A Local Approach to Dimensional Reduction: II. Conformal Invariance in Minkowski Space

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Abstract

We consider the problem of obtaining conformally invariant differential operators in Minkowski space. We show that the conformal electrodynamics equations and the gauge transformations for them can be obtained in the frame of the method of dimensional reduction developed in the first part of the paper. We describe a method for obtaining a large set of conformally invariant differential operators in Minkowski space.

Subj. Class.: Differential geometry; Geometrical approaches to partial differential equations; Field theory.

2000 MSC: 37J15, 58Z05, 53Z05, 78A25, 81T40.

Key words: Differential geometry; Group action and symmetry; Non-compact dimensional reduction; Conformal invariance.

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1 Introduction

The conformal group is one of the most important symmetry groups in physics. It appears naturally in many physical problems, such as high-energy limit of quantum field theory equations [9], phase transitions [16,14], the geometry of anti-de Sitter space [1], the problem of the electromagnetic field of a charged

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particle moving with a constant relativistic acceleration [8], the geometry of the classical Kepler problem [12]. In electrodynamics, the conformal invariance of Maxwell's equations in vacuum was noticed as early as in 1909 by Bateman [2] and Cunningham [3].

The connected component $C_0(1,3)$ of the 15-parameter conformal group in Minkowski space M consists of the 10-parameter Poincaré group, the dilatations $x \mapsto x' = dx$ $(d > 0)$ and the 4-parameter nonlinear continuous group of *special conformal transformations* which are compositions of an inversion I, a translation, and again an inversion:

$$x \mapsto I(x) \mapsto I(x) - a \mapsto I(I(x) - a) = \frac{x - |x|^2 a}{1 - 2a \cdot x + |a|^2 |x|^2} \,. \tag{1}$$

The presence of the nonlinear transformations (1) makes the problem of constructing conformally invariant fields and especially conformally invariant differential operators very complicated.

In the present paper, we study the problem of constructing conformally invariant fields and differential operators in Minkowski space as well as gauge transformations that preserve this invariance. We use the geometric construction of Dirac [4] in which the crucial fact is that $C_0(1,3)$ is locally isomorphic to $O_0(2,4)$ – the connected component of the linear orthogonal group in the space $\mathbb{R}^6$ endowed with a $(-++++-)$ metric. In the Dirac's construction, each point in M is identified with an isotropic straight line in $\mathbb{R}^6$, i.e., Minkowski space (more precisely, its conformal compactification $\overline{M}$ – see Sec. 2) is realized as the projected light cone $Q\mathbb{P}^5 \subset \mathbb{R}^6$. Because of this embedding of M into $\mathbb{R}^6$, the action of $C_0(1,3)$ on M corresponds exactly to the action of $O_0(2,4)$ on the isotropic straight lines in $\mathbb{R}^6$. This is why, it is tempting to consider the relation between the manifestly $O_0(2,4)$-invariant differential operators in $\mathbb{R}^6$ and the $C_0(1,3)$-invariant differential operators in M.

This relation, however, is not straightforward. The Dirac's construction consists of two steps, namely, projecting $\mathbb{R}^6$ onto $\mathbb{RP}^5$, followed by the restriction of $\mathbb{RP}^5$ onto some realization of $Q\mathbb{P}^5$. Both steps are (in general) not natural for differential operators in the sense that, in order to perform them, one needs some additional information.

To overcome these difficulties, we apply the methods developed in the first part of this article [19] (which we will call Part I).

The plan of the article is the following. In Section 2 we describe Dirac's construction, in Section 3 we discuss the dimensional reduction of $\tau(\mathbb{R}^6)$ and $\tau^*(\mathbb{R}^6)$, Sections 4 and 5 are devoted to the reduction of the 6-dimensional Maxwell equations, and finally, in Section 6, we give a complete set of splitting relations for the Dirac's construction.

2 General Description of the Dirac's Construction

We start with the conformal compactification $\overline{M} := (S^1 \times S^3)/\mathbb{Z}_2$ of Minkowski space (for details see [21]). It is based on the Dirac's construction [4] in which the points of $\overline{M}$ are identified with the isotropic straight lines in the 6-dimensional real space $\mathbb{R}^6$ with a diagonal metric tensor $(\eta_{mn}) = \mathrm{diag}(-1,1,1,1,1,-1)$. Let Q^6 be the light cone in $\mathbb{R}^6$ and $\mathbb{R}$ be the multiplicative group of nonzero real numbers acting on $\mathbb{R}^6$ (here and elsewhere, $\mathbb{R}^6$ and Q^6 stand for $\mathbb{R}^6 \setminus \{0\}$ and $Q^6 \setminus \{0\}$, respectively). Therefore $\overline{M}$ is the projected light cone:

$$\overline{M} \cong Q\mathbb{P}^5 := Q^6/\mathbb{R} \ .$$

Let $(\xi^m) := (\xi^\mu, \xi^5, \xi^6)$ $(m, n, \ldots = 0,1,2,3,5,6;\ \mu, \nu, \ldots = 0,1,2,3)$ be the canonical coordinates in $\mathbb{R}^6$. The isomorphism $Q\mathbb{P}^5 \cong (S^1 \times S^3)/\mathbb{Z}_2$ is evident from

$$S^1 \times S^3 = \{\, \xi \in \mathbb{R}^6 \ \mid\ (\xi^0)^2 + (\xi^6)^2 = (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^5)^2 = 1 \,\} \ .$$

It is convenient to introduce in $\mathbb{R}^6$ coordinates adapted to the light cone Q^6. Following [17,10,7,13], we set

$$\begin{aligned} x^\mu &:= \frac{\xi^\mu}{\xi^5 + \xi^6} & \qquad \xi^\mu &= k\, x^\mu \\ k &:= \xi^5 + \xi^6 & \qquad \xi^5 &= k \left(\frac{1 - x^2}{2} + \phi \right) \\ \phi &:= \frac{\xi^m \xi_m}{2(\xi^5 + \xi^6)^2} & \qquad \xi^6 &= k \left(\frac{1 + x^2}{2} + \phi \right) \ , \end{aligned}$$

where $x_\mu := \eta_{\mu\nu} x^\nu$, $\eta_{\mu\nu} := \mathrm{diag}(-1,1,1,1)$, $x^2 := x^\mu x_\mu$. The Jacobian is

$$\frac{\partial(\xi^\mu, \xi^5, \xi^6)}{\partial(x^\mu, k, \phi)} = -k^2 \ ,$$

hence this change of variables is nondegenerate outside the hyperplane $k = 0$. The coordinates (x^μ, k, ϕ) will be referred to as Q-coordinates (the equation of the light cone Q^6 in Q-coordinates is $2k^2\phi = 0$), whereas (ξ^m) will be called ξ-coordinates. We use the indices "$-$" for k, "$+$" for ϕ, and $M, N, \ldots$ for

$(\mu, -, +)$, etc. In Q-coordinates the components of the metric tensor are

$$(g_{MN}) := \begin{pmatrix} g_{\mu\nu} & g_{\mu-} & g_{\mu+} \\ g_{-\nu} & g_{--} & g_{-+} \\ g_{+\nu} & g_{+-} & g_{++} \end{pmatrix} = \begin{pmatrix} k^2\eta_{\mu\nu} & 0 & 0 \\ 0 & 2\phi & k \\ 0 & k & 0 \end{pmatrix}$$

and the nonzero Christoffel symbols are $\Gamma^{\mu}_{\nu-} = \frac{1}{k}\,\eta^{\mu}_{\nu}$, $\Gamma^{+}_{\mu\nu} = -\eta_{\mu\nu}$, $\Gamma^{+}_{-+} = \frac{1}{k}$.

