A Local Approach to Dimensional Reduction: I. General Formalism

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Abstract

We present a formalism for dimensional reduction based on the local properties of invariant cross-sections ("fields") and differential operators. This formalism does not need an ansatz for the invariant fields and is convenient when the reducing group is non-compact.

In the approach presented here, splittings of some exact sequences of vector bundles play a key role. In the case of invariant fields and differential operators, the invariance property leads to an explicit splitting of the corresponding sequences, i.e., to the reduced field/operator. There are also situations when the splittings do not come from invariance with respect to a group action but from some other conditions, which leads to a "non-canonical" reduction.

In a special case, studied in detail in the second part of this article, this method provides an algorithm for construction of conformally invariant fields and differential operators in Minkowski space.

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1 Introduction

Symmetry and reduction have always – sometimes implicitly – been among the main tools in the arsenal of theoretical physics. Many methods that utilize the symmetry of physical systems have been proposed.

In this paper we develop a technique for dimensional reduction of invariant vector fields and one-forms, as well as for reduction of invariant differential operators. We use the methods of differential geometry which allow us to make the reduction procedure simple and natural. In the second part of the paper [6] (briefly referred to as Part II), we apply our method to the problem of obtaining conformally invariant fields and differential operators in Minkowski space.

The main ideas of our method are the following. Let the Lie group G act by bundle morphisms on the vector bundle ξ over a smooth finite-dimensional manifold B, and let $C^\infty(\xi)_G$ stand for the set of all G-invariant sections of ξ. Our goal is to construct the *reduced bundle* ξ_G, i.e., a bundle the space of whose sections, $C^\infty(\xi_G)$, is in a bijective correspondence with $C^\infty(\xi)_G$. We give the construction of the reduced bundle ξ_G (if certain conditions are satisfied).

The construction of the reduced bundle is easily applicable to the important particular cases of the tangent and cotangent bundles of B. In these two cases, the action of G naturally yields certain G-intertwining short exact sequences of vector bundles. It turns out that these sequences are very useful for the classification of the G-invariant vector fields and one-forms.

The dimensional reduction of differential operators is based on the jet bundle picture. The language of jet bundles reveals the geometry of the differential operators and – on the technical side – reduces the operations on differential operators to simple algebraic manipulations. The essential ingredient of the dimensional reduction of a differential operator in this formalism is the restriction of the operator to a submanifold of B. To perform this restriction, one has to find splittings of certain short exact sequences of jet bundles. In the case of G-invariant differential operators, this splitting is provided automatically by the G-invariance.

Our method is purely geometric, using only the geometrically natural structures on the manifold. It applies to any smooth finite dimensional manifold B and to any Lie group G. We note that the method we propose can be applied whether or not the group G is compact or not. If G is non-compact, some interesting phenomena may occur – a simple example is given in Appendix A of Part II.

This method is applicable to tensor products of vector bundles and to spinor bundles, which we plan to study in near future. We believe that our technique can be useful in Kaluza-Klein-type theories.

The plan of the first part of the paper is the following. In Section 2 we describe the method for dimensional reduction, paying special attention to the case of dimensional reduction of the tangent and cotangent bundles. In Section 3 we briefly introduce some notions from the theory of differential operators on vector bundles, and in Sections 4 and 5 we develop the technique for dimensional reduction of invariant differential operators. In Section 6 we explain how to reduce the action of a Lie group K whose action on ξ commutes with that of the reducing group G.

In the second part of the paper, we apply our method to obtain conformally invariant vector fields, one-forms and differential operators in Minkowski space. This construction is based on the observation of Dirac from the 1930's that the conformal group in Minkowski space is locally isomorphic to the orthogonal group in the six-dimensional space with signature $(2,4)$. Our method allows us to find the global transformation laws of the fields, the invariant subbundles, and the so-called equations of conformal electrodynamics as well as the admissible gauge transformations.

2 Dimensional Reduction of Tangent and Cotangent Bundles

Here we describe the method of dimensional reduction, paying special attention to the dimensional reduction of the tangent and cotangent bundles of a smooth manifold. All manifolds, bundles and mappings in the text are supposed to be smooth (C^∞).

Let B be a finite dimensional manifold and $\xi = (E, \pi, B)$ be a finite-dimensional vector bundle over B; let $\xi_b = \pi^{-1}(b)$ denote the fiber of ξ over the point $b \in B$. Let ξ_C (or $\xi|_C$) stand for the bundle $i^*\xi$ induced by the natural embedding $i : C \hookrightarrow B$; in other words, $\xi_C = (E', \pi', C)$, where $E' := \pi^{-1}(C)$ and π' is the restriction of π to E'. By "dimension of the bundle" we will mean the dimension of the typical fiber. The vector space of all sections of ξ will be denoted by $C^\infty(\xi)$.

Let the Lie group G act from the left on ξ by vector bundle morphisms, and let $T : G \times E \to E$ and $t : G \times B \to B$ be the actions of G on the total space and the base respectively (i.e., $\pi \circ T_g = t_g \circ \pi$ and the restriction $T_g : \xi_b \to \xi_{t_g(b)}$ be a linear isomorphism for each $g \in G$). This action naturally induces an action of G on $C^\infty(\xi)$ by

$$g(\psi) := T_g \circ \psi \circ t_g^{-1} , \qquad g \in G , \quad \psi \in C^\infty(\xi) .$$

The subspace of $C^\infty(\xi)$ consisting of all G-invariant sections will be denoted by $C^\infty(\xi)_G$.

