The quantum electrodynamics of relativistic bound states with cutoffs.I.

Mouez Dimassi and Jean-Claude Guillot
Université Paris-Nord
Institut Galilée
Département de Mathématiques
UMR 7539
F-93430 Villetaneuse, France

Abstract

In this note we consider an Hamiltonian with cutoffs describing the interaction of relativistic electrons and positrons in a Coulomb potential with transversal photons in Coulomb gauge. We prove that the Hamiltonian is self-adjoint in the Fock space and has a ground state for a sufficiently small coupling constant.

0 Introduction

This note initiates the study of an Hamiltonian with cutoffs describing relativistic electrons and positrons in a Coulomb potential interacting with transversal photons in Coulomb gauge. Starting with the Dirac equation for an electron in a Coulomb potential we first introduce the quantized Dirac-Coulomb field in the Fock space of electrons and positrons.

In order to define the Hamiltonian in the Fock space of electrons, positrons and photons, we have to introduce appropriate cutoff functions. Thus we are studying an Hamiltonian with cutoff functions with respect to the momenta of electrons, positrons and photons.

In this note we prove that this Hamiltonian is self-adjoint and has a ground state when the coupling constant is sufficiently small.

It is important to remark that the methods developped here can be applied to many other cases as well, as soon as we start with the Dirac Hamiltonian for an electron for which we explicitely know a spectral representation. For example, this would be the case for a relativistic electron in an uniform magnetic field.

In the next future, we will study the absolutely continuous spectrum of the Hamiltonian, together with a first approach of resonances.

Our methods follow closely those of [2,3,4], in which the spectral theory of the Pauli-Fierz Hamiltonian is carefully studied.

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1 The quantization of the Dirac-Coulomb field

The Dirac electron of mass m_0 in a Coulomb potential is described by the following Hamiltonian:

$$H_D := H_D(e) = c\alpha \cdot D + \beta m_0 c^2 - \frac{e^2 Z}{|x|}, \tag{1}$$

acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with domain $D(H_D) = H^1(\mathbb{R}^3; \mathbb{C}^4)$, the Sobolev space of order 1. We refer to [11] for a discussion of the Dirac operator (see also [12]). Here e is the negative charge of the electron, $-Ze$ is the positive charge of the nucleus. $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Dirac matrices in the standard form:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

where σ_i are the usual Pauli matrices.

It is well known that $H_D(e)$ is self-adjoint for $Z \leq 68$ ([11, Theorem 4.2]). The eigenstates of the Dirac-Coulomb operator H_D are labelled by the angular momentum quantum numbers $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots,\ m_j = -j, -j+1, \ldots, j-1, j$, by the spin orbit quantum number $k_j = \pm(j + \frac{1}{2}) = \pm 1, \pm 2, \ldots$ and by the quantum number $n = 0, 1, 2, \ldots$, counting the non-degenerate eigenvalues of the Dirac radial operator associated with k_j. From now on we set $\gamma = (j, m_j, k_j)$.

Let $\psi_{\gamma,n}$ denote the eigenstates of the Hamiltonian H_D. We have

$$H_D(e)\psi_{\gamma,n} = E_{\gamma,n}\psi_{\gamma,n}$$

with

$$E_{\gamma,n} = m_0 c^2 \left(1 + \frac{(Ze^2)^2/c^2}{(n + \sqrt{k_j^2 - (Ze^2)^2/c^2})^2} \right)^{-1/2} \tag{2}$$

Each eigenvalue is degenerate. The infinimum of the discrete spectrum is an isolated eigenvalue of multiplicity two. Together with the bound state energy levels the continuum energy levels are given by

$$\pm\omega(p), \qquad \omega(p) = (m_0^2 c^4 + p^2)^{\frac{1}{2}}, \qquad p = |\mathbf{p}| \tag{3}$$

Here $\mathbf{p}$ is the momentum of the electron. Let $\psi_{\gamma,\pm}(p,x)$ denote the continuum eigenstates of $H_D(e)$. We then have

$$H_D\psi_{\gamma,\pm}(p,x) = \pm(m_0^2c^4 + p^2)^{\frac{1}{2}}\psi_{\gamma,\pm}(p,x)$$

$\psi_{\gamma,n}$ and $\psi_{\gamma,\pm}(p,x)$ are normalized in such a way that

$$\begin{aligned}
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,n}(x)\psi_{\gamma',n'}(x)d^3x &= \delta_{nn'}\delta_{\gamma\gamma'}, \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,\pm}(p,x)\psi_{\gamma',\pm}(p',x)d^3x &= \delta_{\gamma\gamma'}\delta(p-p'), \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,\pm}(p,x)\psi_{\gamma',\mp}(p',x)d^3x &= 0, \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,n}(x)\psi_{\gamma',\mp}(p,x)d^3x &= 0.
\end{aligned}$$