The action of the group $\mathbb{R}$ on $\mathbb{R}^6$ in Q-coordinates is

$$\rho(x^\mu, k, \phi) = (x^\mu, \rho k, \phi) \ ,$$

so as a chart of $\mathbb{RP}^5 \setminus \{k = 0\}$ we use the hyperplane

$$U := \{\xi \in \mathbb{R}^6 \mid k = 1\} \cong \mathbb{RP}^5 \setminus \{k = 0\} \ , \tag{2}$$

transversal to the $\mathbb{R}$-orbits of the points of $\mathbb{R}^6 \setminus \{k = 0\}$. The image of the line $[(x^\mu, k, \phi)]$ through the point (x^μ, k, ϕ), $k \neq 0$ (i.e., of the equivalence class of this point with respect to the action of $\mathbb{R}$) is the point $(x^\mu, 1, \phi) \in U$.

The points of the conformal compactification $\overline{M}$ of Minkowski space are realized as elements of the projected light cone $Q\mathbb{P}^5$, i.e., as isotropic straight lines in $\mathbb{R}^6$, and the points of Minkowski space M correspond to the isotropic straight lines in $\mathbb{R}^6$ lying outside the hyperplane $k = 0$. We use the manifold

$$M := \{\, \xi \in \mathbb{R}^6 \ \mid \ k = 1,\ \phi = 0 \,\} \cong Q\mathbb{P}^5 \setminus \{k = 0\} \tag{3}$$

and the mapping

$$\chi : Q\mathbb{P}^5 \setminus \{k = 0\} \longrightarrow M : [(x^\mu, k, \phi)] \longmapsto (x^\mu, 1, 0) =: (x^\mu)$$

as a chart of $Q\mathbb{P}^5 \setminus \{k = 0\}$.

In the Dirac's construction the isomorphism $C_0(1,3) \cong O_0(2,4)/\mathbb{Z}_2$ is used in the following way. The conformal transformations in $\overline{M}$ (in general nonlinear) correspond to the natural action of the linear group $O_0(2,4)$ on the straight lines in $\mathbb{R}^6$. The 15 generators of $O_0(2,4)$ in ξ-coordinates are

$$(X_{mn})^p{}_q = \eta_{mq}\,\eta^p_n - \eta_{nq}\,\eta^p_m \ , \qquad m < n \ ,$$

and the corresponding fundamental vector fields are

$$X_{mn} = (X_{mn})^p{}_q\,\xi^q\,\frac{\partial}{\partial \xi^p} = \xi_m\,\frac{\partial}{\partial \xi^n} - \xi_n\,\frac{\partial}{\partial \xi^m} \ , \qquad m < n \ .$$

The "physical fundamental vector fields" [20] are

$$
\begin{aligned}
M_{\mu\nu} &:= X_{\mu\nu} && = x_\mu \partial_\nu - x_\nu \partial_\mu \ , \qquad\qquad 0 \le \mu < \nu \le 3 \\
P_\mu &:= X_{\mu 6} - X_{\mu 5} && = \partial_\mu \\
K_\mu &:= -X_{\mu 5} - X_{\mu 6} && = 2x_\mu (x \cdot \partial) + (2\phi - x^2)\partial_\mu - 2k x_\mu \partial_- + 4\phi x_\mu \partial_+ \\
D &:= -X_{56} && = x \cdot \partial - k\partial_- - 2\phi\partial_+ \ ,
\end{aligned}
$$

where $x \cdot \partial := x^\mu \partial_\mu$, etc.

The transformations $\exp(\omega^{\mu\nu} M_{\mu\nu})$ do not change k and ϕ and act on x^μ as Lorentz transformations, whereas the transformations generated by P_μ, K_μ and D act as follows:

$$
P(t) := \exp(t^\mu P_\mu) : \begin{pmatrix} x^\mu \\ k \\ \phi \end{pmatrix} \longmapsto \begin{pmatrix} x^\mu + t^\mu \\ k \\ \phi \end{pmatrix}
$$

$$
K(c) := \exp(c^\mu K_\mu) : \begin{pmatrix} x^\mu \\ k \\ \phi \end{pmatrix} \longmapsto \begin{pmatrix} p^{-1}(c, x, \phi)\left[x^\mu - c^\mu(x^2 - 2\phi)\right] \\ p(c, x, \phi) k \\ p^{-2}(c, x, \phi)\phi \end{pmatrix}
$$

$$
D(d) := \exp(dD) : \begin{pmatrix} x^\mu \\ k \\ \phi \end{pmatrix} \longmapsto \begin{pmatrix} e^d x^\mu \\ e^{-d} k \\ e^{2d}\phi \end{pmatrix} \ ,
$$

where $p(c, x, \phi) := 1 - 2(c \cdot x) + c^2(x^2 - 2\phi)$.

With respect to their actions on x^μ, the vector fields $M_{\mu\nu}$, P_μ and D generate respectively Lorentz transformations, translations and dilatations; when $\phi = 0$, i.e., on Q^6, K_μ generate special conformal transformations. The physical fundamental vector fields satisfy the well-known commutation relations of the Lie algebra of $C_0(1, 3)$.

3 Dimensional Reduction of $\tau(\mathbb{R}^6)$ and $\tau^*(\mathbb{R}^6)$

We define lifts of the action of $\mathbb{R}$ on $\mathbb{R}^6$ to actions $T^{(\lambda)}$ of $\mathbb{R}$ on $\tau(\mathbb{R}^6)$ by bundle morphisms for every real number λ:

$$T_\rho^{(\lambda)}(\xi, u) := (\rho\xi, \rho^\lambda u) \ , \tag{4}$$

where $\xi \in \mathbb{R}^6$, $(\xi, u) \in \tau(\mathbb{R}^6)_\xi$, $\rho \in \mathbb{R}$. The number λ, specifying the action of $\mathbb{R}$ on $\tau(\mathbb{R}^6)$, is usually called a *conformal dimension* of the fields. Since the case $\lambda = 1$ corresponds to the tangent lift of the action of $\mathbb{R}$, $\lambda = 1$ is called a *canonical conformal dimension*. An $\mathbb{R}$-invariant vector field $\mathcal{X} \in C^\infty(\tau(\mathbb{R}^6))_\mathbb{R}$ satisfies the equation

$$T_\rho^{(\lambda)} \mathcal{X}(\rho^{-1}(\xi)) = \mathcal{X}(\xi) \ . \tag{5}$$