Under certain conditions, there exists a vector bundle ξ_G, called a *reduced vector bundle*, such that the space $C^\infty(\xi_G)$ of all its sections is in a bijective correspondence

$$\theta : C^\infty(\xi_G) \longrightarrow C^\infty(\xi)_G \tag{1}$$

with the space $C^\infty(\xi)_G$ of all G-invariant sections of ξ. The group G will be referred to as a *reducing group* and the procedure as a *G-dimensional reduction*. Below we discuss the conditions imposed on the action of G on ξ.

Condition A. *The G-orbits on B are of one and the same type and form a locally trivial bundle*

$$p : B \longrightarrow B/G \ , \tag{2}$$

where p is the natural projection.

The fibers of (2) are homogeneous G-spaces of type G/H where H is some closed subgroup of G. The group G acts naturally on G/H from the left by $g\,[g'] := [gg']$. The local triviality of (2) means that each point $x \in B/G$ has a neighborhood $V \subseteq B/G$ such that there exists an isomorphism $\Phi : p^{-1}(V) \to V \times (G/H)$ satisfying the relation

$$\Phi \circ t_g(b) = (p(b),\, g\,[\pi_2 \circ \Phi(b)]) \ ,$$

where $b \in p^{-1}(V)$, $g \in G$, and π_2 is the canonical projection

$$\pi_2 : V \times (G/H) \longrightarrow G/H : (v,\, [g]) \longmapsto [g] \ .$$

Condition B imposes restrictions on the action T of G on the total space E. If G_b is the stationary group of $b \in B$, then the restriction $T : G_b \times \xi_b \to \xi_b$ determines a linear representation of G_b in ξ_b. Let $\mathrm{st}\,\xi_b$ be the subspace of ξ_b consisting of all vectors fixed with respect to the representation of G_b in it:

$$\mathrm{st}\,\xi_b := \{\, u \in \xi_b \mid T_g(u) = u \quad \forall g \in G_b \,\} \ .$$

The importance of the subspaces $\mathrm{st}\,\xi_b$ in the construction of ξ_G is due to the fact that if ψ is a G-invariant section of ξ, then $\psi(b) \in \mathrm{st}\,\xi_b$ for each $b \in B$.

Condition B. *The set of vector subspaces $\mathrm{st}\,\xi_b \subseteq \xi_b$ form a vector subbundle $\mathrm{st}\,\xi$ which will be called a* stationary subbundle *of ξ.*

If the above two conditions are satisfied, the vector bundle ξ will be called a G-*reducible vector bundle*. Its reduced vector bundle, ξ_G, is a vector bundle over B/G and has the same dimension as $\operatorname{st}\xi$. The local coordinate realizations of $\operatorname{st}\xi$ can be constructed as follows. Let $\{V_\alpha\}$ be a cover of B/G, and $\phi_\alpha : V_\alpha \to B$ be local sections of the bundle (2). Let $N_\alpha := \phi_\alpha(V_\alpha)$ be the graphs of ϕ_α (N_α are submanifolds of B transversal to the orbits of G and of dimension equal to the one of B/G), and $i_\alpha : N_\alpha \hookrightarrow B$ be the natural embeddings. Then the restrictions $\xi_G|_{N_\alpha} := \operatorname{st}\xi|_{N_\alpha} = i^*_\alpha(\operatorname{st}\xi)$ are the coordinate realizations of ξ_G. The cocycle gluing these representations can be naturally constructed with the help of the action of G on ξ [5].

Let us consider in detail the case of the tangent bundle $\tau(B) = (T(B), \pi, B)$ of a finite-dimensional manifold B. Let the action of the Lie group G on B satisfy condition A. Then the tangent lift of t, $t_* : G \times T(B) \to T(B)$, defines an action (t_*, t) of G on $\tau(B)$ which turns $\tau(B)$ into a G-reducible vector bundle. Let $\tau^{\mathrm{v}}(B)$ be the *vertical subbundle* of $\tau(B)$ which by definition consists of the vectors tangent to the fibers of the bundle (2):

$$\tau^{\mathrm{v}}(B)_b := \tau\left(p^{-1}(p(b))\right)_b \ .$$

Let $p^*\tau(B/G)$ be the bundle over B induced from $\tau(B/G)$ by the projection $p : B \to B/G$; clearly, G acts trivially on $p^*\tau(B/G)$. Then it is known [2, Sec. IX.1] that there exists a natural short exact sequence (SES)

$$0 \longrightarrow \tau^{\mathrm{v}}(B) \xrightarrow{\ i\ } \tau(B) \xrightarrow{\ j\ } p^*\tau(B/G) \longrightarrow 0 \ , \tag{3}$$

where i is the natural embedding and j is the natural projection. This SES is G-*intertwining*, i.e., the action of G commutes with the morphisms i and j. Below, we explain how to perform a G-dimensional reduction of (3); we consider only the case of the bundle (2) being globally trivial since it is sufficient for our purposes.

Let us choose a global section of (2) and denote its graph by N. Then the restrictions of $\operatorname{st}\tau^{\mathrm{v}}(B)$, $\operatorname{st}\tau(B)$ and $\operatorname{st}p^*\tau(B/G)$ to N are coordinate realizations of the reduced bundles $\tau^{\mathrm{v}}(B)_G$, $\tau(B)_G$ and $(p^*\tau(B/G))_G$, respectively. Obviously, the SES

$$0 \longrightarrow \tau^{\mathrm{v}}(B)_b \xrightarrow{\ i\ } \tau(B)_b \xrightarrow{\ j\ } p^*\tau(B/G)_b \longrightarrow 0 \tag{4}$$

of G_b-modules over $b \in N$ is (trivially) intertwining with respect to the corresponding representations of the stationary subgroup G_b.