Here $\psi^\dagger_{\gamma,\pm}(p,x)$ (resp. $\psi^\dagger_{\gamma,n}(x)$) is the adjoint spinor of $\psi_{\gamma,\pm}(p,x)$ (resp. $\psi_{\gamma,n}(x)$). $(\psi_{\gamma,n})_{\gamma,n}$ and $(\psi_{\gamma,\pm})_\gamma$ generate a spectral representation of $H_D(e)$.

According to the hole theory [18], the absence in the Dirac theory of an electron with energie $E < 0$ and charge e is equivalent to the presence of a positron with energy $-E > 0$ and charge $-e$. Indeed, by the charge conjugation $C\psi = i\beta\alpha_2\overline{\psi}$, one obtains

$$CH_D(e)C^{-1} = -\left(c\alpha\cdot D + \beta m_0c^2 + \frac{e^2Z}{|x|}\right).$$

Let us split the Hilbert space $H = L^2(\mathbb{R}^3;\mathbb{C}^4)$ into

$$H_{c^-} = P_{(-\infty,0]}(H_D), \qquad H_d = P_{(0,m_0c^2)}(H_D), \qquad H_{c^+} = P_{[m_0c^2,+\infty)}(H_D).$$

Here $P_I(H_D)$ denotes the spectral projection of H_D corresponding to the interval I. Since $(\psi_{n,\gamma})_{\gamma,n}$ is a basis in H_d, every $\psi \in H_d$ can be written as

$$\psi(x) = \text{L.i.m} \sum_{\gamma,n} b_{\gamma,n}\,\psi_{n,\gamma}(x).$$

with

$$\sum_{\gamma,n} |b_{\gamma,n}|^2 < \infty.$$

Hence we can identify H_d with the Hilbert space $\tilde{H}_d = \oplus_\gamma F_\gamma$, where F_γ is a closed subspace of $L^2(\mathbb{R}_+)$. Precisely, there exists a unitary operator U_d from H_d onto $\tilde{H}_d$ given by

$$U_d \psi = \left(\sum_n b_{\gamma,n} \, 1\!\!1_{[n,n+1)}(p) \right)_\gamma , \qquad \psi \in H_d$$

where $p \in \mathbb{R}_+$.

Similarly, $H_{c\pm}$ can be identified with $\tilde{H}_{c\pm} := \oplus_\gamma L^2(\mathbb{R}_+)$ by using the unitary operators $U_{c\pm}$:

$$U_{c\pm}\psi = \left(\text{L.i.m} \int \psi^\dagger_{\gamma,\pm}(p,x)\, \psi(x) dx \right)_\gamma ,$$

where $\psi \in H_{c\pm}$.

1.1 The Fock space for electrons and positrons

Let $\mathcal{H}$ be any separable Hilbert space. Let $\otimes_a^n \mathcal{H}$ denote the antisymmetric n-th tensor power of $\mathcal{H}$ appropriate to Fermi-Dirac statistics. We define the Fermi-Fock space over $\mathcal{H}$, denoted by $\mathcal{F}_a(\mathcal{H})$, to be the direct sum

$$\mathcal{F}(\mathcal{H}) = \overset{\infty}{\underset{n=1}{\oplus}} \otimes_a^n \mathcal{H}$$

where $\otimes_a^0 \mathcal{H} = \mathbb{C}$. Ω will denote the vacuum vector, i.e., the vector $(1, 0, 0, \cdots)$. Let $\mathcal{F}_{a,d}$, $\mathcal{F}_{a,+}$ and $\mathcal{F}_{a,-}$ be the Fermi-Fock spaces over $\tilde{H}_d$, $\tilde{H}_{c+}$ and $\tilde{H}_{c-}$ respectively. Ω_d, Ω_+ and Ω_- denote the associated vacua.

The Fermi-Fock space for electrons and positrons, denoted by $\mathcal{F}_D$, is then the following Hilbert space

$$\mathcal{F}_D = \mathcal{F}_{a,d} \otimes \mathcal{F}_{a,+} \otimes \mathcal{F}_{a,-} \tag{4}$$

The vector $\Omega_d \otimes \Omega_+ \otimes \Omega_-$ is the vacuum of electrons and positrons.