Let $\mathcal{X}^m$, $\mathcal{A}_m$, $\mathcal{F}_{mn}$, $\mathcal{J}_m$ be the components of the corresponding tensors in ξ-coordinates and $(X^M) := (X^\mu, X^-, X^+)$, etc. be their components in Q-coordinates. In coordinate-free notation, we use $\mathcal{X}$, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{J}$. Then (5) is equivalent to $\mathcal{X}^m(\rho\xi) = \rho^\lambda \mathcal{X}^m(\xi)$, which in Q-coordinates reads

$$\begin{aligned} X^{\mu,+}(x, \rho k, \phi) &= \rho^{\lambda-1} X^{\mu,+}(x, k, \phi) \\ X^-(x, \rho k, \phi) &= \rho^\lambda X^-(x, k, \phi) \ . \end{aligned}$$

The first step of the dimensional reduction of $\tau(\mathbb{R}^6)$ is the restriction of its base to the light cone Q^6, which is simultaneously $O_0(2,4)$- and $\mathbb{R}$-invariant submanifold of $\mathbb{R}^6$. As a result we obtain the $O_0(2,4)$- and $\mathbb{R}$-intertwining SES

$$0 \longrightarrow \tau(Q^6) \xrightarrow{m} \tau(\mathbb{R}^6)_{Q^6} \xrightarrow{n} \nu(Q^6) \longrightarrow 0 \tag{6}$$

(cf. Eq. (7) in Part I). The subbundle $m(\tau(Q^6)) \subset \tau(\mathbb{R}^6)_{Q^6}$, consisting of the vectors with components $(X^\mu, X^-, 0)$ in Q-coordinates, is $O_0(2,4)_\mathbb{R}$-invariant.

The second step of the dimensional reduction is the $\mathbb{R}$-reduction of the SES (6). After this process, (6) converts into the $O_0(2,4)_\mathbb{R}$-, i.e., $C_0(1,3)$-intertwining SES

$$0 \longrightarrow \tau(Q^6)_\mathbb{R} \xrightarrow{m} (\tau(\mathbb{R}^6)_{Q^6})_\mathbb{R} \xrightarrow{n} \nu(Q^6)_\mathbb{R} \longrightarrow 0 \tag{7}$$

consisting of vector bundles over M (3). The action of $\mathbb{R}$ on Q^6 is free, therefore (7) is equivalent to the $C_0(1,3)$-intertwining SES

$$0 \longrightarrow \tau(Q^6)_M \xrightarrow{m} \tau(\mathbb{R}^6)_M \xrightarrow{n} \nu(Q^6)_M \longrightarrow 0 \ ,$$

hence the subbundle $m(\tau(Q^6)_M) \subset \tau(\mathbb{R}^6)_M$, consisting of all vectors of the form $(X^\mu, X^-, 0)$, is $C_0(1,3)$-invariant.

On the other hand, the $\mathbb{R}$-reduction of $\tau(Q^6)$ (the left term of (6)) gives the $O_0(2,4)$-intertwining SES

$$0 \longrightarrow \tau^{\mathrm{v}}(Q^6)_{\mathbb{R}} \xrightarrow{q} \tau(Q^6)_{\mathbb{R}} \xrightarrow{r} \tau(M) \longrightarrow 0 \ ,$$

hence the subbundle $q(\tau^{\mathrm{v}}(Q^6)_{\mathbb{R}}) \subset \tau(Q^6)_{\mathbb{R}}$, consisting of all vectors of the form $(0, X^-)$, is $O_0(2,4)_{\mathbb{R}}$-invariant.

Therefore, we obtain the following filtration of $C_0(1,3)$-invariant bundles over M:

$$m\left(q(\tau^{\mathrm{v}}(Q^6)_{\mathbb{R}})\right) \ \subset \ m(\tau(Q^6)_M) \ \subset \ \tau(\mathbb{R}^6)_M$$

or, in Q-coordinates,

$$\left\{(0, X^-, 0)\right\} \ \subset \ \left\{(X^\mu, X^-, 0)\right\} \ \subset \ \left\{(X^\mu, X^-, X^+)\right\} \ ,$$

where $\{\ldots\}$ represents the form of the vectors in the corresponding subbundles.

In the case of the cotangent bundle $\tau^*(\mathbb{R}^6)$, the lift (4) leads to

$$T_\rho^{(\lambda)}(\xi, w) := (\rho\xi, \rho^{-\lambda} w) \ , \tag{8}$$

where $(\xi, w) \in \tau^*(\mathbb{R}^6)_\xi$. An $\mathbb{R}$-invariant differential 1-form $\mathcal{A} \in C^\infty(\tau^*(\mathbb{R}^6))_{\mathbb{R}}$ satisfies $\mathcal{A}_m(\rho\xi) = \rho^{-\lambda}\mathcal{A}_m(\xi)$, which in Q-coordinates reads

$$\begin{aligned} A_{\mu,+}(x, \rho k, \phi) &= \rho^{1-\lambda} A_{\mu,+}(x, k, \phi) \\ A_-(x, \rho k, \phi) &= \rho^{-\lambda} A_-(x, k, \phi) \ . \end{aligned}$$

In Q-coordinates, the Lorentz transformations do not change A_- and A_+ and act on A_μ in the usual way; the translations do not change any of the components of $\mathcal{A}$; the actions of the special conformal transformations and dilatations are given in Appendix B.

In the case of $\tau(Q^6)$, we first perform $\mathbb{R}$-dimensional reduction, which yields the $O_0(2,4)_{\mathbb{R}}$-intertwining SES

$$0 \longleftarrow \tau^{\mathrm{v}}(\mathbb{R}^6)^*_U \xleftarrow{i^*} \tau^*(\mathbb{R}^6)_U \xleftarrow{j^*} \tau^*(U) \longleftarrow 0 \tag{9}$$

of vector bundles over U. In the process of reduction, we have used that the action of $\mathbb{R}$ on $\mathbb{R}^6$ is free, hence $(\tau^{\mathrm{v}}(\mathbb{R}^6)^*)_{\mathbb{R}} = (\tau^{\mathrm{v}}(\mathbb{R}^6)_{\mathbb{R}})^* = \tau^{\mathrm{v}}(\mathbb{R}^6)^*_U$, $\tau^*(\mathbb{R}^6)_{\mathbb{R}} = \tau^*(\mathbb{R}^6)_U$, and $(p^*\tau^*(\mathbb{RP}^5))_{\mathbb{R}} = \tau^*(U)$.

