System of Model Functions for the Two-Dimensional Periodic Magnetic Schrödinger Operator.

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ABSTRACT For the periodic magnetic Schrödinger operator in two dimensions we describe a set of model functions, which solve the equations for eigenfunctions approximately . There exist the model functions of two types: a weak diffraction type and a strong diffraction type. It is shown that the model functions are mutually "almost orthogonal" and the model set is complete in the high energy region – all eigenfunctions with eigenvalues large enough can be described in terms of the model functions. The present paper contains the construction of the system. This is the first of two papers designed to prove that in the high energy region each eigenfunction is close to exactly one of the model functions for a rich set of quasimomenta, for the rest of quasimomenta it is close to a linear combination of the model functions. Information about the isoenergetic surface, a proof of the Bethe-Sommerfeld conjecture and an asymptotic of the integral density of states in the high energy region are going to be obtained as corollaries of the formulae for eigenfunctions.

1 Introduction.

We consider the operator

$$H=\left(\frac{1}{i}\frac{\partial}{\partial x_1}+a_1(x)\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial x_2}+a_2(x)\right)^2+\alpha a_0(x) \tag{1}$$

in $L_2(R^2)$, where $a_0(x)$, $a_1(x)$, $a_2(x)$ are real periodic functions, α is a real parameter. We assume that $a_0(x)$, $a_1(x)$, $a_2(x)$ have the same periods b_1, b_2. For the sake of simplicity, we consider the periods being orthogonal, however all the results are also valid for nonorthogonal periods. Without the loss of generality we assume that $b_1 \geq b_2$,

$$\int_Q a_s(x)dx=0, \quad s=0,1,2, \quad Q=[0,b_1]\times[0,b_2]. \tag{2}$$

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$$\mathrm{div}\vec{a}(x)=0,\ \ \mathrm{where}\vec{a}(x)=(a_1(x),a_2(x)). \tag{3}$$

We use the Fourier representation of the functions $a_0(x)$, $a_1(x)$, $a_2(x)$:

$$a_s(x)=\sum_{m\in Z^2,\ m\neq 0} a_{sm}\exp i(\vec{p}_m(0),x),\quad s=0,1,2, \tag{4}$$

where $(\cdot,\cdot)$ is the inner product in R^2 and $\vec{p}_m(0)$ is a vector of the dual lattice:

$$\vec{p}_m(0)=2\pi(m_1b_1^{-1},m_2b_2^{-1}). \tag{5}$$

We use the notation $\vec{A}(x)=(a_0(x),a_1(x),a_2(x))$. In this paper we consider the potential to be a trigonometric polynomial. It is known (see f.e. [Rs]) that spectral analysis of H can be reduced to analysis of a family of operators $H(t)$, $t\in K$, where K is the elementary cell of the dual lattice,

$$K=[0,2\pi b_1^{-1})\times[0,2\pi b_2^{-1}).$$

The vector t is called the *quasimomentum.* The operator $H(t)$, $t\in K$, acts in $L_2(Q)$. It is described by formula (1) and the quasiperiodic conditions:

$$u(b_1,x_2)=\exp(it_1b_1)u(0,x_2),\quad u(x_1,b_2)=\exp(it_2b_2)u(x_1,0). \tag{6}$$

$$u_{x_1}(b_1,x_2)=\exp(it_1b_1)u_{x_1}(0,x_2),\quad u_{x_2}(x_1,b_2)=\exp(it_2b_2)u_{x_2}(x_1,0).$$

Each operator $H(t)$, $t\in K$, has a discrete semibounded below spectrum $\Lambda(t)$:

$$\Lambda(t)=\cup_{n=1}^{\infty}\lambda_n(t),\ \lambda_n(t)\to_{n\to\infty}\infty.$$

The spectrum Λ of the operator H is the union of the spectra of the operators $H(t)$: $\Lambda=\cup_{t\in K}\Lambda(t)=\cup_{n\in N,t\in K}\lambda_n(t)$. The functions $\lambda_n(t)$ are continuous, so Λ has a band structure:

$$\Lambda=\cup_{n=1}^{\infty}[q_n,Q_n],\ q_n=\min_{t\in K}\lambda_n(t),\ Q_n=\max_{t\in K}\lambda_n(t).$$

It is proven in [BiSu] that the spectrum of operator H is absolutely continuous, i.e. $[q_n,Q_n]$ can not degenerate to a point. The Bethe-Sommerfeld conjecture for the two-dimensional situation is proved in [Mo] by methods of microlocal analysis. Other important properties of the spectrum are proved in [HH].

The eigenfunctions of $H(t)$ and H are simply related. Extending all the eigenfunctions of the operators $H(t)$ quasiperiodically (see (6)) to R^2, we obtain the complete system of eigenfunctions of H.

In the case of $\vec{A}=0$, the eigenvalues and eigenfunctions of the corresponding operators $H_0(t)$, $t\in K$, are naturally indexed by the points of Z^2: $\Psi_m^0(t,x)=\exp i(\vec{p}_m(t),x)$, $\lambda_m^0(t)=p_m^2(t)$, $m\in Z^2$, here and below $\vec{p}_m(t)=\vec{p}_m(0)+t$, $\vec{p}_m(0)$ being given by (5), and $p_m^2(t)=|\vec{p}_m(t)|^2$.

The goal of this paper is to construct the system of of model functions. Model functions solve the equations for eigenfunctions approximately and will be described explicitly. There exist model functions of two types: a weak diffraction type and a strong diffraction type.

It is shown that they are "almost orthogonal" (Theorem 1) and that the model set is complete in the high energy region – all eigenfunctions with eigenvalues large enough can be described in terms of the model functions (Theorem 2).

The plan of the paper is the following. First we consider the case when $\vec{A} = (a_0(x), a_1(x), a_2(x))$ is a trigonometric polynomial of the length R_0:

$$a_s(x) = \sum_{m \in Z^2, 0 < |m| \leq R_0} a_{sm} \exp i(\vec{p}_m(0), x), \quad s = 0, 1, 2. \tag{7}$$

For such a potential we construct a set of model functions (Section 2), which solve the equations for eigenfunctions approximately and satisfy the quasiperiodic conditions with a quasimomentum $t \in K$. We prove that the model functions form a complete system in $L_2(Q)$.

The model functions are not close to the plane waves $e^{i(p_j(t),x)}$. This makes the magnetic case different from the Schrödinger operator with only an electric potential ($\vec{a}(x) = 0, \alpha = 1$), where perturbed eigenfunctions are close to unperturbed ones for a rich set of quasimomenta t (see [K]). The new phenomena is due to the fact that the perturbation by a magnetic field is much stronger than the perturbation only by an electric field and, therefore, gives more complicated diffraction picture. To make the system of model functions complete we have to construct two types of model functions in the high energy region. The first type corresponds to a relatively small refraction inside the crystal (it only changes the phase of the wave without a significant change of its amplitude), the second type corresponds to a strong refraction, which changes the amplitude significantly. The construction of the model functions of the first type includes the construction of an approximate solution of the Eikonal equation by perturbative methods. To construct the model functions of the second type we use the directional representation of the potential, which already turned out to be useful for the Schrödinger operator with an electric potential (see i.e [EsRaTr1], [EsRaTr2], [BäKnTr], [KnTr],[K]). Each directional component of the potential generates the corresponding "branch" of the essential diffraction – it can be described mainly in terms of this component. Hence, the model functions of the second type are splited into several subclasses, each being related to only one directional component and being described essentially in terms of this component. The model functions of the second type correspond to a significant diffraction and can not be constructed by pure perturbative methods.

To prove that the model functions are close to real eigenfunctions of $H(t)$ it is necessary and sufficient to use the perturbation theory, however not in its elementary form, since the small denominator problem is present. The technic which we will use to overcome this problem is a development of the methods suggested for the Schrödinger operator with an electric potential (see [K]). We will do this in a following paper.

Let us introduce some notations. First, let f be a function $f : Q \to C$. We denote by $\|f\|_\nu$ the following norm:

$$\|f\|_\nu = |f_0| + \sum_{r \in Z^2 \setminus \{0\}} |f_r||r|^\nu,$$

f_r being the Fourier coefficients of $f(x)$. If $\vec{f}(x)$ is a vector function: $f : Q \to \mathbf{C^N}$, $\vec{f}(x) = (f_1(x), ..., f_N(x))$, then

$$\|\vec{f}\|_\nu = \sum_{s=1}^{N} \|f_s\|_\nu.$$

Obviously, $\|D^{\vec{\nu}} f\|_0 \leq \left(\frac{2\pi}{b_2}\right)^\nu \|f\|_\nu$, $\vec{\nu} = (\nu_1, \nu_2)$, $\nu = \nu_1 + \nu_2$.

We denote by $\|L\|_*$ the following norm of a matrix L:

$$\|L\|_* = \frac{1}{2} \sup_i \sum_j |L_{ij}| + \frac{1}{2} \sup_j \sum_i |L_{ij}|. \tag{8}$$

Next, let $|Q| = b_1 b_2$, $b_0 = \max\{2\pi/b_2, 1\}$, and let $\vec{\nu}_q$ be the unit vector in the direction of $\vec{p}_q(0)$: $\vec{\nu}_q = \vec{p}_q(0) p_q^{-1}(0)$ and $b_q = 2\pi/p_q(0)$. Last, $(\cdot, \cdot)$ is a scalar product in R^2, $< \cdot, \cdot >$ is a scalar product in $L_2(Q)$.

We denote by c an absolute constant independent of the parameters of H, by c_b a constant defined only by b_1, b_2, by $c_{b,\vec{A}}$ a constant defined only by $b_1, b_2, \vec{A}$, etc.

2 Model Functions.

2.1 Model Functions of the First Type.

Let $0 < \beta < 1/2$, $0 < 2\varepsilon < 1/2 - \beta$, k be a large parameter and $Z_0(k, \beta, \varepsilon)$ be the set of $j \in Z^2$ satisfying the conditions:

1. $$k/4 < p_j(t) < 7k/4. \tag{9}$$

2. For any $q \in Z^2 : 0 < |q| \leq k^\varepsilon$:

$$|(\vec{p}_j(t), \vec{p}_q(0))| > \hat{c} k^{1/2+\beta}, \tag{10}$$

here $\hat{c}$ is a technical constant, it will be defined later. If $j \in Z_0(k, \beta, \varepsilon)$, we define a model function Ψ_j by the formula

$$\Psi_j(x) = \frac{1}{|Q|^{1/2}} e^{i(\vec{p}_j(t), x) + S_j(x)}. \tag{11}$$

with a periodic function $S_j(x)$, which we define next. We want Ψ_j to satisfy the equation $H(t)\Psi_j = \lambda_j \Psi_j$ as precise as possible, λ_j being a constant. This consideration gives us an asymptotic equation to define a periodic function $S_j(x)$:

$$2i\,(\vec{p}_j(t), \nabla S_j(x)) + \Delta S_j(x) + (\nabla S_j(x), \nabla S_j(x)) + 2i\,(\nabla S_j(x), \vec{a}(x)) =$$

$$2\left(\vec{p}_j(t),\vec{a}(x)\right)+\left(\vec{a}(x),\vec{a}(x)\right)+\alpha a_0(x)+C_{j,t,N}+\epsilon_N(x), \tag{12}$$

$$\|\epsilon_N(x)\|_0<c_{\vec{A}}k^{1-2(N+2)\beta}, \tag{13}$$

$C_{j,t,N}$ being a constant with respect to x, N is a fixed natural number, larger N obviously corresponding to a better approximation.

To satisfy this equation we define $S_j(x)$ by the formula:

$$S_j(x)=\sum_{n=0}^{N+1}S_j^{(n)}(x), \tag{14}$$

where

$$S_j^{(0)}(x)=-\sum_{r\in Z^2\setminus\{0\}}\frac{2(\vec{p}_j(t),\vec{a}_r)+\alpha a_{0r}}{p_{j+r}^2(t)-p_j^2(t)}e^{i(\vec{p}_r(0),x)}, \tag{15}$$

$\vec{a}_r$, a_{0r} being the Fourier coefficients of $\vec{a}(x)$ and $a_0(x)$ correspondingly. Next,

$$S_j^{(1)}(x)=\sum_{r\in Z^2\setminus\{0\}}\frac{\tilde{a}_{jr}^{(1)}}{p_{j+r}^2(t)-p_j^2(t)}e^{i(\vec{p}_r(0),x)}, \tag{16}$$

where $\tilde{a}_{jr}^{(1)}$ are the Fourier coefficients of the function

$$\tilde{a}_j^{(1)}(x)=2i\left(\nabla S_j^{(0)}(x),\vec{a}(x)\right)+\left(\nabla S_j^{(0)}(x),\nabla S_j^{(0)}(x)\right)-\left(\vec{a}(x),\vec{a}(x)\right). \tag{17}$$

The functions $S_j^{(n)}$, $n\geq 2$ are defined by the recurrent formula:

$$S_j^{(n)}(x)=\sum_{r\in Z^2\setminus\{0\}}\frac{\tilde{a}_{jr}^{(n)}}{p_{j+r}^2(t)-p_j^2(t)}e^{i(\vec{p}_r(0),x)}, \tag{18}$$

where $a_{jr}^{(n)}$ are the Fourier coefficients of the function

$$\tilde{a}_j^{(n)}=2i\left(\nabla S_j^{(n-1)},\vec{a}\right)+2\left(\nabla S_j^{(n-1)},\sum_{s=0}^{n-2}\nabla S_j^{(s)}\right)$$
$$+(\nabla S_j^{(n-1)},\nabla S_j^{(n-1)}). \tag{19}$$

Lemma 1 *Suppose* $0<\beta<1/2$, $0<2\varepsilon<1/2-\beta$, $j\in Z_0(k,\beta,\varepsilon)$, *the parameter* α *in the definition of* H *satisfies the estimate* $|\alpha|\leq k$, $\vec{A}$ *is a trigonometric polynomial of the length* R_0 *(see (7)) and* $N\in\mathbf{N}$. *Then, the functions* $S_j^{(n)}(x)$, *n=0,1,...,N+1, and* $\tilde{a}_j^{(n)}(x)$, $n=1,...,N+1$, *have the following properties:*

$$S_{jr}^{(n)}=\tilde{a}_{jq}^{(n)}=0 \quad \textit{if}\ |r|>2^nR_0, \tag{20}$$

$$\left\|S_j^{(n)}\right\|_\nu\leq c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}k^{1/2-(1+2n)\beta}, \tag{21}$$

$$\left\|\Re S_j^{(n)}\right\|_\nu \le \tilde{c}_{n,\nu}\left\|\vec{A}\right\|_{\nu+n+2}^{n+1} k^{-2\beta(n+1)}, \tag{22}$$

$$\left\|\left(S_j^{(n)}-S_m^{(n)}\right)\right\|_\nu \le \hat{c}_{n,\nu}\left\|\vec{A}\right\|_{\nu+n+1}^{n+1}|m-j|k^{-2(n+1)\beta}, \tag{23}$$

$$\textit{if } m\in Z_0(k,\beta,\varepsilon),$$

$$\left\|\tilde{a}_j^{(n)}\right\|_\nu \le c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1} k^{1-2n\beta}, \tag{24}$$

$$\left\|\tilde{\Im} a_j^{(n)}\right\|_\nu \le \tilde{c}_{n,\nu}\left\|\vec{A}\right\|_{\nu+n+2}^{n+1} k^{1/2-(1+2n)\beta}, \tag{25}$$

$$\left\|\left(\tilde{a}_j^{(n)}-\tilde{a}_m^{(n)}\right)\right\|_\nu \le \hat{c}_{n,\nu}\left\|\vec{A}\right\|_{\nu+n+1}^{n+1}|m-j|k^{1/2-(2n+1)\beta}, \tag{26}$$

$$\textit{if } m\in Z_0(k,\beta,\varepsilon),$$

where $c_{n,\nu}=(5b_0)^{n(n+1+\nu)+1}$, $\tilde{c}_{n,\nu}=2c_{n,\nu}$, $\tilde{c}_{n,\nu}=4b_0c_{n,\nu}$ *The estimates (21) – (26) hold for any order* ν *of derivatives and* k *large enough, namely* $k: k^\varepsilon > 2^{N+2}R_0b_0^2$, $k^{2\beta} > (5b_0)^{N+2+\nu}\|\vec{A}\|_{\nu+N}R_0+b_0^2$.

Corollary 1

$$\|S_j\|_\nu \le c_\nu\left\|\vec{A}\right\|_\nu k^{1/2-\beta}, \tag{27}$$

$$\|\Re S_j\|_\nu \le c_\nu\left\|\vec{A}\right\|_{\nu+2} k^{-2\beta}, \tag{28}$$

$$\left\|e^{S_j}-1\right\|_0 < c\left\|\vec{A}\right\|_2 k^{-2\beta}, \tag{29}$$

$$\|(S_j-S_m)\|_\nu \le c_\nu\left\|\vec{A}\right\|_{\nu+1}|m-j|k^{-2\beta}. \tag{30}$$

Proof. By (7), $\vec{A}_r=0$ when $|r|>R_0$. From the definition of $S_j^{(0)}$ we get $S_{jr}^{(0)}=0$ when $|r|>R_0$. It follows now from (17) that $\tilde{a}_{jr}^{(1)}=0$ when $|r|>2R_0$. Using (18) we immediately obtain $S_{jr}^{(1)}=0$ when $|r|>2R_0$. Further, using (19) and (18) we obtain (20) for all n by mathematical induction.

Next we prove the pair of estimates (21), (24) by mathematical induction. Using (10) and the given inequality $2^{N+2}R_0<k^\varepsilon$, we get

$$\left|p_{j+r}^2(t)-p_j^2(t)\right| > k^{1/2+\beta} \quad \text{for all } r\in Z^2, 0<|r|\le 2^{N+2}R_0. \tag{31}$$

Using the definition (15) of $S_j^{(0)}(x)$, inequality (31) and the estimates $|\alpha|\le k$, we easily get (21) for $n=0$. Next, we get (24) for $\tilde{a}_j^{(1)}$. In fact, it is easy to see that for any $\vec{f},\vec{g}:Q\to R^2$

$$\left\|\left(\vec{f}(x),\vec{g}(x)\right)\right\|_\nu \le 2^\nu\|\vec{f}\|_\nu\|\vec{g}\|_\nu. \tag{32}$$

Applying (32) in (17) we obtain:

$$\|\tilde{a}_j^1(x)\|_\nu \le 2^\nu\left(2\|\nabla S_j^0(x)\|_\nu\|\vec{a}(x)\|_\nu+\|\nabla S_j^{(0)}(x)\|_\nu^2+\|\vec{a}(x)\|_\nu^2\right).$$

Using already proven estimate (21) for $S_j^{(0)}$, we get (24) for $\tilde{a}_j^1$. It follows from (18) and (31) that for all $\nu \geq 0$ and $n = 1, ..., N+1$,

$$\|S_j^{(n)}(x)\|_\nu \leq \|\tilde{a}_j^{(n)}(x)\|_\nu k^{-1/2-\beta}. \tag{33}$$

Formulae (19) and (32) yield:

$$\|\tilde{a}_j^{(n)}(x)\|_\nu \leq$$

$$2^\nu \left(\|\nabla S_j^{(n-1)}(x)\|_\nu \|\vec{a}(x)\|_\nu + 2\|\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S_j^{(s)}(x)\|_\nu \right)$$

for any $n \geq 2$. Using (33), we obtain

$$\|\tilde{a}_j^{(n)}(x)\|_\nu \leq 2^\nu b_0 k^{-1/2-\beta} \|\tilde{a}_j^{(n-1)}(x)\|_{\nu+1} \times$$

$$\left(\|\vec{a}(x)\|_\nu + 2\|\nabla S_j^{(0)}(x)\|_\nu + 2k^{-1/2-\beta} \sum_{s=1}^{n-1} \|\tilde{a}_j^{(s)}(x)\|_{\nu+1} \right).$$

Suppose (24) holds for $s = 1, ..., n-1$. Using (24) on the right-hand side of the last inequality and taking into account that $(5b_0)^{N+\nu+2} \|\vec{A}\|_{N+\nu} R_0 k^{-2\beta} < 1$ and inequality (21) for $n = 0$, we obtain

$$\|\vec{a}\|_\nu + 2\|\nabla S_j^{(0)}(x)\|_\nu + 2k^{-1/2-\beta} \sum_{s=1}^{n-1} \|\tilde{a}_j^{(s)}(x)\|_{\nu+1} \leq 15 b_0 k^{1/2-\beta} \|\vec{A}\|_{\nu+1}.$$

Thus, $\|\tilde{a}_j^{(n)}(x)\|_\nu < 15 \cdot 2^\nu b_0^2 k^{-2\beta} \|\vec{A}\|_{\nu+1} \|\tilde{a}_j^{(n-1)}(x)\|_{\nu+1}$, the second term giving the main contribution to the sum. Using (24) for $\tilde{a}_j^{(n-1)}(x)$, we get (24) for $\tilde{a}_j^{(n)}(x)$. Estimate (21) follows from (24) and (33).