If condition A is satisfied, after the restriction of the G_b-modules in (4) to their stationary submodules, the SES (4) converts into the SES

$$0 \longrightarrow \mathrm{st}\,\tau^{\mathrm{v}}(B)_b \xrightarrow{\;i\;} \mathrm{st}\,\tau(B)_b \xrightarrow{\;j\;} p^*\tau(B/G)_b \longrightarrow 0\ ,$$

since the representation of G_b in $\tau(B)_b$ is decomposable, i.e., $\tau(B)_b$ is a direct sum of two G_b-invariant subspaces. The proof of the decomposability goes as follows. Since $i(\tau^{\mathrm{v}}(B))$ is a G-invariant subbundle of $\tau(B)$, $i(\tau^{\mathrm{v}}(B)_b)$ is a G_b-invariant subspace of $\tau(B)_b$. Condition A guarantees that for each $b \in B$, there exists a submanifold of B, namely

$$W^{(b)} := \{\, \Phi^{-1}\,(x,\, \pi_2 \circ \Phi(b)) \mid x \in V \subseteq B/G\}$$

(V is an open subset of B/G containing $p(b)$), which is transversal to the orbits of G in B, contains b, and consists of points with one and the same stationary group G_b. This means that G_b acts trivially on $W^{(b)}$ and, hence, on $\tau(W^{(b)})$, so $\tau(W^{(b)})_b$ is the invariant complement of $i(\tau^{\mathrm{v}}(B)_b)$ with respect to the representation of G_b in $\tau(B)_b$. Thus, the representation of G_b in $\tau(B)_b$ is decomposable.

Finally, taking into account that $(p^*\tau(B/G))_G \cong \tau(N)$, we obtain that after a G-dimensional reduction, (3) converts into the SES

$$0 \longrightarrow \tau^{\mathrm{v}}(B)_G \xrightarrow{\;i\;} \tau(B)_G \xrightarrow{\;j\;} \tau(N) \longrightarrow 0\ . \tag{5}$$

The G-dimensional reduction of the cotangent bundle $\tau^*(B)$ is similar. The action t of G on B generates a natural contragradient action $(\hat{t}, t) := (t^{*-1}, t)$ of G on $\tau^*(B)$. The decomposability of the action (t_*, t) implies the decomposability of $(\hat{t}, t)$, hence after a G-dimensional reduction the dual of (3) G-intertwining SES goes into the SES

$$0 \longleftarrow (\tau^{\mathrm{v}}(B)^*)_G \xleftarrow{\;i^*\;} \tau^*(B)_G \xleftarrow{\;j^*\;} \tau^*(N) \longleftarrow 0\ .$$

In general, $(\tau^{\mathrm{v}}(B)^*)_G \neq (\tau^{\mathrm{v}}(B)_G)^*$ (an example is given in Appendix A of Part II). However, if all finite-dimensional representations of G_b are decomposable (in particular, if G_b is compact), it is easy to prove that these two bundles coincide.

The set $C^\infty(\tau(B))_G$ of all G-invariant vector fields is in bijective correspondence with the set of all sections of the reduced bundle, $C^\infty(\tau(B)_G)$. But (5) implies that

$$\tau(B)_G \cong \tau^{\mathrm{v}}(B)_G \oplus \tau(N)\ ,$$

hence, there exists a bijective correspondence between the G-invariant vector fields and the couples of a "scalar field" (a section of $\tau^{\mathrm{v}}(B)_G$) and a vector field on N. The above construction, however, does not determine a fixed splitting of (5) because $\tau(N)$ is not canonically embedded in $\tau(B)_G$. One needs additional information to fix a certain splitting. The case of G-invariant differential forms is completely analogous.

Remark 2.1 *If C is a G-invariant submanifold of B, the restriction to C yields naturally the SES*

$$0 \longrightarrow \tau(C) \xrightarrow{\ m\ } \tau(B)_C \xrightarrow{\ n\ } \nu(C) \longrightarrow 0 \tag{6}$$

of vector bundles over C; here $\nu(C)$ is the quotient bundle, $\tau(B)_C/\tau(C)$. The SES (6) and its dual are G-intertwining, which implies the G-invariance of $m(\tau(C))$ and $n^*(\nu^*(C))$ as subbundles of $\tau(B)_C$ and $\tau^*(B)_C$, respectively.

Remark 2.2 *Let $\zeta \in C^\infty(\tau(B))$ and $\psi \in C^\infty(\tau^*(B))$ be a G-invariant vector field and one-form, respectively. The local coordinates (x^μ) of B ($\mu = 1, \ldots, \dim B$) induce local coordinates $(x^\mu,\, dx^\mu)$ of $\tau(B)$ and $(x^\mu,\, \partial/\partial x^\mu)$ of $\tau^*(B)$. In these coordinates, the conditions for G-invariance of the sections of $\tau(B)$ and $\tau^*(B)$ read*

$$\begin{aligned} g(\zeta)^\mu(b) &:= (t_{g*}\zeta)^\mu(b) = \frac{\partial {t_g}^\mu\,(t_{g^{-1}}(b))}{\partial x^\nu}\,\zeta^\nu\,(t_{g^{-1}}(b)) = \zeta^\mu(b)\ , \\ g(\psi)_\mu(b) &:= (\hat{t}_g\psi)_\mu(b) = \frac{\partial {t_{g^{-1}}}^\nu(b)}{\partial x^\mu}\,\psi_\nu\,(t_{g^{-1}}(b)) = \psi_\mu(b)\ , \end{aligned}$$

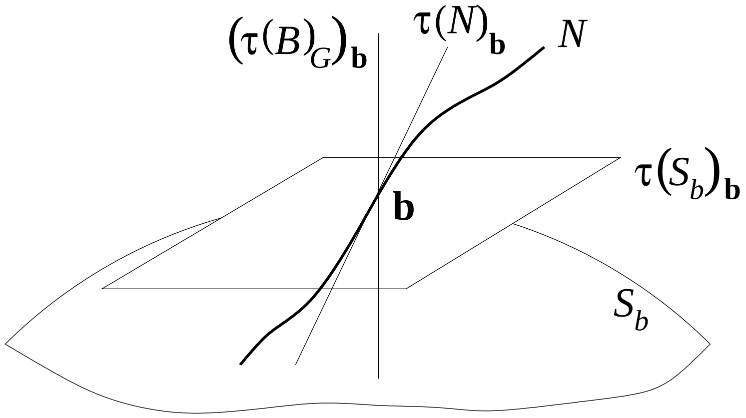
where $g \in G$.