One has

$$\mathcal{F}_D = \overset{\infty}{\underset{q,r,s=0}{\oplus}} \mathcal{F}_a^{(q,r,s)}$$

where $\mathcal{F}_d^{(q,r,s)} = (\otimes_a^q \tilde{H}_d) \otimes (\otimes_a^r \tilde{H}_{c+}) \otimes (\otimes_a^s \tilde{H}_{c-})$.

Let us remark that $\mathcal{F}_D$ is unitarily equivalent to $\mathcal{F}_a(\tilde{H}_d \oplus \tilde{H}_{c+} \oplus \tilde{H}_{c-})$. (See [5] and [17]).

1.2 Creation and annihilation operators

For every $\varphi \in \mathcal{H}$ we define in $\mathcal{F}_a(\mathcal{H})$ the annihilation operator, denoted by $b(\varphi)$, which maps $\otimes_a^{n+1}\mathcal{H}$ into $\otimes_a^n\mathcal{H}$:

$$\begin{aligned} & b(\varphi)\,(A_{n+1}(\varphi_1 \otimes \ldots \otimes \varphi_{n+1})) \\ & \quad = \frac{\sqrt{n+1}}{(n+1)!}\sum_\sigma sgn(\sigma)\,(\varphi, \varphi_{\sigma(1)})\,\varphi_{\sigma(2)} \otimes \ldots \otimes \varphi_{\sigma(n+1)} \end{aligned}$$

where $\varphi_i \in \mathcal{H}$.

The creation operator, denoted by $b^*(\varphi)$, is the adjoint of $b(\varphi)$. $b^*(\varphi)$ and $b(\varphi)$ are bounded operators in $\mathcal{F}_a(\mathcal{H})$ and $||b(\varphi)|| = ||b^*(\varphi)|| = ||\varphi||$.

We now define the annihilation and creation operators in $\mathcal{F}_D$.

1.2.1 Bound states

For every (γ, n), we define in $\mathcal{F}_D$ the annihilation operator, denoted by $b_{\gamma,n}$, which maps $\mathcal{F}^{(q+1,r,s)}$ into $\mathcal{F}^{(q,r,s)}$:

$$b_{\gamma,n}\,(A_{q+1}(f_1 \otimes \ldots \otimes f_{q+1}) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_s(h_1 \otimes \ldots \otimes h_s))$$

$$= [b(U_d\,\psi_{\gamma,n})A_{q+1}(f_1 \otimes \ldots \otimes f_{r+1})] \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_s(h_1 \otimes \ldots \otimes h_s)$$

where $f_i \in \tilde{H}_d$, $g_j \in \tilde{H}_{c+}$ and $h_k \in \tilde{H}_{c^-}$.

The creation operator $b^*_{\gamma,n}$ is the adjoint of $b_{\gamma,n}$. $b^*_{\gamma,n}$ and $b_{\gamma,n}$ are bounded operators in $\mathcal{F}_D$.

1.2.2 Electrons in the continuum

For every $g \in \tilde{H}_{c+}$, we define in $\mathcal{F}_D$ the annihilation operator, denoted by $b_+(g)$, which maps $\mathcal{F}^{(p,q+1,r)}$ into $\mathcal{F}^{(p,q,r)}$:

$$b_+(g)\left(A_q(f_1 \otimes \ldots \otimes f_q) \otimes A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1}) \otimes A_s(h_1 \otimes \ldots \otimes h_s)\right)$$

$$= A_q(f_1 \otimes \ldots \otimes f_q) \otimes [(-1)^q\, b(g) A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1})] \otimes A_s(h_1 \otimes \ldots \otimes h_s)$$

The creation operator $b_+^*(g)$ is the adjoint of $b_+(g)$. $b_+^*(g)$ and $b_+(g)$ are bounded operators in $\mathcal{F}_D$.

We set, for every $\psi \in \tilde{H}_{c+}$,

$$\begin{aligned} b_{\gamma,+}(\psi) &= b_+(P_\gamma^+ \psi) \\ b_{\gamma,+}^*(\psi) &= b_+^*(P_\gamma^+ \psi) \end{aligned}$$

where P_γ^+ is the projection of $\tilde{H}_{c+}$ onto the γ-th component.

1.2.3 Positrons

For every $h \in \tilde{H}_{c-}$, we define in $\mathcal{F}_D$ the annihilation operator, denoted by $b_-(h)$, which maps $\mathcal{F}^{(q,r,s+1)}$ into $\mathcal{F}^{(q,r,s)}$:

$$b_-(h)(A_q(f_1 \otimes \ldots \otimes f_q) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_{s+1}(h_1 \otimes \ldots \otimes h_{s+1}))$$

$$= A_q(h_1 \otimes \ldots \otimes f_q) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes [(-1)^{q+r} b(h) A_{s+1}(h_1 \otimes \ldots \otimes h_{s+1})]$$

The creation operator $b_-^*(h)$ is the adjoint of $b_-(h)$. $b_-^*(h)$ and $b_-(h)$ are bounded operators in $\mathcal{F}_D$.

