so that $u(x_1) > 0$. We now construct a polygonal, starting as close as we wish to zero, along which u grows up at least linearly. Set

$$d_1 := \operatorname{dist}(x_1, \partial\{u > 0\}) \leq |x_1| \leq \varepsilon \,.$$

By Lemma 4.1.4,

$$u(x_1) \geq c^* d_1 \,, \tag{4.2}$$

for a suitable constant c^*. Let $z_1 \in \partial B_{d_1}(x_1) \cap \{u = 0\}$. Define

$$\rho_1 := \sup\{\rho \text{ s.t. } B_\rho(z_1) \subseteq B_1 \text{ and } \sup_{B_\rho(z_1)} u \geq c^* \rho\} \,.$$

By (4.2), $\rho_1 \geq d_1$. Also, if $B_{\rho_1}(z_1)$ touches ∂B_1, then

$$\rho_1 = 1 - |z_1| \geq 1 - d_1 - \varepsilon \geq 1 - 2\varepsilon \geq \frac{1}{2} \,,$$

if $\varepsilon \leq 1/4$, so

$$\sup_{B_1} u \geq \sup_{B_{\rho_1}(z_1)} u \geq \frac{c^*}{2} \,,$$

and the claim would be proven. Thus, we may restrict ourselves to the case in which

$$\sup_{B_{\rho_1}(z_1)} u = c^* \rho_1 \,. \tag{4.3}$$

Take $\tilde{x}_1 \in \bar{B}_{\rho_1}(z_1)$ attaining the sup in (4.3). In fact, by Maximum Principle, $\tilde{x}_1 \in \partial B_{\rho_1}(z_1)$. We want now to show that there exist some universal $\delta > 0$ such that

$$\sup_{B_{2\rho_1}(z_1)} u \geq (1+\delta)\, u(\tilde{x}_1)\,, \tag{4.4}$$

provided $B_{2\rho_1}(z_1) \subseteq B_1$. To prove this, assume

$$\sup_{B_{2\rho_1}(z_1)} u < (1+\delta)\, u(\tilde{x}_1)$$

and define

$$v(x) := \frac{1}{\rho_1} u(z_1 + \rho_1 x)\,, \qquad \forall x \in B_2\,.$$

Then

$$||v||_{\mathrm{L}^\infty(B_2)} \leq \frac{(1+\delta)\, u(\tilde{x}_1)}{\rho_1} = (1+\delta)c^*\,, \tag{4.5}$$

having used (4.3). Assuming $\delta \leq 1$, we get a uniform bound on $||v||_{\mathrm{L}^\infty(B_2)}$. So, by the Theorem of Giaquinta and Giusti (see [GG] or [G2]), $v \in \mathrm{C}^\alpha(B_{3/2})$ and

$$||v||_{\mathrm{C}^\alpha(B_{3/2})} \leq \tilde{c}\,, \tag{4.6}$$

for a universal $\tilde{c}$. Set

$$w := \frac{2v}{c^*(1+\delta)} - 1\,,$$

so that $\partial_i(a_{i,j}\partial_j w) \geq 0$ in $B_{3/2}$ and $w \leq 1$ because of (4.5). Also, by (4.6) and the fact that $w(0) = -1$, we deduce that $\mathcal{L}^n(\{w \leq 0\} \cap B_{3/2}) \geq$ const. Then, by De Giorgi Oscillation Lemma (see, for instance the Appendix of [C] or Theorem 4.9 in [HL] and references therein), we infer $\sup_{B_1} w \leq \lambda$, for a universal $\lambda \in (0,1)$. In particular, since $\tilde{x}_1 \in \bar{B}_{\rho_1}(z_1)$,

$$\lambda \geq w\left(\frac{\tilde{x}_1 - z_1}{\rho_1}\right) = \frac{2}{1+\delta} - 1\,,$$

which is a contradiction if δ is small enough. This proves (4.4). By construction, $u(\tilde{x}_1) \geq u(x_1)$. Hence, there exists a point $x_2 \in \bar{B}_{2\rho_1}(z_1)$ (and in fact $x_2 \in \partial B_{2\rho_1}(z_1)$ by Maximum Principle) such that

$$u(x_2) \geq u(\tilde{x}_1) + \delta c^* \rho_1 \geq u(x_1) + \theta |x_1 - x_2| \,,$$

for universal $\delta,\ \theta > 0$, provided that $B_{3\rho_1}(x_1) \subseteq B_1$.

Iterating, we obtain a polygonal of vertexes $x_1, x_2, \ldots, x_k$ such that: for $2 \leq j \leq k$, $x_j \in \partial B_{3\rho_{j-1}}(x_{j-1})$, $|x_j - x_{j-1}| \geq \rho_j$ and

$$\begin{aligned} u(x_j) &\geq (1+\delta) u(x_{j-1}) \geq && (4.7) \\ &\geq u(x_{j-1}) + \theta |x_j - x_{j-1}| \,, && (4.8) \end{aligned}$$

provided $B_{3\rho_{j-1}}(x_{j-1}) \subseteq B_1$. In particular, iterating (4.8),

$$u(x_j) \geq \theta \ell_j$$

where ℓ_j is the length of the polygonal joining $x_1, \ldots, x_j$. We want to prove that we can take j big enough, and make ℓ_j bigger than some universal quantity. This would finish the proof of the lemma.

Indeed, we must be able to iterate the scheme and find a suitable j^* such that $|x_{j^*}| \geq 1/20$. Otherwise, if $|x_j| < 1/20$ for any j, then $\rho_j \leq |x_j - x_{j-1}| < 1/10$ for any j; and so $B_{3\rho_j}(x_j) \subset B_1$ for any j, and iterating (4.7) we would get $\sup_{B_1} u = +\infty$.

Therefore, $\ell_{j^*} \geq |x_{j^*}| - |x_1| \geq \frac{1}{20} - \varepsilon$. □

We can now sharpen the result of Lemma 4.1.2:

Lemma 4.1.6. *Let $u \in \mathcal{M}_M^\omega$. Assume that $u(\bar{x}) \in (-1,1)$. Then, there exist r_0 and r_1 (depending only on n, the bounds on $a_{i,j}$ and Q) such that u takes both the values 1 and -1 in $B_{r_0}(\bar{x})$, provided that $B_{r_0+r_1}(\bar{x}) \subset S_M^\omega$.*

Proof. By Lemma 4.1.2, u takes, say, the value -1 in $B_{r_0}(\bar{x})$. Without loss of generality, we assume $r_0 \geq 3/C$, where C is the constant in Lemma 4.1.5. If u did not assume the value 1 too, using Lemmas 4.1.3 and 4.1.5, we get

$$2 \geq \sup_{B_{r_0}(x_0)} u + 1 \geq C r_0 \geq 3\,,$$

which is a contradiction. $\square$

We are now in position to improve the previous density estimates:

Lemma 4.1.7. *Let $u \in \mathcal{M}_M^\omega$. Assume that $x \in \{|u| < 1\}$. Then, there exist positive constants r_0, r_1 and c, depending only on n and on the bounds on $a_{i,j}$ and Q, such that*

$$\mathcal{J}_{B_r(x)}(u) \geq c r^{n-1}\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S_M^\omega$.

Proof. By Lemma 4.1.6, Giaquinta and Giusti's Theorem and Theorem 4.1.1

$$\mathcal{L}^n\Big(B_r(x) \cap \{u > 9/10\}\Big) \geq \text{ const}\, r^n$$

and

$$\mathcal{L}^n\Big(B_r(x) \cap \{u < -9/10\}\Big) \geq \text{ const}\, r^n\,.$$

Let a be the average of u in $B_r(x)$. If $a \leq 0$,

$$\begin{aligned}\int_{B_r(x)} |u-a| \ &\geq \ \int_{B_r(x)\cap\{u>9/10\}} (u-a) \geq \frac{9}{10}\mathcal{L}^n\Big(B_r(x)\cap\{u>9/10\}\Big) \geq \\ &\geq \ \text{const}\, r^n\,.\end{aligned}$$

Analogously, if $a \geq 0$,

$$\int_{B_r(x)} |u-a| \geq \frac{9}{10}\mathcal{L}^n\Big(B_r(x) \cap \{u < -9/10\}\Big) \geq \text{ const } r^n .$$

So, in any case, using Poincaré and Cauchy Inequalities:

$$\begin{aligned} \text{const}\, r^{n-1} &\leq \frac{1}{r}\int_{B_r(x)} |u-a| \leq \text{ const } \int_{B_r(x)} |\nabla u| \leq \\ &\leq \text{ const } \sqrt{\int_{B_r(x)} |\nabla u|^2} \sqrt{\mathcal{L}^n\Big(B_r(x) \cap \{|u| < 1\}\Big)} \leq \\ &\leq \text{ const } \Big(\int_{B_r(x)} a_{i,j}\partial_i u \partial_j u + Q\chi_{(-1,1)}(u)\Big) . \end{aligned}$$

□

Let us summarize some of the density estimates obtained in this section:

Proposition 4.1.8. *Let $u \in \mathcal{M}^{\omega}_{M}$. Assume that $x \in \{|u| < 1\}$. Then, there exist positive constants r_0, r_1, c and C, depending only on n and on the bounds on $a_{i,j}$ and Q, such that:*

$$\begin{aligned} &\mathcal{J}_{B_r(x)}(u) \geq cr^{n-1} , \\ &\mathcal{L}^n\Big(B_r(x) \cap \{u > -1\}\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(B_r(x) \cap \{u < 1\}\Big) \geq cr^n , \end{aligned}$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$. Moreover, for any $y \in S^{\omega}_{M}$,

$$\mathcal{J}_{B_r(y)} \leq Cr^{n-1} ,$$

for any $r \geq r_0$, provided $B_{r+r_1}(y) \subset S^{\omega}_{M}$.

Proof. Put together Theorem 4.1.1, Lemma 4.1.6 and Lemma 4.1.7. □

4.2 Density estimates for Ginzburg-Landau

We define $\tilde{\mathbf{R}}^n := \mathbf{R}^n/\sim$ (topologically, $\tilde{\mathbf{R}}^n$ is the product between an $(n-1)-$dimensional torus and a real line). We consider the space

$$\mathcal{Y}^\omega_M := \{u \in \mathrm{H}^1_{\mathrm{loc}}(\tilde{\mathbf{R}}^n)$$

$$\text{with } u(x) \geq 99/100 \text{ for any } x \text{ s.t. } \omega\cdot x \leq 0 \text{ and}$$

$$u(x) \leq -99/100 \text{ for any } x \text{ s.t. } \omega \cdot x \geq M\}\,.$$

We denote by $\mathcal{M}^\omega_M(\mathcal{G})$ the set of minimizers in $\mathcal{Y}^\omega_M$ for the functional $\mathcal{G}$. First we recall:

Theorem 4.2.1 (Caffarelli and Córdoba). *Assume $u \in \mathcal{M}^\omega_M(\mathcal{G})$. Then:*

(i) there exist positive constants c, r_0, r_1 (depending only on n and on the bounds on $a_{i,j}$ and F) such that

$$\mathcal{G}_{B_r(x)}(u) \leq c\, r^{n-1}\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^\omega_M$;

(ii) for any $\theta_0 \in [0,1)$, $\theta \in [-\theta_0, \theta_0]$ and $\mu_0 > 0$, if $\mathcal{L}^n(B_K(x) \cap \{u > \theta\}) \geq \mu_0$, there exist positive constants $c^\star$, r_0, r_1 (depending on n, K, μ_0, θ_0 and the bounds on $a_{i,j}$ and F) such that

$$\mathcal{L}^n\Big(\{u > \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^\omega_M$. Analogously, if $\mathcal{L}^n(B_K(x) \cap \{u < \theta\}) \geq \mu_0$,

$$\mathcal{L}^n\Big(\{u < \theta\} \cap B_r(x)\Big) \geq c^\star r^n\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^\omega_M$.