Example 2.1 *Let us consider the example of the dimensional reduction of the $O_0(3)$-invariant vector fields on $\mathbb{R}^3$, i.e., let $B := \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and let $G := O_0(3)$ act by its tangent lift on $\tau(\mathbb{R}^3)$. (The subscript "0" means "the connected component of the unit element".) The orbits of the action of $O_0(3)$ on $\mathbb{R}^3$ are the spheres centered at the origin $\mathbf{0} := (0,0,0)$.*

For a base N of the reduced bundle (2) we can choose any submanifold diffeomorphic to $B/G \cong \mathbb{R}_+$, e.g., $N := \{\, (\chi(z), 0, z) \mid z > 0 \,\}$, where $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\chi'(z) > 0$ for any $z > 0$.

For each $\mathbf{b} := (x, y, z) \in B$, the vertical subspace, $\tau^{\mathrm{v}}(B)_{\mathbf{b}}$, is the plane $\tau(S_b)_{\mathbf{b}}$ tangent to the sphere S_b of radius $b := \sqrt{x^2+y^2+z^2}$ at point $\mathbf{b}$. The 1-dimensional quotient space $p^*\tau(B/G)_{\mathbf{b}} = \tau(B)_{\mathbf{b}}/\tau^{\mathrm{v}}(B)_{\mathbf{b}}$ can be realized as any subspace of $\tau(B)_{\mathbf{b}}$ transversal to $\tau^{\mathrm{v}}(B)_{\mathbf{b}}$.

The stationary subgroup $G_{\mathbf{b}}$ consists of the rotations around the straight line through the points $\mathbf{0}$ and $\mathbf{b}$. It acts freely on $\tau(S_b)_{\mathbf{b}}$, so $\mathrm{st}\,\tau^{\mathrm{v}}(B)_{\mathbf{b}} = \mathrm{st}\,\tau(S_b)_{\mathbf{b}}$ consists only of the zero vector at $\mathbf{b}$. Thus, $(\tau^{\mathrm{v}}(B)_G)_{\mathbf{b}}$ consists only of the zero vector at $\mathbf{b}$, $(\tau(B)_G)_{\mathbf{b}}$ is the 1-dimensional subspace of the radial vectors at $\mathbf{b}$, and $((p^*\tau(B/G))_G)_{\mathbf{b}}$ can be realized as $\tau(N)_{\mathbf{b}}$ (see the figure).



3 Differential Operators on Vector Bundles

In this section we briefly introduce the necessary facts and notations concerning the theory of differential operators on vector bundles in terms of jets of sections (for more details see, e.g., [1,3,7–13]).

Let ξ be a vector bundle over V, $I_b(B)$ be the ideal of the ring $C^\infty(B)$ consisting of all functions vanishing at $b \in B$, and $I_b^k(B) \subset C^\infty(B)$ be the ideal of functions representable as a product of k functions from $I_b(B)$. Let $Z_b^k(\xi)$ stand for the vector space of those sections of ξ which are products of elements of $I_b^{k+1}(B)$ and $C^\infty(\xi)$, and $J^k(\xi)_b$ be the quotient space

$$J^k(\xi)_b := C^\infty(\xi)/Z_b^k(\xi) \ .$$

The canonical linear mapping $C^\infty(\xi) \to J^k(\xi)_b : \psi \mapsto J^k(\psi)(b)$ maps a section ψ into its *k-jet* at $b \in B$, $J^k(\psi)(b)$. The k-jet is the coordinate-free concept for the section ("the field") ψ and its derivatives up to order k at b. Hence, $J^k(\psi)(b) = J^k(\phi)(b)$ means that in some (and hence in all) local coordinates, the Taylor series of ψ and ϕ around b agree up through order k. The mapping

$$J^k : C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi)) : \psi \longmapsto J^k(\psi)$$

is called a *k-jet* of the section ψ.

The vector bundle structure of the union

$$J^k(\xi) := \bigcup_{b\in B} J^k(\xi)_b$$

is natural: the local coordinates (x^μ, u^a) $(\mu = 1, \ldots, \dim B;\ a = 1, \ldots, \dim \xi)$ of ξ generate local coordinates $(x^\mu, u^a, u^a_{\mu_1}, \ldots, u^a_{\mu_1\ldots\mu_k})$, $(1 \le \mu_1 \le \ldots \le \mu_i \le$

$\dim B$; $i = 1, \ldots, k$) of $J^k(\xi)$, where

$$u^a_{\mu_1 \ldots \mu_i} \left(J^k(\psi)(b) \right) := \frac{\partial^i}{\partial x^{\mu_1} \ldots \partial x^{\mu_i}} \, \psi^a(b) \ , \tag{7}$$

and the transition functions follow from the well-known formulae for transformation of partial derivatives under a change of variables. By definition, $J^0(\xi) := \xi$. The dimension of the vector bundle $J^k(\xi)$ is

$$\dim J^k(\xi) = \binom{\dim B + k}{k} \dim \xi \ .$$

Let $\pi^{k,l} : J^k(\xi) \to J^l(\xi)$, $k > l \geq 0$ denote the natural projections ("cutting off" all derivatives of order $k+1$, …, l).