Let us prove the pair (22), (25). By elementary calculation we obtain from (15):

$$\Re S_j^{(0)} = - \sum_{r \in Z^2 \setminus \{0\}} \frac{(2(\vec{p}_j(t), \vec{a}_r) + \alpha a_{0r})\, p_r^2(0)}{(p_{j+r}^2(t) - p_j^2(t))(p_{j-r}^2(t) - p_j^2(t))} e^{i(\vec{p}_r(0), x)}, \tag{34}$$

The estimate (22) for $n = 0$ easily follows from the last formula and (31). Using this estimate and the formula (17) for $\tilde{a}_j^{(1)}$, and taking into account (22) and (21) for $n = 0$, it is easy to show that $\|\Im \tilde{a}_j^{(1)}\|_\nu \leq 35 \cdot 2^\nu b_0^2 \|\vec{A}\|_{\nu+3} k^{1/2-3\beta}$, i.e., (25) holds for $n = 1$. Formula (18) for $S_j(x)$ yields

$$\|\Re S_j^{(n)}\|_\nu < \|\tilde{a}_j^{(n)}\|_{\nu+2} k^{-1-2\beta} + \|\Im \tilde{a}_j^{(n)}\|_\nu k^{-1/2-\beta}.$$

Using the estimate (24), we obtain:

$$\|\Re S_j^{(n)}\|_\nu < c_{n,\nu} \|A\|_{\nu+n+2}^{n+1} k^{-2\beta(n+1)} + \|\Im \tilde{a}_j^{(n)}\|_\nu k^{-1/2-\beta}. \tag{35}$$

Formulae (19) and (32) yield:

$$\|\Im \tilde{a}_j^{(n)}(x)\|_\nu \leq$$

$$2^\nu \left(\|\Re\nabla S_j^{(n-1)}(x)\|_\nu \|\vec{a}\|_\nu + 2\|\Re\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S_j^{(s)}(x)\|_\nu + \right. \tag{36}$$

$$\left. 2\|\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-2} \|\Re\nabla S_j^{(s)}(x)\|_\nu \right).$$

Suppose (22) holds for $n = s - 1$, and (25) for $n = s$, $s \geq 1$. Then, using (35), we get that (22) holds for n=s, and using (36), we get (25) for $n = s + 1$.

Let us prove (23) and (26). Using formula (15) for $S_j^{(0)}$ it is easy to check (23) when $n = 0$. Considering (17) we readily check (26) for $n = 1$. It is easy to show that

$$\|S_j^{(n)} - S_m^{(n)}\|_\nu < 2b_0|j - m|\|\tilde{a}_j^{(n)}\|_{\nu+1}k^{-1-2\beta} + \|\tilde{a}_j^{(n)} - \tilde{a}_j^{(m)}\|_\nu k^{-1/2-\beta}.$$

Using the estimate (24), we obtain:

$$\|S_j^{(n)} - S_m^{(n)}\|_\nu < 2b_0c_{n,\nu}|j - m|\|A\|_{\nu+n+1}^{n+1}k^{-2\beta(n+1)} + \|\tilde{a}_j^{(n)} - \tilde{a}_j^{(m)}\|_\nu k^{-1/2-\beta}. \tag{37}$$

Formulae (19) and (32) yield:

$$\|\tilde{a}_j^{(n)}(x) - \tilde{a}_m^{(n)}(x)\|_\nu \leq$$

$$2^\nu \left(\|\nabla S_j^{(n-1)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \|\vec{a}\|_\nu + \right.$$

$$2\|\nabla S_j^{(n-1)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S_j^{(s)}(x)\|_\nu +$$

$$\left. 2\|\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-2} \|\nabla S_j^{(s)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \right). \tag{38}$$

Suppose (23) holds for $n = s - 1$, and (26) for $n = s$, $s \geq 1$. Then, using (37), we get (23) holds for n=s, and using (38), we get (26) for $n = s + 1$. The lemma is proved.

Lemma 2 *Under the conditions of Lemma 1 the function $S_j(x)$ defined by (14) – (19) satisfies the relations (12), (13), where*

$$C_{j,t,N} = \sum_{n=1}^{N+1} \tilde{a}_{j0}^{(n)}. \tag{39}$$

Corollary 2

$$C_{j,t,N} < c_b\|\vec{A}\|_1^2 k^{1-2\beta}. \tag{40}$$

Estimate (40) follows from (39) and (24).

Corollary 3 *If $j, m \in Z_0(k,\beta,\varepsilon)$, then*

$$|C_{j,t,N} - C_{m,t,N}| < c_b \left\|\vec{A}\right\|_1^2 |m-j| k^{1/2-3\beta}. \tag{41}$$

This corollary follows from formulas (39) and (26).

Proof. Formula (15) and a direct calculation yield:

$$2i\left(\vec{p}_j(t), \nabla S_j^{(0)}\right) + \Delta S_j^{(0)} - 2(\vec{p}_j(t), \vec{a}) - \alpha a_0 = 0 \tag{42}$$

Using the definition of $S_j^{(1)}$ (see (16) and (17)) it is easy to check that

$$2i(\vec{p}_j(t), \nabla S_j^{(1)}) + \Delta S_j^{(1)} + 2i(\nabla S_j^{(0)}, \vec{a}) + \left(\nabla S_j^{(0)}, \nabla S_j^{(0)}\right) - (\vec{a}, \vec{a}) = \tilde{a}_{j0}^1, \tag{43}$$

we have got $\tilde{a}_{j0}^1$ because the term corresponding to $r = 0$ is omitted in formula (16). Formulae (18) and (19) easily lead to the equation for $S_j^{(n)}$:

$$2i(\vec{p}_j(t), \nabla S_j^{(n)}) + \Delta S_j^{(n)} + 2i(\nabla S_j^{(n-1)}, \vec{a})$$

$$+ \left(\sum_{s=0}^{n-1} \nabla S_j^{(s)}, \sum_{s=0}^{n-1} \nabla S_j^{(s)}\right) - \left(\sum_{s=0}^{n-2} \nabla S_j^{(s)}, \sum_{s=0}^{n-2} \nabla S_j^{(s)}\right) = \tilde{a}_{j0}^{(n)}. \tag{44}$$

Adding formulae (42) – (44) for $n = 0, 1, 2, ...N+1$ and using (21) for $n = 0, ..., N+1$ and $\nu = 1$, we get (12), (13) and (39). The lemma is proved.

Lemma 3 *Under the conditions of Lemma 1 the function $\Psi_j(t,x)$ satisfies the equation:*

$$H\Psi_j = \left(p_j^2(t) - C_{j,t,N}\right)\Psi_j + \delta\Psi_j, \tag{45}$$

$\delta\Psi_j$ obeying the estimate

$$\|\delta\Psi_j\|_0 \le c_{\vec{A}} k^{1-2(N+2)\beta}. \tag{46}$$

We prove this formula by direct substitution of (11) into equation and using (12). To get (46) from (13) we use (29).

2.2 Model Functions of the Second Type.

Let k be a large parameter and $q \in Z^2$, $|q| < k^\varepsilon$. We define a set $Z_q(k,\beta) \subset Z^2$ as the set of j satisfying the inequalities:

$$k/4 < p_j(t) < 7k/4, \tag{47}$$

$$|(\vec{p}_j(t), \vec{p}_q(0))| \le \hat{c} k^{1/2+\beta}, \tag{48}$$

where ε, β are parameters: $0 < \beta < 1/2$, $0 < 2\varepsilon < 1/2 - \beta$, and the constant $\hat{c}$ is technical. The range of the possible values of $\hat{c}$ depends on $q, \beta, \varepsilon, \vec{A}$ and will be specified later. Without loss of generality we assume further that q is the minimal representative in Z^2 of its direction, i.e, that any $m \in Z^2$ colinear with q can be represented as $m = nq$, $n \in Z$.

Proposition 1 *If* $0 < \beta < 1/2$, $0 < 2\varepsilon < 1/2 - \beta$ *and* $q, q' \in Z^2 : q \neq q', |q|, |q'| < k^{\varepsilon}$, *then*

$$Z_q(k,\beta) \cap Z_{q'}(k,\beta) = \emptyset. \tag{49}$$

Proof. Suppose $j \in Z_q(k,\beta)$ In our two-dimensional situation (47) and (48) yield:

$$|(\vec{p}_j(t), \vec{p}_{q'}(0))| > c_b k^{1-2\varepsilon}, \tag{50}$$

for any $j \in Z_q(k,\beta)$ and $q' : 0 < |q'| < k^{\varepsilon}$ which is not a scalar multiple of q. Therefore, $j \notin Z_{q'}(k,\beta)$. The proposition is proved.

We will construct model functions of the second type, corresponding to $j \in Z_q(k,\beta)$, $q : 0 < |q| < k^{\varepsilon}$.

2.2.1 New notations.

Suppose $f \in L_2(Q)$. Then $f(x)$ can be represented as a series:

$$f(x) = \sum_{m \in Z^2} f_m e^{i(p_m(0),x)}. \tag{51}$$

Let $q \in Z^2 \setminus \{0\}$ and q is the minimal representative in Z^2 of its direction. We denote by $f^{(q)}(x)$ the "component" of $f(x)$ in the direction $\vec{p}_q(0)$:

$$f^{(q)}(x) = \sum_{n \in Z} f_{nq} e^{in(\vec{p}_q(0),x)}. \tag{52}$$

Obviously, a function $f^{(q)}(x)$ depends only on $(p_q(0), x)$. Naturally, by $f^{(q)}(y)$, $y \in R$, we denote a periodic function on R with a period $b_q = 2\pi/p_q(0)$ such that $f^{(q)}(x) = f^{(q)}(y)|_{y=(\nu_q,x)}$, i.e.,

$$f^{(q)}(y) = \sum_{n \in Z} f_{nq} e^{inp_q(0)y}. \tag{53}$$

There is the one-to-one correspondence between $f^{(q)}(x)$ and $f^{(q)}(y)$ given by (52), (53). Further we will denote by $f^{(q)}$ both of this functions, depending on situation. Let

$$f^{(\text{non } q)}(x) = f(x) - f^{(q)}(x) = \sum_{m \in Z^2, m \neq nq, n \in Z} f_m e^{i(p_m(0),x)}. \tag{54}$$

If $f(x)$ is a trigonometric polynomial with "length" R_0, such that , $\int_Q f(x)dx = 0$, then, obviously, there is a finite set $\Gamma(R_0)$, $\Gamma(R_0) \subset \{q \in Z^2, 0 < |q| \leq R_0\}$, such that each vector of $\Gamma(R_0)$ is the minimal representative in Z^2 of its direction and

$$f(x) = \sum_{q \in \Gamma(R_0)} f^{(q)}(x). \tag{55}$$

In particular, $\vec{a}(x) = \sum_{q \in \Gamma(R_0)} \vec{a}^{(q)}(x)$. Further, we denote by $a^{(q)\|}(x)$ the component of $\vec{a}^{(q)}(x)$ in the direction of $\vec{p}_q(0)$: $a^{(q)\|}(x) = \left(\vec{a}^{(q)}(x), \nu_q\right)$, and by $a^{(q\perp)}(x)$ we denote the component of $\vec{a}^{(q)}(x)$ in the direction orthogonal to $\vec{p}_q(0)$.

Let $j \in Z_q(k,\beta)$. We decompose $\vec{p}_j(t)$ into two components, in the direction of $\vec{p}_q(0)$ and the orthogonal one: $\vec{p}_j(t) = T_j(t)\vec{p}_q(0) + \vec{d}_j^{(q)}(t)$, here $T_j(t) \in R$ and $\vec{d}_j^{(q)}(t)$ is orthogonal to $\vec{p}_q(0)$. Further, we write $T_j(t)$ in the form $T_j(t) = \tau_j + l_j$, $0 \le \tau_j < 1$, $l_j \in Z$, $\tau_j = \tau_j(q,t)$, $l_j = l_j(q,t)$. Thus,

$$\vec{p}_j(t) = (\tau_j + l_j)\vec{p}_q(0) + \vec{d}_j^{(q)}(t). \tag{56}$$

Obviously, there is a one-to-one correspondence between $\{j,t\}$ and $\{\tau_j, l_j, \vec{d}_j^{(q)}\}$ for a fixed q. We will often use the above representation for $\vec{p}_j(t), j \in Z_q(k,\beta)$.

Proposition 2 *Let $m, j \in Z^2$ and $\vec{d}_m^{(q)}(t) = \vec{d}_j^{(q)}(t)$, then $\tau_m = \tau_j$.*

Proof. If $\vec{d}_m^{(q)}(t) = \vec{d}_j^{(q)}(t)$, then obviously, $\vec{p}_m(t) - \vec{p}_j(t) = (\tau_m + l_m - \tau_j - l_j)\vec{p}_q(0)$, i.e., the vector $\vec{p}_{m-j}(0) = \vec{p}_m(t) - \vec{p}_j(t)$ is colinear with $\vec{p}_q(0)$. Note that $\vec{p}_q(0)$ is a minimal representative of the family of vectors $\vec{p}_l(0)$ colinear with $\vec{p}_q(0)$. Hence, $\tau_m + l_m - \tau_j - l_j \in Z$. Taking into account that $0 \le \tau_m, \tau_j < 1$, we obtain $\tau_m = \tau_j$. The proposition is proved.

2.2.2 Preliminary considerations.

If j satisfies inequality (48), then it may be not possible to solve the equation $H(t)\Psi_j = \lambda_j \Psi_j$ using the perturbative approach (11), (14) – (19), since for $j \in Z_q(k,\beta)$ the relation (31) does not hold, and, therefore, the function $S_j(x)$ can become arbitrary large. To find a way to construct the second type model functions, let us start with some observations about the first type functions. We see that for any given $q \in \Gamma(R_0)$ a first type function $\Psi_j(x)$ (see (11)) can be represented in the form:

$$\Psi_j(x) = f_{l_j}(\vec{d}_j(t), \tau_j, x) e^{(\vec{d}_j(t),x) + S_j^{(\text{non } q)}(x)}, \tag{57}$$

where

$$f_{l_j}(\vec{d}_j(t), \tau_j, x) = e^{i(l_j+\tau_j)(\vec{p}_q(0),x) + S_j^{(q)}(x)}, \tag{58}$$

(56) has been used. We have chosen the notation $f_{l_j}(\vec{d}_j(t), \tau_j, x)$ instead of a shorter one $f_j(t,x)$ in order to stress the special role of index l_j. When only dependence on l_j is important we will shorten the notation to $f_{l_j}(x)$. Since $S_j^{(q)}(x)$ as a function of x depends only on $(\vec{p}_q(0), x)$, the function $f_{l_j}(\vec{d}_j(t), \tau_j, x)$ depends only on $(\vec{p}_q(0), x)$ too, and, hence, there is a function $f_{l_j}(\vec{d}_j(t), \tau_j, y)$ on the real axis such that

$$f_{l_j}(\vec{d}_j(t), \tau_j, x) = f_{l_j}(\vec{d}_j(t), \tau_j, y)\Big|_{y=(\nu_q,x)}. \tag{59}$$

Since $S_j^{(q)}(y)$ is periodic with the period $b_q = 2\pi p_q^{-1}(0)$, the function $f_{l_j}(\vec{d}_j(t), \tau_j, y)$ is quasiperiodic with the quasimomentum $2\pi\tau_j$, i.e.,

$$f_{l_j}(\vec{d}_j(t), \tau_j, b_q) = \exp(i2\pi\tau_j) f_{l_j}(\vec{d}_j(t), \tau_j, 0),$$

$$f'_{l_j}(\vec{d_j}(t), \tau_j, b_q) = \exp(i2\pi\tau_j) f'_{l_j}(\vec{d_j}(t), \tau_j, 0). \tag{60}$$

In Appendix 3 it is shown that function $f_{l_j}(y)$ satisfies the differential equation:

$$\left(\frac{1}{i}\frac{\partial}{\partial y} + a^{(q\|)}(y)\right)^2 f_{l_j}(y) + w^{(q,\vec{d}_j^{(q)}(t))}(y) f_{l_j}(y) = (\mu_{l_j} + \zeta_j(y)) f_{l_j}, \tag{61}$$

where $a^{(q\|)}(y)$ and $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ are periodic potentials, $a^{(q\|)}(y)$ has been defined above, $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ is an "effective" potential in the direction of $\vec{p}_q(0)$, $\mu_{l_j} = \mu_{l_j}(\tau_j, \vec{d}_j^{(q)}(t))$ is a constant with respect to y, a function $\zeta_j(y)$ admits the estimate: $\|\zeta_j\|_0 = o(1)$. The effective potential $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ depends on $\vec{A}$ and $\vec{d}_j^{(q)}(t)$, but does not depend on τ_j, l_j (see formula (74)). Formulae (61) and (60) mean that up to the term ζ_j the function $f_{l_j}(y)$ satisfies an equation an eigenfunction of the operator $H^{(q,\vec{d}_j^{(q)}(t))}$ in $L_2(0, b_q)$, $H^{(q,\vec{d}_j^{(q)}(t))}$ being described by the differential expression on the left-hand side of (61) and by the quasiperiodic conditions (60). Without the loss of generality [**?**] we can consider μ_{l_j} to be eigenvalues of $H^{(q,\vec{d}_j^{(q)}(t))}$. It is remarkable that the differential expression on the left-hand side of (61) depends only on $\vec{d}_j^{(q)}(t)$, but not on τ_j, l_j. This means that functions $f_{l_j}(\vec{d_j}(t), \tau_j, y)$, $f_{l_m}(\vec{d_j}(t), \tau_j, y)$ with different l_j, l_m satisfy the equations (61) with the same differential expression on the left-hand side and the same quasiperiodic conditions.

Let $\phi_{l_j}(\vec{d_j}(t), \tau_j, y)$ be an eigenfunction of $H^{(q,\vec{d}_j^{(q)}(t))}(y)$, corresponding to μ_{l_j}, i.e., it satisfies (61) with $\zeta_j(y) = 0$. Let us introduce function $\tilde{\Psi}_j(x)$ by replacing f_{l_j} by ϕ_{l_j} in (57):

$$\tilde{\Psi}_j(x) = \psi_j^{(q)}(x) e^{(\vec{d_j}(t),x) + S_j^{(\text{non } q)}(x)}, \tag{62}$$

$$\psi_j^{(q)}(x) = \phi_{l_j}\left(\vec{d}_j^{(q)}(t), \tau_j . y\right)\Big|_{y=(\nu_q, x)}, \tag{63}$$

We observe that $\tilde{\Psi}_j(x)$ satisfies (45), with $\|\delta\Psi_j\|_0 = o(k)$. Comparing this approximate solution of (45) with the plane wave $\exp i(\vec{p_j}(t), x)$, we see that it gives a better approximation then the plane wave, which generates an error of order $O(k)$. Therefore $\tilde{\Psi}_j(x)$ can be used as a model function for $j \in Z_0(k, \beta, \varepsilon)$, at least as the first approximation of the model function. We are not going to use (62) for $j \in Z_0(k, \beta, \varepsilon)$, because formula (11) is more simple and precise. The use of formula (62) is that it gives us a hint how to construct a model function in the case when $j \in Z_q(k, \beta)$. As the first step, we define model functions of the second type by formula (62).

Note that the formula (62) is not the most convenient one to use. Inconvenience is that $S_j^{(\text{non } q)}(x)$ depends on $\vec{d}_j^{(q)}(t), \tau_j, l_j, x$, while it would be much easier technically to handle a function which does not depend on l_j, τ_j. Such function $\tilde{S}_j(x)$ exists and will be presented later. It depends only on $\vec{d}_j^{(q)}(t), x$ and $\tilde{S}_j(x) \approx S_j^{(\text{non } q)}(x)$ for large $|j|$. Thus, we define the second type functions by the formula

$$\Psi_j(x) = \psi_j(t, x) e^{i(\vec{d}_j^{(q)}, x) + \tilde{S}_j(x)}. \tag{64}$$

This is the final definition. Further in this section, we will show that the functions Ψ_j of the second type are almost orthogonal to each other and satisfy the equation

$$(H(t)\Psi_j)(x) = \lambda_j \Psi_j(x) + \delta\Psi_j(x), \tag{65}$$

where $\lambda_j \approx p_j^2(t)$ and $\delta\Psi_j$ satisfies the estimate

$$\|\delta\Psi_j(x)\|_0 < c_{\vec{A}} k^{1/2+\beta}. \tag{66}$$

From the first sight the relation (65) does not look good, because $\delta\Psi_j$ is not small. However, (65) can be improved. We will show that $\delta\Psi_j(x)$ admits the following expansion:

$$\delta\Psi_j(x) = \sum_{m:|m-j|<k^{\varepsilon}} B_{jm}\Psi_m + O(k^{-k^{\varepsilon-\beta}}).$$

The matrix B_{mj} has the following properties:

$$\|B\|_* < c_{\vec{A}} k^{1/2+\beta}, \tag{67}$$

$$B_{jm} = 0, \text{ if } m - j = nq, \ n \in Z. \tag{68}$$

The property (68) enable us to improve the approximation (65) in one simple step. Indeed, let us consider the function:

$$\hat{\Psi}_j(x) = \Psi_j(x) + \sum_{m:|m-j|<k^{\varepsilon}} \frac{B_{jm}}{\lambda_j(t) - \lambda_m(t)} \Psi_m(x). \tag{69}$$

Taking into account that $|p_j^2(t) - p_m^2(t)| > c_b k^{1-2\varepsilon}$ when $j - m \neq nq, n \in Z$ (see (48)), we will prove that $|\lambda_j(t) - \lambda_m(t)| > \tilde{c}_b k^{1-2\varepsilon}$, and, hence, it can be easily seen that the function $\hat{\Psi}_j(x)$ gives a better approximation:

$$\|H(t)\hat{\Psi}_j(x) - \lambda_j \hat{\Psi}_j(x)\|_{L_2(Q)} = O(k^{-2\varepsilon+2\beta}),$$

the second term in (69) being small:

$$\|\hat{\Psi}_j(x) - \Psi_j(x)\|_{L_2(Q)} = O(k^{-1/2+2\varepsilon+\beta}).$$

Thus $\hat{\Psi}_j(x)$ gives already a good approximation, however it is possible to proceed even further.