This statement was proved in [CC] (see Lemmas 2 and 3 there) in the case $a_{i,j} = \delta_{i,j}$. The proof of this result, which contains the one of Theorem 4.1.1 as a particular case, will be presented in Section 4.3. There, we indeed simplify a bit the proof of [CC] and correct some minor flawns.

We now derive some easy consequences from the previous result:

Corollary 4.2.2. *Fix $\theta_0 \in [0,1)$. Assume $u \in \mathcal{M}^\omega_M(\mathcal{G})$ and $u(x) \in [-\theta_0, \theta_0]$. Then, there exist positive constants c, r_0, r_1 (depending only on θ_0, n and on the bounds on $a_{i,j}$ and F) such that*

$$\mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(\{u < -\theta_0\} \cap B_r(x)\Big) \geq cr^n\,,$$

for any $r \geq r_0$, provided $B_{r+r_1}(x) \subset S^\omega_M$.

Proof. Define $\theta^* := (1+\theta_0)/2$. From the fact that u is Hölder continuous and (ii) of Theorem 4.2.1,

$$\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) \geq cr^n \qquad \text{and} \qquad \mathcal{L}^n\Big(\{u < \theta^*\} \cap B_r(x)\Big) \geq cr^n\,, \tag{4.9}$$

so that

$$\begin{aligned} &\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) - \mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) = \\ = \;& \mathcal{L}^n\Big(\{-\theta^* < u \leq \theta_0\} \cap B_r(x)\Big) \leq \frac{1}{\gamma(\theta^*)} \int_{\{-\theta^* < u \leq \theta_0\} \cap B_r(x)} F\,, \end{aligned}$$

thus, by (i) of Theorem 4.2.1,

$$\mathcal{L}^n\Big(\{u > -\theta^*\} \cap B_r(x)\Big) - \mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \leq cr^{n-1}\,.$$

From this and (4.9),

$$\mathcal{L}^n\Big(\{u > \theta_0\} \cap B_r(x)\Big) \geq \text{const}\, r^n - \text{const}\, r^{n-1} \geq \text{const}\, r^n\,,$$

if r big enough. $\square$

In the sequel, following a standard notation in Geometric Measure Theory (see, for instance [G1]), we denote the perimeter of a Borel set E in an open set Ω as

$$P(E,\Omega) := \sup\{\int_E \operatorname{div}\varphi \ \text{s.t.}\ \varphi \in C^1_0(\Omega,\mathbf{R}^n)\,,\ |\varphi(x)| \le 1\}\,.$$

The following result complete the density estimates we need here:

Lemma 4.2.3. *Fix $\theta_0 \in [0,1)$. Assume $u \in \mathcal{M}^{\omega}_{M}(\mathcal{G})$ and $u(x) \in [-\theta_0, \theta_0]$. Then, there exist positive constants c, r_0, r_1 (depending only on θ_0, n and on the bounds on $a_{i,j}$ and F) such that*

$$\mathcal{L}^n\Big(\{|u| < \theta_0\} \cap B_r(x)\Big) \ge c r^{n-1}\,,$$

for any $r \ge r_0$, provided $B_{r+r_1}(x) \subset S^{\omega}_{M}$.

Proof. Define $u^*(x)$ be the function that agrees with u if $|u(x)| < \theta_0$ and that is constantly equal to θ_0 (respectively, $-\theta_0$) if $u(x) \ge \theta_0$ (respectively, if $u(x) \le -\theta_0$).

Using the Coarea Formula and the Isoperimetric Inequality (see, for instance, [G1], p. 20 and p. 25), we infer from Corollary 4.2.2 that

$$
\begin{aligned}
& \int_{B_r(x)\cap\{|u|<\theta_0\}} |\nabla u| = \int_{B_r(x)} |\nabla u^*| \geq \\
\geq \ & \int_{-\theta_0}^{\theta_0} \Big(P(\{u^* < t\}, B_r(x)) \Big)\, dt \geq \\
\geq \ & \text{const} \int_{-\theta_0}^{\theta_0} \Big(\min \Big\{ \mathcal{L}^n \Big(\{u^* < t\} \cap B_r(x) \Big) , \\
& \mathcal{L}^n \Big(\{u^* \geq t\} \cap B_r(x) \Big) \Big\}^{(n-1)/n} \Big)\, dt = \\
= \ & \text{const} \int_{-\theta_0}^{\theta_0} \Big(\min \Big\{ \mathcal{L}^n \Big(\{u < t\} \cap B_r(x) \Big) , \\
& \mathcal{L}^n \Big(\{u \geq t\} \cap B_r(x) \Big) \Big\}^{(n-1)/n} \Big)\, dt \geq \\
\geq \ & \text{const}\ r^{n-1} .
\end{aligned}
$$

Therefore, by Cauchy Formula and (i) of Theorem 4.2.1,

$$
\begin{aligned}
& \text{const}\, r^{n-1} \leq \frac{1}{A} \int_{B_r(x)} |\nabla u|^2 + A \cdot \mathcal{L}^n \Big(\{|u| < \theta_0\} \cap B_r(x) \Big) \leq \\
\leq \ & \frac{\text{const}}{A} r^{n-1} + A \cdot \mathcal{L}^n \Big(\{|u| < \theta_0\} \cap B_r(x) \Big) .
\end{aligned}
$$

Choosing A suitably big, this proves the claim. ☐

Let us summarize the density estimates we developed towards the application of the Geometric Lemma:

Proposition 4.2.4. *Fix $\theta_0 \in [0,1)$. Let $u \in \mathcal{M}_M^\omega$. Assume that $x \in \{|u| < \theta_0\}$. Then, there exist positive constants r_0, r_1, c and C, depending only on θ_0, n and on the bounds on $a_{i,j}$ and F, such that:*

$$
\begin{aligned}
& \mathcal{L}^n \Big(B_r(x) \cap \{|u| < \theta_0\} \Big) \geq c r^{n-1} , \\
& \mathcal{L}^n \Big(B_r(x) \cap \{u > -\theta_0\} \Big) \geq c r^{n} \qquad \text{and} \\
& \mathcal{L}^n \Big(B_r(x) \cap \{u < \theta_0\} \Big) \geq c r^{n} ,
\end{aligned}
$$

for any $r \geq r_0$*, provided* $B_{r+r_1}(x) \subset S^{\omega}_M$*. Moreover, for any* $y \in S^{\omega}_M$*,*

$$\mathcal{L}^n\Big(B_r(y) \cap \{|u| < \theta_0\}\Big) \leq C r^{n-1}\,,$$

for any $r \geq r_0$*, provided* $B_{r+r_1}(y) \subset S^{\omega}_M$*.*

4.3 Proof of Theorems 4.1.1 and 4.2.1

This proof is a variation of the one in [CC], where the result was proved in the case $a_{i,j} = \delta_{i,j}$. Here, we simplify a bit the proof and correct some flawns of [CC]. Let g be a radial smooth function that is identically equal to -1 in B_{r-1} and takes value 1 on ∂B_r. Define $u^\star = \min\{u, g\}$. Since u is a minimizer,

$$\begin{aligned}
\mathcal{J}_{B_r}(u) &\leq \mathcal{J}_{B_r}(u^\star) \leq \text{const}\,(\int_{B_r - B_{r-1}} |\nabla u^\star|^2 + r^{n-1}) \leq \\
&\leq \text{const}\,(\int_{B_r - B_{r-1}} (|\nabla u|^2 + |\nabla g|^2) + r^{n-1}) \leq \\
&\leq \text{const} \int_{B_r - B_{r-1}} |\nabla u|^2 + r^{n-1}\,.
\end{aligned}$$

Let us now cover $B_r - B_{r-1}$ with balls $\mathcal{B}_1, \dots \mathcal{B}_K$ of radius 2; obviously, one can take $K \leq \text{const}\, r^{n-1}$. Let $\mathcal{B}'_i$ be the ball concentric to $\mathcal{B}_i$ with radius 3. Then, from the estimate above and Caccioppoli Inequality,

$$\begin{aligned}
\mathcal{J}_{B_r}(u) &\leq \text{const}\,(\sum_{i=1}^{K} \int_{\mathcal{B}_i} |\nabla u|^2 + r^{n-1}) \leq \\
&\leq \text{const}\,(\sum_{i=1}^{K} \int_{\mathcal{B}'_i} |u|^2 + r^{n-1})\,.
\end{aligned}$$

As observed at page 17, $|u| \leq 1$: this finishes the proof of (i).

We now prove (ii). Notice that it is enough to prove it for any θ close to -1. Indeed: assume the result to be true for $\theta^\star$ close to -1, and let

$\theta \in [-\theta_0, \theta_0]$, with $\theta^\star \leq -\theta_0$. Then

$$\mu_0 \leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_K\Big) \leq \mathcal{L}^n\Big(\{u > \theta^\star\} \cap B_K\Big)\,,$$

therefore, using the result for $\theta^\star$, our hypotheses on F and (i), we infer that

$$\begin{aligned}
Cr^n &\leq \mathcal{L}^n\Big(\{u > \theta^\star\} \cap B_r\Big) \leq \\
&\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + \mathcal{L}^n\Big(\{\theta^\star < u \leq \theta\} \cap B_r\Big) \leq \\
&\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + \frac{1}{\inf_{u \in [\theta^\star, \theta_0]} F} \int_{B_r} F \leq \\
&\leq \mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) + Cr^{n-1}\,,
\end{aligned}$$

and so $\mathcal{L}^n\Big(\{u > \theta\} \cap B_r\Big) \geq Cr^n$. From these consideration, in the rest of the proof, we may and do assume that θ is close to -1.

We distinguish two cases:

Case 1: if[1] $0 \leq d < 2$. We define

$$\begin{aligned}
\mathcal{V}(r) &= \mathcal{L}^n(B_r \cap \{u > \theta\})\,, \\
\mathcal{A}(r) &= \int_{B_r \cap \{u \leq \theta\}} F(x, u)\,.
\end{aligned}$$

We want to prove that

$$\begin{aligned}
&C\Big((\mathcal{V}(r-1))^{\frac{n-1}{n}} + \mathcal{A}(r-1)\Big) \leq \\
&\leq \mathcal{V}(r) - \mathcal{V}(r-1) + \mathcal{A}(r) - \mathcal{A}(r-1)\,. \qquad (4.10)
\end{aligned}$$

[1] An easy modification of this argument in the case $d = 0$ allows also to consider bounded measurable (instead of Lipschitz) coefficients. In this way, Theorem 5.0.3 is also valid without the assumption on the $a_{i,j}$ to be Lipschitz. I suspect the same to happen in Theorem 6.1.1, but I do not see how to prove it.

Indeed, if (4.10) holds, then, by induction, one sees that $\mathcal{V}(k)+\mathcal{A}(k) \geq Ck^n$; recalling (i), we get $\mathcal{V}(k) \geq Ck^n$, for k big enough, proving (ii) – notice that we used here the hypothesis $\mathcal{V}(K) \geq \mu_0$ to start the induction.