Let ξ and η be vector bundles over B. A *differential operator* (DO) of order k from ξ to η is a mapping

$$D : C^\infty(\xi) \longrightarrow C^\infty(\eta) : \psi \longmapsto D\psi$$

such that $J^k(\psi)(b) = 0$ implies $D\psi(b) = 0$. If D is a linear mapping, it is called a *linear differential operator*. Let $\mathrm{Diff}_k(\xi, \eta)$ and $\mathrm{LDiff}_k(\xi, \eta)$ stand respectively for the vector spaces of DOs and linear DOs of order k from ξ to η.

There exists a bijective correspondence between the set $\mathrm{LDiff}_k(\xi, \eta)$ and the space $\mathrm{Hom}(J^k(\xi), \eta)$ of vector bundle morphisms over the identity in B. This correspondence is based on the fact that every vector bundle morphism $T : \zeta \to \eta$ over the identity in B generates naturally a mapping

$$T_* : C^\infty(\zeta) \longrightarrow C^\infty(\eta) : \phi \longmapsto T_*(\phi) := T \circ \phi \ ,$$

and each linear DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ corresponds to a vector bundle morphism $T \in \mathrm{Hom}(J^k(\xi), \eta)$ such that $D = T_* \circ J^k$, i.e., that the diagram

$$\begin{tikzcd} C^\infty(\xi) \arrow[r, "J^k"] \arrow[d, "D"'] & C^\infty(J^k(\xi)) \arrow[dl, "T_*"] \\ C^\infty(\eta) & \end{tikzcd}$$

commutes. The morphism T is called the *total symbol* of D. For a nonlinear DO this construction is analogous, but in this case T is simply a fiber preserving mapping.

Parenthetically, in the language of category theory (see, e.g., [4]), the couple $(J^k, J^k(\xi))$ is a universal element for the covariant functor $\mathcal{F} = (\mathcal{F}_{\mathrm{Ob}}, \mathcal{F}_{\mathrm{Mor}})$ from the category $\mathcal{V}(B)$ of vector bundles over B to the category of linear

DOs of order k from $\xi \in \mathcal{V}(B)$ to another vector bundle over B. Namely, if $\eta,\ \zeta \in \mathcal{V}(B)$ and $T \in \mathrm{Hom}(\eta, \zeta)$, then

$$\mathcal{F}_{\mathrm{Ob}}(\eta) = \mathrm{LDiff}_k(\xi, \eta), \qquad \mathcal{F}_{\mathrm{Mor}}(T) = T_* = T \circ\ .$$

One can differentiate simultaneously both sides of the differential equation $D\psi = \phi$, thus obtaining a differential operator of higher order acting on ψ. The formal definition of this is the following. Let $D = T_* \circ J^k \in \mathrm{Diff}_k(\xi, \eta)$. There exists a unique fiber preserving mapping $P^l(T) : J^{k+l}(\xi) \to J^l(\eta)$ such that the diagram

$$\begin{array}{ccc} C^\infty(J^{k+l}(\xi)) & \xrightarrow{P^l(T)_*} & C^\infty(J^l(\eta)) \\ {\scriptstyle J^{k+l}}\uparrow & & \uparrow{\scriptstyle J^l} \\ C^\infty(\xi) & \xrightarrow[D]{} & C^\infty(\eta) \end{array}$$

commutes. The *l-th prolongation* of D is by definition

$$P^l(D) := P^l(T)_* \circ J^{k+l} = J^l \circ T_* \circ J^k \in \mathrm{Diff}_{k+l}(\xi, J^l(\eta))\ .$$

In general, $R^{k,l} := \ker P^l(T)$ is a family of linear subspaces of the vector bundle $J^{k+l}(\xi)$. A DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ said to be *formally integrable* if for each $l \geq 0$ the following conditions are satisfied

(a) $R^{k,l}$ is a vector subbundle of $J^{k+l}(\xi)$;
(b) the natural projection $\pi^{k+l+1,\,k+l} : R^{k,\,l+1} \longrightarrow R^{k,l}$ is an epimorphism.

For a formally integrable DO, the subbundle $R^{k,0} \subset J^k(\xi)$ is called its *equation*.

The meaning of the formal integrability of a DO is that for a formally integrable DO $D \in \mathrm{Diff}_k(\xi, \eta)$, by differentiating l times the equation $D\psi = 0$, one can never obtain a condition on the derivatives of ψ that contains no derivatives of order $k + l$ but only lower order derivatives (see the example below).

The formal integrability of a differential operator can be established by using methods of homological algebra (for references see, e.g., [1]).

Example 3.1 *An example of a differential operator that is not formally integrable is the following.*

Let ξ and η be vector bundles over $\mathbb{R}^3$ with typical fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively, and

$$D := \begin{pmatrix} \partial_{zz} - y\partial_{xx} \\ \partial_{yy} \end{pmatrix} \in \mathrm{LDiff}_2(\xi, \eta)$$

(where x, y and z are the coordinates in $\mathbb{R}^3$). Then $\pi^{3,2} : R^{2,1} \to R^{2,0}$ is an epimorphism, but $\pi^{4,3} : R^{2,2} \to R^{2,1}$ is not, because the second prolongation of D yields the condition $u_{xxy} = 0$ which was not present in $R^{2,1}$. In fact, there are infinitely many conditions of this kind that appear in the higher prolongations, and the general solution of $D\psi = 0$ contains only 12 parameters:

$$\begin{aligned}\psi(x,y,z) = {} & xy(\alpha_1 z^3 + 3\alpha_2 z^2 + \alpha_3 z + \alpha_4) + y(\alpha_5 z^3 + \alpha_6 z^2 + \alpha_7 z + \alpha_8) \\ & +(\alpha_1 x^3 + 3\alpha_5 x^2 + \alpha_9 x + \alpha_{10})z + \alpha_2 x^3 + \alpha_6 x^2 + \alpha_{11} x + \alpha_{12} \ .\end{aligned}$$

This example is studied in detail in the Introduction of [11].