2.2.3 Construction of the model functions.

For $j \in Z_q(k,\beta)$ we define a model function Ψ_j by the formula

$$\Psi_j(x) = \psi_j^{(q)}(x) e^{i(\vec{d}_j^{(q)},x) + \tilde{S}_j(x)}, \tag{70}$$

$\psi_j^{(q)}(t,x)$, $\tilde{S}_j(x)$ being defined as follows. Function $\tilde{S}_j(x)$ is given by the formula:

$$\tilde{S}_j(x) = -\sum_{r\in Z^2\setminus\{0\}, r\neq nq, n\in Z} \frac{2(\vec{d}_j^{(q)}, \vec{a}_r) + \alpha a_{0r}}{2(\vec{d}_j^{(q)}(t), \vec{p}_r(0))} e^{i(\vec{p}_r(0),x)}. \tag{71}$$

It is shown in Appendix 1 that $\tilde{S}_j(x) \approx S_j^{(0,\text{non }q)}$. We could have taken $S_j^{(0,\text{non }q)}$ instead of $\tilde{S}_j(x)$ - it would give eventually the same result, however $\tilde{S}_j(x)$ has an important technical advantage: $\tilde{S}_j(x) = \tilde{S}_m(x)$ when $j - m = nq$, $n \in Z$.

We define the functions $\psi_j^{(q)}(t,x)$ by the formula (63), where τ_j, l_j are defined by (56), $\phi_{l_j}\left(\vec{d}_j^{(q)}(t), \tau_j, y\right)$ is the l_j-th eigenfunction of the operator $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ in $L_2(0,b_q)$, the operator $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ being defined by the differential expression

$$H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)u(y) = \left(\frac{1}{i}\frac{\partial}{\partial y} + a^{(q\|)}(y)\right)^2 u(y) + w^{(q,\vec{d}_j^{(q)}(t))}(y)u(y) \tag{72}$$

and the quasiperiodic boundary conditions:

$$u(b_q) = \exp(i2\pi\tau_j)u(0), \quad u'(b_q) = \exp(i2\pi\tau_j)u'(0). \tag{73}$$

In formula (72) the periodic potential $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ is defined as follows:

$$w^{(q,\vec{d}_j^{(q)}(t))}(y) = 2|\vec{d}_j^{(q)}(t)|a_q^{(q\perp)}(y) + \alpha a_0^{(q)}(y) + (a,a)^{(q)}(y) -$$

$$2i(\nabla\tilde{S}_j, \vec{a})^{(q)}(y) - (\nabla\tilde{S}_j, \nabla\tilde{S}_j)^{(q)}(y). \tag{74}$$

It is shown in Appendix 4 that $w^{(q,\vec{d}_j^{(q)}(t))} = 2|\vec{d}_j^{(q)}(t)|a_q^{(q\perp)}(y) + \alpha a_0^{(q)} + O(k^{4\varepsilon})$. Taking into account that $|\vec{d}_j^{(q)}(t)| \approx k$ and $|\alpha| \le k$, we see that the main part $2|\vec{d}_j^{(q)}(t)|a_q^{(q\perp)}(y) + \alpha a_0^{(q)}$ of the effective potential is of order k and depends only on $\vec{A}^{(q)}$. The additional part of order $O(k^{4\varepsilon})$ depends on $\vec{A}^{\text{non}(q)}$ too. The additional part is much smaller, but nevertheless gives essential contribution, which plays particularly important role for smaller eigenvalues.

The operator $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ depends only on $q, \vec{d}_j^{(q)}(t), \tau_j$. This means that if $\vec{p}_j(t) - \vec{p}_m(t) = n\vec{p}_q(0)$, then according to (56) $\tau_j = \tau_m$ and $\vec{d}_j^{(q)}(t) = \vec{d}_m^{(q)}(t)$, and, therefore, $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j) = H_1^{q,\vec{d}_m^{(q)}(t)}(\tau_m)$.

If $w^{(q,\vec{d}_j^{(q)}(t))}(y) = 0$, then the eigenvalues and eigenfunctions of $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ are naturally parameterized by index $l \in Z$ and given by the formulae:

$$\mu_l = (\tau + l)^2 p_q^2(0), \; l \in Z,$$

$$\phi_l(\tau, y) = \exp\left(i(\tau + l)p_q(0)y - i\int_0^y a^{(q\|)}(t)dt\right).$$

If $w^{(q,\vec{d}_j^{(q)}(t))}(y) \ne 0$ we enumerate eigenvalues μ_l and the corresponding eigenfunctions ϕ_l in such a way that μ_l depend continuously on $w^{(q,\vec{d}_j^{(q)}(t))}(y)$. We consider the functions ϕ_l to be normalized, i.e., $b_q^{-1}\int_0^{b_q} |\phi_l(y)|^2 dy = 1$.

Considering that $\phi_{l_j}(\vec{d}_j^{(q)}, \tau_j, y)$ is an eigenfunction of $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ with $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ being a trigonometric polynomial of the length $2R_0 < 2k^{\varepsilon}$, we easily obtain

$$\|D^{\nu}\psi_j^{(q)}(t,x)\|_{L_2(Q)} < c^{\nu}_{\|\vec{A}\|_0} k^{(1/+\beta)\nu} \tag{75}$$

if $\nu < k^{1/2+\beta-\varepsilon}$.

Lemma 4 *Let* $0 < 3\varepsilon < 1/2 - \beta$, $|q| < k^{\varepsilon}$, $j \in Z_q(k,\beta)$, $R_0 < k^{\varepsilon}$. *Then,*

$$|2(\vec{d}_j^{(q)}(t), \vec{p}_r(0))|^2 > c_b k^{1-2\varepsilon} \ \ \textit{for any } |r| < k^{\varepsilon},\ r \neq nq,\ n \in Z, \tag{76}$$

and

$$\|\tilde{S}_j\|_{\nu} < c_b \|\vec{A}\|_{\nu} k^{2\varepsilon}, \tag{77}$$

$$\|D^{\vec{\nu}}\left(\tilde{S}_j - \tilde{S}_m\right)\|_0 < c_b\|\vec{A}\|_{\nu+1}|\vec{d}_j^{(q)}(t) - \vec{d}_m^{(q)}(t)|k^{-1+4\varepsilon},\ \textit{if } m \in Z_q(k,\beta), \tag{78}$$

$$\Re\tilde{S}_j = 0. \tag{79}$$

<u>Proof.</u> Clearly, $(\vec{d}_j^{(q)}(t), \vec{p}_r(0)) = (\vec{p}_j(t), \vec{p}_r(0)) - (\vec{p}_j(t), \vec{\nu}_q)(\vec{p}_r(0), \vec{\nu}_q)$. Taking into account (48), (50) we get $|(\vec{d}_j^{(q)}(t), \vec{p}_r(0))| > c_b k^{1-2\varepsilon}$ for any $r \neq nq$, $n \in Z$. Considering formula (71) for $\tilde{S}_j$, we obtain (77). We get (78) in a similar way. Considering that $(\vec{d}_j^{(q)}(t), \vec{a}(x))$ and $\alpha a_0(x)$ are real-valued functions we obtain $\Re\tilde{S}_j(x) = 0$ <u>The lemma is proved.</u>

Let us prove that function Ψ_j described above satisfies (65) – (68).

Lemma 5 *Under the conditions of Lemma 4 the function* $\Psi_j(t,x)$, $j \in Z_q(k,\beta)$, *satisfies the equation*

$$H\Psi_j = \left(|(\vec{d}_j^{(q)}|^2 + \mu_{l_j}(\tau_j)\right)\Psi_j +$$

$$B_{1j}(x)\Psi_j(x) + i\left(\nabla\psi_j^{(q)}(x), \vec{B}_{2j}(x)\right)e^{i(\vec{d}_j^{(q)},x)+\tilde{S}_j(x)} \tag{80}$$

where

$$B_{1j}(x) = -(\nabla\tilde{S}_j(x), \nabla\tilde{S}_j(x) + 2i\vec{a})^{(non\ q)}(x) + (\vec{a}(x), \vec{a}(x))^{(non\ q)} \tag{81}$$

$$\vec{B}_{2j}(x) = -2i\left(\nabla\tilde{S}_j(x) + i\vec{a}^{(non\ q)}\right), \tag{82}$$

the functions $\vec{B}_1(x)$, $B_2(x)$ *having the properties:*

$$(\vec{B}_{1j})_r = (B_{2j})_r = 0, \quad \textit{if } r = nq,\ n \in Z, \tag{83}$$

$(\vec{B}_1)_r$, B_{2r} $r \in Z^2$ *being Fourier coefficients,*

$$\|\vec{B}_1(x)\|_0 < c\|\vec{A}\|_1^2 k^{4\varepsilon}, \tag{84}$$

$$\|\vec{B}_2(x)\|_0 < c\|\vec{A}\|_1 k^{2\varepsilon}. \tag{85}$$

Corollary 4 *The function* $\Psi_j(t,x)$ *satisfies the equation*

$$H\Psi_j = \left(|(\vec{d}_j^{(q)}|^2 + \mu_{l_j}(\tau_j)\right)\Psi_j + \delta\Psi_j,$$

where the function $\delta\Psi_j$ *obeys the estimate:*

$$\|\delta\Psi_j(x)\|_0 < c\|\vec{A}\|_1 k^{1/2+\beta+2\varepsilon}. \tag{86}$$

The corollary follows from formula (80) and estimates (84), (85) and (75).

The proof of the lemma can be found in Appendix 2.

2.3 Properties of the set of model functions.

Let us define the set of model functions as follows:

$$\Psi_j(x) = \frac{1}{|Q|^{1/2}} e^{i(\vec{p}_j(t),x)+S_j(x)}, \quad \text{if } j \in Z_0(k,\beta,\varepsilon), \tag{87}$$

$$\Psi_j(x) = \frac{1}{|Q|^{1/2}} \psi_j^{(q)}(x) e^{i(\vec{d}_j^{(q)},x)+\tilde{S}_j(x)}, \text{ if } j \in Z_q(k,\beta). \tag{88}$$

It is easy to see that

$$Z_0 \cap Z_q = \emptyset,\ Z_q \cap Z_{q'} = \emptyset \text{ if } q \neq q',\ q,q' \in \Gamma(k^\varepsilon) \tag{89}$$

$$Z_0 \cup \cup_{q\in\Gamma(k^\varepsilon)} Z_q = J$$

$$J = \{j : k/4 \leq |\vec{p}_j(t)| \leq 7k/4\},$$

given $0 < 3\varepsilon < 1/2 - \beta$. Thus, we have defined model functions for all j in J.

Let G be the Gramm matrix of the set $\Psi_j(x)$, $j \in J$, and I_J be the identity matrix for $j \in J$:

$$G_{jm} =< \Psi_m, \Psi_j >, \quad (I_J)_{jm} = \delta_{jm},\ j,m \in J.$$

Theorem 1 *Let* $0 < \beta < 1/2$, $0 < 7\varepsilon < \min\{\frac{1/2-\beta}{4}, 14\beta\}$ *. Then,* G *admits the representation* $G = I_J + \delta G$, *where* δG *obeys the estimate:*

$$\|\delta G\|_* =_{k\to\infty} O(k^{-\gamma_0}), \quad \gamma_0 = \min\{2\beta, 1/2 - \beta - 7\varepsilon\}. \tag{90}$$

Remark . Obviously, $0 < \gamma_0 < 1/3$.

First, let us prove a lemma.

Lemma 6 *Any model function Ψ_j, $j \in J$, can be represented in the form:*

$$\Psi_j = F_j(x)e^{i(\vec{p}_j(t),x)}, \tag{91}$$

where

$$\|D_\nu F_j\|_0 < \left(c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2-\beta}\right)^\nu, \ \text{if} \quad j \in Z_0(k,\beta,\varepsilon) \text{ and } \nu < k^{1/2-\beta-\varepsilon}, \tag{92}$$

$$\|D_\nu F_j\|_{L_2(Q)} < \left(c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2+\beta}\right)^\nu, \ \text{if} \quad j \in Z_q(k,\beta) \text{ and } \nu < k^{1/2+\beta-\varepsilon}. \tag{93}$$

Proof. If $j \in Z_0(k,\beta)$, then $F_j(x) = |Q|^{-1/2}e^{S_j(x)}$ by formula (11). Considering (27), (29) and taking into account that $S_j(x)$ is a trigonometric polynomial of the length $NR_0 < k^\varepsilon$, we obtain by the method of mathematical induction:

$$\|D^{\vec{\nu}}e^{S_j(x)}\| < c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2-\beta}\left(\nu R_0 N + c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2-\beta}\right)^{\nu-1}. \tag{94}$$

Choosing $\nu < k^{1/2-\beta-\varepsilon}$, we obtain $\nu R_0 N < k^{1/2-\beta}$. Inequality (92) easily follows from (94) and the last inequality.

If $j \in Z_q(k,\beta)$, then it is easy to show that $F_j(x) = \psi_j^{(q)}(x)e^{-i(p_q(0),x)(l_j+\tau_j)+\tilde{S}_j}$, l_j, τ_j being defined by (56). Using the inequality $|l_j+\tau_j|p_q(0)^2 < \hat{c}k^{1/2+\beta}$, which is equivalent to the assumption $j \in Z_q(k,\beta)$, and formulae (75), (77), (79), we obtain (Appendix 12):

$$\|D_\nu F_j\|_{L_2(Q)} < \left(\nu R_0 N + c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2+\beta}\right)^\nu \tag{95}$$

for $\nu = 1, 2, \ldots$. Considering $\nu < k^{1/2+\beta-\varepsilon}$, we easily obtain (93) from (95). The lemma is proved.

The proof of Theorem 1. Suppose $|\vec{p}_j(t) - \vec{p}_m(t)| > k^{1/2+\beta+\varepsilon}$. Then

$$< \Psi_m, \Psi_j >= \int_Q F_m(x)\bar{F}_j(x)e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)}dx. \tag{96}$$

Using Lemma 6, we obtain

$$\|D^\nu(F_m\bar{F}_j)\|_{L_1(Q)} < (c_{\|A\|_1,\beta,\varepsilon}k^{1/2+\beta})^\nu$$

for all $\nu < k^{1/2-\beta-\varepsilon}$. Integrating by parts ν times in the direction of $\vec{p}_m(t) - \vec{p}_j(t)$ yields:

$$| < \Psi_m, \Psi_j > | < \left(c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2+\beta}|\vec{p}_m(t) - \vec{p}_j(t)|^{-1}\right)^\nu \tag{97}$$

for any $\nu < k^{1/2-\beta-\varepsilon}$. Considering that $|\vec{p}_j(t) - \vec{p}_m(t)| > k^{1/2+\beta+\varepsilon}$ we arrive at the inequality:

$$| < \Psi_m, \Psi_j > | < c^\nu_{\|\vec{A}\|_1,\beta,\varepsilon}k^{-\varepsilon(\nu-4)+2+4\beta}|\vec{p}_m(t) - \vec{p}_j(t)|^{-4}. \tag{98}$$

Choosing ν large enough, say $\nu > 10/\varepsilon$, we obtain:

$$\sigma_1 = \sup_j \sum_{m:|\vec{p}_m(t)-\vec{p}_j(t)|>k^{1/2+\beta+\varepsilon}} | < \Psi_m, \Psi_j > | < c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{-2}. \tag{99}$$

Next, we consider the case:

$$|\vec{p}_m(t)-\vec{p}_j(t)|\leq k^{1/2+\beta+\varepsilon}. \tag{100}$$

We consider two types of functions: (87) and (88).

1. Suppose $j,m\in Z_0(k,\beta,\varepsilon)$, i.e., both Ψ_m,Ψ_j are given by (87). Then,

$$<\Psi_m,\Psi_j>=\frac{1}{|Q|}\int_Q \exp\left(i(\vec{p}_m(t)-\vec{p}_j(t),x)+\left(S_m+\bar{S}_j\right)(x)\right)dx. \tag{101}$$

For $m=j$, obviously,

$$<\Psi_j,\Psi_j>=\frac{1}{|Q|}\int_Q e^{2\Re S_j(x)}dx=1+O_2\left(k^{-2\beta}\right), \tag{102}$$

(28) has been used. Suppose $m\neq j$. Let us consider $S_m(x)+\bar{S}_j(x)$. Using formulae (28) and (30) we obtain:

$$\left\|D^{\vec{\nu}}\left(S_m(x)+\bar{S}_j(x)\right)\right\|_0<c_{\nu,N}\|\vec{A}\|_{\nu+2}|m-j|k^{-2\beta},\ \ m\neq j. \tag{103}$$

We represent the integral (101) in the form $\int_Q vDudx$, where u is just the integrand itself, D is the derivative in the direction of $\vec{p}_m(t)-\vec{p}_j(t)$ and

$$v=\frac{1}{|\vec{p}_m(t)-\vec{p}_j(t)|+D\left(S_m+\bar{S}_j\right)(x)}.$$

Integrating (101) by parts and considering (103), we arrive at inequality:

$$<\Psi_m,\Psi_j>< \ c_{\|A\|_4}k^{-2\beta}|m-j|^{-1}. \tag{104}$$

Repeating integration by parts 4 times yields:

$$<\Psi_m,\Psi_j>< \ c_{\|A\|_7}k^{-2\beta}|j-m|^{-4}, \tag{105}$$

if both Ψ_m,Ψ_j are given by (87), and $m\neq j$.

Therefore, for Ψ_m,Ψ_j given by (87) and satisfying (100):

$$\sigma_2=\sup_j\sum_{m\neq j}|<\Psi_m,\Psi_j>|<c_{\|A\|_7}k^{-2\beta}. \tag{106}$$

2. Suppose both Ψ_m,Ψ_j are given by (88). This means that $j\in Z_q(k,\beta)$, $m\in Z_q(k,\beta)$ with the same $q:|q|<k^{\varepsilon}$, because otherwise we have the relation $|\vec{p}_m(t)-\vec{p}_j(t)|>c_bk^{1-2\varepsilon}$, contradicting to (100). Clearly,

$$<\Psi_m,\Psi_j>=$$

$$\frac{1}{|Q|}\int_Q \psi_m^{(q)}(x)\bar{\psi}_j^{(q)}(x)\exp\left(i(\vec{d}_m^{(q)}(t)-\vec{d}_j^{(q)}(t),x)+\left(\tilde{S}_m+\bar{\tilde{S}}_j\right)(x)\right)dx. \tag{107}$$

Considering (78) and (79), we obtain:

$$\left\|D^{\vec{\nu}}\left(\tilde{S}_m+\bar{\tilde{S}}_j\right)\right\|_0 < c\|\vec{A}\|_{\nu+1}|\vec{d}_m(t)) - \vec{d}_j(t)|k^{-1+4\varepsilon}. \qquad (108)$$

Suppose $|\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)| > k^{\varepsilon}$. We represent the integral (107) in the form $\int_Q \tilde{v}\tilde{D}\tilde{u}dx$, where $\tilde{u}$ is just the integrand itself, $\tilde{D}$ is the derivative in the direction of $\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)$ and

$$v = \frac{1}{\left|\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)\right| + \tilde{D}\left(\tilde{S}_m+\bar{\tilde{S}}_j\right)(x)}.$$

calculatibg $\tilde{v}$ we have taken into account that $\tilde{D}\left(\psi^{(q)}_m\bar{\psi}^{(q)}_j\right)(x) = 0$, since $\psi^{(q)}_m(x)$ and $\bar{\psi}^{(q)}_j(x)$ depend only on $(\vec{p}_q(0), x)$. Integrating (107) by parts 4 times and considering (108), we arrive at inequality:

$$| < \Psi_m, \Psi_j > | < c_{\|\vec{A}\|_7,\beta,\varepsilon}k^{-1+4\varepsilon}|\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)|^{-4}. \qquad (109)$$

We want to estimate $\sum_{m\in Z_q:|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|>k^{2\varepsilon}} | < \Psi_m, \Psi_j > |$. Since $\vec{p}_m(t)$ can be represented in the form: $\vec{p}_m(t) = \vec{d}^{(q)}_m(t) + (\tau_m + l_m)p_q(0)$, we first summarize terms corresponding to the same $\vec{d}^{(q)}_m(t)$ and and then summarize the results over all $\vec{d}^{(q)}_m(t) : |\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)| > k^{2\varepsilon}$:

$$\sum_{m\in Z_q:|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|>k^{2\varepsilon}} | < \Psi_m, \Psi_j > | = \sum_{\vec{d}^{(q)}_m(t):|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|>k^{2\varepsilon}} \sum_{l_m} | < \Psi_m, \Psi_j > |.$$

Note that the sum over l_m contains no more than $c_b k^{1/2+\beta}$ terms, since $m \in Z_q(k,\beta)$. Using (109), we obtain:

$$\sum_{l_m} | < \Psi_m, \Psi_j > | \ < \ c_{\|\vec{A}\|_7,\beta,\varepsilon}k^{-1/2+\beta+4\varepsilon}|\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)|^{-4}.$$

Taking into account that points $\vec{d}^{(q)}_m(t)$ differ from lattice points only by a vector $\tau_m\vec{p}_q(0)$, $|\tau_m\vec{p}_q(0)| < c_b k^{\varepsilon}$, we obtain the estimate for the sum:

$$\sup_{j\in Z_q} \sum_{m\in Z_q:|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|>k^{2\varepsilon}} | < \Psi_m, \Psi_j > | < C_{\|\vec{A}\|_6}k^{-1/2+\beta+4\varepsilon}. \qquad (110)$$

Let $|\vec{d}^{(q)}_m(t) - \vec{d}^{(q)}_j(t)| \leq k^{2\varepsilon}$. From (108) we obtain that Fourier coefficients of $\tilde{S}_m(x) + \bar{\tilde{S}}_j(x)$ are bounded by $c\|A\|_1 k^{-1+4\varepsilon+\delta}$ uniformly in m, j. Considering also that $\tilde{S}_m(x) + \bar{\tilde{S}}_j(x)$ is a trigonometric polynomial of the length $R_0 < k^{\varepsilon}$, we easily get:

$$e^{\tilde{S}_m(x)+\bar{\tilde{S}}_j(x)} = 1 + \sum_{r\in Z^2} s_{mjr}\exp i(\vec{p}_r(0), x), \qquad (111)$$

$$\sum_{r\in Z^2}\sup_{j,m}|s_{mjr}| < c_{\|A\|_1}k^{-1+6\varepsilon+\delta}. \qquad (112)$$