We now prove (4.10). Let h be an auxiliary function to be chosen later in such a way that $-1 \leq h \leq 1$, $h = 1$ on ∂B_r and $h = -1$ in B_{r-1}. We will denote by C suitable constants and by A a free parameter, to be chosen in the sequel. Let $\sigma = \min\{u, h\}$ and $\beta = \min\{u - \sigma, 1 + \theta\}$ Applying Sobolev Inequality to β^2 and then Cauchy Inequality and the fact that $\nabla\beta$ vanishes outside $\{u - \sigma \leq 1 + \theta\}$,

$$\begin{aligned}
&\Big(\int_{B_r} \beta^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}} \leq \\
\leq\ & C\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} |\beta|\,|\nabla\beta| \leq \\
\leq\ & CA\Big(\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} (a_{i,j}\partial_i u\partial_j u - a_{i,j}\partial_i\sigma\partial_j\sigma - 2a_{i,j}\partial_i(u-\sigma)\partial_j\sigma)\Big) + \\
& + \frac{C}{A}\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} (u-\sigma)^2 . \qquad (4.11)
\end{aligned}$$

Integration by parts, together with the minimality property of u, implies

$$\begin{aligned}
&\Big(\int_{B_r} \beta^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}} \leq \\
\leq\ & CA\Big[\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} (F(x,\sigma) - F(x,u)) + \\
& +2\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} \Big(a_{i,j}\partial^2_{i,j}\sigma\,(u-\sigma) + (\partial_i a_{i,j})\partial_j\sigma\,(u-\sigma)\Big)\Big] + \\
& + \frac{C}{A}\int_{B_r\cap\{u-\sigma\leq 1+\theta\}} (u-\sigma)^2 . \qquad (4.12)
\end{aligned}$$

Since $\sigma = h = -1$ in B_{r-1}, the left hand side of (4.12) is bigger than

$$C\left(\mathcal{L}^n(B_{r-1}\cap\{u > \theta\})\right)^{\frac{n-1}{n}} .$$

We now estimate the right hand side of (4.12) in B_{r-1}, in which $\sigma = -1$: Notice now that, if θ is close to -1 and $u \leq \theta$, we have, by our assumptions on F, that

$$F(x,u) \geq C\,(1+u)^d \geq \frac{C}{2^{2-d}}(1+u)^2 \tag{4.13}$$

so that

$$\frac{C}{A}\int_{B_{r-1}\cap\{u\leq\theta\}}(1+u)^2 - \frac{CA}{2}\int_{B_{r-1}\cap\{u\leq\theta\}}F(x,u) \leq 0$$

if we choose A suitably big. Therefore, the right hand side of (4.12) in B_{r-1} is negative and less than $-C\,\mathcal{A}(r-1)$.

On the integration over $B_r - B_{r-1}$, we split the domain of integration. First of all, since θ is close to -1 and $\sigma \leq u$, our assumptions on F imply that $F(x,\sigma) \leq F(x,u)$ if $u \leq \theta$. Thus,

$$CA\int_{B_r-B_{r-1}} F(x,\sigma) - F(x,u) \leq CA\,\mathcal{L}^n(\{u>\theta\}\cap(B_r - B_{r-1}))\,.$$

Also, recalling (4.13),

$$\begin{aligned} &\frac{C}{A}\int_{B_r-B_{r-1}}(u-\sigma)^2 \leq \\ \leq\ &\frac{C}{A}\int_{(B_r-B_{r-1})\cap\{u\leq\theta\}}(u+1)^2 + \frac{C}{A}\mathcal{L}^n(\{u>\theta\}\cap(B_r - B_{r-1})) \leq \\ \leq\ &\frac{C}{A}\int_{(B_r-B_{r-1})\cap\{u\leq\theta\}}F(x,u) + \frac{C}{A}\mathcal{L}^n(\{u>\theta\}\cap(B_r - B_{r-1}))\,. \end{aligned}$$

Finally, we define, $d' = \max\{1,d\}$ and, for $\rho \in [r-1,r]$,

$$h(\rho) = 2(\rho - r + 1)_+^{\frac{2}{2-d'}} - 1\,.$$

Notice that this setting is possible for our assumption on d. With this definition, we can complete the estimate on the right hand side of (4.12): Indeed, from a direct calculation one sees that, in $B_r - B_{r-1}$,

$$|\nabla h| + |D^2 h| \le C(\rho - r + 1)^{\frac{2(d'-1)}{2-d'}} \le C(h+1)^{d'-1}\,.$$

Also, in the set $\{u > \sigma\}$ we have $\sigma = h$ and then the last formula implies

$$\begin{aligned} (|\nabla h| + |D^2 h|)\cdot(u-\sigma) &\le C(h+1)^{d'-1}\cdot(u-h) \le \\ &\le C\,(u+1)^{d'}\,, \end{aligned} \tag{4.14}$$

in $(B_r - B_{r-1}) \cap \{u > \sigma\}$. Therefore, recalling that $\sigma = -1$ in B_{r-1}, splitting the domain of integration into the sets $\{u \le \theta\}$ and $\{u > \theta\}$, using (4.14) and our hypotheses on F, we get

$$\begin{aligned} &\int_{B_r} \Big(a_{i,j}\partial^2_{i,j}\sigma\,(u-\sigma) + (\partial_i a_{i,j})\partial_j\sigma\,(u-\sigma)\Big) = \\ &= \int_{(B_r - B_{r-1})\cap\{u>\sigma\}} \Big(a_{i,j}\partial^2_{i,j}\sigma\,(u-\sigma) + (\partial_i a_{i,j})\partial_j\sigma\,(u-\sigma)\Big) \le \\ &\le C\mathfrak{L}^n\Big((B_r - B_{r-1})\cap\{u>\theta\}\Big) + C\int_{(B_r - B_{r-1})\cap\{u\le\theta\}} (u+1)^{d'} \le \\ &\le C\mathfrak{L}^n\Big((B_r - B_{r-1})\cap\{u>\theta\}\Big) + C\int_{(B_r - B_{r-1})\cap\{u\le\theta\}} F(x,u)\,. \end{aligned}$$

Collecting all these estimates, the proof of (4.10) follows. This finishes the proof of (ii) in the first case.

Case 2: if $d = 2$. We use suitable positive parameters Θ and T: we will fix Θ small enough and then choose T suitably large.

Let $k \in \mathbf{N}$. We use a barrier function $h = h_k \in \mathrm{C}^2(B_{(k+1)T})$ so that $-1 \le h \le 1$, $h = 1$ on $\partial B_{(k+1)T}$,

$$(h+1) + |\nabla h| + |D^2 h| \le C(h+1) \le Ce^{-\Theta T} \tag{4.15}$$

in B_{kT}, and

$$|\nabla h| + |D^2 h| \leq C\Theta(h+1) \tag{4.16}$$

in $B_{(k+1)T}$. From the last formula and our assumptions on F, we deduce, if Θ is suitably small,

$$|\nabla h| + |D^2 h| \leq \sqrt{\Theta} F_u(x,h) \tag{4.17}$$

in $B_{(k+1)T} \cap \{u \leq \theta\}$. We postpone the explicit construction of such h to the end of the proof. Define $\theta' = \theta - Ce^{-\Theta T}$; if T is large enough, $\theta' > -1$. Define also

$$\sigma = \min\{u, h\} \quad \text{and} \quad \beta = \min\{u - \sigma, 1 + \theta'\}\,.$$

As in (4.11)–(4.12), we have that

$$\begin{aligned} & \Big(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \Big)^{\frac{n-1}{n}} \leq \\ \leq \ & CA\Big[\int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} \Big(F(x,\sigma) - F(x,u) + a_{i,j} \partial^2_{i,j} \sigma (u-\sigma) + \\ & + \partial_i a_{i,j} \partial_j \sigma (u - \sigma) \Big) \Big] + \\ & + \frac{C}{A} \int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} (u-\sigma)^2 \,, \end{aligned} \tag{4.18}$$

where A is a free parameter, to be chosen suitably large in what follows. We now estimate the left hand side of (4.18). From (4.15), if T is big enough, we get $\theta - h \geq (1 - \theta_0)/2$ in B_{kT}. Thus, the left hand side of (4.18) is bigger than $C\mathcal{V}(kT)^{\frac{n-1}{n}}$.

Let us now estimate the right hand side of (4.18). First of all, we consider the contribution in $\{u \leq \theta\}$. Let us observe that, since $-1 \leq \sigma \leq$

$u \leq 1$,

$$\begin{aligned}
& (u+1)^2 - (\sigma+1)^2 - \frac{1}{2}(u-\sigma)^2 = \\
= \;& (u+\sigma)\,(u-\sigma) + 2(u-\sigma) - \frac{1}{2}(u-\sigma)^2 = \\
= \;& (u-\sigma)\,(\frac{1}{2}u + \frac{3}{2}\sigma + 2) \geq 0\,;
\end{aligned}$$

accordingly, in $\{\sigma < u \leq \theta\}$,

$$\begin{aligned}
& F(x,u) - F(x,\sigma) = \int_h^u F_u(x,\zeta)\,d\zeta \geq \\
& \geq C\int_h^u (\zeta+1)\,d\zeta = C\Big[(u+1)^2 - (h+1)^2\Big] \geq \\
& \geq C(u-h)^2\,.
\end{aligned} \tag{4.19}$$

Consequently, choosing A suitably large and recalling (4.17), the contribution of the right hand side of (4.18) in $\{u \leq \theta\}$ is controlled by

$$\int_{B_{(k+1)T}\cap\{\sigma<u\leq\theta\}} \Big(F(x,\sigma) - F(x,u) + C\sqrt{\Theta}F_u(x,\sigma)(u-\sigma)\Big)\,. \tag{4.20}$$

We now show that this quantity is indeed negative. Since we ruled out θ to be close to -1, we have that F and F_u are monotone in $(-1,\theta)$. Hence, in $\{\sigma < u \leq \theta\}$, $F(x,\sigma) - F(x,u)$ is negative and, furthermore,

$$|F_u(x,\sigma)(u-\sigma)| \leq |F(x,\sigma) - F(x,u)|\,.$$

Since we assumed Θ to be small, we have shown that the quantity in (4.20) is negative.

Let us now bound the right hand side of (4.18) in $\{u > \theta\}$. First, we notice that this term has no contribution in B_{kT}: indeed, from (4.15),

$$B_{kT} \cap \{\sigma < u \leq \sigma + 1 + \theta'\} \subseteq B_{kT} \cap \{\sigma < u \leq \theta\}\,.$$

Thus, the right hand side of (4.18) is bounded by

$$\int_{(B_{(k+1)T}-B_{kT})\cap\{u>\theta\}} \Big(F(x,\sigma)-F(x,u)+(\sigma+1)(u-\sigma)+(u-\sigma)^2\Big)\,, \tag{4.21}$$

and this term can be bounded by

$$\mathcal{L}^n\Big(\{u>\theta\}\cap(B_{(k+1)T}-B_{kT})\Big)\,.$$

Collecting all the estimates, we get

$$C(\mathcal{V}(kT))^{\frac{n-1}{n}} \le \mathcal{V}((k+1)T)-\mathcal{V}(kT)\,.$$

Let us now define $\alpha_k = \mathcal{V}(kT)-\mathcal{V}((k-1)T)$. Notice that $\sum_{1\le j\le k}\alpha_j = \mathcal{V}(kT)$ and therefore

$$C(\sum_{1\le j\le k}\alpha_j)^{\frac{n-1}{n}} \le \alpha_{k+1}\,.$$

By induction over k, one easily gets that $\alpha_k \ge ck^{n-1}$ and thus $\mathcal{V}(kT)\ge ck^n$, which is the desired result for the case $d=2$.

We now construct explicitly the barrier introduced at page 50. Define the following functions $\Phi:[0,1]\longrightarrow\mathbf{R}$, $\Psi:[1,(k+1)T]\longrightarrow\mathbf{R}$:

$$\begin{aligned}\Phi(\rho) &= 2e^{\Theta[\frac{3}{8}\rho^6-\frac{10}{8}\rho^4+\frac{15}{8}\rho^2-(k+1)T]}-1 \quad \text{and}\\ \Psi(\rho) &= 2e^{\Theta[\rho-(k+1)T]}-1\,.\end{aligned}$$

By explicit computations,

$$\Phi(1)=\Psi(1)\,,\quad \Phi'(1)=\Psi'(1)\quad\text{and}\quad \Phi''(1)=\Psi''(1)\,.$$

Thus, the function $\bar{h}$ agreeing with Φ in $[0,1]$ and with Ψ in $[1,(k+1)T]$ belongs to $\mathrm{C}^2([0,(k+1)T],\mathbf{R})$. Define $h(x)=\bar{h}(|x|)$. Notice that

$$|\Phi'(\rho)|\le C\Theta\rho(\Phi+1)\,,\qquad |\Phi''(\rho)|\le C\Theta(\Phi+1) \tag{4.22}$$

in $[0,1]$ and

$$|\Psi'(\rho)| + |\Psi''(\rho)| \leq C\Theta(\Psi + 1) \tag{4.23}$$

in $[1,(k+1)T]$. Moreover,

$$\begin{aligned} & (h+1) + |\nabla h| + |D^2 h| \leq \\ \leq \quad & (\bar{h}+1) + \Big(1 + \frac{2}{|x|}\Big)|\bar{h}'| + |\bar{h}''|\,. \end{aligned} \tag{4.24}$$

By means of (4.22), we bound the right hand side of (4.24) in B_1 by $C(\Phi+1) \leq Ce^{\Theta(C-(k+1)T)}$. Similarly, in $B_{(k+1)T} - B_1$, using (4.23), we bound (4.24) by $C(\Psi+1) \leq Ce^{-\Theta T}$. This proves (4.15). In a similar way, one can prove (4.16).