4 Restriction of a Differential Operator to a Submanifold

The dimensional reduction of an invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)_G$ is closely related to the problem of restricting D to N where N is a submanifold of B. By "restricting D to N", we mean constructing a DO $D_N \in \mathrm{Diff}_k(\xi_N,\eta_N)$ from D by means of the natural embedding $i : N \hookrightarrow B$. This procedure is not naturally defined, so in this section we will explain how it can be performed.

To understand the problem, let us first consider it in local coordinates. Let $\dim B = n$, $\dim N = \nu$, and let the local coordinates x^1, ..., x^n are such that

$$N = \{x^{\nu+1} = \cdots = x^n = 0\} \ .$$

We will call x^1, ..., x^ν "internal for N", and $x^{\nu+1}$, ..., x^n "external for N" coordinates. Let $\psi \in C^\infty(\xi)$, and let $\psi \circ i \in C^\infty(\xi_N)$ be its restriction to N.

If we calculate the k-jet of ψ, and then restrict $J^k(\psi)$ to N, we obtain $J^k(\psi) \circ i \in C^\infty(J^k(\xi)_N)$. For each $b \in N$, $J^k(\psi)(b)$ contains derivatives with respects to all coordinates, x^1, ..., x^n. At the same time, if we first restrict ψ to N and then take the k-jet of $\psi \circ i$, we obtain $J^k(\psi \circ i) \in C^\infty(J^k(\xi_N))$. Since $\psi \circ i$ depends only on the internal for N coordinates, x^1, ..., x^ν, its k-jet, $J^k(\psi \circ i)$, contains derivatives with respect to these coordinates only. Thus, the dimensions of $J^k(\xi)_N$ and $J^k(\xi_N)$ are $\binom{n+k}{k} \dim \xi$ and $\binom{\nu+k}{k} \dim \xi$, respectively.

If D contains only differentiations with respect to internal for N coordinates (in which case we will say that D is *internal for* N), the restricted DO D_N is simply equal to D. What to do, however, if D is not internal for N?

Let us first note that there exists a natural projection $j^k : J^k(\xi)_N \to J^k(\xi_N)$ which simply is "cutting off" all non-internal derivatives, i.e., those containing at least one external for N partial derivative. The relationship between the total symbols of D and D_N can be simply expressed with the help of j^k by means of the diagram

$$\begin{tikzcd} C^\infty(J^k(\xi)_N) \arrow[r, "j^k_*"] \arrow[d, "(T_*)_N"'] & C^\infty(J^k(\xi_N)) \arrow[dl, "(T_N)_*"] \\ C^\infty(\eta_N) & \end{tikzcd}$$

Clearly, for an internal DO D, the total symbol of D_N is defined naturally by $T = T_N \circ j^k$. If the DO is not internal, one needs some additional information.

Let I^k_N be the subbundle of $J^k(\xi)_N$ consisting of the k-jets of all sections of ξ that vanish on N:

$$I^k_N := \{J^k(\psi)(b) \ \mid \ \psi \in C^\infty(\xi) \ \text{ s.t. } \ \psi(b) = 0 \ \ \forall\, b \in N\} \ .$$

In other words, I^k_N consists of those elements of $J^k(\xi)_N$ all internal for N derivatives of which (including the zeroth derivatives) are 0. Using the notations (7), the coordinates of the elements of I^k_N satisfy

$$u^a = 0 \ , \quad u^a_{\mu_1 \ldots \mu_i} = 0 \ , \qquad 1 \le \mu_1 \le \ldots \le \mu_i \le \nu \ , \quad i = 1, \ldots, k \ .$$

If $i^k : I^k_N \hookrightarrow J^k(\xi)_N$ is the natural embedding, then clearly $j^k \circ i^k$, so the horizontal sequence in the diagram below is exact:

$$\begin{tikzcd} 0 \arrow[r] & I^k_N \arrow[r, "i^k"] & J^k(\xi)_N \arrow[r, "j^k"] \arrow[d, "T"'] & J^k(\xi_N) \arrow[r] \arrow[l, bend right, "S"'] \arrow[dl, "T_N"] & 0 \ . \\ & & \eta_N & & \end{tikzcd} \tag{8}$$

The essence of the problem of restricting a DO to N is that while $J^k(\xi)_N \cong I^k_N \oplus J^k(\xi_N)$, the bundle $J^k(\xi_N)$ is not naturally embedded in $J^k(\xi)_N$. Therefore, to define T_N, one needs to choose a fiber preserving mapping $S : J^k(\xi_N) \to J^k(\xi)_N$ over the identity in N such that

$$j^k \circ S = \text{identity in } J^k(\xi_N) \ . \tag{9}$$

Then the total symbol of D_N is defined by

$$T_N = T \circ S \ . \tag{10}$$

The condition (9) on S guarantees that for an internal for N DO, (10) yields a DO coinciding with the natural restriction of D to N. The mapping S is called a *splitting relation*.

One can define S by specifying its image $S(J^k(\xi_N)) \subset J^k(\xi)_N$, which can be defined as a kernel of the total symbol $\mathcal{T}$ of some DO $\mathcal{D}$ of order k acting on $C^\infty(\xi)$.

To clarify the matters, let us consider the following example (more examples will be given in Part II).

Example 4.1 *Let $B := \mathbb{R}^2$ and (x, y) be the Cartesian coordinates, let ξ be a globally trivial bundle over B with fiber $\mathbb{R}$, and let $N := \{y = 0\} \subset B$.*

Then the fiber coordinates of $J^2(\xi)_N$ are $(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$, while the fiber coordinates of $J^2(\xi_N)$ are only the ones that do not contain y-derivatives, i.e., (u, u_x, u_{xx}).