Substituting (111) in (107), we obtain:

$$< \Psi_m, \Psi_j >= A^{(1)}_{mj} + A^{(2)}_{mj} \qquad (113)$$

where

$$A^{(1)}_{mj} = \frac{1}{|Q|}\int_Q \psi^{(q)}_m(x)\bar{\psi}^{(q)}_j(x)e^{i(\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t),x)}dx,$$

$$A^{(2)}_{mj} = \sum_{r\in Z^2} s_{mjr}A^{(2)}_{mjr}, \tag{114}$$

$$A^{(2)}_{mjr} = \frac{1}{|Q|}\int_Q \psi^{(q)}_m(x)\bar{\psi}^{(q)}_j(x)e^{i(\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)+\vec{p}_r(0),x)}dx.$$

Thus,

$$\sup_{j\in Z_q}\sum_{m\in Z_q:|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|<k^{\delta}} |<\Psi_m,\Psi_j>| \le$$

$$\sup_j\sum_m |A^{(1)}_{mj}| + \sup_j\sum_{m,r}|s_{jmr}||A^{(2)}_{mjr}|. \tag{115}$$

Let us consider $A^{(1)}_{mj}$. The functions $\psi^{(q)}_m$ and $\psi^{(q)}_j$ depend only on $(\vec{p}_q(0),x)$. Moreover, they admit expansion in the series of exponents $c_n e^{i(\tau_m+l_m+n)(\vec{p}_q(0),x)}$, $n\in Z$ and $c_n e^{i(\tau_j+l_j+n)(\vec{p}_q(0),x)}$, $n\in Z$, correspondingly. This means that the integrand of $A^{(1)}_{mj}$ can be represented as a linear combination of $e^{i(\vec{p}_m(t)-\vec{p}_j(t)+r_0\vec{p}_q(0),x)}$, $r_0\in \mathbf{N}$. Hence, the integral can differ from zero only if $\vec{d}^{(q)}_m(t)=\vec{d}^{(q)}_j(t)$. In this case,

$$A^{(1)}_{mj} = \frac{1}{|Q|}\int_Q \psi^{(q)}_m(x)\bar{\psi}^{(q)}_j(x)dx.$$

Considering again that $\psi^{(q)}_m(x), \bar{\psi}^{(q)}_j(x)$ depend only on $(\vec{p}_q(0),x)$ and $\tau_m=\tau_j$ (see Proposition 2), we obtain

$$A^{(1)}_{mj} = b_q^{-1} < \phi_{l_m}(\vec{d}^{(q)}_j(t),\tau_j,y), \phi_{l_j}(\vec{d}^{(q)}_j(t),\tau_j,y) >_1 = \delta_{l_m l_j},$$

$<\cdot,\cdot>_1$ being a scalar product in $L_2(0,b_q)$. Recalling that $\vec{d}^{(q)}_m(t)=\vec{d}^{(q)}_j(t)$ and $\tau_j=\tau_m$, we get

$$A^{(1)}_{mj} = \delta_{jm}. \tag{116}$$

Let us consider $A^{(2)}_{mjr}$. The functions $\psi^{(q)}_m$ and $\psi^{(q)}_j$ depend only on $(\vec{p}_q(0),x)$. Moreover, they admit expansion in the series of exponents $c_n e^{i(\tau_m+l_m+n)(\vec{p}_q(0),x)}$, $n\in Z$ and $c_n e^{i(\tau_j+l_j+n)(\vec{p}_q(0),x)}$, $n\in Z$, correspondingly. This means that the integrand of $A^{(2)}_{mjr}$ can be represented as a linear combination of the exponents $e^{i(\vec{p}_m(t)-\vec{p}_j(t)+n\vec{p}_q(0)+\vec{p}_r(0)),x)}$, $n\in \mathbf{N}$. Hence, the integral can differ from zero only if can differ from zero only if $\vec{p}_m(t)-\vec{p}_j(t)+\vec{p}_r(0)=r_0\vec{p}_q(0), r_0\in Z$. Note that in this case the integrand is a linear combination of exponents $e^{im(\vec{p}_q(0),x)}$, $m\in Z$. Thus,

$$\sup_{j\in Z_q}\sum_{m\in Z_q:|\vec{d}^{(q)}_m(t)-\vec{d}^{(q)}_j(t)|<k^{\delta}} |A^{(2)}_{mj}| < \sup_{j\in Z_q}\sum_{r\in Z^2, m\in\sigma_{jr}} |s_{mjr}||A^{(2)}_{mjr}| <$$

$$\left(\sum_{r\in Z^2}\sup_{j\in Z_q, m\in\sigma_{jr}}|s_{mjr}|\right)\left(\sup_{j\in Z_q, r\in Z^2}\sum_{m\in\sigma_{jr}}|A^{(2)}_{mjr}|\right), \tag{117}$$

$$\sigma_{jr} = \Big\{ m \in Z_q(k,\beta) : |\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t)| < k^{\delta},$$

$$\vec{p}_{m+r}(t) - \vec{p}_j(t) = r_0 \vec{p}_q(0),\ r_0 \in Z \big\} . \tag{118}$$

It follows from the definition of σ_{jr} that $\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t) + \vec{p}_r(0) = (\vec{p}_r(0), \nu_q)\nu_q$. Therefore,

$$A^{(2)}_{mjr} = \frac{1}{|Q|} \int_Q \psi_j^{(q)} \bar{\psi}_j^{(q)} e^{i(\vec{p}_r(0),\nu_q)(\nu_q,x)} dx,$$

the integrand being a linear combination of the exponents $\exp in(\vec{p}_q(0), x)$, $n \in Z$. This integral obviously can be reduced to the one-dimensional form:

$$A^{(2)}_{mjr} = b_q^{-1} < \phi_{l_m}(\vec{d}_m^{(q)}(t), \tau_m, y), \phi_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, y) e^{-i(\vec{p}_r(0),\nu_q)y} >_1, \tag{119}$$

Considering that the functions $\phi_l, l \in Z$ form an orthonormal system in $L_2(0, b_q)$ and that all the elements m of σ_{jr} differ only by their l_m, we obtain:

$$\sum_{m \in \sigma_{jr}} |A^{(2)}_{mjr}|^2 \leq 1. \tag{120}$$

Taking into account that σ_{jr} contains about $ck^{1/2+\beta}$ terms, and using the Cauchy-Schwartz inequality, we obtain:

$$\sup_{j \in Z_q, r \in Z^2} \sum_{m \in \sigma_{jr}} |A^{(2)}_{mjr}| < ck^{1/4+\beta/2}. \tag{121}$$

Using (117), (112) and the last estimate, we obtain:

$$\sup_{j \in Z_q} \sum_{r \in Z^2, m \in \sigma_{jr}} |s_{mjr}||A^{(2)}_{mjr}| = c_{\|A\|_6} k^{-3/4+\beta/2+6\varepsilon+\delta}. \tag{122}$$

Considering (110), (115), (116) and (122), we obtain

$$\sup_{m \in Z_q} \sum_{j \in Z_q} | < \Psi_m, \Psi_j > - \delta_{mj}| < c_{\|A\|_6} k^{-1/2+\beta+4\varepsilon+\delta}) = c_{\|A\|_6} k^{-\gamma_0}. \tag{123}$$

Indices m and j are symmetric in this case, so we can exchange them in (123).

3. Suppose Ψ_m is given by (88), Ψ_j – by (87). According to (56),

$$\vec{p}_m(t) = (\tau_m + l_m)\vec{p}_q(0) + \vec{d}_m^{(q)}(t), \tag{124}$$

$$\vec{p}_j(t) = (\tau_j + l_j)\vec{p}_q(0) + \vec{d}_j^{(q)}(t), \tag{125}$$

where $|(\vec{p}_m(t), \vec{p}_q(0))| < \hat{c}k^{1/2+\beta}$. Since $|\vec{p}_j(t) - \vec{p}_m(t)| < k^{1/2+\beta+\delta}$, the inequality

$$\hat{c}k^{1/2+\beta} < |(\vec{p}_j(t), \vec{p}_q(0))| < c_b p_q(0) k^{1/2+\beta+\delta}, \tag{126}$$

which is equivalent to the inequality

$$\hat{c}p_q^{-1}(0)k^{1/2+\beta} < p_q(0)(l_j + \tau_j) < c_b k^{1/2+\beta+\delta}. \tag{127}$$

We split S_j into two parts: $S_j = S_j^{(q)} + S_j^{(\text{non } q)}$. Hence,

$$\Psi_j(x) = \frac{1}{|Q|^{1/2}} f_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, x) e^{i(\vec{d}_j^{(q)}(t),x)+S_j^{(\text{non } q)}},$$

where f_{l_j} depends only on $(\vec{p_q}(0), x)$:

$$f_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, x) = \exp\left(i(\tau_j + l_j)(\vec{p_q}(0), x) + S_j^{(q)}(x)\right). \tag{128}$$

Thus,

$$\Psi_m \bar{\Psi}_j =$$

$$\frac{1}{|Q|} \psi_m^{(q)}(x) \bar{f}_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, x) \exp\left(i(\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t), x)) + \delta S_{jm}\right), \tag{129}$$

$$\delta S_{mj} = \bar{S}_j^{(\text{non } q)} + \tilde{S}_m.$$

Hence,

$$< \Psi_m, \Psi_j >= \frac{1}{|Q|} \int_Q \psi_m^{(q)}(x) \bar{f}_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, x) e^{i(\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t), x) + \delta S_{jm}} dx. \tag{130}$$

We estimate $\sup_j \sum_m | < \Psi_m, \Psi_j > |$ and $\sup_m \sum_j | < \Psi_m, \Psi_j > |$. The considerations are, in fact, similar to those in the case 2 with the difference that instead of $\psi_j^{(q)}(x)$ satisfying the equation for an eigenfunction exactly, we have the function $f_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, x)$, satisfying the analogous equation approximately (see Appendix 3).

We obtain an estimate for δS_{mj}. It is easy to see that

$$\delta S_{jm} = (\bar{S}_j^{(\text{non } q)} - \bar{\tilde{S}}_j) + (\bar{\tilde{S}}_j + \tilde{S}_m).$$

It is shown in Appendix 1 that

$$\|S_j^{(\text{non } q)} - \tilde{S}_j\|_\nu < c_{\|A\|_\nu} k^{-1/2+\beta+5\varepsilon}. \tag{131}$$

By (79), (78)

$$\|\bar{\tilde{S}}_j + \tilde{S}_m\|_\nu < c_{\|A\|_\nu+1} \left(|(\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t)| k^{-1+4\varepsilon}\right). \tag{132}$$

Adding (131) – (132) we obtain

$$\|\delta S_{mj}\|_0 = c_{\|A\|_\nu} k^{-1/2+\beta+5\varepsilon} + c_{\|A\|_\nu+1} \left(|(\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t)| k^{-1+4\varepsilon}\right). \tag{133}$$

Similar estimates hold for derivatives of $\delta S_{mj}(x)$.

Suppose first that $|\vec{d}_j^{(q)}(t) - \vec{d}_m^{(q)}(t)| > k^{1/2}$. Integrating by parts 4 times in the direction of $\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t)$, we obtain $< \Psi_j, \Psi_m >= O(k^{-3+\beta+5\varepsilon})$ and, therefore,

$$\sup_{j \in Z_q} \sum_{m \notin Z_q : |\vec{d}_j^{(q)}(t) - \vec{d}_m^{(q)}(t)| > k^{1/2}} | < \Psi_j, \Psi_m > | = O(k^{-1+\beta+5\varepsilon}), \tag{134}$$

$$\sup_{m\notin Z_q} \sum_{j\in Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|>k} |<\Psi_j,\Psi_m>| = O(k^{-1+\beta+5\varepsilon}). \tag{135}$$

Let $|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|\leq k^{1/2}$. From (133) we obtain:

$$\|\delta S_{mj}\|_0 = O(k^{-1/2+\beta+5\varepsilon}). \tag{136}$$

Noting that δS_{mj} is a trigonometric polynomial of the length $NR_0<k^{\varepsilon}$ and considering (136) we get:

$$e^{\delta S_{jm}} = 1+\sum_{r\in Z^2}\tilde{s}_{jmr}\exp i(\vec{p}_r(0),x), \tag{137}$$

$$\sum_{r\in Z^2}|r|\sup_{jm}|\tilde{s}_{jmr}| = O(k^{-1/2+\beta+7\varepsilon}). \tag{138}$$

Using (130) and (137), we represent $<\Psi_m,\Psi_j>$ in the form:

$$<\Psi_m,\Psi_j>= A_{mj}^{(1)}+A_{mj}^{(2)} \tag{139}$$

where

$$A_{mj}^{(1)}=\frac{1}{|Q|}\int_Q \psi_m^{(q)}(x)\bar{f}_{l_j}(\vec{d}_j^{(q)}(t),\tau_j,x)e^{i(\vec{d}_m^{(q)}(t)-\vec{d}_j^{(q)}(t),x)}dx,$$

$$A_{mj}^{(2)}=\sum_{r\in Z^2}\tilde{s}_{jmr}A_{jmr}^{(2)}$$

$$A_{mjr}^{(2)}=\frac{1}{|Q|}\int_Q \psi_m^{(q)}(x)\bar{f}_{l_j}(\vec{d}_j^{(q)}(t),\tau_j,x)e^{i(\vec{d}_m^{(q)}(t)-\vec{d}_j^{(q)}(t)+\vec{p}_r(0),x)}dx.$$

Let us consider $A_{mj}^{(1)}$. The integrand is a periodic function, more precisely a linear combination of exponents $e^{(\vec{p}_m(t)-\vec{p}_j(t)+n\vec{p}_q(0),x)}$, $n\in \mathbf{N}$. This means that the integral can differ from zero only if $\vec{p}_m(t)-\vec{p}_j(t)=n\vec{p}_q(0)$ for some $n\in Z$. In this case, $\vec{d}_m^{(q)}(t)=\vec{d}_j^{(q)}(t)$, and $\tau_m=\tau_j$. Thus, we obtain:

$$A_{mj}^{(1)}=\frac{1}{|Q|}\int_Q \psi_m^{(q)}(x)\bar{f}_{l_j}(\vec{d}_m^{(q)}(t),\tau_m,x))dx.$$

Using the fact that $f_{l_j}(\vec{d}_m^{(q)}(t),\tau_m,x)$ and $\psi_m^{(q)}(x)$ depend only on $(\vec{p}_q(0),x)$, and satisfy the same quasiperiodic conditions we get

$$A_{mj}^{(1)}=b_q^{-1}<\phi_{l_m}(\vec{d}_m^{(q)}(t),\tau_m,y),f_{l_j}(\vec{d}_m^{(q)}(t),\tau_m,y))>_1 .$$

It is shown in Appendix 3 that $f_{l_j}(\vec{d}_j^{(q)}(t),\tau_j,y)$ is an approximate solution of the equation for eigenfunctions:

$$H_1^{(q,\vec{d}_j^{(q)}(t))}f_{l_j}=\Big(\mu_{l_j}+\zeta(\tau_j,l_j,y)\Big)f_{l_j}, \tag{140}$$

where $H_1^{(q,\vec{d}_j^{(q)}(t))}$ is defined by (72), $\mu_{l_j}=\mu_{l_j}(\tau_j,\vec{d}_j^{(q)}(t))$ is an eigenvalue. The function $\zeta(\tau_j,l_j,y)$ obeys the estimate

$$\|\zeta(\tau_j,l_j,y)\|_0<c_{\|A\|_1}k^{-1/2+\beta+7\varepsilon}. \tag{141}$$

By definition, $\phi_{l_m}(\vec{d}_m^{(q)}(t), \tau_m, y)$ satisfies

$$H_1^{(q, \vec{d}_m^{(q)}(t))} \phi_{l_m} = \mu_{l_m} \phi_{l_m}. \tag{142}$$

Since $\vec{d}_j^{(q)}(t) = \vec{d}_m^{(q)}(t)$, the operators on the left hand sides of (140) and (142) are the same. According to Lemma 27 in Appendix 5, we can choose the technical constant $\hat{c}$ in such a way that

$$\mu_{l_j}(\tau_m) - \mu_{l_m}(\tau_j) > k^{1/2+\beta} p_q^2(0)/32 \text{ for all } j, m : \tag{143}$$
$$|(\vec{p}_j(t), \vec{p}_q(0))| \geq \hat{c} k^{1/2+\beta}, \ |(\vec{p}_m(t), \vec{p}_q(0))| < \hat{c} k^{1/2+\beta}.$$

Using (140) – (142), we obtain

$$A_{mj}^{(1)} = 0, \text{ if } \vec{d}_j^{(q)}(t) \neq \vec{d}_m^{(q)}(t).$$

$$A_{mj}^{(1)} = \frac{O\left(k^{-1/2+\beta+7\varepsilon}\right)}{\mu_{l_j}(\tau_m) - \mu_{l_m}(\tau_m)}, \text{ if } \vec{d}_j^{(q)}(t) = \vec{d}_m^{(q)}(t).$$

It is shown in Appendix 3 that μ_{l_j} admits the representation: $\mu_{l_j} = p_q^2(0)(\tau_j + l_j)^2 + \Delta\mu_{l_j}$, where $p_q^2(0)(\tau_j + l_j)^2$ obviously is the unperturbed eigenvalue corresponding to given τ_j, l_j, and $\Delta\mu_{l_j} = \Delta\mu_{l_j}(\tau_j, \vec{d}_j^{(q)}(t))$ is the shift of eigenvalue under the perturbation; it satisfies the estimate $\Delta\mu_{l_j} = O(k^{1-2\beta})$. Considering (143), we easily get

$$\sup_{m \in Z_q} \sum_{j \notin Z_q : |\vec{d}_j^{(q)}(t) - \vec{d}_m^{(q)}(t)| < k^{1/2}} |A_{mj}^{(1)}| = O(k^{-1+8\varepsilon}). \tag{144}$$

In Appendix 6 we show that

$$\sum_{l \in Z} \frac{1}{|E - \mu_l|} < c k^{-1/2-\beta} \ln k. \tag{145}$$

for any E such that $\min_{l \in Z} |E - \mu_l| > c k^{1/2+\beta}$, $E > c k^{1+2\beta}$. Taking into account the last inequality, we obtain

$$\sup_{j \notin Z_q} \sum_{m \in Z_q : |\vec{d}_j^{(q)}(t) - \vec{d}_m^{(q)}(t)| < k^{1/2}} |A_{mj}^{(1)}| = O(k^{-1+7\varepsilon}) \ln k. \tag{146}$$

Note that $A_{mjr}^{(2)}$ can differ from zero only if $\vec{p}_{m+r}(t) - \vec{p}_j(t) = r_0 \vec{p}_q(0)$, $r_0 \in Z$. It is easy to see that

$$\sup_{j \in Z_q} \sum_{m \in Z_q : |\vec{d}_m^{(q)}(t) - \vec{d}_j^{(q)}(t)| < k^{\delta}} |A_{mj}^{(2)}| < \sup_{j \in Z_q} \sum_{r \in Z^2, m \in \sigma_{jr}} |\tilde{s}_{mjr}||A_{mjr}^{(2)}| <$$
$$\left(\sum_{r \in Z^2} |r| \sup_{j \in Z_q, m \in \sigma_{jr}} |\tilde{s}_{mjr}| \right) \left(\sup_{j \in Z_q, r \in Z^2} |r|^{-1} \sum_{m \in \sigma_{jr}} |A_{mjr}^{(2)}| \right), \tag{147}$$

σ_{jr} being given by (118). If $m \in \sigma_{jr}$, then

$$A_{mjr}^{(2)} = \frac{1}{|Q|} \int_Q \psi_m^{(q)} \bar{f}_{l_j}(\vec{d}_j^{(q)}(t), \tau_j, (\vec{p}_q(0), x)) e^{i(\vec{p}_r(0), \nu_q)(\nu_q, x)} dx =$$

$$\frac{1}{b_q}\int_0^{b_q}\phi_{l_m}(\vec{d}_m^{(q)}(t),\tau_m,y)\bar{f}_{l_j}(\vec{d}_j^{(q)}(t),\tau_j,y)e^{i(\vec{p}_r(0),\nu_q)y}dy, \qquad (148)$$

the integrand being a periodic function. Let us break σ_{jr} into two parts: $\sigma_{jr}=\sigma'_{jr}\cup\sigma''_{jr}$,

$$\sigma'_{jr}=\{m\in\sigma_{jr},m:|(\vec{p}_j(t),\vec{p}_q(0))|-|(\vec{p}_m(t),\vec{p}_q(0))|<2|(\vec{p}_r(0),\vec{p}_q(0))|\}.$$

$$\sigma''_{jr}=\{m\in\sigma_{jr},m:|(\vec{p}_j(t),\vec{p}_q(0))|-|(\vec{p}_m(t),\vec{p}_q(0))|\geq 2|(\vec{p}_r(0),\vec{p}_q(0))|.\}$$

It is easy to show that the number of elements in σ'_{jr} does not exceed $c_b|r|$. Obviously, $|A_{mjr}|<c$ and, therefore,

$$\sum_{m\in\sigma'_{jr}}|A_{mjr}|<c_b|r|. \qquad (149)$$

Let us consider

$$\sum_{m\in\sigma''_{jr}}|A_{mjr}|. \qquad (150)$$

It is easy to check (Appendix 7) that the function $g_{jr}(y)=f_{l_j}(\vec{d}_j^{(q)}(t),\tau_j,y)e^{-i(\vec{p}_r(0),\nu_q)y}$, obeying the quasiperiodic conditions with quasimomentum τ_m, which approximately satisfies the equation for eigenfunctions:

$$H_1^{(q,\vec{d}_{j-r}^{(q)}(t))}g_{jr}(y)=\left(\mu_{l_{j-r}}(\tau_{j-r})+\zeta_{jr}\right)g_{jr}(y),\quad \|\zeta_{jr}\|_0=O(|r|k^{1/2-\beta}). \qquad (151)$$