Restricting (9) to the $O_0(2,4)_{\mathbb{R}}$-invariant submanifold $M \subset U$, we obtain the following $C_0(1,3)$-intertwining SES of vector bundles over M

$$0 \longleftarrow \tau^{\mathrm{v}}(\mathbb{R}^6)^*_M \xleftarrow{\;i^*\;} \tau^*(\mathbb{R}^6)_M \xleftarrow{\;j^*\;} \tau^*(U)_M \longleftarrow 0 \, . \tag{10}$$

On the other hand, the $O_0(2,4)_{\mathbb{R}}$-invariance of M as a submanifold of U gives us the $C_0(1,3)$-intertwining SES

$$0 \longleftarrow \tau^*(M) \xleftarrow{\;k^*\;} \tau^*(U)_M \xleftarrow{\;l^*\;} \nu^*(M) \longleftarrow 0 \, , \tag{11}$$

where $k : \tau(M) \to \tau(U)_M$ is the natural embedding and $\nu^*(M) := (\tau(U)_M/\tau(M))^*$ (cf. Eq. (7) in Part I)).

From (10) and (11), we obtain the following $C_0(1,3)$-invariant filtration of vector bundles over M:

$$j^*(l^*(\nu^*(M))) \;\subset\; j^*(\tau^*(U)_M) \;\subset\; \tau^*(\mathbb{R}^6)_M \; ,$$

or, in Q-coordinates,

$$\{(0,0,A_+)\} \;\subset\; \{(A_\mu,0,A_+)\} \;\subset\; \{(A_\mu,A_-,A_+)\} \; .$$

This allows us to impose the conformally invariant conditions

$$A_- = 0 \qquad \text{or} \qquad A_\mu = A_- = 0 \tag{12}$$

(which can be noticed from the transformation laws given in Appendix B).

4 Dimensional Reduction of the 6-Dimensional Maxwell Equations

Let $\mathcal{A} \in C^\infty(\tau^*(\mathbb{R}^6))$ be the 6-dimensional electromagnetic vector potential, $\mathcal{J} \in C^\infty(\tau^*(\mathbb{R}^6))$ be the electromagnetic current density and

$$\mathcal{F}_{mn} = (\mathrm{d}\mathcal{A})_{mn} = \frac{\partial \mathcal{A}_n}{\partial \xi^m} - \frac{\partial \mathcal{A}_m}{\partial \xi^n}$$

be the field strength tensor. Then the 6-dimensional Maxwell equations (6DME) are

$$\frac{\partial \mathcal{F}_{mn}}{\partial \xi_m} = \Box_6 \mathcal{A}_n - \frac{\partial}{\partial \xi^n} (\nabla_6 \cdot \mathcal{A}) = \mathcal{J}_n \, . \tag{13}$$

Note that since the homogeneous Maxwell equations $\mathrm{d}\mathcal{F} = 0$ are automatically conformally invariant, we consider only the inhomogeneous ones and for the sake of brevity we omit the adjective "inhomogeneous".

In Q-coordinates, (13) read

$$\begin{aligned}
\nabla^N F_{N\mu} &= \frac{1}{k^2}\,\partial^\nu F_{\nu\mu} - \frac{1}{k}\,\partial_+ F_{\mu-} - \frac{1}{k}\,\partial_- F_{\mu+} + \frac{2\phi}{k^2}\,\partial_+ F_{\mu+} = J_\mu \\
\nabla^N F_{N-} &= \frac{1}{k^2}\,\partial^\nu F_{\nu-} - \frac{1}{k}\,\partial_- F_{-+} + \frac{2\phi}{k^2}\,\partial_+ F_{-+} - \frac{3}{k^2}\,F_{-+} = J_- \\
\nabla^N F_{N+} &= \frac{1}{k^2}\,\partial^\nu F_{\nu+} + \frac{1}{k}\,\partial_+ F_{-+} = J_+ \ ,
\end{aligned}$$

or, in terms of vector potential,

$$\begin{aligned}
\nabla^N F_{N\mu} &= \frac{1}{k^2}\,\Box A_\mu - \frac{1}{k^2}\,\partial_\mu(\partial\cdot A) + \frac{2}{k}\,\partial_-\partial_+ A_\mu - \frac{2\phi}{k^2}\,\partial_+\partial_+ A_\mu \\
&\quad -\frac{1}{k}\,\partial_\mu\partial_+ A_- - \frac{1}{k}\,\partial_\mu\partial_- A_+ + \frac{2\phi}{k^2}\,\partial_\mu\partial_+ A_+ = J_\mu \\
\nabla^N F_{N-} &= \frac{1}{k^2}\,\Box A_- - \frac{1}{k^2}\,\partial_-(\partial\cdot A) + \frac{1}{k}\,\partial_-\partial_+ A_- - \frac{2\phi}{k^2}\,\partial_+\partial_+ A_- \\
&\quad -\frac{1}{k}\,\partial_-\partial_- A_+ + \frac{2\phi}{k^2}\,\partial_-\partial_+ A_+ + \frac{3}{k^2}\,\partial_+ A_- - \frac{3}{k^2}\,\partial_- A_+ = J_- \\
\nabla^N F_{N+} &= \frac{1}{k^2}\,\Box A_+ - \frac{1}{k^2}\,\partial_+(\partial\cdot A) - \frac{1}{k}\,\partial_+\partial_+ A_- + \frac{1}{k}\,\partial_-\partial_+ A_+ = J_+ \ ,
\end{aligned}$$

where $\Box A_\mu := \eta^{\rho\nu}\partial_\rho\partial_\nu A_\mu$, $\partial\cdot A := \eta^{\mu\nu}\partial_\mu A_\nu$.

Obviously, the 6DME are $O_0(2,4)$-invariant. They are also $\mathbb{R}$-invariant in the canonical case when $\mathcal{A}$ and $\mathcal{J}$ change respectively according to the actions $T^{(1)}$ and $T^{(3)}$ of $\mathbb{R}$.

We perform the $\mathbb{R}$-reduction of 6DME, following the general procedure described in Sec. 5 in Part I. From the vanishing of the Lie derivative of the action (8) of $\mathbb{R}$ on the $\mathbb{R}$-invariant differential 1-forms $\mathcal{A} \in C^\infty(\tau^*(\mathbb{R}^6))_{\mathbb{R}}$, we obtain

$$\xi^m\,\frac{\partial \mathcal{A}_n}{\partial \xi^m} + \lambda\,\mathcal{A}_n = 0 \ .$$

In the case $\lambda = 1$ this equation reads

$$\partial_- A_{\mu,+}(x,k,\phi) = 0 \qquad \partial_- A_-(x,k,\phi) = \frac{1}{k}\,A_-(x,k,\phi) \ . \tag{14}$$

We restrict the first prolongation of (14) to the submanifold U (2), transversal to the $\mathbb{R}$-orbits in $\mathbb{R}^6$, and obtain the splitting relations

$$\begin{aligned} \partial_- A_{\mu,+}(x,\phi) &= 0 & \partial_- A_-(x,\phi) &= -A_-(x,\phi) \\ \partial_-\partial_- A_{\mu,+}(x,\phi) &= 0 & \partial_-\partial_- A_-(x,\phi) &= 0 \ , \end{aligned}$$

where $A_{\mu,-,+}(x,\phi) := A_{\mu,-,+}(x,1,\phi)$.