The Laplacian $\Delta := \partial_{xx} + \partial_{yy} \in \mathrm{LDiff}_2(\xi, \xi)$ is not internal for N because it contains y-derivatives.

Let us choose the splitting condition S to be defined by specifying its image, $S(J^k(\xi_N)) := \ker \mathcal{T}$, where $\mathcal{T}$ is the total symbol of the DO $\mathcal{D} := \partial_{xx} - \partial_{yy} + 3\partial_x$. Then u_{yy} can be expressed as $u_{yy} = u_{xx} + 3u_x$, so the restricted to N Laplacian becomes $\Delta_N = 2\partial_{xx} + 3\partial_x \in \mathrm{LDiff}_2(\xi_N, \xi_N)$.

Remark 4.1 *In general, the order of D_N can be higher that the order of D; for example, if in the above example we had chosen $S = u_{xxx} - 2u_{xx} - u_{yy} - 2u_x$, then the reduced Laplacian would have become $\Delta_N = \partial_{xxx} - \partial_{xx} - 2\partial_x \in \mathrm{LDiff}_3(\xi_N, \xi_N)$. This can happen when S contains derivatives of order higher than the order of D. However, a DO $D \in \mathrm{Diff}_m(\xi, \eta)$ can be considered as an element of $\mathrm{Diff}_l(\xi, \eta)$ for all $l > m$. Therefore, all constructions from this section hold if by k we mean the maximum of the ofder of D and the order of the highest derivative in the splitting relation S.*

5 Dimensional Reduction of Invariant Differential Operators

Let ξ and η be G-reducible vector bundles over the manifold B with the same action t of G on B. Then the actions of $g \in G$ on $C^\infty(\xi)$ and $C^\infty(\eta)$,

$$g^\xi : \psi \longmapsto T_g^\xi \circ \psi \circ t_g^{-1} \ , \quad g^\eta : \chi \longmapsto T_g^\eta \circ \chi \circ t_g^{-1} \ ,$$

generate an action of g on the DOs:

$$g(D) := g^\eta \circ D \circ (g^\xi)^{-1} \ , \quad D \in \mathrm{Diff}_k(\xi, \eta)$$

A DO $D \in \mathrm{Diff}_k(\xi, \eta)$ is G-*invariant* if $g(D) = D$. The space of all G-invariant DOs will be denoted by $\mathrm{Diff}_k(\xi, \eta)_G$. If $\psi \in C^\infty(\xi)_G$ and $D \in \mathrm{Diff}_k(\xi, \eta)_G$, then

$D\psi \in C^\infty(\eta)_G$, therefore each G-invariant DO D generates a reduced DO

$$D_G := \theta^{-1} \circ D \circ \theta : C^\infty(\xi_G) \longrightarrow C^\infty(\eta_G) \ .$$

To construct D_G, one must do the following:

(a) find the stationary subbundles $\mathrm{st}\,\xi$ and $\mathrm{st}\,\eta$;
(b) choose a submanifold $N \cong B/G$ transversal to the G-orbits in B; and the main problem is
(c) restrict the DO $D : C^\infty(\mathrm{st}\,\xi) \to C^\infty(\mathrm{st}\,\eta)$ to N.

As pointed out in section 4, to perform step (c) in this algorithm, one needs a fiber preserving mapping $S : J^k(\xi_N) \to J^k(\xi)_N$. In the case of dimensional reduction of a G-invariant DO, there exists a natural splitting of the SES (8) due to the possibility to extend uniquely by G-invariance each section of $\mathrm{st}\,\xi_N$ to a section over a neighborhood of N. In the rest of this section, we explain how this can be done.

The *Lie derivative* $L : C^\infty(\xi) \to C^\infty(\mathfrak{g}^* \otimes \xi)$ of the action (T,t) of the Lie group G on the section $\psi \in C^\infty(\xi)$ is defined by

$$L\psi(Y) := \frac{d}{dt}\, e^{tY}(\psi)\Big|_{t=0} = \frac{d}{dt}\, T_{\exp(tY)} \circ \psi \circ t_{\exp(-tY)}\Big|_{t=0} \ ,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G, and $Y \in \mathfrak{g}$. The stationary subbundle $\mathrm{st}\,\xi$ is invariant under the action of G on ξ, hence the Lie derivative has a natural restriction to $\mathrm{st}\,\xi$ which will also be denoted by L.

Let

$$\widetilde{L} : J^1(\mathrm{st}\,\xi) \longrightarrow \mathfrak{g}^* \otimes \mathrm{st}\,\xi$$

be the total symbol of the Lie derivative $L \in \mathrm{LDiff}_1(\mathrm{st}\,\xi, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$. The $(k-1)$st prolongation of L provides a splitting relation for each G-invariant DO $D \in \mathrm{Diff}_k(\xi,\eta)_G$. Indeed, let

$$P^{k-1}(\widetilde{L}) : J^k(\mathrm{st}\,\xi) \longrightarrow \mathfrak{g}^* \otimes J^{k-1}(\mathrm{st}\,\xi)$$

be the symbol of the $(k-1)$st prolongation of L,

$$R^k_N := (\ker P^{k-1}(\widetilde{L}))_N \subset J^k(\mathrm{st}\,\xi)_N$$

be the restriction of its kernel to N, and i^k and I^k_N have the same meaning as in (8) (with $\mathrm{st}\,\xi$ instead of ξ). Then it can be shown that R^k_N defines a splitting of (8), i.e., it is a complementary subbundle of $i^k(I^k_N)$ in $J^k(\mathrm{st}\,\xi)_N$:

$$J^k(\mathrm{st}\,\xi)_N = i^k(I^k_N) \otimes R^k_N \ .$$

This means that for each $b \in N$, all non-internal for N derivatives in $J^k(\psi)(b)$ can be expressed in terms of the purely internal derivatives by solving the system

$$P^{k-1}(\widetilde{L})_N = 0 \ ,$$

thus providing us with an algorithm for restriction to N, and hence for dimensional reduction of each G-invariant DO of order k.