Thus, $g_{jr}(y)$ has properties similar to those of $f_{l_{j-r}}(\vec{d}_m^{(q)}(t),\tau_m,y)$. Considering that $|(\vec{p}_r(0),\vec{p}_q(0))|>\hat{c}k^{1/2+\beta}$, and $|(\vec{p}_j(t),\vec{p}_q(0))|-|(\vec{p}_m(t),\vec{p}_q(0))|\geq 2|(\vec{p}_r(0),\vec{p}_q(0))|$ we easily obtain that $|(\vec{p}_{j-r}(0),\vec{p}_q(0))|>k^{1/2+\beta}/2$, i.e., $|l_{j-r}|>k^{1/2+\beta}/4$, and, hence, $\mu_{l_{j-r}}(\tau_m)-\mu_{l_m}(\tau_m)>ck^{1/2+\beta}$. Thus,

$$A_{mj}^{(2)}=0,\ \text{if}\ \vec{d}_{j-r}^{(q)}(t)\neq\vec{d}_m^{(q)}(t).$$

$$A_{mjr}^{(2)}=\frac{O\left(|r|k^{1/2-\beta}\right)}{\mu_{l_{j-r}}(\tau_m)-\mu_{l_m}(\tau_m)},\ \text{if}\ \vec{d}_{j-r}^{(q)}(t)=\vec{d}_m^{(q)}(t).$$

Taking into account (143), we obtain

$$\sup_{j\in Z_q}\sum_{m\in\sigma''_{jr}}|A_{mjr}^{(2)}|=O(|r|). \qquad (152)$$

Using (149), we obtain:

$$\sup_{j\notin Z_q}\sum_{m\in Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|<k^{1/2}}|A_{mjr}^{(2)}|=O(|r|). \qquad (153)$$

Considering (147) and (138), we obtain:

$$\sup_{j\notin Z_q}\sum_{m\in Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|<k^{1/2}}|A_{mj}^{(2)}|=O(k^{-1/2+\beta+7\varepsilon}). \qquad (154)$$

Similarly,

$$\sup_{m\in Z_q}\sum_{j\notin Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|<k^{1/2}}|A_{mj}^{(2)}|=O(k^{-1/2+\beta+7\varepsilon}). \tag{155}$$

Considering (139), (144) and the last two estimates, we obtain:

$$\sup_{j\notin Z_q}\sum_{m\in Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|<k^{1/2}}|<\Psi_m,\Psi_j>|=O(k^{-1/2+\beta+7\varepsilon}), \tag{156}$$

$$\sup_{m\in Z_q}\sum_{j\notin Z_q:|\vec{d}_j^{(q)}(t)-\vec{d}_m^{(q)}(t)|<k^{1/2}}|<\Psi_m,\Psi_j>|=O(k^{-1/2+\beta+7\varepsilon}). \tag{157}$$

Taking into account (134), (135), and the last two estimates we get

$$\sup_{m\notin Z_q}\sum_{j\in Z_q}|<\Psi_m,\Psi_j>|=O(k^{-1/2+\beta+7\varepsilon}), \tag{158}$$

$$\sup_{j\in Z_q}\sum_{m\notin Z_q}|<\Psi_m,\Psi_j>|=O(k^{-1/2+\beta+7\varepsilon}). \tag{159}$$

Considering (99), (102), (106), (123), (158) and (159), we get (90). The theorem is proved.

Let $J_0\subset J$,

$$J_0=\{j\in Z^2:k/2<p_j(t)<3k/2\}.$$

Let us denote by L a finite dimensional subspace in $L_2(R^3)$ spaned on the vectors $\{\Psi_j\}_{j\in J}$, and by L_0 the finite dimensional subspace spaned by $\{\Psi_j^0\}_{j\in J_0}$, $\Psi_j^0(x)=|Q|^{-1/2}e^{(\vec{p}_j(t),x)}$, We denote by P and P_0 projections in $L_2(Q)$ on L and L_0 correspondingly.

Theorem 2 *If* $0<\beta<1/2$, $0<7\varepsilon<\min\{(1/2-\beta)/5,14\beta\}$ $\beta/7<\varepsilon<(1-6\beta)/32$, *then the following asymptotic relation holds:*

$$P_0P=P_0+\delta P,\quad \|\delta P\|=O(k^{-M}),\ M=k^{1/2-\beta}. \tag{160}$$

Proof. We break J_0 into subsets: $J_0=\hat{Z}_0\cup\left(\cup_{q\in\Gamma(k^\varepsilon)}\hat{Z}_q\right)$, where

$$\hat{Z}_0=J_0\cap Z_0(k,\gamma,\varepsilon),\quad \hat{Z}_q=J_0\cap Z_q(k,\gamma,\varepsilon),$$

$$(1-\gamma_0)/2+2\varepsilon<\gamma<1/2-3\varepsilon. \tag{161}$$

Considering that $\gamma_0<1/3$, we obtain that $\gamma>1/3$. Using the definition of γ_0, it is easy to show that $\gamma>\beta$. Therefore, the set $\hat{Z}_0(k,\gamma,\varepsilon)$ and, moreover its $(k^{1/2+\gamma})$-neighborhood belong to $Z_0(k,\beta,\varepsilon)$.

I We prove that each $\Psi_j^0(x)$, $j\in Z_0(k,\gamma,\varepsilon)$, can be represented as a linear combination of Ψ_m, $m\in Z_0(k,\beta,\varepsilon)$ with a high accuracy:

$$\Psi_j^0(x)=\sum_{|m-j|<k^{1/2-\gamma+\mu}}c_{jm}\Psi_m(x)+\epsilon_j(x), \tag{162}$$

$$\|\epsilon_j\|_0 = O_2(k^{-\zeta k^{\mu-2\varepsilon}}), \tag{163}$$

where

$$2\varepsilon \le \mu < 2\gamma, \quad \zeta = \gamma_0/2 - 1/2 + \gamma - 2\varepsilon.$$

Note that $1/2 - \gamma + \mu < 1/2 + \gamma$, therefore all indexes m in the sum above belong to $Z_0(k, \beta, \varepsilon)$ and all Ψ_m are given by (11).

Step 1. First we check that each $\Psi_j^0(x)$, $j \in \hat{Z}_0(k, \gamma, \varepsilon)$, can be represented as a linear combination of Ψ_m with a good accuracy:

$$\Psi_j^0(x) = \sum_{m:|m-j|<k^{1/2-\gamma+2\varepsilon}} c_{jm}\Psi_m(x) + \epsilon_j(x),. \tag{164}$$

$$\|\epsilon_j\|_{L_2(Q)} = O_2(k^{-\gamma_0/2}), \tag{165}$$

To prove (164), (165) it suffices to check that the projection of $\Psi_j^0(x)$ on L has the L_2-norm greater than $1 - k^{-\gamma_0}$. This readily follows from the inequality

$$\sum_{m:|m-j|<k^{1/2-\gamma+2\varepsilon}} | < \Psi_m, \Psi_j^0 > |^2 > 1 - c_{\|A\|_2} k^{-\gamma_0}, \tag{166}$$

since the system of functions Ψ_m is almost orthogonal (Theorem 1). Let us prove (166). If $j \in \hat{Z}_0(k, \gamma, \varepsilon)$ and $|m - j| < k^{1/2-\gamma+2\varepsilon}$, then, obviously, $m \in Z_0(k, \beta, \varepsilon)$ and $\Psi_m(x)$ is given by (11). Hence

$$< \Psi_m(x), \Psi_j^0(x) >= |Q|^{-1} \int_Q e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)+S_m(x)} dx.$$

Setting β being equal to γ in formula (103) (by the definition of $\hat{Z}_0(k, \gamma, \varepsilon)$ we can do this) and using (29), we replace $S_m(x)$ by $-\bar{S}_j(x)$, with a small error following :

$$< \Psi_m(x), \Psi_j^0(x) >= |Q|^{-1} \int_Q e^{-\bar{S}_j(x)+i(\vec{p}_m(t)-\vec{p}_j(t),x)} dx +$$

$$O_2\left(|m - j| k^{-2\gamma} + k^{-2\gamma}\right). \tag{167}$$

Considering that $|j - m| \le k^{1/2-\gamma+2\varepsilon}$ we obtain $O_2\left(|m - j| k^{-2\gamma} + k^{-2\gamma}\right) = O_2(k^{1/2-3\gamma+2\varepsilon})$. Note that the above integral is the $(j - m)$-th Fourier coefficient of the function $e^{-\bar{S}_j(x)}$. We will denote it by $(e^{-\bar{S}_j(x)})_{j-m}$. Thus,

$$\sum_{m:|m-j|<k^{1/2-\gamma+\delta}} | < \Psi_m(x), \Psi_j^0(x) > |^2 =$$

$$\sum_{m:|m-j|<k^{1/2-\gamma+2\varepsilon}} \left|(e^{-\bar{S}_j(x)})_{j-m} + O_2(k^{1/2-3\gamma+2\varepsilon})\right|^2.$$

Using the Cauchy-Schwarz inequality and considering that the sum contains about $ck^{1-2\gamma+4\varepsilon}$ terms, we obtain:

$$\sum_{m:|m-j|<k^{1/2-\gamma+2\varepsilon}} | < \Psi_m(x), \Psi_j^0(x) > |^2 > \sum_{m:|m-j|<k^{1/2-\gamma+2\varepsilon}} |(e^{-\bar{S}_j(x)})_{j-m}|^2 +$$

$$O_2(\|e^{-\bar{S}_j}\|_{L_2(Q)}k^{1-4\gamma+4\varepsilon}) + O_2(k^{2-8\gamma+8\varepsilon}). \tag{168}$$

Now we estimate the incomplete sum of the Fourier coefficients on the right-hand side. We have an obvious formula for the complete sum:

$$\sum_{m\in Z^2} |(e^{-\bar{S}_j(x)})_{j-m}|^2 = \|e^{-\bar{S}_j}\|^2_{L_2(Q)} = 1 + O_2(k^{-2\gamma}), \tag{169}$$

here (29) with $\beta = \gamma$ has been used. Thus, it remains to estimate the tail of the series. Let us consider $(e^{-\bar{S}_j(x)})_{j-m}$ for $|j-m| \geq k^{1/2-\gamma+2\varepsilon}$. Integrating M times by parts the corresponding integral (see the right-hand side of (167)) in the direction of $\vec{p}_m(t) - \vec{p}_j(t)$ and considering estimate (27) with $\beta = \gamma$ (for details see Appendix 11), we obtain:

$$|(e^{-\bar{S}_j(x)})_{j-m}| < \left(c_{\|\vec{A}\|_0,\beta,\varepsilon} k^{(1/2-\gamma+\varepsilon)}|m-j|^{-1}\right)^M \tag{170}$$

for any $M < k^{1/2-\gamma-\varepsilon}$. Choosing $M > 3/\varepsilon$, we arrive at the inequality:

$$\sum_{m:|j-m|\geq k^{1/2-\gamma+2\varepsilon}} | < \Psi_m(x), \Psi_j^0(x) > |^2 < c_{\|\vec{A}\|_0,\beta,\varepsilon} k^{-2}. \tag{171}$$

It is not difficult to show that $1-4\gamma+4\varepsilon < -\gamma_0$ and $-2\gamma < -\gamma_0$, therefore, we can replace the $O_2(\cdot)$ in (168), (169) by $O_2(k^{-\gamma_0})$. The estimate (166) follows from (168), (169) and (171).

Step 2. We prove that the function ε_j in (162) satisfies the estimate:

$$\|\epsilon_j\|_0 = O_2(k^{-\gamma_0/2+1/2-\gamma+2\varepsilon}). \tag{172}$$

Let us consider the set W_1: $W_1 : \{r : |r-j| < \mathbf{2}k^{1/2-\gamma+2\varepsilon}\}$ and let P_1^0 be the projection on the linear span of Ψ_r^0, $r \in W_1$. Estimate (172) follows from two estimates:

$$\|P_1^0\epsilon_j\|_0 = O_2(k^{-\gamma_0/2+1/2-\gamma+2\varepsilon}), \tag{173}$$

$$\|(I-P_1^0)\epsilon_j\|_0 < \left(c_{\|A\|_0,\beta,\varepsilon}k^{-\varepsilon}\right)^M, \ M < k^{1/2-\beta-\varepsilon}. \tag{174}$$

First we consider $P_1^0\epsilon_j$. The estimate for ϵ_j in (164) yields $\|P_1^0\epsilon_j\|_{L_2(Q)} = O_2(k^{-\gamma_0/2})$. Considering that W_1 contains about $ck^{1-2\gamma+4\varepsilon}$ elements and using the Cauchy-Schwarz inequality, we get (173). To prove (174), we apply the projection $I - P_1^0$ to both sides of (164). Note that

$$(I-P_1^0)\Psi_j^0 = 0 \ \text{ and } \ |(\Psi_m)_r| < \left(c_{\|A\|_0,\varepsilon,\beta}k^{1/2-\gamma+\varepsilon}|m-r|^{-1}\right)^M, \ M < k^{1/2-\gamma-\varepsilon}, \tag{175}$$

for any $r \notin W_1$ (by analogy with (170)). Using Theorem 1 we obtain that $|c_{mj}| < 2$. Considering (175) and the last estimate we easily obtain (174). Thus, (172) holds.

Step 3. Let us consider sets W_n: $W_n : \{m : |m-j| < (n+1)k^{1/2-\gamma+2\varepsilon}\}$ and let P_n^0 be the corresponding projections on the linear spans of Ψ_m^0, $m \in W_n$. Clearly, $\cup_{n=1}^{N_1} W_n$, $N_1 = [k^{\mu-2\varepsilon}]$ belongs to the $(k^{1/2+\gamma})$-neighborhood of $\hat{Z}_0(k,\gamma,\varepsilon)$, which is up to technical multiplies is the same as $\hat{Z}_q(k,\gamma,\varepsilon)$. Let us consider formulae (164), (172). Applying once more formulae (162), (172) to each $m \in W_1$, we get

$$P_1^0 \varepsilon_j = \sum_{m \in W_1} (\epsilon_j)_m \Psi_m^0(x) = \sum_{r \in W_2} c_{jr} \Psi_r(x) + \sum_{m \in W_1} (\epsilon_j)_m \epsilon_m(x), \tag{176}$$

here $c_{jr} = \sum_m (\epsilon_j)_m c_{mr}$. Clearly,

$$\| \sum_{m \in W_1} (\epsilon_j)_m \epsilon_m(x) \|_0 \le \|\epsilon_j\|_0 \sup_m \|\epsilon_m\|_0 = O_2(k^{-2(\gamma_0/2-1/2+\gamma-2\varepsilon)}).$$

Using (164), (174) (176) and the last estimate, we obtain:

$$\Psi_j^0(x) = \sum_{m \in W_2} c_{jr} \Psi_r(x) + \epsilon_j(x), \quad \|\epsilon_j\|_0 = O(k^{-2(\gamma_0/2-1/2+\gamma-2\varepsilon)}). \tag{177}$$

Repeating this procedure N_1 times, we get (162).

II. We prove that each $\Psi_j^0(x)$, $j \in \hat{Z}_q(k,\gamma)$, can be represented as a linear combination of Ψ_m with a high accuracy:

$$\Psi_j^0(x) = \sum_{|\vec{d}_m(t) - \vec{d}_j(t)| < k^{1/2-\beta+\mu'},\ m \in Z_q(k,\gamma+2\zeta)} c_{jm} \Psi_m(x) + \epsilon_j(x), \tag{178}$$

$$\|\epsilon_j\|_0 = O(k^{-\tilde{\gamma} k^{\zeta}}), \quad 3\tilde{\gamma} = \beta\mu'(\gamma - \beta),$$

here

$$\mu' > 0 : 1/2 - \beta + \mu' < 1/2 + \gamma - 2\varepsilon, \quad 0 < 3\zeta < 1/2 - \gamma - 5\varepsilon.$$

We prove formula (178) in two steps. First, let us define functions $\hat{\Psi}_r$, $r \in Z_q(k, 1-3\varepsilon)$ as follows:

$$\hat{\Psi}_r = \hat{\phi}_{l_r}^{(q)}(\vec{d}_r(t), \tau_r, x) e^{i(\vec{d}_r(t),x) + \tilde{S}_r}, \tag{179}$$

here $\hat{\phi}_{l_r}^{(q)}(\vec{d}_r(t), \tau_r, x)$ depends only on $(\vec{p}_q(0), x)$ and is defined by the formula:

$$\hat{\phi}_{l_r}^{(q)}(\vec{d}_r(t), \tau_r, x) = \hat{\phi}_{l_r}^{(q)}((\vec{d}_r(t), \tau_r, y)\Big|_{y=(\vec{p}_q(0),x)},$$

where

$$\hat{\phi}_{l_r}(\vec{d}_r^{(q)}(t), \tau_r, y) = \begin{cases} \phi_{l_r}(\vec{d}_r^{(q)}(t), \tau_r, y), & \text{if } p_q^2(0)\,|\tau_j + l_r| < \hat{c} k^{1/2+\beta}, \\ f_{l_r}(\vec{d}_r^{(q)}(t), \tau_r, y), & \text{otherwise.} \end{cases} \tag{180}$$

First, we show that each $\Psi_j^0(x)$, $j \in \hat{Z}_q(k,\gamma)$, can be represented as a linear combination of functions $\hat{\Psi}_r(x)$ with a high accuracy:

$$\Psi_j^0(x) = \sum_{r : |\vec{d}_r(t) - \vec{d}_j(t)| < k^{1/2-\beta+\mu'},\ r \in Z_q(k, 1/2+\gamma+\zeta)} \hat{c}_{jr} \hat{\Psi}_r(x) + \hat{\epsilon}_j(x), \tag{181}$$

$$\|\hat{\epsilon}_j\|_0 = O(k^{-\tilde{\gamma} k^{\zeta}}).$$

Second, we prove that any $\hat{\Psi}_r(x)$ in (181) can be represented as a linear combination of $\Psi_m(x)$ with a high accuracy:

$$\hat{\Psi}_r(x) = \sum_{r:|\vec{d_m}(t)-\vec{d_r}(t)|<k^{1/2-\beta+\mu'},\ r\in Z_q(k,\gamma+2\zeta)} d_{rm}\Psi_m(x) + \epsilon_r^*(x), \quad \|\epsilon_r^*\|_0 = O(k^{-2\tilde{\gamma}k^{\zeta}}). \quad (182)$$

Substituting (182) in (181), we obtain (178).

Proof of formula (181). We can not prove formula (162) in the case $j \in Z_q(k,\gamma)$, since estimates (27) – (30) do not hold for $S_j(x)$. However, the estimates similar to (27) – (30), namely the estimates (77) – (79), hold for $\tilde{S}_j(x)$. Estimates (77) – (79) are pretty strong, they are analogous to (27) – (30) for $\beta = 1/2 - 2\varepsilon$. This is related to the fact that $\tilde{S}_m(x)$ does not "feel" the $\vec{a}^{(q)}(x)$-component of $\vec{a}(x)$, therefore, the condition $j \in Z_q(k,\gamma)$ can not destroy good estimates for $\tilde{S}_j(x)$. Let us introduce the functions

$$\tilde{\Psi}_m(x) = e^{i(\vec{p}_m(t),x)+\tilde{S}_m(x)}, \quad m \in Z_q(k, 1-3\varepsilon).$$

Considering in the same way as in **I** and using (77) – (79), we prove the formula analogous to (162):

$$\Psi_j^0(x) = \sum_{|r-j|<k^{2\varepsilon+\mu}} c_{jm}\tilde{\Psi}_m(x) + \epsilon_j(x), \quad (183)$$

$$\|\epsilon_j\|_0 = O(k^{-\zeta k^{\mu-2\varepsilon}}),$$

μ being in the interval $(2\varepsilon, 1-4\varepsilon)$. We chose $\mu : 2\varepsilon + \mu = 1/2 - \beta + \mu'$. Obviously, $\sum_m |c_{jm}|^2 = O(1)$, since functions $\tilde{\Psi}_m(x)$ are almost orthogonal (the proof is completely analogous to the first part of the proof of Theorem 1).

The next step is to show that any $\tilde{\Psi}_m(x)$ in (183) can be represented as a linear combination of functions $\hat{\Psi}_r(x)$ with a high accuracy:

$$\tilde{\Psi}_m(x) = \sum_{r\in Z_q(k^{1/2+\gamma+\zeta}),\ \vec{d_r}(t)=\vec{d_m}(t)} \tilde{c}_{mr}\hat{\Psi}_r(x) + \tilde{\epsilon}_m(x), \quad \|\tilde{\epsilon}_m\|_0 = O(k^{-(\gamma-\beta)\beta k^{\zeta}}). \quad (184)$$

It is easy to show (Appendix 9) that an exponent $\phi_{l_m}^0(\tau_r, y) = e^{i(l_m+\tau_m)p_q(0)y}$, $|l_m| < 2k^{1/2+\gamma}$, $y \in [0, b_q]$ can be represented with a high accuracy as a linear combination of functions $\hat{\phi}_n(\vec{d_m}(t), \tau_m, y)$, $|n| < k^{1/2+\gamma+\zeta}$:

$$\phi_{l_m}^0(\tau_m, y) =$$

$$\sum_{n\in Z,|n|<k^{1/2+\gamma+\zeta}} \tilde{c}_{mn}\hat{\phi}_n(\vec{d_m}(t), \tau_m, y) + \tilde{\varepsilon}_{l_m}(y),$$

$$\|\tilde{\varepsilon}_{l_m}\|_0 = O(k^{-\beta(\gamma-\beta)k^{\zeta}}).$$

Setting $y = (x, \vec{p}_q(0))p_q^{-1}(0)$ and multiplying by $\tilde{S}_m(x)$, which is equal to $\tilde{S}_r(x)$ for all $r : \vec{d_r}(t) = \vec{d_m}(t)$, we obtain (184). Substituting (184) in (183) and taking into account that $1/2 - \beta + \mu' - 2\varepsilon > 1/2 - \gamma - 2\varepsilon > 3\zeta$ and $\zeta > 3\tilde{\gamma}$, $\beta(\gamma - \beta) > 3\tilde{\gamma}$, we obtain (181).