With these relations, the $\mathbb{R}$-reduced 6DME read

$$\begin{aligned} &\Box A_\mu - \partial_\mu(\partial\cdot A) - 2\phi\,\partial_+\partial_+ A_\mu - \partial_\mu\partial_+ A_- + 2\phi\,\partial_\mu\partial_+ A_+ = J_\mu \\ &\Box A_- - 2\phi\,\partial_+\partial_+ A_- + 2\,\partial_+ A_- = J_- \\ &\Box A_+ - \partial_+(\partial\cdot A) - \partial_+\partial_+ A_- = J_+ \ . \end{aligned} \tag{15}$$

The restriction to the submanifold M is complicated because after setting $\phi = 0$ in (15) we obtain the equations

$$\begin{aligned} &\Box A_\mu - \partial_\mu(\partial\cdot A) - \partial_\mu\partial_+ A_- = J_\mu \\ &\Box A_- + 2\,\partial_+ A_- = J_- \\ &\Box A_+ - \partial_+(\partial\cdot A) - \partial_+\partial_+ A_- = J_+ \end{aligned} \tag{16}$$

which contain ∂_+ derivatives, i.e., the $\mathbb{R}$-reduced 6DME are not internal for M. To restrict the non-internal DO in (16) to M, we need some additional information, e.g., a splitting relation for the SES (8) in Part I. Besides, we want the restricted DO to be conformally invariant. Splitting relations of this kind can be obtained by considering the kernel of a manifestly $O_0(2,4)$-invariant DO after its $\mathbb{R}$-reduction and restriction to M.

Let us choose as a splitting relation the manifestly $O_0(2,4)$-invariant equation

$$\nabla_6\cdot\mathcal{A} = \frac{\partial}{\partial\xi_m}\mathcal{A}_m = 0 \ . \tag{17}$$

Its $\mathbb{R}$-reduction reads

$$\partial\cdot A(x,\phi) + \partial_+ A_-(x,\phi) - 2\phi\,\partial_+ A_+(x,\phi) + 2\,A_+(x.\phi) = 0 \ . \tag{18}$$

The first prolongation of (18), restricted to M, yields

$$\begin{aligned} &\partial\cdot A + \partial_+ A_- + 2A_+ = 0 \\ &\partial_\mu(\partial\cdot A) + \partial_\mu\partial_+ A_- + 2\,\partial_\mu A_+ = 0 \\ &\partial_+(\partial\cdot A) + \partial_+\partial_+ A_- = 0 \ . \end{aligned} \tag{19}$$

This is not a splitting of the SES (8) in Part I, but provides us with a sufficient set of splitting relations for it. Combining (16) with (19), we obtain the equations

$$\begin{aligned}
&\Box A_\mu(x) + 2\,\partial_\mu A_+(x) = J_\mu(x)\\
&\Box A_-(x) - 2\,\partial \cdot A(x) - 4\,A_+(x) = J_-(x)\\
&\Box A_+(x) = J_+(x)\ ,
\end{aligned}\tag{20}$$

often referred to as *conformal electrodynamics equations* [7,5].

Numerous modifications of (20), discussed in the literature, can be easily obtained from them. For example, eliminating $A_+(x)$, we obtain

$$\begin{aligned}
&\partial^\mu F_{\mu\nu}(x) + \frac{1}{2}\,\partial_\nu \Box A_-(x) = J_\nu(x) + \frac{1}{2}\,\partial_\nu J_-(x)\\
&\frac{1}{4}\,\Box^2 A_-(x) - \frac{1}{2}\,\Box\partial \cdot A(x) = J_+(x) - \frac{1}{4}\,\Box J_-(x)\ .
\end{aligned}\tag{21}$$

Some of the conformally invariant conditions (12) can be imposed independently on $\mathcal{A}$ and $\mathcal{J}$. For example, setting $J_-(x) = 0$ in (21), we arrive at the system [7,22,11]

$$\begin{aligned}
&\partial^\mu F_{\mu\nu}(x) + \frac{1}{2}\,\partial_\nu \Box A_-(x) = J_\nu(x)\\
&\frac{1}{4}\,\Box^2 A_-(x) - \frac{1}{2}\,\Box\partial \cdot A(x) = J_+(x)\ .
\end{aligned}$$

If, in addition, $A_-(x) = 0$, we obtain the equations [18,5]

$$\begin{aligned}
&\partial^\mu F_{\mu\nu}(x) = J_\nu(x)\\
&-\frac{1}{2}\,\Box\partial \cdot A(x) = J_+(x)\ .
\end{aligned}$$

5 Conformally Invariant Gauge Transformations

In this section, we derive the most general gauge transformations preserving the conformal invariance. Let $\mathcal{A}' = \mathcal{A} + \mathrm{d}\mathcal{S}$ be a gauge transformation in $\mathbb{R}^6$. The $\mathbb{R}$-invariance of $\mathcal{A}$ and $\mathcal{A}'$ imposes the conditions

$$\partial_- \partial_{\mu,+} S(x,k,\phi) = 0$$

$$\partial_- \partial_- S(x,k,\phi) = -\frac{1}{k}\,\partial_- S(x,k,\phi)$$

on $\mathcal{S}$ (cf. (14)). Their solution is

$$S(x,k,\phi) = s\ln|k| + t(x,\phi)\ , \tag{22}$$

where s is a constant and t is an arbitrary function of x and ϕ. The dimensional reduction of (22) yields

$$\begin{aligned} S(x,\phi) &:= S(x,1,\phi) = t(x,\phi)\ , \\ \partial_- S(x,\phi) &:= \partial_- S(x,k,\phi)|_{k=1} = s\ . \end{aligned} \tag{23}$$

The actions of $K(c)$ and $D(d)$ on $S(x,k,\phi)$ (22) are

$$\begin{aligned} D(d)(S)(x,k,\phi) &= s\,d + S(e^{-d}x, k, e^{-2d}\phi) \\ K(c)(S)(x,k,\phi) &= s\ln|p(-c,x,\phi)| + S('x,k,'\phi)\ , \end{aligned}$$

where $'x^\mu := p^{-1}(-c,x,\phi)[x^\mu + c^\mu(x^2 - 2\phi)]$, $'\phi := p^{-2}(-c,x,\phi)\,\phi$. The infinitesimal form of these relations is

$$\begin{aligned} \delta S(x,k,\phi) &= d\,[s - (x\cdot\partial + 2\phi\,\partial_+)S(x,k,\phi)] \\ \delta S(x,k,\phi) &= c^\mu \left\{2sx_\mu + \left[(x^2-2\phi)\partial_\mu - 2x_\mu x\cdot\partial - 4\phi\,x_\mu\partial_+\right] S(x,k,\phi)\right\}\ . \end{aligned}$$

After the $\mathbb{R}$-reduction and restriction to M we obtain

$$\begin{aligned} D(d)(S)(x,k,\phi) &= s\,d + S(e^{-d}x) \approx S(x) + d\,[s - x\cdot\partial\, S(x)] \\ K(c)(S)(x,k,\phi) &= s\ln|p(-c,x)| + S('x) \\ &\approx S(x) + c^\mu\left[2sx_\mu + (x^2\partial_\mu - 2x_\mu x\cdot\partial)S(x)\right]\ , \end{aligned}$$

where $S(x) := S(x,0)$, $'x^\mu := p(-c,x,0)(x^\mu + c^\mu x^2)$.