A very important for this algorithm fact is that the Lie derivative is a formally integrable differential operator. Due to the formal integrability of L, in order to reduce a G-invariant DO of order k, one does not need to consider $P^l(\widetilde{L})_N = 0$ for $l > k-1$, so the above theory always yields an algorithm consisting of finitely many steps. This, in particular, implies that in the process of reduction of a G-invariant DO the order of the operator does not increase, i.e., if $D \in \mathrm{Diff}_k(\xi,\eta)_G$, then $D_G \in \mathrm{Diff}_m\,(\xi_G,\eta_G)$, where $m \leq k$.

In the examples below we illustrate the method of dimensional reduction of invariant differential operators on simple examples. Notice how simple the calculations are (as opposed to, say, change of variables), and the fact that, although the reducing group in the second example is non-compact, the reduction procedure is essentially the same.

Example 5.1 *As an example, let us consider the $O_0(3)$-reduction ot the Laplace operator acting on scalar functions in $\mathbb{R}^3$.*

For a base of the reduced bundle let us choose the transversal to the orbits of $O_0(3)$ manifold $N := \{(0,0,z)\,|\, z>0\}$.

The generators of $O_0(3)$ are

$$X_1 := x\partial_y - y\partial_x \ , \quad X_2 := x\partial_z - z\partial_x \ , \quad X_3 := y\partial_z - z\partial_y \ .$$

The infinitesimal symmetry condition $X_2 f = 0$ implies $\partial_x f = \frac{x}{z}\partial_z f$. The restriction to the submanifold N of the x-component of the first prolongation of this condition yields $\partial_{xx} f = \frac{1}{z}\partial_z f$ on N. Similarly, the condition $X_3 f = 0$ gives $\partial_{yy} f = \frac{1}{z}\partial_z f$ on N. Plugging these equations in $\Delta := \partial_{xx} + \partial_{yy} + \partial_{zz}$, we obtain the reduced Laplace operator,

$$\Delta_{O_0(3)} = \frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} \ ,$$

acting on functions defined on N – as expected, for this choice of N, $\Delta_{O_0(3)}$ is the radial part of the Laplacian.

Example 5.2 *As an example of reduction with a non-compact reducing group consider the $O_0(1,2)$-reduction of the $(1+2)$-dimensional D'Alembertian*

$$\Box_3 := \partial_{xx} + \partial_{yy} - \partial_{zz}$$

in the interior of the "future" light cone. Let $N := \{(0,0,z)\,|\, z>0\}$.

The generators of $O_0(1,2)$ are $x\partial_y - y\partial_x$, $x\partial_z + z\partial_x$, and $y\partial_z + z\partial_y$. The same calculation as in the previous example yields $\partial_{xx}f = -\frac{1}{z}\partial_z f$, $\partial_{yy}f = -\frac{1}{x}\partial_z f$ on N, so the reduced wave operator is

$$(\Box_3)_{O_0(1,2)} = -\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz}\right) \ .$$

Note that the $O_0(3)$-invariant harmonic functions satisfy the same equation as the $O_0(1,2)$-invariant solutions of the wave equation.

6 Dimensional Reduction of a Group Action

We need some details concerning the dimensional reduction of a group action. Let $\xi = (E, \pi, B)$ be a G-reducible vector bundle, the Lie group K act on ξ by vector bundle morphisms (F, f), and the actions of G and K commute. Due to the mutual commutativity of the actions (T, t) and (F, f) of G and K on ξ, the action of $k \in K$ maps the G-invariant sections of ξ into G-invariant ones (i.e., $C^\infty(\xi)_G$ is a K-invariant subset of $C^\infty(\xi)$). This fact allows us to define a natural action (F_G, f_G) of K on the reduced bundle $\xi_G = (\mathrm{st}\,\xi_N, \pi_G, N)$ as follows. Let $k \in K$, $b \in N$, $\sigma \in (\xi_G)_b = \mathrm{st}\,\xi_b$, and let $g \in G$ is such that $t_g \circ f_k(b) \in N$, then

$$\begin{aligned}(t_G)_k(b) &:= t_g \circ f_k(b) \ , \\ (F_G)_k(\sigma) &:= T_g \circ F_k(\sigma) \in (\xi_G)_{(t_G)_k(b)} \ .\end{aligned}$$

The action k_G of $k \in K$ on $C^\infty(\xi_G)$ is

$$k_G : C^\infty(\xi_G) \longrightarrow C^\infty(\xi_G) : S \longmapsto k_G(S) := \theta^{-1} \circ k \circ \theta \circ S \ .$$

Induced representations are an example of dimensional reduction of a group action.

Let ξ and η be G-reducible vector bundles over B with the same action t of G on B. Let the Lie group K act on ξ and η by vector bundle morphisms with the action f of K on B. Let the action (T^ξ, t) of G on ξ commute with the action (F^ξ, f) of K on ξ, and the same be valid for the actions (T^η, t) and (F^η, f) on η. It is easy to see that if a DO $D \in \mathrm{Diff}_k(\xi, \eta)$ is simultaneously K- and G-invariant, then the reduced DO $D_G \in \mathrm{Diff}_k(\xi_G, \eta_G)$ is invariant under the reduced actions (F^ξ_G, f_G) and (F^η_G, f_G) of K on ξ_G and η_G.

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