This finishes the proof of the case $d = 2$. The proof of Theorems 4.1.1 and 4.2.1 is completed.

4.4 Application of the Geometric Lemma

Following is the application to the fluid jets:

Proposition 4.4.1. *Let $u \in \mathcal{M}^{\omega}_{M}$. There exist positive constants κ, r_0 and r_1, depending only on n and on the bounds on $a_{i,j}$ and Q, such that: for any $x \in \{|u| < 1\}$ and any $r \geq r_0$, there exist z_1 and z_2, such that*

$$B_{\kappa r}(z_1) \subseteq \{u = 1\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u = -1\} \cap B_r(x)\,,$$

provided $B_{2r+r_1}(x) \subset S^{\omega}_{M}$.

Proof. Apply the Geometric Lemma to the sets $E := \{u > -1\}$, or $E := \{u < 1\}$, and $\beta := \{|u| < 1\}$. Take $\mu := \mathcal{L}^n$ and $\nu(B) := \mathcal{J}_B(u^{\omega}_{M})$. The hypotheses of the Geometric Lemma are fulfilled in light of Proposition 4.1.8. $\square$

Here is the application of the Geometric Lemma to the Ginzburg-Landau functional:

Proposition 4.4.2. *Fix $\theta_0 \in [0,1)$. Let $u \in \mathcal{M}^{\omega}_{M}(\mathcal{G})$. There exist positive constants κ, r_0 and r_1, depending only on θ_0, n and on the bounds on $a_{i,j}$ and F, such that: for any $x \in \{|u| \le \theta_0\}$ and any $r \ge r_0$, there exist z_1 and z_2, such that*

$$B_{\kappa r}(z_1) \subseteq \{u \ge \theta_0\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u \le -\theta_0\} \cap B_r(x)\,,$$

provided $B_{2r+r_1}(x) \subset S^{\omega}_{M}$.

Proof. Apply the Geometric Lemma to the sets $E := \{u > -\theta_0\}$ or $E := \{u < \theta_0\}$ and $\beta := \{|u| < \theta_0\}$; also, the measure μ and ν appearing in the "abstract" Geometric Lemma will be taken here to be the Lebesgue measure and the restriction of the Lebesgue measure to the set $\{|u| < \theta_0\}$, respectively. $\square$

The reader can compare Propositions 4.4.1 and 4.4.2, and the results in Proposition 4.2.4 with the ones in Proposition 4.1.8. Note that for the general Ginzburg-Landau functional the constants in the estimates depend on θ_0 (i.e., roughly, on the value attained at the center). On the other side, for the functional (1.1) one has uniform bounds.

Chapter 5

Plane-like minimizers for jets of fluid

In this Chapter, we consider the functional (1.1). We will prove the following

Theorem 5.0.3. *There exists a universal constant M_0, depending only on n, the bounds on $a_{i,j}$ and Q stated above*[1]*, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{J}$ for which the set $\{|u| < 1\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.*

Moreover, such u enjoys the following property of "quasi-periodicity":

- *if $\omega \in \mathbf{Q}^n$, then u is periodic (with respect to the identification induced by ω);*
- *if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.*

The precise definition of "class A minimizer" will be given in Section 5.5: roughly, we say that u is a class A minimizer if the functional does not increase under any compact (but not necessarily small) modification.

[1] I would like to remark that one can indeed prove Theorem 5.0.3 for bounded measurable (instead of Lipschitz) coefficients: see footnote at page 47.

The proof of Theorem 5.0.3 will make use of the results of *existence and rigidity* of minimizers of [AC], the *regularity result* of [GG], several *density estimates* of [CC] and a *geometric construction* of [CL]. Later, we will extend the results of Theorem 5.0.3 to more general Ginzburg-Landau-type functionals (see below Section 6.1).

We remark that the fluid jet functional is a particular case of the general Ginzburg-Landau. However, we preferred to treat the case of the "fluid jets" separately both because of its physical relevance and because some of the arguments are easier or more explicit in this case. Moreover, the main result of Chapter 5 (Theorem 5.0.3) is stronger than Theorem 6.1.1.

The detailed study of the functional $\mathcal{J}$ also showed some of the properties of the fluid jets that we found interesting in themselves, as the growth from the free boundary and the regularity issues.

The scheme of this chapter is the following: first of all, we consider the "compact" case in which $\omega \in \mathbf{Q}^n$. In Section 5.1, we recall some results of existence and regularity of constrained minimizers. In particular, minimizers are Hölder continuous, with a uniform control on the Hölder constant. We also point out that minimizers are indeed Lipschitz.

Also, minimizers satisfy the following rigidity property: if a minimizer takes value $\theta \in (-1, 1)$ at some point x, it must take both the values 1 and -1 at some point y, whose distance from x can be uniformly bounded.

In Section 5.2, we discuss the notion of "minimal minimizer". In Section 5.3, we prove that the minimal minimizer satisfies the so called Birkhoff property. In Sections 5.4 and 5.5 we prove that, for rational frequencies, the minimal minimizer of strips wide enough is indeed unconstrained and hence class A. In Section 5.6, we extend the result to irrational frequencies.

Before starting with the technical statements and proofs, we would like to discuss a heuristic (absolutely non-mathematical!) "proof" of the existence of plane-like minimizers. For simplicity, we will consider a two-dimensional case; even if obviously *non* rigorous, this argument will allow us to introduce some of the ideas we will develop during the actual proof.

Given any vector $v \in \mathbf{R}^2$, let us construct two "walls" in direction v and at distance M from each other. Let us consider a fluid flowing through the "pipe" given by these two walls. This fluid can be thought as being "constrained" by the walls.

In some part of the space, the fluid is probably not touching the constraining walls, while, somewhere else, it might stick to them.

Let us now consider bigger and bigger M's, i.e. let us enlarge the pipe. Our "hope" is that, for a suitable (universal!) M, the fluid will not stick to the walls anymore. At this point, the fluid is actually "unconstrained": we can remove the pipe and obtain a free "plane-like" fluid jet.

Obviously, some work needs to be done to make this argument rigorous, especially to show that M is indeed a uniform quantity, independent on the given direction v. Making this argument rigorous will be precisely the target of the following pages.

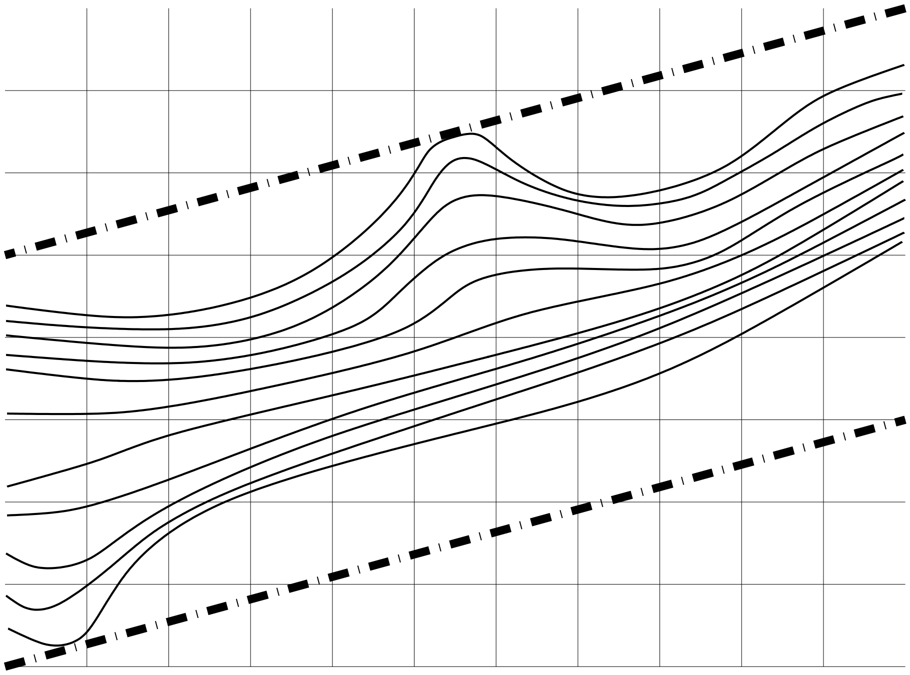

Figure 5.1: A 2−dimensional fluid constrained by two walls

5.1 Existence and regularity of constrained minimizers

In this section, we recall some known properties of existence and regularity of a minimizer for the functional $\mathcal{J}$ when we prescribe boundary values for u in a strip. The results of this section are an offspring of [AC] and [GG]. In the following, for a vector $\omega \in \mathbf{R}^n$ and $M \in \mathbf{R}$, we denote the strip of size M by

$$S_M^\omega := \{x \in \mathbf{R}^n \text{ s.t. } x \cdot \omega \in [0, M]\} .$$

When $\omega \in \mathbf{Q}^n$, we can recover compactness by an identification of the space. Namely, we consider the following relation $\sim$ induced by ω:

$$x \sim y \qquad \text{iff} \qquad x - y = k \in \mathbf{Z}^n \quad \text{with} \quad \omega \cdot k = 0 . \tag{5.1}$$

We will denote by $\tilde{S}^\omega_M = S^\omega_M/\sim$ the quotient space induced by such relation (which is topologically equivalent to the product of an $(n-1)-$dimensional torus and a real interval).

We also set

$$\mathcal{X}^\omega_M := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}^\omega_M) \text{ s.t. } \nabla u \in \mathrm{L}^2(\tilde{S}^\omega_M)$$
$$\text{with } u(x) = 1 \text{ for any } x \text{ s.t. } \omega \cdot x = 0 \text{ and}$$
$$u(x) = -1 \text{ for any } x \text{ s.t. } \omega \cdot x = M\} .$$

We start by discussing the existence of such constrained minimizers:

Theorem 5.1.1 (Alt and Caffarelli). *For any $\omega \in \mathbf{Q}^n$ and for any $M > 0$, there exist an absolute minimum of $\mathcal{J}_{S^\omega_M}$ in the space $\mathcal{X}^\omega_M$. Such minimizer is periodic (with respect to the identification $\sim$).*

For the proof see Theorem 1.3 of [AC] or the proof of Theorem 6.2.1 here below. The assumption $\omega \in \mathbf{Q}^n$ has been made in order to restrict ourselves to a compact domain, and have that $\mathcal{J}$ is finite when evaluated, for instance, on the admissible linear function.

With a little abuse of notation, we sometimes identify the functions in $\mathcal{X}^\omega_M$ with their constant extension outside the strip S^ω_M. We also denote by $\mathcal{M}^\omega_M$ the set of the minima found by Theorem 5.1.1.

In order to apply Theorem 5.1.1, we will now restrict our attention to rational frequencies, i.e., if not differently stated, we will assume $\omega \in \mathbf{Q}^n$. This restriction will be dropped in Section 5.6, by an approximation argument.

It is a standard consequence of Poincaré inequality that, indeed, $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^n)$ (see, for instance [S]). Further regularity is addressed by the following

Theorem 5.1.2 (Giaquinta and Giusti). *If $u \in \mathcal{M}^\omega_M$, then u is C^α for any $\alpha \in [0, 1)$. Furthermore, given any $K \subset\subset S^\omega_M$, the C^α norm of u in K is bounded by a constant depending only on α, n, the bounds on $a_{i,j}$ and Q and on* $\operatorname{dist}(K, \partial S^\omega_M)$.