Proof of formula (182). First, let us check that $\Psi_m(x)$, $m\in Z_q(k,\gamma+2\zeta)$ can be represented as a linear combination of $\hat\Psi_r$ with a high accuracy:

$$\Psi_m(x)=\sum_{r:|\vec d_r(t)-\vec d_m(t)|<k^{1/2-\beta+\mu'},\ r\in Z_q(k,\gamma+3\zeta)} c_{mr}\hat\Psi_r(x)+\epsilon_m(x), \tag{185}$$

$$\|\epsilon_m(x)\|_0<k^{-3\tilde\gamma k^{\zeta}}. \tag{186}$$

In fact, if $m\in Z_q(k,\beta)$, then the representation is trivial: $c_{mr}=\delta_{mr}$, $\epsilon_m(x)=0$. Suppose $m\in Z_q(k,\gamma+2\zeta)\cap Z_0(k,\beta,\varepsilon)$. We use Fourier representation of $\Psi_m(x)$:

$$\Psi_m(x)=\sum_{s\in Z^2}a_{ms}\Psi_s^{(0)}(x). \tag{187}$$

Integrating by parts, it is easy to show (Appendix 11) that

$$\sum_{s:|m-s|>k^{1/2-\beta+\mu'}}|a_{ms}|<ck^{-\mu' k^{1/2-\beta-\varepsilon}}k^2. \tag{188}$$

Therefore,

$$\Psi_m(x)=\sum_{s:|m-s|<k^{1/2-\beta+\mu'}}a_{ms}\Psi_s^{(0)}(x)+\epsilon_m(x), \tag{189}$$

$\epsilon_m(x)$ obeys the estimate:

$$\|\epsilon_m(x)\|_0<k^{-\mu' k^{1/2-\beta-\varepsilon}}k^2. \tag{190}$$

Using (181) in (189), and taking into account that $1/2-\beta-\varepsilon>1/2-\gamma>3\zeta$, we obtain (185), (186).

Now we want to invert (185), i.e., to prove (182). First, we find a formula for coefficients c_{mr}. Let us consider the matrix

$$\tilde G_{ms}=<\hat\Psi_s,\Psi_m>,\quad m,s\in Z_q(k,\gamma+3\zeta).$$

Matrix elements $\tilde G_{ms}$ are analogous to matrix elements G_{ms} and have similar properties. It is shown in Appendix 11 that

$$\tilde G=I+\delta\tilde G, \tag{191}$$

$\delta\tilde G$ obeying the estimate:

$$\|\delta\tilde G\|_*<k^{-\gamma_0}. \tag{192}$$

Multiplying both sides of (185) by Ψ_s, $s\in Z_q(k,\gamma+3\zeta)$, we obtain

$$<\Psi_m,\Psi_s>=\sum_{r\in Z_q(k,\gamma+3\zeta)}c_{mr}<\hat\Psi_r,\Psi_s>+<\epsilon_m,\Psi_s>,\quad m\in Z_q(k,\gamma+2\zeta).$$

We rewrite the last formula in matrix form:

$$G_{sm}=(\tilde GC+E)_{sm}\ \text{ for all } s\in Z_q(k,\gamma+3\zeta),\ \ m\in Z_q(k,\gamma+2\zeta),$$

here $C_{rm}=c_{mr}$, $E_{sm}=<\epsilon_m,\Psi_s>$. Therefore,

$$C_{rm}=\left(\tilde G^{-1}(G-E)\right)_{rm},\quad \text{for all } r\in Z_q(k,\gamma+3\zeta) \text{ and } m\in Z_q(k,\gamma+2\zeta). \tag{193}$$

To invert (185) we consider a linear combination: $\sum_{m\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\Psi_m$, $n\in Z_q(k,\gamma+\zeta)$. Substituting (185) for Ψ_m, we obtain:

$$\sum_{m\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\Psi_m=\sum_{r\in Z_q(k,\gamma+3\zeta)}s_{nr}\hat{\Psi}_r+\tilde{\epsilon}_n(x), \tag{194}$$

where

$$s_{nr}=\sum_{m\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}c_{mr},$$

$$\tilde{\epsilon}_n(x)=\sum_{m\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\epsilon_m(x).$$

Let us estimate $\|\tilde{\epsilon}_n(x)\|_0$. It is easy to see that

$$\|\tilde{\epsilon}_n(x)\|_0<\|G^{-1}\tilde{G}\|_*\sup_m\|\epsilon_m(x)\|_0.$$

Using (90), (192) and (186) we obtain:

$$\|\tilde{\epsilon}_n(x)\|_0<ck^{-3\tilde{\gamma}k^{\zeta}}. \tag{195}$$

Using (193), we calculate coefficients s_{nr}:

$$s_{nr}=\delta_{nr}-\delta_1 s_{nr}-\delta_2 s_{nr}, \tag{196}$$

$$\delta_1 s_{nr}=\sum_{m\in Z_q(k,\gamma+3\zeta)\setminus Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\left(\tilde{G}^{-1}G\right)_{rm},$$

$$\delta_2 s_{nr}=\sum_{m\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\left(\tilde{G}^{-1}E\right)_{rm}.$$

Let us estimate $\delta_1 s_{nr}$. We consider $n\in Z_q(k,\gamma+\zeta)$. Therefore, $|n-m|>k^{1/2+\gamma+2\zeta}$ for all $m\in Z_q(k,\gamma+3\zeta)\setminus Z_q(k,\gamma+2\zeta)$. It is shown in Appendix 11 that under this condition $\left|(G^{-1}\tilde{G})_{nm}\right|<k^{-k^{\gamma-\beta+2\zeta}}$. Using the last inequality , we obtain:

$$\delta_1 s_{nr}=O(k^{-k^{\gamma-\beta+2\zeta}}k^2).$$

From the estimate (186) we obtain $|E_{ms}|=O(k^{-3\tilde{\gamma}k^{\zeta}})$. Therefore,

$$\delta_2 s_{nr}=O(k^{-3\tilde{\gamma}k^{\zeta}}k^2).$$

Considering that $\gamma-\beta+2\zeta>2\zeta$, we obtain:

$$\delta_1 s_{nr}+\delta_2 s_{nr}=O(k^{-3\tilde{\gamma}k^{2\zeta}}k^2).$$

Using (179), (180) it is easy to show that $\|\hat{\Psi}_r\|_0<k^2$. Therefore,

$$\left\|\sum_r(\delta_1 s_{nr}+\delta_2 s_{nr})\hat{\Psi}_r(x)\right\|_0=O(k^{-3\tilde{\gamma}k^{\zeta}}k^4). \tag{197}$$

Substituting (196) in (194) and using (195), (197), we obtain:

$$\hat{\Psi}_n=\sum_{j\in Z_q(k,\gamma+2\zeta)}(G^{-1}\tilde{G})_{mn}\Psi_m+\epsilon_n(x),$$

$$\|\epsilon_n^*(x)\|=O(k^{-3\tilde{\gamma}k^{\zeta}}k^4)=O(k^{-2\tilde{\gamma}k^{\zeta}}).$$

Replacing index n by r, and denoting $(\hat{G}G^{-1})_{mr}$ by d_{mr}, we obtain (182).

The theorem is proved.

3 Appendices.

3.1 Appendix 1. Proof of the estimate (131).

Lemma 7 *Suppose j satisfies the conditions (10) and (9). Then,*

$$||S_j^{0,(non\ q)}-\tilde{S}_j||_\nu \leq c_b||A||_\nu k^{-1/2+\beta+5\varepsilon}), \tag{198}$$

$$||S_j^{0,(non\ q)}||_\nu = c_b||A||_\nu k^{2\varepsilon}), \tag{199}$$

$$||S_j^{n,(non\ q)}||_\nu < c_{||A||_{\nu+n}} k^{-1/2-\beta(2n-1)+4\varepsilon}). \tag{200}$$

Corollary 5

$$||S_j^{(non\ q)}-\tilde{S}_j||_\nu < c_b||A||_\nu k^{-1/2+\beta+5\varepsilon}). \tag{201}$$

Proof. First we prove (198). In fact, using the definitions of $S_j^{(0,\text{non}\ q)}$ and $\tilde{S}_j$ (see (15), (51) – (54), and (71)), we get

$$S_j^{(0,\text{non}\ q)}(x)-\tilde{S}_j(x)=h_1(x)+h_2(x),$$

where

$$h_n(x)=\sum_{r\in Z^2, 0<|r|<R_0, r\neq sq, s\in Z}(h_n)_r e^{i(\vec{p}_r(0),x)},\quad n=1,2,$$

$$(h_1)_r=-\left(2(\vec{p}_j(t),\vec{a}_r)+\alpha a_{0r}\right)\left(\frac{1}{|\vec{p}_{j+r}(t)|^2-|\vec{p}_j(t)|^2}-\right.$$

$$\left.\frac{1}{2(\vec{d}_j^{(q)}(t),\vec{p}_r(0))}\right).$$

$$(h_2)_r=-\frac{2(\vec{p}_j(t)-\vec{d}_j^{(q)}(t),\vec{a})}{2(\vec{d}_j^{(q)}(t),\vec{p}_r(0))}.$$

Using relations (56) and (126), we obtain

$$|(\vec{p}_j(t)-\vec{d}_j^{(q)}(t)|=|(\vec{p}_j(t),\vec{p}_q(0))|p_q^{-1}(0)=O(k^{1/2+\beta+\delta}), \tag{202}$$

$$2(\vec{d}_j^{(q)}(t),\vec{p}_r(0))-|\vec{p}_{j+r}(t)|^2+|\vec{p}_j(t)|^2=$$

$$2(\vec{d}_j^{(q)}(t),\vec{p}_r(0))-2(\vec{p}_j(t),\vec{p}_r(0))-p_r^2(0)=$$

$$-2(\vec{p}_j(t),\nu_q)(\vec{p}_r(0),\nu_q)-p_r^2(0)=O(k^{1/2+\beta+\varepsilon}),$$

the estimate $|r|<R_0<k^\varepsilon$ has been used. Considering the last estimate, and (76), we get

$$\left|\frac{1}{|\vec{p}_{j+r}(t)|^2-|\vec{p}_j(t)|^2}-\frac{1}{2(\vec{d}_j^{(q)}(t),\vec{p}_r(0))}\right|<$$

$$\frac{c_b k^{1/2+\beta+\varepsilon}}{\left(\vec{d}_j^{(q)}(t), \vec{p}_r(0)^2\right)} < c_b k^{-3/2+\beta+5\varepsilon}.$$

Taking into account that $\|2(\vec{p}_j, \vec{a}) + \alpha a_0\|_0 < \|A\|_0 k$, we obtain:

$$\|h_1(x)\|_\nu < c_b \|A\|_\nu k^{-1/2+\beta+5\varepsilon}).$$

Again, considering (202) and (76) it is easy to see that

$$\|h_2(x)\|_\nu < c_b \|A\|_\nu k^{-1/2+\beta+3\varepsilon}).$$

Adding the last two estimates, we get (198). Formula (199) follows from (77) and (198).

Now we prove (200). Using (17), we obtain:

$$\|\tilde{a}_j^{(1,\text{non } q)}\|_\nu \leq 2^{\nu+1} \left(\|S_j^{(0)}\|_{\nu+1} \|\vec{a}\|_\nu\| + \|S_j^{(0,\text{non } q)}\|_\nu \|S_j^{(0}\|_\nu + \|a\|_\nu^2 \right). \qquad (203)$$

Using (199) and (21) for n=0, we obtain

$$\|\tilde{a}_j^{(1,\text{non } q)}\|_\nu \leq c_{\|\vec{A}\|_{\nu+1}} k^{1/2-\beta+2\varepsilon} \qquad (204)$$

Estimate (16) yields

$$\|S_j^{(1,\text{non } q)}\|_\nu < ck^{-1+2\varepsilon} \|\tilde{a}_j^{(1,\text{non } q)}\|_\nu.$$

From the last two estimates we obtain (**??**) for $n = 1$. Further we proceed with the method of mathematical induction. Further, using formulae (18) and (19), we get (200). The lemma is proved.

3.2 Appendix 2 (Proof of Lemma 4).

Let us rewrite the operator H in the form

$$H = (-i\nabla + \vec{a}, -i\nabla + \vec{a}) + \alpha a_0,$$

and Ψ_j in the form $\Psi_j = \psi_j e^{S(x)}$, where

$$S(x) = i(\vec{d}_j^{(q)}, x) + \tilde{S}_j(t). \qquad (205)$$

It is easy to see that

$$(-i\nabla + \vec{a})\,\Psi_j = (-i\nabla\psi_j - i\psi_j \nabla S + \vec{a}\psi_j)\, e^S,$$

and further,

$$\begin{gathered} (-i\nabla + \vec{a}, -i\nabla + \vec{a})\Psi_j^0 + \alpha a_0(x)\Psi_j^0 = \\ (-\Delta\psi_j - (\nabla\psi_j \nabla S) - \psi_j \Delta S - i(\nabla\psi_j, \vec{a}))\, e^S \\ + (-i\nabla\psi_j - i\psi_j\nabla S + \vec{a}\psi_j, -i\nabla S + \vec{a})\, e^S \end{gathered} \qquad (206)$$

$$+\alpha a_0(x)\psi_j e^S.$$

By formula (205),

$$\nabla S(x) = i\vec{d}_j^{(q)} + \nabla\tilde{S}_j(x), \tag{207}$$

$$\Delta S(x) = \Delta\tilde{S}_j(x), \tag{208}$$

By definition $\psi_j(x)$ depends only on $(\vec{p}_q(0), x)$ and $\vec{d}_j^{(q)}(t)\perp\vec{p}_q(0)$, therefore,

$$\left(\vec{d}_j^{(q)}, \nabla\psi_j(x)\right) = 0. \tag{209}$$

Using (207) – (209), we simplify (206):

$$\begin{gathered}
(-i\nabla + \vec{a}, -i\nabla + \vec{a})\Psi_j^0 + \alpha a_0\Psi_j^0 = \\
\left(-\Delta\psi_j - 2(\nabla\psi_j\nabla\tilde{S}_j) - \psi_j\Delta\tilde{S}_j - 2i(\nabla\psi_j, \vec{a})\right) e^S \\
+ \left(|\vec{d}_j^{(q)}|^2 - 2i(\vec{d}_j^{(q)}, \nabla\tilde{S}_j) - (\nabla\tilde{S}_j, \nabla\tilde{S}_j) - 2i(\nabla\tilde{S}_j, \vec{a})\right)\psi_j e^S \\
\left(2(\vec{d}_j^{(q)}, \vec{a}) + (\vec{a}, \vec{a}) + \alpha a_0\right)\psi_j e^S.
\end{gathered} \tag{210}$$

Taking into account (71) we readily show that

$$2i\left(\vec{d}_j^{(q)}(t), \nabla\tilde{S}_j(x)\right) = 2\left(\vec{d}_j^{(q)}(t), \vec{a}^{(\text{non }q)}(x)\right) + \alpha a_0^{(\text{non }q)}(x). \tag{211}$$

Using the last relation we simplify (210) further:

$$\begin{gathered}
(-i\nabla + \vec{a}, -i\nabla + \vec{a})\Psi_j^0 + \alpha a_0(x)\Psi_j^0 = \\
\left(-\Delta\psi_j - 2(\nabla\psi_j\nabla\tilde{S}_j) - \psi_j\Delta\tilde{S}_j - 2i(\nabla\psi_j, \vec{a})\right) e^S \\
+ \left(|\vec{d}_j^{(q)}(t)|^2 - (\nabla\tilde{S}_j, \nabla\tilde{S}_j) - 2i(\nabla\tilde{S}_j, \vec{a}) + 2(\vec{d}_j^{(q)}, \vec{a}^{(q)}) + \right. \\
\left. \alpha a_0^{(q)}(x) + (\vec{a}, \vec{a})\right)\psi_j e^S
\end{gathered} \tag{212}$$

By (63),

$$(\nabla\psi_j(x), \vec{a}^{(q)}) = a^{(q\|)}(x)\ \phi'_{l_j}(\vec{d}_j^{(q)}, \tau_j, y)\Big|_{y=(x,\nu_q)},$$

$$\Delta\psi_j(x) = \phi''_{l_j}(\vec{d}_j^{(q)}, \tau_j, y)\Big|_{y=(x,\nu_q)}.$$

Using the last two relations and (74), we get:

$$\begin{gathered}
(-i\nabla + \vec{a}, -i\nabla + \vec{a})\Psi_j + \alpha a_0(x)\Psi_j = \\
\left(|\vec{d}_j^{(q)}(t)|^2 + \mu_{l_j}\right)\psi_j e^S - 2\left(\nabla\psi_j, \nabla\tilde{S}_j + i\vec{a}^{(\text{non }q)}\right) e^S \\
- \left((\nabla\tilde{S}_j, \nabla\tilde{S}_j + 2i\vec{a}) - (\vec{a}, \vec{a})\right)^{(\text{non }q)}\psi_j e^S.
\end{gathered}$$

Using notations (82) and (81), we get (80) and (83). Estimates (84) and (85) follow from (77). The lemma is proved.

3.3 Appendix 3. The equation for $f_{l_j}(\tau_j, \vec{d}_j^{(q)}(t), y)$.

Lemma 8 *If* $|\tau_j + l_j|p_q^2(0) > \hat{c}k^{1/2+\beta}$, *then the function*

$$f_{l_j}(\tau_j, \vec{d}_j^{(q)}(t), y) = e^{i(\tau_j + l_j) + S_j^{(q)}(y)},$$

satisfies the equation

$$H_1^{(q,\vec{d}_j^{(q)}(t))} f_{l_j} = \left(\mu_{l_j} + \zeta_j(y)\right) f_{l_j}, \tag{213}$$

where $H_1^{(q,\vec{d}_j(t))}$ *is defined by (72),* $\mu_{l_j} = \mu_{l_j}(\tau_j, \vec{d}_j(t))$ *is an eigenvalue of* $H_1^{(q,\vec{d}_j(t))}$. *The eigenvalue* μ_{l_j} *admits the representation:*

$$\mu_{l_j} = p_q^2(0)(\tau_j + l_j)^2 + \Delta\mu_{l_j},$$

where $p_q^2(0)(\tau_j + l_j)^2$ *is the unperturbed eigenvalue corresponding to* $w^{(q,\vec{d}_j^{(q)}(t))}(y) = 0$ *and* $\Delta\mu_{l_j} = \Delta\mu_{l_j}(\tau_j, l_j)$ *is the shift of the eigenvalue under the perturbation satisfying the estimate:*

$$|\Delta\mu_{l_j}| < C_{\|A\|_1} k^{1-2\beta}. \tag{214}$$

The function $\zeta_j(y)$ *in (213) obeys the estimate*

$$\|\zeta_j(y)\|_0 < c_{\|A\|_1} k^{-1/2+\beta+7\varepsilon}). \tag{215}$$

<u>Proof.</u> Let $\vec{p}_j(t) = (\tau_j + l_j)\vec{p}_q(0) + \vec{d}_j^{(q)}(t)$, $|\tau_j + l_j|p_q^2(0) > \hat{c}k^{1/2+\beta}$. First we take ${}^{(q)}$-part of the both sides of the equation (12). We obtain

$$2i\left(\vec{p}_j(t), \nabla S_j^{(q)}(x)\right) + \Delta S_j^{(q)}(x) + (\nabla S_j(x), \nabla S_j(x))^{(q)} + 2i\,(\nabla S_j(x), \vec{a}(x))^{(q)}$$

$$= 2\left(\vec{p}_j(t), \vec{a}^{(q)}(x)\right) + (\vec{a}(x), \vec{a}(x))^{(q)} + \alpha a_0(x)^{(q)} + C_{j,t,N} + \epsilon_N^{(q)}(x),$$

$$\|\epsilon_N^{(q)}(x)\|_0 < c_{\|A\|_{N+2}} k^{1-2(N+2)\beta}.$$

Note that

$$(\nabla S_j(x), \nabla S_j(x))^{(q)} = \left(\nabla S_j^{(q)}(x), \nabla S_j^{(q)}(x)\right) + \left(\nabla S_j(x)^{(\text{non } q)}, \nabla S_j^{(\text{non } q)}(x)\right)^{(q)},$$

since

$$\left(\nabla S_j^{(q)}(x), \nabla S_j^{(\text{non } q)}(x)\right)^{(q)} = 0.$$

Similarly,

$$(\nabla S_j(x), \vec{a}(x))^{(q)} = \left(\nabla S_j^{(q)}(x), \vec{a}^{(q)}(x)\right) + \left(\nabla S_j^{(\text{non } q)}(x), a^{(\text{non } q)}(x)\right)^{(q)},$$

$$(\vec{a}(x), \vec{a}(x))^{(q)} = \left(\vec{a}^{(q)}(x), \vec{a}^{(q)}(x)\right) + \left(\vec{a}^{(\text{non } q)}(x), a^{(\text{non } q)}(x)\right)^{(q)}.$$