The gauge transformations for the conformal electrodynamics equations (20) must be compatible with the condition (17) used as a splitting relation for obtaining (20). Therefore, the gauge function must satisfy the 6-dimensional "wave equation" $\Box_6\mathcal{S} = 0$, which in Q-coordinates reads

$$\frac{1}{k^2}\Box S(x,k,\phi) + \frac{2}{k}\,\partial_-\partial_+ S(x,k,\phi) - \frac{2\phi}{k^2}\,\partial_+\partial_+ S(x,k,\phi) + \frac{2}{k^2}\,\partial_+ S(x,k,\phi) = 0\ .$$

After the $\mathbb{R}$-reduction, we obtain for $t(x,\phi)$ the condition

$$\Box t(x,\phi) - 2\phi\,\partial_+\partial_+ t(x,\phi) + 2\,\partial_+ t(x,\phi) = 0\ ,$$

whose first prolongation, restricted to M, yields the relations

$$\partial_+ t(x,0) + \frac{1}{2}\Box t(x,0) = 0\ , \tag{24}$$

$$\partial_+ \Box t(x,0) = 0\ . \tag{25}$$

Acting on (24) with $\Box$ and taking into account (25), we obtain the internal for M condition $\Box^2 t(x) = 0$, where $t(x) := t(x,0)$. Due to (23) and (24), the conformally invariant "gauge transformations" in M are

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \partial_\mu S(x) \\ A'_-(x) &= A_-(x) + s \\ A'_+(x) &= A_+(x) - \frac{1}{2}\Box S(x)\ , \end{aligned}$$

where the gauge function $S(x)$ is a solution to

$$\Box^2 S(x) = 0 \tag{26}$$

[7,11,5]. It is easy to check that (26) is sufficient for the gauge invariance of (20). If the conformally invariant condition $A_-(x) = 0$ is imposed, the constant s must be zero [15].

6 A Complete Set of Splitting Relations for the Dirac's Construction

When one wants to obtain a conformally invariant DO in Minkowski space using the Dirac's construction and starting from a manifestly $O_0(2,4)$-invariant DO in $\mathbb{R}^6$, the restriction of this DO to the submanifold M (3) plays a crucial role. To obtain a conformally invariant DO in M, one needs an $O_0(2,4)$-invariant splitting relation. We shall show that a complete set of splitting relations for noncanonical conformal dimensions can be obtained from the prolongation of the 6-dimensional D'Alembertian $\Box_6$.

Let us consider the equation

$$\Box_6 \mathcal{A}_m = 0\ , \tag{27}$$

and let $\mathbb{R}$ act on $C^\infty(\tau^*(\mathbb{R}^6))$ according to (8). In Q-coordinates, (27) reads

$$\begin{aligned}
&\frac{1}{k^2}\,\Box A_\mu + \frac{2}{k}\,\partial_-\partial_+ A_\mu - \frac{2\phi}{k^2}\,\partial_+\partial_+ A_\mu + \frac{2}{k^2}\,\partial_\mu A_+ = 0\\
&\frac{1}{k^2}\,\Box A_- + \frac{2}{k}\,\partial_-\partial_+ A_- - \frac{2\phi}{k^2}\,\partial_+\partial_+ A_- - \frac{2}{k^3}\,(\partial\cdot A)\\
&\qquad + \frac{2}{k^2}\,\partial_+ A_- - \frac{2}{k^2}\,\partial_- A_+ + \frac{4\phi}{k^3}\,\partial_+ A_+ - \frac{4}{k^3}\,A_+ = 0\\
&\frac{1}{k^2}\,\Box A_+ + \frac{2}{k}\,\partial_-\partial_+ A_+ - \frac{2\phi}{k^2}\,\partial_+\partial_+ A_+ = 0\ .
\end{aligned} \tag{28}$$

After $\mathbb{R}$-reduction and restriction of (28) to M, we obtain

$$\begin{aligned}
&\Box A_\mu(x) + 2(1-\lambda)\partial_+ A_\mu(x) + 2\partial_\mu A_+(x) = 0\\
&\Box A_-(x) + 2\partial\cdot A(x) + 2(1-\lambda)\partial_+ A_-(x) + 2(\lambda-3)A_+(x) = 0\\
&\Box A_+(x) + 2(1-\lambda)\partial_+ A_+(x) = 0\ .
\end{aligned} \tag{29}$$

For the canonical conformal dimension $\lambda = 1$, the DO in the left-hand side of (29) is internal for M. If $\lambda \neq 1$, (29) gives conformally invariant splitting relations for first-order DOs on $C^\infty(\tau(\mathbb{R}^6))$:

$$\begin{aligned}
\partial_+ A_\mu(x) &= \frac{1}{2(\lambda-1)}[\Box A_\mu(x) + 2\partial_\mu A_+(x)]\\
\partial_+ A_-(x) &= \frac{1}{2(\lambda-1)}[\Box A_-(x) - \partial\cdot A(x) + 2(\lambda-3)A_+(x)]\\
\partial_+ A_\mu(x) &= \frac{1}{2(\lambda-1)}\,\Box A_+(x)\ .
\end{aligned} \tag{30}$$

The first prolongation of (28) gives a complete set of conformally invariant splitting relations for second-order DOs if $\lambda \neq 0, 1$:

$$\begin{aligned}
\partial_+\partial_+ A_\mu(x) &= \frac{1}{4(\lambda-1)\lambda}\left[\Box^2 A_\mu(x) + 2\partial_\mu \Box A_+(x)\right]\\
\partial_+\partial_+ A_-(x) &= \frac{1}{4(\lambda-1)\lambda}\left[\Box^2 A_-(x) - 4\Box\partial\cdot A(x) + 4(\lambda-3)\Box A_+(x)\right]\\
\partial_+\partial_+ A_+(x) &= \frac{1}{4(\lambda-1)\lambda}\,\Box^2 A_+\ .
\end{aligned}$$

The second prolongation of (28) gives a complete set of conformally invariant splitting relations for third-order DOs if $\lambda \neq -1, 0, 1$:

$$\partial_+\partial_+\partial_+ A_\mu(x) = \frac{1}{8(\lambda-1)\lambda(\lambda+1)}\left[\Box^3 A_\mu(x) + 4\partial_\mu\Box^2 A_+(x) + 4\lambda\partial_\mu\Box A_+(x)\right]$$
$$\partial_+\partial_+\partial_+ A_-(x) = \frac{1}{8(\lambda-1)\lambda(\lambda+1)}\left[\Box^3 A_-(x) - 6\Box^2\partial\cdot A(x) + 6(\lambda-3)\Box^2 A_+(x)\right]$$
$$\partial_+\partial_+\partial_+ A_\mu(x) = \frac{1}{8(\lambda-1)\lambda(\lambda+1)}\,\Box^3 A_+(x)\ .$$

In general, the $(k-1)$st prolongation of (28), followed by a restriction to M, provides a complete set of conformally invariant splitting relations for DOs of order k if $\lambda \neq 2-k, 3-k, \ldots, 1$. In general, the order of the DOs in the process of restriction increases.