Theorem 5.1.2 was addressed here in Section 3.1. The interested reader may also see [GG], or Theorem 7.6 of [G2], or Chapter 5 of [G].

Also, by using the Lipschitz regularity of the coefficients $a_{i,j}$, we will see that the minimizers are actually Lipschitz (Theorem 3.2.1).

5.2 The minimal minimizer

Following the ideas developed in [CL], we now turn our attention to a special minimizer of $\mathcal{J}$, namely the *minimal minimizer*. The minimal minimizer is defined by

$$u^\omega_M(x) := \inf_{u \in \mathcal{M}^\omega_M} u(x) .$$

By definition, for any $u \in \mathcal{M}^\omega_M$ and $\theta \in \mathbf{R}$,

$$\{u^\omega_M > \theta\} \subseteq \{u > \theta\} . \tag{5.2}$$

We now want to prove that u^ω_M is still a minimizer: this will be accomplished in the following Proposition 5.2.3. For this, we need some easy auxiliary observation.

Lemma 5.2.1. *For any $u,\, v \in \mathrm{L}^1_{\mathrm{loc}}(\tilde{S}^\omega_M)$ with $\nabla u,\, \nabla v \in \mathrm{L}^2(\tilde{S}^\omega_M)$,*

$$\mathcal{J}_{S^\omega_M}(u) + \mathcal{J}_{S^\omega_M}(v) \geq \mathcal{J}_{S^\omega_M}(\min\{u,v\}) + \mathcal{J}_{S^\omega_M}(\max\{u,v\}) \,.$$

Proof. It is a straightforward computation. □

Corollary 5.2.2. *If $u,\, v \in \mathcal{M}^\omega_M$, then* $\min\{u,v\}$, $\max\{u,v\} \in \mathcal{M}^\omega_M$.

Proof. Since $\mathcal{J}_{S^\omega_M}(u) \leq \mathcal{J}_{S^\omega_M}(\min\{u,v\})$ and $\mathcal{J}_{S^\omega_M}(v) \leq \mathcal{J}_{S^\omega_M}(\max\{u,v\})$, by Lemma 5.2.1 we deduce

$$\mathcal{J}_{S^\omega_M}(u) = \mathcal{J}_{S^\omega_M}(\min\{u,v\}) \quad \text{and} \quad \mathcal{J}_{S^\omega_M}(v) = \mathcal{J}_{S^\omega_M}(\max\{u,v\}) \,.$$

□

Proposition 5.2.3. $u^\omega_M \in \mathcal{M}^\omega_M$.

The proof of Proposition 5.2.3 is now a standard application of the Theorem of Ascoli.

5.3 Birkhoff property for the minimal minimizer

Proposition 5.3.1. *Let $\theta \in \mathbf{R}$. Then, the set $\{u^\omega_M > \theta\}$ satisfies the Birkhoff property with respect to ω. Namely:*

- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \leq 0$, then $\{u^\omega_M > \theta\} \supseteq \{\mathcal{T}_k u^\omega_M > \theta\}$;*
- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \geq 0$, then $\{u^\omega_M > \theta\} \subseteq \{\mathcal{T}_k u^\omega_M > \theta\}$.*

Proof. Set $v := \min\{u_M^\omega, \mathcal{T}_k u_M^\omega\}$. If $k \cdot \omega \leq 0$, by Corollary 5.2.2, v is a minimizer in the strip $\mathcal{T}_k S_M^\omega$. By construction, $\mathcal{T}_k u_M^\omega$ is the minimal minimizer in $\mathcal{T}_k S_M^\omega$. Thus, from (5.2),

$$\{\mathcal{T}_k u_M^\omega > \theta\} \subseteq \{v > \theta\} \subseteq \{u_M^\omega > \theta\}\,.$$

Analogously, if $k \cdot \omega \geq 0$, v is a minimizer in S_M^ω, therefore

$$\{u_M^\omega > \theta\} \subseteq \{v > \theta\} \subseteq \{\mathcal{T}_k u_M^\omega\}\,.$$

□

5.4 The minimal minimizer is unconstrained

Proposition 5.4.1. *There exists $M_0 > 0$, depending only on n and on the bounds on $a_{i,j}$ and Q, such that if $M \geq M_0|\omega|$, then the set $\{|u_M^\omega| < 1\}$ does not touch the upper constraint $\{\omega \cdot x = M\}$. Also, $\forall \alpha \geq 0$, $u_M^\omega = u_{M+\alpha}^\omega$.*

Proof. Let S be a strip of width 3 around the hyperplane $L := \{x \cdot \omega = M/2\}$. First of all, we will prove that there exists a strip H of width 1 parallel to the constraints and at distance universally bounded from L, on which

$$\text{either } u_M^\omega = -1 \text{ or } u_M^\omega = 1\,. \tag{5.3}$$

For that, we consider two cases:

Case 1: there exists a point $x^* \in S$ so that $u_M^\omega(x^*) \in (-1, 1)$. Then, from Proposition 4.4.1 there exists a ball (whose radius is big if M is big), contained in $\{u_M^\omega = -1\} = \mathcal{C}\{u_M^\omega > -1\}$. By the results in Section 5.3, we obtain a strip of width 1 intersecting such ball and contained in $\{u_M^\omega = -1\}$.

Case 2: $u^\omega_M(S) \subseteq (-\infty,-1]\cup[1,+\infty)$. By the continuity of u^ω_M, either $u^\omega_M(S)\subseteq(-\infty,-1]$ or $u^\omega_M(S)\subseteq[1,+\infty)$.

In any case, (5.3) is proved. With no loss of generality, assume also $\omega_1>0$. Now we infer that it is not possible that $u^\omega_M\Big|_H=1$: Otherwise, since the set $\{u^\omega_M=1\}$ satisfies the Birkhoff property with respect to ω (see Section 5.3), we obtain that

$$\bigcup_{j\in\mathbf{N}}\mathcal{T}_{(-j,0,\ldots,0)}H\subseteq\bigcup_{j\in\mathbf{N}}\mathcal{T}_{(-j,0,\ldots,0)}\{u^\omega_M=1\}\subseteq\{u^\omega_M=1\}\,,$$

thus $u^\omega_M=1$ in the whole portion of space trapped between the lower constraint and H. But then we can consider the integer translation $v:=\mathcal{T}_{(-1,0,\ldots,0)}u^\omega_M$, which has the same energy as u^ω_M, contradicting the fact that u^ω_M is the minimal minimizer.

Therefore, $u^\omega_M\Big|_H=-1$. Then, again, since the set $\{u^\omega_M>-1\}$ satisfies the Birkhoff property with respect to ω, for any $j\in\mathbf{N}$,

$$\mathcal{T}_{(j,0,\ldots,0)}H\cap\{u^\omega_M>-1\}=\emptyset\,,$$

so $u^\omega_M=-1$ in the whole portion of space trapped between the upper constraint and H. $\square$

In order to clearly state the next result, we introduce the following notation. If $a>0$, we define

$$S^\omega_{M,a}:=\{x\in\mathbf{R}^n\ \text{s.t.}\ x\cdot\omega\in[-a,M+a]\}$$

and

$$\mathcal{X}^\omega_{M,a}:=\{u\in\mathrm{L}^1_{\mathrm{loc}}(\tilde{S}^\omega_M)\ \text{s.t.}\ \nabla u\in\mathrm{L}^2(\tilde{S}^\omega_M)$$

$$\text{with } u(x) = 1 \text{ for any } x \text{ s.t. } \omega \cdot x = -a \text{ and}$$

$$u(x) = -1 \text{ for any } x \text{ s.t. } \omega \cdot x = M + a\} .$$

Theorem 5.4.2. *There exists a constant M_0, depending only on n and on the bounds on $a_{i,j}$ and Q, so that if $M \geq M_0|\omega|$, then the minimal minimizer u^ω_M is an unconstrained minimizer, i.e. for any $a > 0$ and any $u \in \mathcal{X}^\omega_{M,a}$, $\mathcal{J}_{S^\omega_{M,a}}(u^\omega_M) \leq \mathcal{J}_{S^\omega_{M,a}}(u)$.*

Proof. Proposition 5.4.1 above establishes that the upper constraint $\{x \cdot \omega = M\}$ is irrelevant. Hence, we have only to show that the lower constraint $L_0 := \{x \cdot \omega = 0\}$ is not necessary either. Given $a > 0$, take $k \in \mathbf{Z}^n$ so that $k \cdot \omega > a$: for instance, if $\omega_1 \neq 0$, take $k := A \cdot \operatorname{sign}(\omega_1)e_1$ with $A \in \mathbf{N}$ sufficiently big. And consider translations $\mathcal{T}_k u$. □

Remark 5.4.1. Following [CL], we would like to sketch an alternative geometric argument for proving that the minimal minimizer is unconstrained. Let ϱ be the constant in Proposition 2.1.1. We assume, without loss of generality, that ϱ, $M/4 \in \mathbf{N}$ and that all the components of ω are multiple of 4ϱ. Let us consider a non-overlapping covering of the strip S^ω_M. If K_ω is the number of cubes of side 4ϱ intersecting the plane $\{\omega \cdot x = 0\}$, we have that the number of cubes in the covering is $MK_\omega/4$. On the other hand, let ν be the number of cubes in the covering whose dialation of a factor $1/2$ intersects the set $\{|u| < 1\}$. Using a one-directional linear function interpolating -1 and 1 as a comparison, one gets that $\mathcal{J}_{S^\omega_M}(u^\omega_M) \leq \text{const}K_\omega$. On the other hand, by the density estimates in Lemmas 4.1.7 and 4.2.3, if a cube Q of size 2ϱ intersects the set $\{|u| < 1\}$, then $\mathcal{J}_{2Q}(u^\omega_M) \geq \text{const}$. From these two estimates, it follows that $\nu \leq \text{const}K_\omega < MK_\omega/4$, provided that M is big enough. Thus, there is a cube of side 2ϱ not touching the set $\{|u| < 1\}$. In the same way as above, this and the Birkhoff property would imply that the minimizer is unconstrained.

A similar geometric construction has been used in [CL] for dealing with manifoolds with negative curvature.

Remark 5.4.2. From Theorem 5.4.2 and Theorem 5.1.2, it follows that, if $M \geq M_0|\omega|$, then, for any $\alpha \in [0, 1)$,

$$||u_M^\omega||_{\mathrm{C}^\alpha(\mathbf{R}^n)} \leq C^\star ,$$

for a suitable $C^\star$, depending only on α, n and on the bounds on $a_{i,j}$ and Q. Also, by Theorem 3.2.1,

$$||u_M^\omega||_{\mathrm{C}^{0,1}(\mathbf{R}^n)} \leq C^\star .$$

5.5 The minimal minimizer is class A

Definition 5.5.1. We say that u is a **class A** minimizer for the functional $\mathcal{J}$ if, for any compact set K, $\mathcal{J}_K(u) < \infty$, and for any $v \in \mathrm{L}^1(K)$ with $\nabla v \in \mathrm{L}^2(K)$ verifying $v = u$ in $\mathcal{C}(K)$, we have $\mathcal{J}_K(u) \leq \mathcal{J}_K(v)$.

Notice that the definition of class A requires the functional not to increase under any compact (but not necessarily small) modification. This property was introduced for geodesics in [Mo], and it appeared also in Aubry-Mather theory.

Theorem 5.5.1. *There exists a constant M_0, depending only on n and on the bounds on $a_{i,j}$ and Q, so that if $M \geq M_0|\omega|$, then the minimal minimizer u_M^ω is a class A minimizer.*

Proof. Given a compact perturbation, we can always consider u_M^ω as periodic with a period larger than the diameter of the perturbation and (in light of Theorem 5.4.2) as a minimizer in a strip of width bigger than the size of the perturbation. $\square$

5.6 The minimal minimizer is class A: irrational frequency

In order to prove Theorem 5.0.3, it suffices now to consider the case of an irrational frequency $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, since the rational frequency case was dealt with in the previous Sections. An irrational frequency ω will now be handled passing to the limit rational frequencies ω_n approximating ω.