Thus,

$$2i\left(\vec{p}_j(t), \nabla S_j^{(q)}(x)\right) + \Delta S_j^{(q)}(x) + \left(\nabla S_j^{(q)}(x), \nabla S_j^{(q)}(x)\right) +$$

$$2i\left(\nabla S_j^{(q)}(x),\vec{a}^{(q)}(x)\right)=2\left(\vec{p}_j(t),\vec{a}^{(q)}(x)\right)+\left(\vec{a}^{(q)}(x),\vec{a}^{(q)}(x)\right)+w_j(x)+$$
$$+C_{j,t,N}+\epsilon_N^{(q)}(x), \tag{216}$$

where

$$w_j(x)=\alpha a_0(x)^{(q)}-\left(\nabla S_j(x)^{(\text{non } q)},\nabla S_j^{(\text{non } q)}(x)\right)^{(q)}-$$
$$2i\left(\nabla S_j^{(\text{non } q)}(x),\vec{a}^{(\text{non } q)}(x)\right)^{(q)}+\left(\vec{a}^{(\text{non } q)}(x),\vec{a}^{(\text{non } q)}(x)\right)^{(q)}, \tag{217}$$

Equation (216), which $S_j^{(q)}(x)$ satisfies, is analogous to (12) up to replacement $\vec{a}$ by $\vec{a}^{(q)}$, $\alpha a_0(x)$ by $w_j(x)$. By analogy with Lemma 3, this means that $F_j(x)=e^{i(\vec{p}_j(t),x)+S_j^{(q)}(x)}$ obeys the equation:

$$\left(\frac{1}{i}\nabla+\vec{a}^{(q)}(x)\right)^2F_j(x)+w_j(x)F_j(x)+\epsilon_N^{(q)}(x)F_j(x)=$$
$$(p_j^2(t)-C_{j,t,N})F_j(x).$$

Let us introduce a new system of orthogonal coordinates (y,z), y being directed along $\vec{p}_q(0)$: $y=(\vec{p}_q(0),x)p_q^{-1}(0)$. In the new variables the last equation has the form:

$$\left(\frac{\vec{\nu}_q}{i}\frac{d}{dy}+\vec{d}_j^{(q)}+\vec{a}^{(q)}(y)\right)^2f_{l_j}(y)+w_j(y)f_{l_j}(y)$$
$$+\epsilon_N^{(q)}(y)f_{l_j}(y)=(p_j^2(t)-+C_{j,t,N})f_{l_j}(y).$$

Considering that $\vec{d}_j^{(q)}(t)\perp\nu_q$ and $p_j^2(t)=|\vec{d}_j^{(q)}(t)|^2+(l_j+\tau_j)^2p_q^2(0)$, we obtain:

$$\left(\frac{1}{i}\frac{d}{dy}+a^{(q\|)}(y)\right)^2f_{l_j}(y)+2|\vec{d}_j^{(q)}(t)|a^{(q\perp)}(y)f_{l_j}(y)+a^{(q\perp)}(y)^2f_{l_j}(y)+$$
$$w_j(y)f_{l_j}(y)=\left((l_j+\tau_j)^2p_q^2(0)-C_{j,t,N}\right)f_{l_j}(y)-\epsilon_N^{(q)}(y)f_{l_j}(y).$$

We show that

$$2|\vec{d}_j^{(q)}(t)|a^{(q\perp)}(y)+a^{(q\perp)}(y)^2+w_j(y)=w^{(q,\vec{d}_j^{(q)}(t))}(y)+\delta w_j(y), \tag{218}$$

where $\|\delta w_j(y)\|_0<c_{\|A\|_1}k^{-1/2+\beta+7\varepsilon})$ and $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ is given by (74). First, we replace $\nabla S_j^{(\text{non } q)}(x)$ by $\tilde{S}_j(x)$ in formula (217) for $w_j(t)$. Using (201) and (199) we obtain:

$$w_j(y)=\tilde{w}_j(y)+\delta w_j(y),$$

where

$$\tilde{w}_j(x)=\alpha a_0(x)^{(q)}-\left(\nabla\tilde{S}_j(x),\nabla\tilde{S}_j(x)\right)^{(q)}-$$
$$2i\left(\nabla\tilde{S}_j(x),\vec{a}^{(\text{non } q)}(x)\right)^{(q)}+\left(\vec{a}^{(\text{non } q)}(x),\vec{a}^{(\text{non } q)}(x)\right)^{(q)},$$
$$\|\delta w_j(y)\|_0<C_{\|A\|_1}k^{-1/2+\beta+7\varepsilon}).$$

Taking into account that

$$\nabla\tilde{S}_j(x)=\nabla\tilde{S}_j^{(\text{non } q)}(x),$$

and, therefore,

$$(\tilde S_j(x), \vec a^{(\mathrm{non}\ q)})^{(q)} = (\tilde S_j(x), \vec a)^{(q)}$$

and

$$(\vec a^{(\mathrm{non}\ q)}, \vec a^{(\mathrm{non}\ q)})^{(q)} + (\vec a^{(q\perp)}, \vec a^{(q\perp)}) = (\vec a, \vec a)^{(q)} - (\vec a^{(q\|)}, \vec a^{(q\|)}),$$

we readily see that $2|\vec d_j^{(q)}(t)|a^{(q\perp)}(y) + a^{(q\perp)}(y)^2 + \tilde w_j(y) = w^{(q,\vec d_j^{(q)}(t))}(y)$. Therefore, (218) holds. Thus,

$$H_1^{(q,\vec d_j^{(q)}(t))} f_{l_j} = \Big((l_j+\tau_j)^2 p_q^2(0) - C_{j,t,N} + \zeta_j(y)\Big) f_{l_j}, \tag{219}$$

where $\zeta_j(y) = -\delta w_j(y) - \epsilon_N^{(q)}(y)$, and, hence, $\|\zeta_j(y)\|_0 < C_{\|A\|_1} k^{-1/2+\beta+7\varepsilon})$. Formula (219) means [?] that there exists an eigenvalue μ_{l_j} such that $\mu_{l_j} = p_q^2(0)(l_j+\tau_j)^2 + \Delta\mu_{l_j}$, where

$$\Delta\mu_{l_j} = -C_{j,t,N} + O(k^{-1/2+\beta+7\varepsilon}). \tag{220}$$

Therefore, we can replace $-\tilde C_j$ in (219) by $\Delta\mu_{l_j} + O(k^{-1/2+\beta+7\varepsilon})$, and then replace $O(k^{-1/2+\beta+7\varepsilon}) + \zeta_j(y)$ just by $\zeta_j(y)$. Using (40), we obtain (214). The lemma is proved.

3.4 Appendix 4. The main part of the potential $w^{(q,\vec d_j^{(q)}(t))}(y)$.

Lemma 9 *The potential $w^{(q,\vec d_j^{(q)}(t))}(y)$ can be represented in the form:*

$$w^{(q,\vec d_j^{(q)}(t))}(y) = 2|\vec d_j^{(q)}(t)|a_q^{(q\perp)}(y) + \alpha a_0^{(q)}(y) + \tilde w(y),$$

$$\|\tilde w(y)\|_0 = O(k^{2\varepsilon}). \tag{221}$$

Proof. Using (74), we easily obtain:

$$\tilde w(y) = (a,a)^{(q)}(y) - a^{(q\|)2}(y) + (\vec a^{(q)}, \vec a^{(q)})_0 - 2i(\nabla\tilde S_j, \vec a)^{(q)}(y) - (\nabla\tilde S_j, \nabla\tilde S_j)^{(q)}(y).$$

Using the estimate (77) for $\tilde S_j$, we easily arrive at (221). The lemma is proved.

3.5 Appendix 5. Eigenvalues and eigenfunctions of operator $H_1^{q,\vec d_j^{(q)}(t)}(\tau_j)$.

According to (72) and (73) the operator $H_1^{q,\vec d_j^{(q)}(t)}(\tau_j)$ is defined by the differential expression

$$H_1^{q,\vec d_j^{(q)}(t)}(\tau_j)u(y) = \left(\frac{1}{i}\frac{\partial}{\partial y} + a^{(q\|)}(y)\right)^2 u(y) + w^{(q,\vec d_j^{(q)}(t))}(y)u(y) \tag{222}$$

and the quasiperiodic boundary conditions:

$$u(b_q) = \exp(i2\pi\tau_j)u(0), \quad u'(b_q) = \exp(i2\pi\tau_j)u'(0), \tag{223}$$

$b_q = 2\pi p_q^{-1}(0)$. It was proven in Appendix 3 that the functions $f_l(\vec{d}_j^{(q)}(t), \tau_j, y)$, $|l| > k^{1/2+\beta}$ approximately satisfy equations for eigenfunctions (213). This means that $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ has eigenvalues μ_l, $|l+\tau_j| > \hat{c}k^{1/2+\beta}$ obeying the asymptotic formula

$$\mu_l = p_q^2(0)(\tau_j + l)^2 + \Delta\mu_l(\vec{d}_j^{(q)}(t), \tau_j) + O(k^{-1/2+\beta+7\varepsilon}), \tag{224}$$

$$\Delta\mu_l(\vec{d}_j^{(q)}(t), \tau_j) = O(k^{1-2\beta}), \tag{225}$$

Relation (225) means that the shift of eigenvalues under the perturbation is of order $k^{1-2\beta}$ Note that it is greater than a typical distance between eigenvalues, which is of order of $k^{1/2+\beta}$. Does it mean that the distance between eigenvalues can change significantly under the perturbation? In the next lemma we show that this can not be the case, since close eigenvalues are moving under the perturbation with approximately the same speed. The distance between them can not change at most by the value $ck^{1/2-3\beta}$ which is obviously greater than $k^{1/2+\beta}$.

Lemma 10 *If* $|l+\tau_j|p_q^2(0), |m+\tau_m|p_q^2(0) > \hat{c}k^{1/2+\beta}$, *then there are the estimates:*

$$\Delta\mu_l - \Delta\mu_m = O(|l-m|k^{\gamma_1}),, \tag{226}$$

$$\Delta\mu_l - \Delta\mu_{-l} = O(k^{\gamma_1}),, \tag{227}$$

$$\gamma_1 = \max\{1/2 - 3\beta, -1/2 + \beta + 7\varepsilon\}.$$

Proof. Estimate (226) easily follows from (220) and (41). To prove (227) let us check that

$$-\nabla S_j(x)^{(q)} - i\vec{a}^{(q\|)} = \overline{\nabla S_m(x)^{(q)} + i\vec{a}^{(q\|)}} + O(k^{-2\beta}), \tag{228}$$

in the norm $\|\cdot\|_0$ where $m : \vec{d}_m^{(q)}(t) = \vec{d}_j^{(q)}(t)$, $\tau_m = \tau_j$, $l_m = -l_j$. In fact, using (15), we obtain that

$$-\nabla S_j^{(0)(q)} - i\vec{a}^{(q\|)} = \overline{\nabla S_m^{0(q)} + i\vec{a}^{(q\|)}} + O(k^{-2\beta}). \tag{229}$$

It follows from (18) that

$$\|\nabla S_j^{(n)(q)} - \overline{\nabla S_m^{(n)(q)}}\|_0 < c_b\|\Im\tilde{a}_j^{(n)}\|_1 k^{-1/2-\beta} + c_b\|\tilde{a}_j^{(n)}\|_2 k^{-1-2\beta}, \quad n \geq 1.$$

Using (24) and (25) we obtain

$$-\nabla S_j^{(n)(q)} = \overline{\nabla S_{\tilde{j}}^{(n)(q)}} + O(k^{-(2n+2)\beta}), \quad n \geq 1. \tag{230}$$

Adding (229) and (230), we obtain (228). Using (17), (22) and (229), it is easy to show that $|a_{j0}^{(1)} - a_{m0}^{(1)}| < c_{\|A\|_2}k^{1/2-3\beta}$. Using (19) and (230) we obtain $|a_{j0}^{(n)} - a_{m0}^{(n)}| < c_{\|A\|_2}k^{1/2-(2n+1)\beta}$. Using formula (39) for $C_{j,t,N}$, we obtain $C_{j,t,N} - C_{m,t,N} = O(k^{1/2-3\beta})$. Taking into account (220), we arrive at (227) .The lemma is proved.

It is not difficult to show [1] that formula (224) describes all eigenvalues in the high energy region. Let $\hat{c}$ be chosen in such a way that

$$\min_{l} \left| \hat{c}^2 k^{1+2\beta} - p_q^2(0)(\tau_j + l)^2 \right| > k^{1/2+\beta} p_q^2(0)/16. \tag{231}$$

Let us consider the operator $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j)$ with the potential $\xi w^{(q,\vec{d}_j^{(q)}(t))}(y)$, $0 \leq \xi \leq 1$. It follows from (224), (220)¿ and (40) that eigenvalues $\mu_l(\xi)$ obey the estimate $\mu_l(\xi) = p_q^2(0)(\tau_j + l)^2 + O(k^{1-2\beta})$. Thus the shift of eigenvalue $\mu_l(\xi)$ under the perturbation is of order $O(k^{1-2\beta})$. Formulae (226) and (227) mean that closely situated eigenvalues μ_l are moving simultaneously under the perturbation, i.e., distance between them can change only by the value of order $O(k^{\gamma_1})$, which is less then $O(k^{1-2\beta})$. Therefore, for every ξ there is $\hat{c}(\xi)$, depending continuously on ξ such that $\hat{c}(0) = \hat{c}$, $\hat{c}(\xi) - \hat{c} = O(k^{1-2\beta})$ and

$$\min_{l} \left| \hat{c}^2(\xi) k^{1+2\beta} - \mu_l(\xi) \right| > k^{1/2+\beta} p_q^2(0)/16. \tag{232}$$

Since (232) holds for any ξ, the number of eigenvalues satisfying the inequality $\mu_l < c^2(\xi) k^{1+2\beta}$ does not change with ξ. This means that for any ξ they can be enumerated by indices $l \in Z$, which satisfy the inequality $p_q^2(0)(\tau_j + l)^2 < \hat{c}^2(\xi) k^{1+2\beta}$. The rest of eigenvalues obey formula (224), where $l : p_q^2(0)(\tau_j + l)^2 \geq \hat{c}^2(\xi) k^{1+2\beta}$.

Lemma 11 *If* $\vec{d}_j(t) = \vec{d}_m(t)$,

$$|(\vec{p}_j(t), \vec{p}_q(0))| \geq \hat{c} k^{1/2+\beta}, \ |(\vec{p}_m(t), \vec{p}_q(0))| < \hat{c} k^{1/2+\beta},$$

then

$$\hat{b}_q^2(\tau_j + l_j)^2 + c_{l_j} - \mu_{l_m} > k^{1/2+\beta} p_q^2(0)/32. \tag{233}$$

Proof. According to (231), (232)

$$\mu_{l_j} - \mu_{l_m} > k^{1/2+\beta} p_q^2(0)/8$$

if $p_q^2(0)(\tau_m + l_m)^2 < \hat{c}^2 k^{1+2\beta}$, i.e., $|(\vec{p}_m(t), \vec{p}_q(0))| < \hat{c} k^{1/2+\beta}$, and

$$\mu_{l_j}(\xi) - \hat{c}^2(\xi) k^{1+2\beta} > k^{1/2+\beta} p_q^2(0)/16$$

if $p_q^2(0)(\tau_j + l_j)^2 > c^2 k^{1+2\beta}$, i.e., $|(\vec{p}_j(t), \vec{p}_q(0))| > \hat{c} k^{1/2+\beta}$. Adding the last estimates, we obtain $\mu_{l_j}(\xi) - \mu_{l_m}(\xi) > k^{1/2+\beta} p_q^2(0)/16$. The lemma is proved.

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[1] f.e. by reducing the equation $H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_j) = \lambda u$, $\lambda > \|\xi V\|$ to a Schrödinger equation with a smaller periodic potential by the change of variables $dy = \sqrt{\lambda - \xi V} dx$ and applying standard perturbation theory considerations.

3.6 Appendix 6.

Lemma 12 *Suppose $E > k^{1+2\beta}$, $\min_{l\in Z}|E-\mu_l| > ck^{1/2+\beta}$, then (145) holds.*

Proof. Let us consider first $|l| < k^{1/2+\beta/2}$. Then, $E-\mu_l > ck^{1+2\beta}$. We replace potential $w^{(q,\vec{d}_j^{(q)}(t))}(y)$ by a constant potential $w_0 = \max_y w^{(q,\vec{d}_j^{(q)}(t))}(y)$. The eigenvalues $\tilde{\mu}_l$ corresponding to this potential is easy to calculate: $\tilde{\mu}_l = w_0 + (2\pi l + \tau_j)^2 b_q^{-2}$. Obviously, $0 < \tilde{\mu}_l - \mu_l < \max_y w^{(q,\vec{d}_j^{(q)}(t))}(y) - \min_y w^{(q,\vec{d}_j^{(q)}(t))}(y) < c_b k = o(k^{1+2\beta})$. Therefore,

$$\sum_{|l|<k^{1/2+\beta/2}} \frac{1}{|E-\mu_l|} < 2\sum_{|l|<k^{1/2+\beta/2}} \frac{1}{|E-\tilde{\mu}_l|} < ck^{-1/2-\beta}. \qquad (234)$$

If $k^{1/2+\beta/2} \geq |l|$, then $\mu_l = (2\pi l+\tau_j)^2 b_q^{-2} + \Delta\mu_l$, where $|\Delta\mu_l - \Delta\mu_{l'}| < |l-l'|k^{1/2-3\beta}$, $\Delta\mu_l = O(k|l|^{-2})$, since $\|w^{(q,\vec{d}_j^{(q)}(t))}\|_0 < ck$. Using these properties and the estimate $\min_{l\in Z}|E-\mu_l| > ck^{1/2+\beta}$ it is easy to show that

$$\sum_{|l|>k^{1/2+\beta/2}} \frac{1}{|E-\mu_l|} < 2\sum_{|l|>k^{1/2+\beta/2}} \frac{1}{|E-(2\pi l+\tau)^2 b_q^{-2} + \Delta\mu_{l_0}|} < ck^{-1/2-\beta}\ln k, \qquad (235)$$

l_0 being l for which $|E-\mu_l|$ riches its minimum. Adding (234) and (235), we obtain (145). The lemma is proved.

3.7 Appendix 7.

It is easy to check by direct calculation that $g_{jr}(y)$ satisfies the equation

$$H_1^{q,\vec{d}_j^{(q)}(t)}(\tau_m) g_{jr}(y) = \left(p_q^2(0)(l_{j-r}+\tau_m)^2 + \Delta\mu_{l_j}(\tau_j) + \zeta_{jr}(y)\right) g_{jr}(y),$$

$$\|\zeta_{jr}\|_0 = O(k^{1/2-\beta}|r|).$$

Considering (226), we obtain

$$\Delta\mu_{l_j}(\tau_m) - \Delta\mu_{l_{j-r}}(\tau_m) = O(|r|k^{\gamma_1}).$$

The last estimate yields (151).

3.8 Appendix 8

We introduce $\hat{\phi}_l$ by the formula

$$\hat{\phi}_l(\vec{d}_j^{(q)}(t), \tau_j, y) = \begin{cases} \phi_l(\vec{d}_j^{(q)}(t), \tau_j, y), & \text{if } p_q^2(0)|\tau_j + l| < \hat{c}k^{1/2+\beta}, \\ f_l(\vec{d}_j^{(q)}(t), \tau_j, y), & \text{otherwise.} \end{cases} \qquad (236)$$

Let $\hat{G}$ be the Gram matrix of the system $\{\hat{\phi}_l\}_{l\in Z}$ in $L_2(0,b_q)$: $\hat{G}_{ml} =< \hat{\phi}_l, \hat{\phi}_m >_1$.

Lemma 13 *Matrix $\hat{G}$ satisfies to the asymptotic relation:*

$$\hat{G} = I + \delta\hat{G}, \ \|\delta\hat{G}\|^* =_{k\to\infty} O(k^{-2\beta}), \tag{237}$$

Matrix $\delta\hat{G}$ admits the representation: $\delta\hat{G} = \delta_1\hat{G} + \delta_2\hat{G}$, where

$$\left(\delta_1\hat{G}\right)_{ml} = 0 \ \mathit{if}\ |m-l| > k^{1/2+\gamma}, \ \beta < \gamma < 1/2, \tag{238}$$

$$\|\delta_2\hat{G}\|_* < k^{-\beta(\gamma-\beta)k^{1/2-\beta-\varepsilon}}. \tag{239}$$

<u>Proof.</u> Obviously

$$< \phi_l, \phi_m >_1 = \delta_{lm}. \tag{240}$$

Let us consider $< f_l, f_m >_1$. Using formulae for f_l, f_m we obtain:

$$< f_l, f_m >_1 = \int_0^{b_q} \exp\left(i(l-m)y + S_l^{(q)}(y) + \bar{S}_m^{(q)}(y)\right) dy.$$

If $l = m$ then using (28) we get $< f_l, f_l >_1 = 1 + O(k^{-2\beta})$. If $l \neq m$, then (28) and (30) yield: $\|S_l^{(q)}(y) + \bar{S}_m^{(q)}(y)\|_\nu < c\|\vec{A}\|_{\nu+1}|l-m|k^{-2\beta}$. Suppose $|l-m| < k^{4\beta}$. Then, integrating by parts three times and summarizing over m, we obtain:

$$\sup_l \sum_{0<|m-l|<k^{4\beta}} | < f_l, f_m >_1 | < c_{\|\vec{A}\|_4} k^{-2\beta}. \tag{241}$$

Let $|l-m| \geq k^{4\beta}$. Taking into account that $S_l^{(q)} + \bar{S}_m^{(q)}$ is a trigonometric polynomial of the length NR_0, we obtain by induction:

$$\|e^{S_l^{(q)} + \bar{S}_m^{(q)}}\|_\nu < c\|\vec{A}\|_0 |l-m|k^{-2\beta}\left(\nu N R_0 + |l-m|k^{-2\beta}\right)^{\nu-1}, \ \nu \geq 1.$$

Integrating by parts $\nu_m = \left[|l-m|k^{-2\beta}/NR_0\right] + 1$ times, we obtain:

$$| < f_l, f_m >_1 | < \|\vec{A}\|_0 \left(ck^{-2\beta}\right)^{\nu_m}. \tag{242}$$

Summarizing over m, we arrive to the inequality:

$$\sup_l \sum_{m:|m-l|\geq k^{4\beta}} | < f_l, f_m >_1 | < c_{\|A\|_0} k^{-2\beta}. \tag{243}$$

Adding (241) and (243), we obtain:

$$\sup_l \sum_m | < f_l, f_m >_1 | = 1 + O_{\|A\|_4}(k^{-2\beta}). \tag{244}$$

Next, we consider $< \phi_l, f_m >_1$. Considering that ϕ_l satisfies an equation for an eigenfunction and f_m satisfies an analogous equation approximately (Appendix 3), we obtain:

$$| < \phi_l, f_m >_1 | < \frac{c_{\|A\|_0} k^{-1/2+\beta+7\varepsilon}}{p_q^2(0)(\tau_j + m)^2 + \Delta\mu_m - \mu_l}. \tag{245}$$

Taking into account (143) and summarizing over m, we get

$$\sup_l \sum_m | < \phi_l, f_m >_1 | = O(k^{-1+8\varepsilon}). \tag{246}$$

We showed in Appendix 6 that

$$\sum_{l \in Z} \frac{1}{|E - \mu_l|} < ck^{-1/2-\beta} \ln k, \tag{247}$$

for any E such that $\min_{l \in Z} |E - \mu_l| > ck^{1/2+\beta}$, $E > ck^{1+2\beta}$. Setting $E = p_q^2(0)(\tau_j + m)^2 + \Delta\mu_m$ and considering (245), (247), we obtain

$$\sup_m \sum_l | < \phi_l, f_m >_1 | = O(k^{-1+8\varepsilon}). \tag{248}$$

From formulae (240), (246) and (248) we obtain (237).