As an example of application, we reduce the $O_0(2,4)$-invariant 6-current conservation law

$$\nabla_6\cdot\mathcal{J} = 0\ ,$$

which in Q-coordinates reads

$$\frac{1}{k^2}\,\partial\cdot J + \frac{1}{k}\,\partial_+ J_- - \frac{1}{k}\,\partial_- J_+ - \frac{2\phi}{k^2}\,\partial_+ J_+ + \frac{2}{k^2}\,J_+ = 0\ .$$

Performing $\mathbb{R}$-reduction for the canonical for $\mathcal{J}$ conformal dimension $\lambda = 3$, we have

$$\partial\cdot J(x,\phi) + \partial_+ J_-(x,\phi) - 2\phi\,\partial_+ J_+(x,\phi) = 0\ .$$

Applying (30) to this equation, we obtain the conformally invariant condition

$$\partial\cdot J(x) + \frac{1}{2}\,\Box J_-(x) = 0 \tag{31}$$

[7,11,6,13], which can be regarded as a conservation law for $J_\mu(x) + \frac{1}{2}\partial_\mu J_-(x)$. Calculating the divergence of the first equation of (21) and taking into account (31), we have

$$\Box^2 A_- = 0\ . \tag{32}$$

Combining (32) with (21), we get the one-parametric family

$$\partial^\mu F_{\mu\nu}(x) + \frac{1}{2}\,\partial_\nu\Box A_-(x) = J_\nu(x) + \frac{1}{2}\,\partial_\nu J_-(x)$$
$$\beta\,\Box^2 A_-(x) - \frac{1}{2}\,\Box\partial\cdot A(x) = J_+(x) - \frac{1}{4}\,\Box J_-(x)\ .$$

The more popular current conservation law

$$\partial\cdot J(x) = 0\ ,$$

valid for the conformal electrodynamics equations (20) with $J_{-}(x) = 0$, can be derived from (31) by imposing the conformally invariant condition $J_{-} = 0$.

Acknowledgments

We would like to express our gratitude to Lubomir Chorbadjiev for the innumerable discussions on symmetry and reduction. We are indebted to Prof. Luis J. Boya who read the article and made many valuable comments. We are grateful to Prof. Rafael de la Llave for his help and encouragement. N.P. was partially supported by grants DMS 9802156 and TARP-003658-071.

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A APPENDIX A

(We use the notations from Sec. 2 and work in Q-coordinates.)

The manifold Q^6 is an $O_0(2,4)$-invariant submanifold of $\mathbb{R}^6$; the action of $O_0(2,4)$ on it satisfies condition A from Part I and its tangent and cotangent lifts satisfy condition B from Part I. This yields the $O_0(2,4)$-intertwining SESs

$$0 \longrightarrow \tau(Q^6) \xrightarrow{\;m\;} \tau(\mathbb{R}^6)_{Q^6} \xrightarrow{\;n\;} \nu(Q^6) \longrightarrow 0$$

$$0 \longleftarrow \tau^*(Q^6) \xleftarrow{\;m^*\;} \tau^*(\mathbb{R}^6)_{Q^6} \xleftarrow{\;n^*\;} \nu^*(Q^6) \longleftarrow 0\ ,$$

consisting of $O_0(2,4)$-reducible vector bundles over Q^6.

The fiber $\tau(Q^6)_b$ of $\tau(Q^6)$ over $b \in Q^6$, naturally embedded in $\tau(\mathbb{R}^6)_b$, consists of the vectors annihilating on Q^6 the differential form

$$\mathrm{d}(\xi_m \xi^m) = \mathrm{d}(2k^2\phi) = 2k\,(2\phi\,\mathrm{d}k + k\,\mathrm{d}\phi)\ .$$

Therefore $\tau(Q^6)$ is generated by the vector fields ∂_μ and ∂_-, and $\tau^*(Q^6)$ is generated by the differential forms $\mathrm{d}x^\mu$ and $\mathrm{d}k$ ($\mu = 0, 1, 2, 3$). From the above SESs we obtain that

$$m(\tau(Q^6)) = \{(X^\mu, X^-, 0)\} \qquad \text{and} \qquad n^*(\nu^*(Q^6)) = \{(A_\mu, A_-, 0)\}$$

are $O_0(2,4)$-invariant subbundles of $\tau(\mathbb{R}^6)_{Q^6}$ and $\tau^*(\mathbb{R}^6)_{Q^6}$, respectively. The action of $O_0(2,4)$ on Q^6 is transitive and we choose the point $y := (0, 0, 0, 0, 1, 0) \in Q^6$ for a realization of $Q^6/O_0(2,4)$. The action of $O_0(2,4)$ on y is generated by the vector field $t^\mu\partial_\mu - d\partial_-$. Hence, the stationary group $O_0(2,4)_y$ of the point y consists of the Lorentz transformations $M(\omega) := \exp(\omega^{\mu\nu}M_{\mu\nu})$ and the special conformal transformations $K(c)$, and therefore is non compact. The actions of $M(\omega)$ and $K(c)$ on $\tau(Q^6)_y$ and $\tau^*(Q^6)_y$ are

$$M(\omega)\begin{pmatrix} X^\mu \\ X^- \end{pmatrix} = \begin{pmatrix} M^\mu{}_\nu(\omega) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X^\nu \\ X^- \end{pmatrix} \qquad K(c)\begin{pmatrix} X^\mu \\ X^- \end{pmatrix} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ -2c_\nu & 1 \end{pmatrix}\begin{pmatrix} X^\nu \\ X^- \end{pmatrix} ,$$

$$M(\omega)\begin{pmatrix} A_\mu \\ A_- \end{pmatrix} = \begin{pmatrix} M^\nu{}_\mu(-\omega) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_\nu \\ A_- \end{pmatrix} \qquad K(c)\begin{pmatrix} A_\mu \\ A_- \end{pmatrix} = \begin{pmatrix} \delta^\mu_\nu & 2c_\mu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_\nu \\ A_- \end{pmatrix} .$$

Therefore $\mathrm{st}\,\tau(Q^6)_y$ consists of all vectors of the form $X^-(\partial_-)_y$ whereas $\mathrm{st}\,\tau^*(Q^6)_y$ consists of the zero 1-form only. Hence, performing an $O_0(2,4)$-reduction of $\tau(Q^6)$ and $\tau^*(Q^6)$, we obtain respectively the reduced bundles

$$\tau^{\mathrm{v}}(Q^6)_{O_0(2,4)} = \tau(Q^6)_{O_0(2,4)}$$

and

$$(\tau^{\mathrm{v}}(Q^6)^*)_{O_0(2,4)} = (\tau^*(Q^6))_{O_0(2,4)} ,$$

which are *not* dual because

$$\dim\left(\tau^{\mathrm{v}}(Q^6)_{O_0(2,4)}\right)^* = \dim\left(\mathrm{st}\,\tau(Q^6)_y\right)^* = 1 ,$$

whereas

$$\dim\left(\tau^{\mathrm{v}}(Q^6)^*\right)_{O_0(2,4)} = \dim \mathrm{st}\,\tau^*(Q^6)_y = 0 .$$

B APPENDIX B – Actions of the Special Conformal Transformations and Dilatations on $C^\infty(\tau^*(\mathbb{R}^6))$

In many papers on the conformal electrodynamics the authors use slightly different adapted to the light cone in $\mathbb{R}^6$ coordinates, and usually work only on the light cone Q^6, so we present a collection of the basic formulae for the action of the conformal group on the 1-forms in $\mathbb{R}^6$ for different conformal dimensions λ.