Namely, consider minimal minimizers $u_M^{\omega_n}$. By Remark 5.4.2 and the Theorem of Ascoli, $u_M^{\omega_n}$ converges to some u_∞ uniformly on compact sets. The reader may easily convince herself that u_∞ is a class A minimizer. This completes the proof of Theorem 5.0.3.

Chapter 6

Plane-like minimizers for Ginzburg-Landau

6.1 The Ginzburg-Landau-type functional

In this chapter, we will deal with the functional (1.2). We will prove that:

Theorem 6.1.1. *There exists a universal constant M_0, depending only on n, the bounds on $a_{i,j}$ and F stated above, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u for the functional $\mathcal{G}$ for which the set $\{|u| \leq 99/100\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.*

Moreover, such u enjoys the following property of "quasi-periodicity":

- *if $\omega \in \mathbf{Q}^n$, then u is periodic (with respect to the identification induced by ω);*
- *if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.*

The reader may compare Theorems 6.1.1 and 5.0.3: notice, in particular, that in Theorem 5.0.3 it was possible to control the set $\{|u| < 1\}$ in a uniform way.

The definition of "class A minimizer" was given in Section 5.5. The proof of Theorem 6.1.1 will use some of the machinery already developed in chapter 5. The main obstacle in shadowing the proof presented above consists in the fact that we do not expect in general to have the rigidity and non-degeneracy properties stated in Lemmas 4.1.4 and 4.1.5. Namely, for smooth potentials, the minimizers do not reach the values 1 and -1, so, in this case, we are not in presence of a free boundary. However, the arguments involving the minimal minimizer and the Birkhoff property still hold.

In more detail, the proof will go as follows: In Section 6.2 we discuss the existence and regularity of constrained minimizers. The arguments will be in analogy with the ones in Section 5.1; the main difference is that, in this case, we can not exactly prescribe the values on the boundary of the strip to be ± 1: this will be replaced by the weaker constraint of requiring the function to be bigger than, say, $99/100$ if $\omega \cdot x \leq 0$ and smaller that $-99/100$ if $\omega \cdot x \geq M$. Obviously, the explicit choice of the values $\pm 99/100$ is by no means important.

In Section 6.3 we proof that the minimal minimizer is unconstrained and class A, provided that the strip is wide enough: the arguments in these last two sections will be close to the analogous ones presented in Chapter 5.

6.2 Existence and regularity of constrained minimizers

We first turn our attention to a rational frequency ω. The irrational frequency case will then be handled by a limit argument, analogous to the one in Section 5.6.

We recall now the equivalence relation "$\sim$" defined in (1.3), and we define $\tilde{\mathbf{R}}^n := \mathbf{R}^n / \sim$ (topologically, $\tilde{\mathbf{R}}^n$ is the product between an $(n-1)-$dimensional torus and a real line). We consider the space of admissible functions

$$\mathcal{Y}_M^\omega := \{u \in \mathrm{H}^1_{\mathrm{loc}}(\tilde{\mathbf{R}}^n)$$

$$\text{with } u(x) \geq 99/100 \text{ for any } x \text{ s.t. } \omega \cdot x \leq 0 \text{ and}$$

$$u(x) \leq -99/100 \text{ for any } x \text{ s.t. } \omega \cdot x \geq M\}\,.$$

Following is the extension of the existence Theorem 5.1.1 we need:

Theorem 6.2.1. *For any $\omega \in \mathbf{Q}^n$ and for any $M > 0$, there exist an absolute minimum in $\mathcal{Y}_M^\omega$ of the functional*

$$\mathcal{G}_{\tilde{\mathbf{R}}^n}(u) := \int_{\tilde{\mathbf{R}}^n} a_{i,j}\partial_i u \partial_j u + F\,.$$

Such minimizer is periodic (with respect to the identification $\sim$).

Proof. Set $u_0 := -\frac{2\omega\cdot x}{M} + 1$ in $\{\omega \cdot x \in [0, M]\}$, and continuously extend u_0 as a constant to $\tilde{\mathbf{R}}^n$. For any $R > 0$, define $D_R := \{\omega \cdot x \in [-R, R]\}$ and $\tilde{D}_R := D_R/\sim$ (topologically, $\tilde{D}_R$ is the product between an $(n-1)-$dimensional torus and an interval).

Consider a minimizing sequence u_k: by cutting off the values above 1 and below -1, we may assume $-1 \leq u_k \leq 1$. We may also assume $\mathcal{G}_{\tilde{\mathbf{R}}^n}(u_k) \leq \mathcal{G}_{\tilde{\mathbf{R}}^n}(u_0)$, hence $||\nabla u_k||_{\mathrm{L}^2(\tilde{\mathbf{R}}^n)}$ is bounded. From these facts and Poincaré inequality, it follows that $||u_k||_{\mathrm{L}^2(\tilde{D}_R)}$ is bounded.

Therefore, we may assume that u_k converges to a suitable u in $\mathrm{L}^2(\tilde{D}_R)$ (and a.e.) and that ∇u_k converges to ∇u weakly in $\mathrm{L}^2(\tilde{D}_R)$, so that

$$\liminf_{k\longrightarrow\infty} \int_{\tilde{\mathbf{R}}^n} a_{i,j}\partial_i u_k \partial_j u_k \geq \liminf_{k\longrightarrow\infty} \int_{\tilde{D}_R} a_{i,j}\partial_i u_k \partial_j u_k \geq \int_{\tilde{D}_R} a_{i,j}\partial_i u \partial_j u\,,$$

and $\int_{\tilde{D}_R} |F(x, u_k) - F(x, u)|$ goes to zero by the Dominated Convergence Theorem. Sending first k and then R to $+\infty$, the claim follows. $\square$

The set of minimizers obtained via Theorem 6.2.1 will be denoted by $\mathcal{M}^{\omega}_{M}(\mathcal{G})$. In analogy with Theorem 5.1.2, we point out that the minimizers whose existence has just been proven are C^{α}, for a suitable $\alpha \in (0,1)$ with a uniform bound on the C^{α} norm in the interior: this follows by using Giaquinta and Giusti's Theorem. Also, in the case of smooth coefficients $a_{i,j}$ and a smooth potential F, further regularity can be obtained by applying standard elliptic estimates to the Euler-Lagrange equation of the minimizers. Indeed, in this case, by inspecting the first variational of the functional, it is easy to see that minimizers verify

$$2\partial_i(a_{i,j}\partial_j u) = F_u \,.$$

6.3 The minimal minimizer is unconstrained and class A

Proposition 6.3.1. *There exists $M_0 > 0$, depending only on n and on the bounds on $a_{i,j}$ and F, such that if $M \geq M_0|\omega|$, then the set $\{u^{\omega}_{M} > -99/100\}$ does not touch the upper constraint $\{\omega \cdot x = M\}$. Also, $\forall \alpha \geq 0$, $u^{\omega}_{M} = u^{\omega}_{M+\alpha}$.*

The proof of Proposition 6.3.1 is analogous to the one of Proposition 5.4.1.

The rest of the proof of Theorem 6.1.1 follows exactly as in Sections 5.4—5.6.

Chapter 7

Back to the minimal surfaces

7.1 A little bit of Riemannian Geometry

7.1.1 The distance function

In this subsection we consider an $n-$dimensional Riemannian manifold (M, g). The metric g induces the distance $d(x, y)$ between two points $x, y \in M$ that can be defined as the $g-$length of the shortest path in M between x and y.

The distance from a closed nonempty subset K of M can be defined as $d_K(x) = \inf_{y \in K} d(x, y)$. For elementary properties of the distance function in the Euclidean case, in which $(M, g) = (\mathbf{R}^n, \delta_{i,j})$, see the Appendix in [GT] or Section 1 in [Md].

We denote by g^{-1} the inverse metric of g, i.e. $g_{i,k} g^{-1}_{k,j} = \delta_{i,j}$ in local coordinates. The following result will be used in the proof of Theorem 7.2.1 below. Its proof is taken from [MaMe].

Theorem 7.1.1. *The distance function d_K is a viscosity solution of*

$$g^{-1}_{i,j} \partial_i u \partial_j u = 1 \qquad \text{in} \quad M - K\,.$$

Proof. By Lagrange Multipliers one sees that

$$\inf_{g_{i,j} w_i w_j = 1} v \cdot w = -g^{-1}_{i,j} v_i v_j\,. \tag{7.1}$$

Let us show that d_K is a viscosity subsolution. Let ϕ be a C^1 function touching d_K from above at $\bar{x}$ and let $\gamma : \mathbf{R} \longrightarrow M$ be a path parameterized by arclength with $\gamma(0) = \bar{x}$. Then,

$$\begin{aligned} \frac{\phi(\gamma(t)) - \phi(\bar{x})}{t} \quad &\geq \quad \frac{d_K(\gamma(t)) - d_K(\bar{x})}{t} \geq \\ &\geq \quad -\frac{d(\gamma(t), \gamma(0))}{t} \geq -1 \,. \end{aligned}$$

Passing to the limit for $t \longrightarrow 0$ and using (7.1), one deduces that $g^{-1}_{i,j}\partial_i\phi\partial_j\phi \leq 1$.

Let us now show that d_K is a viscosity supersolution. Let ϕ be a C^1 function touching d_K from below at $\bar{x}$. Let γ be the geodesic arc parameterized by arclength which minimizes the distance from $\bar{x}$ to K, with $\gamma(0) = \bar{x}$. Then, for $t \geq 0$,

$$t + \phi(\gamma(t)) - \phi(\bar{x}) \leq t + d_K(\gamma(y)) - d_K(\gamma(0)) \,.$$

The right hand side of the above formula vanishes, since γ also minimizes the distance to K for any other point $\gamma(t)$. Dividing by t, passing to the limit for $t \longrightarrow 0$ and using (7.1), we get $g^{-1}_{i,j}\partial_i\phi\partial_j\phi \geq 1$. □

We now recall the following well known result:

Proposition 7.1.2. *Let ϕ be a continuous function, differentiable at a point x_0. Then, there are two C^1 functions ψ_+ and ψ_- such that $\psi_\pm(x_0) = \phi(x_0)$ and $\psi_+ > \phi > \psi_-$ in a neighborhood of x_0.*

For a proof, see Proposition 1.6 at page 3 of [E]. From Theorem 7.1.1, Proposition 7.1.2 and the fact that d_K is Lipschitz (hence differentiable almost everywhere), we get

Corollary 7.1.3. $g^{-1}_{i,j}\partial_i d_K \partial_j d_K = 1$ *almost everywhere.*

7.1.2 The Coarea Formula

We recall a variant of the Coarea Formula, that can be found in [DM] (in particular, Lemma 2.2). The interested reader may also see the comments after Proposition 2.4 in [B].

Given $f \in \mathrm{L}^1_{\mathrm{loc}}$, we denote by Df the gradient of f in the sense of distributions (as usual, for instance, in [G1]).

Theorem 7.1.4. *Let Ω be a bounded open set of $\mathbf{R}^n$. Let $\Psi(x,s,p)$ a positive Borel function of $\Omega \times \mathbf{R} \times \mathbf{R}^n$, which is sublinear in p. Let $u \in \mathrm{Lip}(\Omega)$. Then,*

$$\int_\Omega \Psi(x,\, u(x),\, \nabla u(x))\, dx = \int_{\mathbf{R}} \Big(\int_\Omega \Psi(x, t, D\chi_{\{u>t\}})\, dx \Big)\, dt\,.$$

7.1.3 The perimeter on $\mathbf{R}^n$ induced by a Riemannian metric

Let us consider a (continuous) Riemannian metric g on $\mathbf{R}^n$, whose inverse is the matrix g^{-1}. Let $\mathcal{H}^{n-1}$ be the usual $(n-1)-$dimensional Haussdorff measure and $\mathcal{L}^n$ be the usual Lebesgue measure on $\mathbf{R}^n$. Following [BPV], at any point $x \in \mathbf{R}^n$, we associate the following quantity:

$$\mathcal{D}(x) \quad = \quad \Big(\mathcal{L}^n(\{v \in \mathbf{R}^n \text{ s.t. } g_{i,j}(x) v_i v_j \leq 1\}) \Big)^{-1}$$

By means of Riemannian Geometry, one can define the perimeter P_g induced by the metric g: from formula (5.8) of [BPV], we know that

$$P_g(A,\Omega) = \mathcal{L}^n(B_1) \cdot \int_{\partial^\star A \cap \Omega} \sqrt{g^{-1}_{i,j}\nu_i\nu_j}\, \mathcal{D}(x)\, d\mathcal{H}^{n-1}\,,$$

where $\partial^\star$ denotes the reduced boundary and ν the unit normal of the set A.