We define $\delta_1 \hat{G}$ as follows:

$$\delta_1 \hat{G} = \begin{cases} \delta \hat{G} & \text{if } |m - l| \le k^{1/2+\gamma}, \\ 0, & \text{otherwise;} \end{cases}$$

and, correspondingly,

$$\delta_2 \hat{G} = \begin{cases} \delta \hat{G} & \text{if } |m - l| > k^{1/2+\gamma}, \\ 0, & \text{otherwise.} \end{cases}$$

It remains to prove the estimate for (239). Obviously, $(\delta_2 \hat{G})_{ml} = 0$, if $|j| < \hat{c}k^{1/2+\beta}$ and $|m| < \hat{c}k^{1/2+\beta}$, since ϕ_l and ϕ_m, $m \ne l$, are orthogonal. Suppose $|m| \ge \hat{c}k^{1/2+\beta}$ and $|l| \ge \hat{c}k^{1/2+\beta}$, then $(\delta_2 \hat{G})_{ml} =< f_l, f_m >_1$. Using (242) and taking into account that $\nu_m > k^{1/2+\gamma-2\beta-\varepsilon}$ when $|l - m| > k^{1/2+\gamma}$ and summarizing over m, we obtain

$$\sup_l \sum_{|m-l| > k^{1/2+\gamma}} | < f_l, f_m >_1 | < ck^{-2\beta k^{1/2+\gamma-2\beta-\varepsilon}}. \tag{249}$$

Suppose $|l| < \hat{c}k^{1/2+\beta}$ and $|m| \ge \hat{c}k^{1/2+\beta}$. Then $(\delta_2 \hat{G})_{ml} =< \phi_l, f_m >_1$. Using the equation for ϕ_l and taking into account that $w^{(q, \vec{d}_j^{(q)}(t))}(y)$ is a trigonometric polynomial of the length $2R_0$, while $\|w^{(q, \vec{d}_j^{(q)}(t))}\|_0 < ck$, we show that

$$\|D^\nu \phi_l\|_{L_2} < ck^{(1/2+\beta)\nu} \text{ if } \nu < k^{1/2+\beta} R_0^{-1}.$$

Taking into account that $S_m^{(q)}$ is a trigonometric polynomial of the length NR_0 and using (27), we obtain by induction:

$$\|D^\nu e^{\bar{S}_m^{(q)}}\|_0 < (ck^{1/2-\beta})^\nu, \text{ if } \nu < k^{1/2-\beta}/NR_0.$$

Therefore,

$$\left\| D^\nu \left(\phi_l \overline{e^{\bar{S}_m^{(q)}}} \right) \right\|_{L_1} < (ck^{1/2+\beta})^\nu, \text{ if } \nu < k^{1/2-\beta}/NR_0.$$

Integrating by parts $\nu \approx k^{1/2-\beta}/NR_0$ times we obtain:

$$| < \phi_l, f_m >_1 | < (ck^{1/2+\beta}|m|^{-1})^{k^{1/2-\beta}/NR_0},$$

and, therefore,

$$\sup_l \sum_{m:|m-l|>k^{1/2+\gamma}} | < \phi_l, f_m >_1 | < ck^{1/2+\beta}k^{-(\gamma-\beta)k^{1/2-\beta-\varepsilon}}. \qquad (250)$$

and similarly,

$$\sup_m \sum_{l:|m-l|>k^{1/2+\gamma}} | < \phi_l, f_m >_1 | < ck^{1+\gamma}k^{-(\gamma-\beta)k^{1/2-\beta-\varepsilon}}. \qquad (251)$$

Summarizing inequalities (249), (250) and (251), and taking into account that $\gamma > \beta$, we obtain (238). The lemma is proved.

3.9 Appendix 9

Lemma 14 *Let* $\phi_l^0 = e^{ip_q(0)(\tau_j+l)y}$, $|l| < k^{1/2+\gamma}$, $\beta+\varepsilon<\gamma<1/2$, *then*

$$\phi_l^0(y) = \sum_{m:|m|<k^{1/2+\gamma+\zeta}} d_{lm}\hat\phi_m(\vec{d_j}(t), \tau_j, y) + \tilde\varepsilon_l, \quad \|\tilde\varepsilon\|_0 = O(k^{-\beta(\gamma-\beta)k^{\zeta}}) \qquad (252)$$

for any $0<\zeta<1/2-\gamma$.

Proof. Since the system $\{\hat\phi_m\}_{m\in Z}$ is complete,

$$\phi_l^0 = \sum_{m\in Z} d_{lm}\hat\phi_m. \qquad (253)$$

Hence, it is enough to prove that

$$\sum_{m:|m|\geq k^{1/2+\gamma+\zeta}} |d_{lm}| = O(k^{-\beta(\gamma-\beta)k^{\zeta}}). \qquad (254)$$

We introduce the notations:

$$\vec{d_l} = \{d_{lm}\}_{m\in Z}, \ \vec{d_l^0} = \{< \phi_l^0, \hat\phi_m >_1\}_{m\in Z}.$$

$\|\cdot\|_{l_1}$ is a norm in l_1, $\hat G$ is the Gram matrix of the system $\{\hat\phi_m\}_{m\in Z}$, $\delta\hat G = \hat G - I$.

Using (253), it is easy to show that $\vec{d_l^0} = G\vec{d_l}$ and $\|\vec{d_l^0}\|_{l_2} < c$, $\|\vec{d_l^0}\|_{l_1} < ck$. Considering Lemma 29, we obtain

$$\vec{d_l} = (I+\delta\hat G)^{-1}\vec{d_l^0} = (I+\delta_1\hat G)^{-1}\vec{d_l^0} + \delta\vec{d_l}, \quad \|\delta\vec{d_l}\|_{l_1} < ckk^{-\beta(\gamma-\beta)k^{1/2-\beta-\varepsilon}}.$$

Considering the property (238) of $\delta_1\hat G$, we obtain:

$$d_{lm} = \left((\delta_1\hat G)^p(I+\delta_1\hat G)^{-1}\vec{d_l^0} + \delta\vec{d_l}\right)_m, \quad p = \left[|l-m|k^{-1/2-\gamma}\right].$$

Using the estimate (237) for $\|\delta_1\hat{G}\|$, we obtain:

$$\left|\left((\delta_1\hat{G})^p(I+\delta_1\hat{G})^{-1}\vec{d}_l^0\right)_m\right| < k^{-2\beta p}.$$

Summarizing over $m : |m| \geq k^{1/2+\gamma+\zeta}$, we arrive at the inequality

$$\|(\delta G')^p(I+\delta G')^{-1}\vec{d}_l^0\|_{l_1} < k^{-2\beta k^{\zeta}}k^{1/2+\gamma}.$$

Using the estimate for $\|\delta\vec{d}_l\|_{l_1}$ and considering that $1/2-\beta-\varepsilon > 1/2-\gamma > \zeta$, we obtain (254). The lemma is proved.

3.10 Appendix 10

Lemma 15 *Under conditions of Theorem 1, matrix* $\tilde{G}$ *admits the representation:*

$$\tilde{G} = I + \delta\tilde{G}, \tag{255}$$

$\delta\tilde{G}$ *obeying the estimate:*

$$\|\delta\tilde{G}\|_* < k^{-\gamma_1}, \tag{256}$$

$$\gamma_1 = \min\{2\beta, 1/2-\gamma-3\zeta-5\varepsilon\}.$$

Proof. It $r \in Z_q(k,\beta)$, then $\hat{\Psi}_r(x) = \Psi_r(x)$, therefore, by Theorem 1,

$$\max_{r\in Z_q(k,\beta)} \sum_{m\in Z_q(k,\gamma+3\zeta)} | < \hat{\Psi}_r, \Psi_m > | < ck^{-\gamma_0}, \tag{257}$$

$$\max_{m\in Z_q(k,\gamma+3\zeta)} \sum_{r\in Z_q(k,\beta)} | < \hat{\Psi}_r, \Psi_m > | < ck^{-\gamma_0}. \tag{258}$$

Let $r \in Z_q(k,\gamma+3\zeta)\setminus Z_q(k,\beta)$ and $|\vec{p}_r(t)-\vec{p}_m(t)| > k^{1/2+\beta+\varepsilon}$. We represent $\hat{\Psi}_r$ in the form $\hat{\Psi}_r = \hat{F}_r(x)e^{i(\vec{p}_r(t),x)}$, $\hat{F}_r(x) = e^{S_r^{(q)}(x)+\tilde{S}_r(x)}$. It is not difficult to show that $\hat{F}_r(x)$ satisfies the estimate analogous to (92). Integrating by parts and taking into account that $\Psi_m(x)$ satisfies (92) or (93), we obtain:

$$< \hat{\Psi}_r, \Psi_m > < \left(c_{\|\vec{A}\|_0,\beta,\varepsilon}k^{1/2+\beta}|\vec{p}_m(t)-\vec{p}_r(t)|^{-1}\right)^{\nu} <$$

$$c^{\nu}_{\|\vec{A}\|_0,\beta,\varepsilon}k^{2+4\beta-\varepsilon(\nu-4)}|\vec{p}_m(t)-\vec{p}_r(t)|^{-4} \tag{259}$$

for any $\nu < k^{1/2-\beta-\varepsilon}$. Choosing ν large enough, say $\nu > 10/\varepsilon$, we get that

$$\sup_{r\in Z_q(k,\gamma+3\zeta)\setminus Z_q(k,\beta)} \sum_{m\in Z_q(k,\gamma+3\zeta):|\vec{p}_m(t)-\vec{p}_r(t)|>k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < c_{\|\vec{A}\|_0,\beta,\varepsilon}k^{-2}. \tag{260}$$

$$\sup_{m\in Z_q(k,\gamma+3\zeta)} \sum_{r\in Z_q(k,\gamma+3\zeta)\setminus Z_q(k,\beta):|\vec{p}_m(t)-\vec{p}_r(t)|>k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < c_{\|\vec{A}\|_0,\beta,\varepsilon}k^{-2}. \tag{261}$$

Next, we consider the case: $r, m \in Z_q(k, \gamma+3\zeta) \setminus Z_q(k, \beta)$, $|\vec{p}_m(t) - \vec{p}_r(t)| \le k^{1/2+\beta+\varepsilon}$. We have

$$< \hat{\Psi}_r, \Psi_m >= \frac{1}{|Q|} \int_Q \exp\left(i(\vec{p}_r(t) - \vec{p}_m(t), x) + \left(S_r^{(q)} + \tilde{S}_r + \bar{S}_m \right)(x) \right) dx. \tag{262}$$

For $m = r$, obviously,

$$< \hat{\Psi}_r, \Psi_r >= \frac{1}{|Q|} \int_Q e^{\Re S_r^{(q)}(x) + \Re \bar{S}_r} dx = 1 + O_2(k^{-2\beta}), \tag{263}$$

(29) and (79) have been used. Suppose $r \neq m$. We represent the integral (262) in the form $\int_Q uDvdx$, where u is just the integrand itself

$$u = \exp\left(i(\vec{p}_r(t) - \vec{p}_m(t), x) + \left(S_r^{(q)} + \tilde{S}_r + \bar{S}_m \right)(x) \right)$$

and

$$v = \frac{1}{|\vec{p}_r(t) - \vec{p}_m(t)| + D\left(S_r^{(q)} + \tilde{S}_r + \bar{S}_m \right)(x)},$$

D is the derivative in the direction of $\vec{p}_r(t) - \vec{p}_m(t)$. Let us consider $S_r^{(q)}(x) + \tilde{S}_r(x) + \bar{S}_m(x)$. Using formula (28), (30) and taking (131) for $\beta = \gamma + 3\zeta$, we obtain:

$$\left\| D^{\vec{\nu}} \left(S_r^{(q)}(x) + \tilde{S}_r(x) + \bar{S}_m(x) \right) \right\|_0 < c_{\nu,N} \|\vec{A}\|_{\nu+2} \left(|m-r| k^{-2\beta} + k^{-1/2+\gamma+3\zeta+5\varepsilon} \right), \quad m \neq r. \tag{264}$$

Integrating (262) by parts and considering (264), we arrive at inequality:

$$< \hat{\Psi}_r, \Psi_m > < \ c_{\|A\|_2} \left(k^{-2\beta} + k^{-1/2+\gamma+3\zeta+5\varepsilon} \right) |r-m|^{-1}. \tag{265}$$

Repeating integration by parts 4 times yields:

$$< \hat{\Psi}_r, \Psi_m > < \ c_{\|A\|_6} \left(k^{-2\beta} + k^{-1/2+\gamma+3\zeta+5\varepsilon} \right) |r-m|^{-3}. \tag{266}$$

Therefore,

$$\sup_{r \in Z_q(k,\gamma+3\zeta) \setminus Z_q(k,\beta)} \sum_{m \in Z_q(k,\gamma+3\zeta) \setminus Z_q(k,\beta) : |\vec{p}_m(t) - \vec{p}_r(t)| \le k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < c_{\|A\|_6} k^{-\gamma_1}. \tag{267}$$

$$\sup_{m \in Z_q(k,\gamma+3\zeta) \setminus Z_q(k,\beta)} \sum_{r \in Z_q(k,\gamma+3\zeta) \setminus Z_q(k,\beta) : |\vec{p}_m(t) - \vec{p}_r(t)| \le k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < c_{\|A\|_6} k^{-\gamma_1}. \tag{268}$$

Let $r \in Z_q(k, \gamma+3\zeta) \setminus Z_q(k, \beta)$, $m \in Z_q(k, \beta)$ and $|\vec{p}_m(t) - \vec{p}_r(t)| \le k^{1/2+\beta+\varepsilon}$. Because of the last inequality the estimate

$$\hat{c} k^{1/2+\beta} < |(\vec{p}_r(t), \vec{p}_q(0))| < c_b p_q(0) k^{1/2+\beta+\varepsilon}, \tag{269}$$

which is equivalent to the estimate

$$\hat{c} p_q^{-1}(0) k^{1/2+\beta} < p_q(0)(l_r + \tau_r) < c_b k^{1/2+\beta+\varepsilon} \tag{270}$$

holds. Obviously,

$$\hat{\Psi}_r \overline{\Psi}_m =$$

$$\frac{1}{|Q|} f_{l_r}(\vec{d}_r^{(q)}(t), \tau_r, x) \overline{\psi_m^{(q)}}(x) e^{i(\vec{d}_r^{(q)}(t) - \vec{d}_m^{(q)}(t), x) + \delta S_{rm}}, \tag{271}$$

$$\delta S_{rm} = \tilde{S}_r + \overline{\tilde{S}}_m.$$

By (79), (78)

$$\|\tilde{S}_r + \bar{\tilde{S}}_m\|_\nu < c_{\|A\|_{\nu+1}} \left(|(\vec{d}_r^{(q)}(t) - \vec{d}_m^{(q)}(t)| k^{-1+4\varepsilon} \right). \tag{272}$$

This estimate is stronger than (133). Similar estimates hold for derivatives of $\delta S_{rm}(x)$. Further considerations are completely the same as in part **3** of Theorem 1 (see (133) – (159). We obtain:

$$\sup_{r \in Z_q(k, \gamma+3\zeta) \setminus Z_q(k,\beta)} \sum_{m \in Z_q(k,\beta) : |\vec{p}_m(t) - \vec{p}_r(t)| \leq k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < k^{-\gamma_0}, \tag{273}$$

$$\sup_{m \in Z_q(k,\beta)} \sum_{r \in Z_q(k, \gamma+3\zeta) \setminus Z_q(k,\beta) : |\vec{p}_m(t) - \vec{p}_r(t)| \leq k^{1/2+\beta+\varepsilon}} | < \hat{\Psi}_r, \Psi_m > | < k^{-\gamma_0}. \tag{274}$$

Adding estimates (257), (258), (260), (261), (267), (268), (273) and (274), and taking into account that $\gamma_1 \leq \gamma_0$, we obtain (256).

The lemma is proved.

Lemma 16 *Under conditions of Theorem 1 matrix G agmits the representation: $G = I + \delta_1 G + \delta_2 G$, matrices $\delta_1 G$ and $\delta_2 G$ having the following properties:*

$$(\delta_1 G)_{mr} = 0, \quad \textit{if } |m - r| > k^{1/2+\beta+\zeta},$$

$$\|\delta_1 G\|_* < ck^{-\gamma_0}, \quad \|\delta_2 G\|_* < \left(c_{\|A\|_0, \beta, \varepsilon} k^{-\zeta} \right)^{k^{1/2-\beta-\varepsilon}}.$$

for any $\varepsilon < \zeta < 1/2 - \beta$.

Proof. We define $\delta_1 G$ as follows

$$(\delta_1 G)_{mr} = \begin{cases} (G - I)_{mr}, & \text{when } |m - r| \leq k^{1/2+\beta+\zeta}, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1, $\|\delta_1 G\|_* < ck^{-\gamma_0}$. Obviously,

$$(\delta_2 G)_{mr} = \begin{cases} G_{mr}, & \text{when } |m - r| \geq k^{1/2+\beta+\zeta}, \\ 0 & \text{otherwise.} \end{cases}$$

Using (97) and taking into account that $|m - r| > k^{1/2+\beta+\zeta}$ we get the estimate for $\delta_2 G$. The lemma is proved.

Lemma 17 *Under conditions of Theorem 1 matrix* $\tilde{G}$ *agmits the representation:* $\hat{G} = I + \delta_1\tilde{G} + \delta_2\tilde{G}$*, matrices* $\delta_1\tilde{G}$ *and* $\delta_2\tilde{G}$ *having the following properties:*

$$(\delta_1\tilde{G})_{mr} = 0, \quad \textit{if } |m-r| > k^{1/2+\beta+\zeta},$$

$$\|\delta_1\tilde{G}\|_* < ck^{-\gamma_0}, \quad \|\delta_2\hat{G}\|_* < \left(c_{\|A\|_0,\beta,\varepsilon}k^{-\zeta}\right)^{k^{1/2-\beta-\varepsilon}}$$

for any $\varepsilon < \zeta < 1/2 - \beta$.

The proof of the lemma is analogous to that of the previous lemma.

3.11 Appendix 11. Fourier coefficients of Ψ_m.

Let us denote by a_{mj} Fourier coefficients of function $\Psi_m(x)$:

$$a_{mj} = \frac{1}{|Q|}\int \Psi_m(x)e^{-i(\vec{p}_j(t),x)}dx.$$

Lemma 18 *If* $m \in Z_0(k,\beta)$ *and* $\mu' > 0$*, then*

$$\sum_{j\in Z^2:|m-j|>k^{1/2-\beta+\mu'}} |a_{mj}| < \left(c_{\|A\|_0,\beta,\varepsilon}k^{-\mu'}\right)^{k^{1/2-\beta-\varepsilon}} k^2. \tag{275}$$

<u>Proof.</u> Using notations of Lemma 7 we obtain:

$$a_{mj} = \frac{1}{|Q|}\int F_m(x)e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)}dx.$$

Integrating by parts $\left[k^{1/2-\beta-\varepsilon}\right]$ times in the direction of the vector $\vec{p}_m(t) - \vec{p}_j(t)$ and applying estimate (92), we arrive at inequality:

$$|a_{mj}| < \left(c_{\|A\|_0,\beta,\varepsilon}k^{1/2-\beta}/|\vec{p}_m(t) - \vec{p}_j(t)|\right)^{\nu_0}.$$

Taking into account that $|m-j| > k^{1/2-\beta+\mu'}$ and summarizing over j, we get (275). <u>The lemma is proved.</u>

3.12 Appendix 12. Proof of estimate (95).

By the definition of $Z_q(k,\beta)$, $|(\vec{p}_j(t), \vec{p}_q(0))| \leq \hat{c}k^{1/2+\beta}$. Considering (56), we obtain: $|l_j + \tau_j|p_q(0)^2 < \hat{c}k^{1/2+\beta}$. Using (77), (79), we arrive by induction to the inequality:

$$\|D_\nu e^{-i(p_q(0),x)l_j+\tilde{S}_j}\|_0 < c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2+\beta}\left(\nu R_0 N + c_{\|\vec{A}\|_1,\beta,\varepsilon}k^{1/2+\beta}\right)^{\nu-1}.$$

Considering (75) we obtain (95).

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