1. Special conformal transformations:

a) finite transformations:

$$
\begin{aligned}
K(c)(A)_\mu(x,k,\phi) &= p^{-2}[p(\delta_\mu^\nu + 2x_\mu c^\nu) - 2(c_\mu + c^2 x_\mu)(x^\nu + c^\nu(x^2 - 2\phi))]A_\nu('x,'k,'\phi) \\
&\quad +2k(c_\mu + c^2 x_\mu)A_-('x,pk,p^{-2}\phi) - 4\phi p^{-3}(c_\mu + c^2 x_\mu)A_+('x,pk,p^{-2}\phi) \\
K(c)(A)_-(x,k,\phi) &= pA_-('x,pk,p^{-2}\phi) \\
K(c)(A)_+(x,k,\phi) &= 2p^{-2}[c^2(x^\nu + c^\nu(x^2 - 2\phi)) - pc^\nu]A_\nu('x,pk,p^{-2}\phi) \\
&\quad -2kc^2 A_-(K(-c)('x,pk,p^{-2}\phi) + p^3(p + 4\phi c^2)A_+('x,pk,p^{-2}\phi)\ ,
\end{aligned}
$$

where $p := p(-c,x,\phi)$, $'x^\mu := p^{-1}[x^\mu - c^\mu(x^2 - 2\phi)]$; after $\mathbb{R}$-reduction and restriction to M:

$$
\begin{aligned}
K(c)(A)_\mu(x) &= p^{-1-\lambda}[p(\delta_\mu^\nu + 2x_\mu c^\nu) - 2(c_\mu + c^2 x_\mu)(x^\nu + x^2 c^\nu)]A_\nu(K(-c)x) \\
&\quad +2(c_\mu + c^2 x_\mu)A_-(K(-c)x) \\
K(c)(A)_-(x) &= p^{1-\lambda}A_-(K(-c)x) \\
K(c)(A)_+(x) &= 2p^{-1-\lambda}[c^2(x^\nu + x^2 c^\nu) - pc^\nu]A_\nu(K(-c)x) \\
&\quad -2p^{-\lambda}c^2 A_-(K(-c)x) + p^{-1-\lambda}A_+(K(-c)x)\ ,
\end{aligned}
$$

where $p := p(-c,x,0)$, $K(-c)x^\mu := p^{-1}(x^\mu + x^2 c^\mu)$;

b) infinitesimal transformations:

$$
\begin{aligned}
\delta A_\mu(x,k,\phi) &= \{[(x^2 - 2\phi)c\cdot\partial - 2c\cdot x(x\cdot\partial + 1) + 2kc\cdot x\partial_- - 4\phi c\cdot x\partial_+]\delta_\mu^\nu \\
&\quad +2(x_\mu c^\nu - c_\mu x^\nu)\}A_\nu(x,k,\phi) + 2kc_\mu A_-(x,k,\phi) - 4\phi c_\mu A_+(x,k,\phi) \\
\delta A_-(x,k,\phi) &= [(x^2 - 2\phi)c\cdot\partial - 2c\cdot x(x\cdot\partial - k\partial_- + 2\phi\partial_+ - 1)]A_-(x,k,\phi) \\
\delta A_+(x,k,\phi) &= -2c^\nu A_\nu(x,k,\phi) + [(x^2 - 2\phi)c\cdot\partial - 2c\cdot\partial(x\cdot\partial + 2) \\
&\quad +2kc\cdot x\partial_- - 4\phi c\cdot x\partial_+]A_+(x,k,\phi)\ ;
\end{aligned}
$$

after $\mathbb{R}$-reduction and restriction to M:

$$
\begin{aligned}
\delta A_\mu(x) &= \{[x^2 c\cdot\partial - 2c\cdot x(x\cdot\partial + \lambda)]\delta_\mu^\nu + 2(x_\mu c^\nu - c_\mu x^\nu)\}A_\nu(x) + 2c_\mu A_-(x) \\
\delta A_-(x) &= [x^2 c\cdot\partial - 2c\cdot x(x\cdot\partial + \lambda - 1)]A_-(x,k,\phi) \\
\delta A_+(x) &= -2c^\nu A_\nu(x) + [x^2 c\cdot\partial - 2c\cdot\partial(x\cdot\partial + \lambda + 1)]A_+(x)\ .
\end{aligned}
$$

2. Dilatations:

a) finite transformations:

$$
\begin{aligned}
D(d)(A)_\mu(x,k,\phi) &= e^{-d} A_\mu(e^{-d}x, e^{d}k, e^{-2d}\phi) \\
D(d)(A)_-(x,k,\phi) &= e^{d} A_-(e^{-d}x, e^{d}k, e^{-2d}\phi) \\
D(d)(A)_+(x,k,\phi) &= e^{-2d} A_+(e^{-d}x, e^{d}k, e^{-2d}\phi) \ ;
\end{aligned}
$$

after $\mathbb{R}$-reduction and restriction to M:

$$
\begin{aligned}
D(d)(A)_\mu(x) &= e^{-\lambda d} A_\mu(e^{-d}x) \\
D(d)(A)_-(x) &= e^{-(\lambda-1)d} A_-(e^{-d}x) \\
D(d)(A)_+(x) &= e^{-(\lambda+1)d} A_+(e^{-d}x) \ ;
\end{aligned}
$$

b) infinitesimal transformations:

$$
\begin{aligned}
\delta A_\mu(x,k,\phi) &= -d\,(x\cdot\partial - k\partial_- + 2\phi\partial_+ + 1) A_\mu(x,k,\phi) \\
\delta A_-(x,k,\phi) &= -d\,(x\cdot\partial - k\partial_- + 2\phi\partial_+ - 1) A_-(x,k,\phi) \\
\delta A_+(x,k,\phi) &= -d\,(x\cdot\partial - k\partial_- + 2\phi\partial_+ + 2) A_+(x,k,\phi) \ ;
\end{aligned}
$$

after $\mathbb{R}$-reduction and restriction to M:

$$
\begin{aligned}
\delta A_\mu(x) &= -d\,(x\cdot\partial + \lambda) A_\mu(x) \\
\delta A_-(x) &= -d\,(x\cdot\partial + \lambda - 1) A_-(x) \\
\delta A_+(x) &= -d\,(x\cdot\partial + \lambda + 1) A_+(x) \ .
\end{aligned}
$$