For further reference, we point out the following easy

Lemma 7.1.5. *If g is Lipschitz, $\mathcal{D}$ is Lipschitz; also, $||\mathcal{D}||_{\mathrm{Lip}}$ depends only on n, on the nondegeneracy of g and on $||g||_{\mathrm{Lip}}$.*

Proof. First notice that $\mathcal{D}$ is universally bounded and bounded from zero by the nondegeneracy of g. Fix $x \in \mathbf{R}^n$: choosing a suitable system of coordinate, we rule out g to be diagonal at x, with positive eigenvalues $\lambda_1, \dots, \lambda_n$. Then, since g is Lipschitz,

$$\begin{aligned}
& \mathcal{L}^n(\{v \in \mathbf{R}^n \text{ s.t. } g_{i,j}(x)v_iv_j \leq 1\}) - \\
& -\mathcal{L}^n(\{v \in \mathbf{R}^n \text{ s.t. } g_{i,j}(y)v_iv_j \leq 1\}) \leq \\
\leq\ & \mathcal{L}^n(\{v \in \mathbf{R}^n \text{ s.t. } \frac{1}{1+\mathrm{const}|x-y|} \leq g_{i,j}(x)v_iv_j \leq 1\}) = \\
=\ & \frac{1}{\sqrt{\lambda_1 \cdot \dots \cdot \lambda_n}} \mathcal{L}^n(\{w \in \mathbf{R}^n \text{ s.t. } \frac{1}{1+\mathrm{const}|x-y|} \leq |w|^2 \leq 1\}) \leq \\
\leq\ & \mathrm{const}|x-y|\,.
\end{aligned}$$

□

7.2 The Theorem of Modica

Fix $\theta \in (0,1]$. In this subsection we consider the space $\tilde{\mathcal{S}} = [-1,1] \times \mathbf{T}^{n-1}$ and functions $u \in \mathrm{H}^1_{\mathrm{loc}}(\tilde{\mathcal{S}})$ with $u(x) \geq \theta$ if $x_1 \geq 1$ and $u(x) \leq -\theta$ if $x_1 \leq -1$. Let us consider the functional

$$\mathcal{E}^\varepsilon(u) = \int_{\tilde{\mathcal{S}}} \varepsilon\, a_{i,j}\partial_i u \partial_j u + F(u)\, dx\,.$$

We will assume on $a_{i,j}$ and F the hypotheses stated in Chapter 1. We denote by $\mathcal{H}^{n-1}$ the usual $(n-1)-$dimensional Haussdorff measure. We define

$$P(A,\Omega) = \int_{\partial^\star A \cap \Omega} \sqrt{a_{i,j}\nu_i\nu_j}\, d\mathcal{H}^{n-1}\,, \tag{7.2}$$

where $\partial^\star$ denotes the reduced boundary and ν the unit normal of the set A.

Notice that, if $a_{i,j} = \delta_{i,j}$, the above definition reduces to the usual perimeter. Also, by means of the discussion in Section 7.1.3, we have that *given any metric g on $\mathbf{R}^n$, we can choose $a_{i,j}$ so that P, as defined in* (7.2)*, is the perimeter induced by the metric g.*

Explicitly,

$$a_{i,j} = \Big(\mathcal{L}^n(B_1) \cdot \mathcal{D}\Big)^2 g^{-1}_{i,j}\,.$$

Hence, if g is Lipschitz, by Lemma 7.1.5, the coefficients $a_{i,j}$ are also Lipschitz. Also, they are uniformly elliptic because of the nondegeneracy of g. Moreover, using the definition of functionals on measures (see, for instance, [B] at page 293), and recalling that

$$D\chi_A = \nu\,|D\chi_A| \qquad \text{and} \qquad |D\chi_A| = \mathcal{H}^{n-1}\Big|_{\partial^\star A}\,,$$

(see, for instance, [G1] pages 44 and 52), we have that

$$P(A,\Omega) = \int_\Omega \sqrt{a_{i,j}\partial_i\chi_A\partial_j\chi_A}\,.$$

We address now the following result:

Theorem 7.2.1 (Modica). *Let u_ε be minimizers for $\mathcal{E}^\varepsilon$. Then there is a subsequence ε_h converging to zero such that u_{ε_h} converges in* $\mathrm{L}^1(\tilde{\mathcal{S}})$ *to a function u_0 so that $u_0(x) = \pm 1$ for almost all $x \in \tilde{\mathcal{S}}$ and the set $E = \{u_0 = -1\}$ satisfies*

$$P(E,\tilde{\mathcal{S}}) = \min\Big\{P(F,\tilde{\mathcal{S}})\,,\ (-\infty,-1] \subset F \subset (-\infty,1)\Big\}\,.$$

Notice that, using the Theorem of Caffarelli and Córdoba (see Chapter 4), from Theorem 7.2.1 it follows that

Corollary 7.2.2. *Let $\tau \in [-\theta, \theta]$. Any level surface $\{u_{\varepsilon_h} = \tau\}$ converges uniformly to the minimal surface ∂E.*

Theorem 7.2.1 has been proved in [Md] in the case $a_{i,j} = \delta_{i,j}$ and widely extended in [OS] and [B], with different proofs. The interested reader may also see [BPV]. Here, we give a proof of Theorem 7.2.1 which is a modification of the one in [Md], by using the tools developed in the subsections above. A useful function will be the primitive of the square root of the potential, namely

$$\phi(t) = \int_0^t \sqrt{F(s)}\, ds\,.$$

It is easy to see that ϕ is invertible, and its inverse, that will be denoted by ψ is bounded and uniformly continuous on bounded sets.

Notice that the existence of the converging subsequence in Theorem 7.2.1 is given by the following compactness argument: by cutting off the values above 1 and below -1, we may assume we may assume $|u_\varepsilon| \leq 1$. Therefore, $v_\varepsilon(x) = \phi(u_\varepsilon(x))$ is bounded in L^1. On the other hand,

$$\begin{aligned} \int_{\tilde{\mathcal{S}}} |\nabla v_\varepsilon| &= \int_{\tilde{\mathcal{S}}} |F^{\frac{1}{2}}(u_\varepsilon)||\nabla u_\varepsilon| \leq \\ &\leq \frac{1}{2}\int_{\tilde{\mathcal{S}}} \varepsilon^{\frac{1}{2}}|\nabla u_\varepsilon|^2 + \varepsilon^{-\frac{1}{2}} F(u_\varepsilon) \leq \operatorname{const} \varepsilon^{-\frac{1}{2}} \mathcal{E}^\varepsilon(u_\varepsilon)\,. \end{aligned} \tag{7.3}$$

By considering functions of one variable that grows from -1 to 1 in an interval of size $\sqrt{\varepsilon}$, it is easy to see that the last term in (7.3) is bounded uniformly in ε. Therefore, by compactness, we get (up to a subsequence) that v_ε converges in L^1 to a function v_0. Hence $u_\varepsilon(x) = \psi(v_\varepsilon(x))$ converges (up to a subsequence) in L^1 to a function u_0.

The fact that u_0 takes only values $+1$ and -1 can be easily proved as follows: since, as observed above, $\varepsilon^{-\frac{1}{2}}\mathcal{E}^\varepsilon(u_\varepsilon)$ is bounded, by Fatou's Lemma

$$\begin{aligned}\int_{\tilde{\mathbb{S}}} F(u_0(x))\,dx \;&\le\; \liminf \int_{\tilde{\mathbb{S}}} F(u_{\varepsilon_h}(x))\,dx \le \\ &\le\; \liminf \mathcal{E}^{\varepsilon_h}(u_{\varepsilon_h}) = 0\,.\end{aligned}$$

Thus, since $F \ge 0$, we infer $F(u_0(x)) = 0$ a.e., and so $|u_0(x)| = 0$ a.e.

The fact that u_0 has minimal interface area is more delicate. To prove it, we start with this auxiliary result:

Lemma 7.2.3. *Let A be an open subset of $\mathbf{R} \times \mathbf{T}^{n-1}$ with ∂A a non-empty, compact, smooth hypersurface. Assume that*

$$(-\infty,-1] \subset A \subset (-\infty,1)\,.$$

Let v_0 be the function that takes value -1 in $A \cap \tilde{\mathbb{S}}$ and that takes value 1 in the complement of A in $\tilde{\mathbb{S}}$. Then, there is a family of Lipschitz continuous functions v_ε such that: $v_\varepsilon(x) \in [-1,1]$, $v_\varepsilon = 1$ if $x_1 \ge 1$, $v_\varepsilon = -1$ if $x_1 \le -1$, v_ε converges to v_0 in $\mathrm{L}^1(\tilde{\mathbb{S}})$, and

$$\limsup \varepsilon^{-\frac{1}{2}}\mathcal{E}^\varepsilon(v_\varepsilon) \le \Big(\int_{-1}^{1} \sqrt{F(s)}\,ds\Big)\, P(A,\tilde{\mathbb{S}})\,. \tag{7.4}$$

Proof. Let h be the signed distance induced by the metric $a^{-1}_{i,j}$ from ∂A, i.e. $h(x) = \operatorname{dist}(x,\partial A)$ if $x \notin A$, and $h(x) = -\operatorname{dist}(x,\partial A)$ if $x \in A$, where "dist" denotes the distance induced by $a^{-1}_{i,j}$. Let us also define $\chi_0(t) = -1$ if $t < 0$ and $\chi_0(t) = 1$ if $t \ge 0$. In this way, $v_0(x) = \chi_0(h(x))$. We will approximate χ_0 in order to construct the sequence v_ε. Define, for $t \in [-1,1]$,

$$\psi_\varepsilon(t) = \int_{-1}^{t} \sqrt{\frac{\varepsilon}{\sqrt{\varepsilon} + F(s)}}\,ds\,,$$

and $\eta_\varepsilon = \psi_\varepsilon(1)$. Let $\phi_\varepsilon : [0, \eta_\varepsilon] \longrightarrow [-1, 1]$ denote the inverse of ψ_ε. Let us consider the function χ_ε that extends ϕ_ε to the entire real line with a constant continuous extension. In this way, χ_ε is a Lipschitz function on $\mathbf{R}$.

Define $v_\varepsilon(x) = \chi_\varepsilon(h(x))$. By construction, v_ε is Lipschitz and takes values in $[-1, 1]$. Also, since $a_{i,j}\partial_i h \partial_j h = 1$ a.e. (Corollary 7.1.3), using Coarea Formula (Theorem 7.1.4), we get

$$\begin{aligned}\int_{\tilde{\mathbb{S}}} |v_\varepsilon - v_0| &= \int_{\tilde{\mathbb{S}}} |\chi_\varepsilon(h(x)) - \chi_0(h(x))| \sqrt{a_{i,j}\partial_i h \partial_j h} = \\ &= \int_{\mathbf{R}} |\chi_\varepsilon(t) - \chi_0(t)| \, P(S_t) \, dt \,,\end{aligned}$$

where $S_t = \{x \in \tilde{\mathbb{S}} \,:\, h(x) > t\}$. Notice that, by construction, χ_ε and χ_0 agree outside $[0, \eta_\varepsilon]$: therefore,

$$\int_{\tilde{\mathbb{S}}} |v_\varepsilon - v_0| \leq 2\eta_\varepsilon \sup_{t \in [0, \eta_\varepsilon]} P(S_t) \,. \tag{7.5}$$

It is known from the properties of h that

$$\lim_{t \longrightarrow 0} P(S_t) = P(A \cap \tilde{\mathbb{S}}) \,. \tag{7.6}$$

Also, by construction, $\eta_\varepsilon \leq 2\varepsilon^{1/4}$. These considerations and (7.5) proves that v_ε converges to v_0 in L^1.

Take now x so that $x_1 \geq 1$. We want to prove that $v_\varepsilon(x) = 1$. Let $m = \min_{x_1 \geq 1} h(x)$: by hypotheses, $m > 0$. If ε is small enough, $m > \eta_\varepsilon$, thus $v_\varepsilon(x) = 1$.

It only remains to prove (7.4). We define

$$\gamma_\varepsilon = \sup_{t \in [0, \eta_\varepsilon]} P(S_t) \,.$$

As above, we can use Coarea Formula to deduce

$$\varepsilon^{-\frac{1}{2}}\mathcal{E}^{\varepsilon}(v_\varepsilon) \le \gamma_\varepsilon \int_0^{\eta_\varepsilon} [\varepsilon^{\frac{1}{2}}(\phi'_\varepsilon(t))^2 + \varepsilon^{-\frac{1}{2}}(\sqrt{\varepsilon} + F(\phi_\varepsilon(t)))]\, dt\,. \tag{7.7}$$

Also, by construction,

$$\varepsilon^{\frac{1}{2}}\phi'_\varepsilon(t) = \sqrt{\sqrt{\varepsilon} + F(\phi_\varepsilon(t))}\,,$$

so that (7.7) becomes

$$\begin{aligned} \varepsilon^{-\frac{1}{2}}\mathcal{E}^{\varepsilon}(v_\varepsilon) &\le 2\gamma_\varepsilon \int_0^{\eta_\varepsilon} \sqrt{\sqrt{\varepsilon} + F(\phi_\varepsilon(t))}\,\phi'_\varepsilon(t)\,dt = \\ &= 2\gamma_\varepsilon \int_{-1}^{1} \sqrt{\sqrt{\varepsilon} + F(s)}\,ds\,. \end{aligned}$$

From this and (7.6), the proof of (7.4) is complete. □

We can now complete the proof of Theorem 7.2.1. It is enough to verify that $P(E, \tilde{\mathbb{S}}) \le P(A, \tilde{\mathbb{S}})$ for any open bounded set containing $(-\infty, -1]$ and contained in $(-\infty, 1)$ with smooth boundary. Using Lemma 7.2.3, we find a sequence of v_ε, converging to the function v_0 defined above, so that $\mathcal{E}^{\varepsilon}(u_\varepsilon) \le \mathcal{E}^{\varepsilon}(v_\varepsilon)$ and

$$P(A, \tilde{\mathbb{S}}) \ge \frac{1}{2c_0} \limsup \varepsilon^{-\frac{1}{2}}\mathcal{E}^{\varepsilon}(v_\varepsilon)\,,$$

where we set $c_0 = \int_{-1}^{1}\sqrt{F}$. We now show that

$$P(E, \tilde{\mathbb{S}}) \le \frac{1}{2c_0} \liminf \varepsilon_h^{-\frac{1}{2}}\mathcal{E}^{\varepsilon_h}(u_{\varepsilon_h})\,, \tag{7.8}$$

where ε_h is the subsequence extracted at page 77. With (7.8) in hand, the proof of Theorem 7.2.1 finishes as follows:

$$\begin{aligned} P(E, \tilde{\mathbb{S}}) &\le \frac{1}{2c_0} \liminf \varepsilon_h^{-\frac{1}{2}}\mathcal{E}^{\varepsilon_h}(u_{\varepsilon_h}) \le \\ &\le \frac{1}{2c_0} \liminf \varepsilon_h^{-\frac{1}{2}}\mathcal{E}^{\varepsilon_h}(v_{\varepsilon_h}) \le P(A, \tilde{\mathbb{S}})\,. \end{aligned}$$

To prove (7.8), we define (for $\varepsilon = \varepsilon_h$) $w_\varepsilon = \phi(u_\varepsilon)$, and use Coarea Formula (Theorem 7.1.4):

$$\int_{\tilde{\mathcal{S}}} \sqrt{a_{i,j}\partial_i w_0 \partial_j w_0} = \int_{\mathbf{R}} P(\{w_0 > t\})\, dt$$

Since $|u_0(x)| = 1$ (as we already proved!), the last term in the relation above is simply $c_0 P(E, \tilde{\mathcal{S}})$. Therefore, by lower semicontinuity,

$$c_0 P(E, \tilde{\mathcal{S}}) \leq \liminf \int_{\tilde{\mathcal{S}}} \sqrt{a_{i,j}\partial_i w_\varepsilon \partial_j w_\varepsilon}\,.$$

Noticing that $\partial_i w_\varepsilon(x) = \phi'(u_\varepsilon(x))\,\partial_i u_\varepsilon$, we get

$$\begin{aligned} c_0 P(E, \tilde{\mathcal{S}}) &\leq \liminf \int_{\tilde{\mathcal{S}}} \sqrt{F(u_\varepsilon)\, a_{i,j}\partial_i u_\varepsilon \partial_j u_\varepsilon} \leq \\ &\leq \frac{1}{2}\int_{\tilde{\mathcal{S}}} [\varepsilon^{\frac{1}{2}}\, a_{i,j}\partial_i u_\varepsilon \partial_j u_\varepsilon + \varepsilon^{-\frac{1}{2}} F(u_\varepsilon)]\,. \end{aligned}$$

This proves (7.8) and completes the proof of Theorem 7.2.1.

7.3 Plane-like minimizers: minimal surfaces as a limiting case

We now point out that the results of [CL] about plane-like minimal surfaces can be recovered as a limiting case of jet flows or Ginzburg-Landau minimizers.

Namely, we now turn our attention to the $\varepsilon-$dependent functional

$$\mathcal{G}^{\varepsilon}_{\Omega}(u) := \int_\Omega \varepsilon a_{i,j}\partial_i u \partial_j u + \frac{1}{\varepsilon} F(u)\, dx\,. \tag{7.9}$$

The assumptions on $a_{i,j}$ and F will be the same as in Section 6.1 and $\varepsilon \in (0, 1]$ is a small parameter[1].

[1]For simplicity, we take here F independent of x: a more general situation can be dealt with using the same techniques, following [B]

Let us consider a Lipschitz Riemannian metric g on $\mathbf{R}^n$. Choosing suitable $a_{i,j}$, we know from [Md] and [CC] (see Section 7.2) that the level sets of the minimizers of (7.9) converge (when $\varepsilon \longrightarrow 0$, up to subsequences) to minimal surfaces (i.e., minimal with respect to the metric g). Thus, the result we now address is the following:

Theorem 7.3.1. *There exists a universal constant M_0, depending only on n, the bounds on $a_{i,j}$ and F, but independent of ε, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u^ε for the functional $\mathcal{G}^\varepsilon$, for which the set $\{|u^\varepsilon| \leq 99/100\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_0|\omega|]\}$.*

Moreover, such u^ε enjoys the following property of "quasi-periodicity":

- *if $\omega \in \mathbf{Q}^n$, then u^ε is periodic (with respect to the identification induced by ω);*
- *fixed $\varepsilon > 0$, if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u^ε can be approximated uniformly on compact sets by periodic class A minimizers of $\mathcal{G}^\varepsilon$.*

Also, in the case $F(u) = \chi_{(-1,1)}(u)$: There exists a universal constant M_1, depending only on n, the bounds on $a_{i,j}$ and Q, but independent of ε, such that: given any $\omega \in \mathbf{R}^n$, there exists a class A minimizer u^ε for the functional $\mathcal{G}^\varepsilon$ for which the set $\{|u^\varepsilon| < 1\}$ is contained in the strip $\{x \text{ s.t. } x \cdot \omega \in [0, M_1|\omega|]\}$.

Such u^ε also satisfies the quasi-periodicity property sated above.

In particular, sending ε to zero, we re-obtain the main result of [CL]: *Let g be a Lipschitz metric on $\mathbf{T}^n$. There exists a universal constant M (depending* *only on n, the nondegeneracy of g and its Lipschitz norm), such that, given any $\omega \in \mathbf{R}^n$, there exists a codimension 1 surface contained in the strip $\{\omega \in [0, M|\omega|]\}$, which is class A minimal with respect to the perimeter induced by g. Also, if ω is rational, the surface obtained is periodic (with respect to the identification induced by ω); if ω is irrational, the surface obtained can be approximated locally uniformly by periodic class A minimal surfaces.*

Following a notation introduced in the previous sections, we will denote by $\mathcal{M}^{\omega}_{M}(\mathcal{G}^{\varepsilon})$ the set of minimizers among the admissible functions in the strip S^{ω}_{M}: see pages 59 and 70 for details.

The proof of Theorem 7.3.1 will follow from the work done in Chapter 5, using the following suitable rescaling: for any function u we define

$$\mathcal{R}^{\varepsilon} u(x) := u(\varepsilon x) . \tag{7.10}$$

Notice that the renormalization $\mathcal{R}^{\varepsilon}$ preserves the ellipticity constants and the properties of the double-well potential, but it does *not* preserve integer periodicity. Therefore we can not apply the arguments involving the Birkhoff property to $\mathcal{R}^{\varepsilon} u$. Thus, we will only use the renormalization for extending the Geometric Lemma, since this result did *not* use the integer periodicity. Namely, the following Proposition 7.3.2 is the analogue of Propositions 4.4.1 and 4.4.2:

Proposition 7.3.2. *Fix $\theta_0 \in [0, 1)$. Let $u \in \mathcal{M}^{\omega}_{M}(\mathcal{G}^{\varepsilon})$. There exist positive constants κ, r_0 and r_1, depending only on θ_0, n and on the bounds on $a_{i,j}$ and F (but independent of ε), such that: for any $x \in \{|u| \le \theta_0\}$ and any $r \ge r_0$,* *there exist* z_1 *and* z_2*, such that*

$$B_{\kappa r}(z_1) \subseteq \{u \geq \theta_0\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u \leq -\theta_0\} \cap B_r(x)\,,$$

provided $B_{2r+r_1}(x) \subset S^{\omega}_{M}$.

In the case of the jet flows, the result above can be sharpened as follows:

Proposition 7.3.3. *Let* $u \in \mathcal{M}^{\omega}_{M}(\mathcal{G}^{\varepsilon})$*. If* $F = \chi_{(-1,1)}$ *and* Q *satisfies the assumptions stated above, then: There exist positive constants* κ*,* r_0 *and* r_1*, depending only on* n *and on the bounds on* $a_{i,j}$ *and* Q *(but independent of* ε*), such that: for any* $x \in \{|u| < 1\}$ *and any* $r \geq r_0$*, there exist* z_1 *and* z_2*, such that*

$$B_{\kappa r}(z_1) \subseteq \{u = 1\} \cap B_r(x)\,, \qquad \text{and} \qquad B_{\kappa r}(z_2) \subseteq \{u = -1\} \cap B_r(x)\,,$$

provided $B_{2r+r_1}(x) \subset S^{\omega}_{M}$.

Proof. Apply the rescaling (7.10). Use Proposition 4.4.2 for proving Proposition 7.3.2 and Proposition 4.4.1 for proving Proposition 7.3.3. □

With these results in hands, the proof of Theorem 7.3.1 readily follows by applying the Birkhoff property to u.

Remark 7.3.1. A natural question arising in this context is whether the minimal surfaces that are plane-like minimizers are always graphs.

The reader may easily convince herself that the answer to such question, in general, is no, just looking at a periodic metric on $\mathbf{R}^2$ which is very small on a set with an “S” shape and very big elsewhere.

Analogously, one can construct examples of metrics whose plane-like minimizers for the functional (7.9) are not monotone in any direction.

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