We set for every $\psi \in \tilde{H}_{c-}$,

$$\begin{aligned} b_{\gamma,-}(\psi) &= b_-(P_\gamma^- \psi) \\ b_{\gamma,-}^*(\psi) &= b_-^*(P_\gamma^- \psi) \end{aligned}$$

where P_γ^- is the projection of $\tilde{H}_{c-}$ onto the γ-th component.

A simple computation shows that the following anticommutation relations hold

$$\{b^*_{\gamma,n}, b_{\beta,m}\} = \delta_{\gamma,\beta}\,\delta_{n,m} \tag{5}$$

$$\{b^*_{\gamma,n}, b^*_{\beta,m}\} = 0 = \{b_{\gamma,n}, b_{\beta,m}\} \tag{6}$$

$$\{b_{\gamma,+}(\psi_1), b^*_{\beta,+}(\psi_2)\} = \delta_{\gamma,\beta}(P^+_\gamma\psi_1, P^+_\gamma\psi_2)_{L^2(\mathbb{R}_+)}, \qquad \psi_i \in \tilde{H}_{c+} \tag{7}$$

$$\{b_{\gamma,-}(\psi_1), b^*_{\beta,-}(\psi_2)\} = \delta_{\gamma,\beta}(P^-_\gamma\psi_1, P^-_\gamma\psi_2)_{L^2(\mathbb{R}_+)}, \qquad \psi_i \in \tilde{H}_{c-} \tag{8}$$

Futhermore

$$\{b^\#_{\gamma,\pm}(\psi), b^\#_{\beta,n}\} = 0. \tag{9}$$

Here $b^\#$ is b or b^*.

$$\{b_{\gamma,+}(\psi_1), b_{\beta,-}(\psi_2)\} = \{b_{\gamma,+}(\psi_1), b^*_{\beta,-}(\psi_2)\} = 0, \tag{10}$$

$$\{b^*_{\gamma,+}(\psi_1), b_{\beta,-}(\psi_2)\} = \{b^*_{\gamma,+}(\psi_1), b^*_{\beta,-}(\psi_2)\} = 0, \tag{11}$$

where $\psi_1 \in \tilde{H}_{c+}$ and $\psi_2 \in \tilde{H}_{c-}$. One should remark that, in contrast to [11], the charge conjugaison operator C is not included in the definition of the annihilation and creation operators for the positrons.

Our definition is close to the hole theory and is the one occuring in many text books in Quantum Field Theory as in [13], [14] and [15]. The other method for the quantization of the Dirac field is close to the so-called symmetric theory of charge. Both approaches are described in [16].

In the first approach the Fock space for electrons and positrons is built upon the subspaces $\tilde{H}_d$, $\tilde{H}_{c+}$ and $\tilde{H}_{c-}$ of the one-particle Hilbert space and, in the second case, upon the subspaces $\tilde{H}_d$, $\tilde{H}_{c+}$ and $C'\tilde{H}_{c-}$ where $C' = U_{c-}\,C\,U_{c-}^{-1}$. As in [10 ; chapter X] we introduce operator-valued distributions $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ such that we write

$$\begin{aligned} b_{\gamma,\pm}(\psi) &= \int_{\mathbb{R}+} dp\; b_{\gamma,\pm}(p)\,\overline{(P^\pm_\gamma\psi)\,(p)} \\ b^*_{\gamma,\pm}(\psi) &= \int_{\mathbb{R}+} dp\; b^*_{\gamma,\pm}(p)\,(P^\pm_\gamma\psi)\,(p) \end{aligned}$$

where $\psi \in \tilde{H}_{c+} \oplus \tilde{H}_{c-}$.

We now give a representation of $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$. Let $\mathcal{D}_D$ denote the set of smooth vectors $\Phi \in \mathcal{F}_D$ for which $\Phi^{(q,r,s)}$ has a compact support and $\Phi^{(q,r,s)} = 0$ for all but finitely many (q, r, s).

For every (p, γ), $b_{\gamma,+}(p)$ maps $\mathcal{F}_a^{(q,r+1,s)} \cap \mathcal{D}_D$ into $\mathcal{F}_a^{(q,r,s)} \cap \mathcal{D}_D$ and is given by

$$(b_{\gamma,+}(p)\Phi)^{(q,r,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_r, \gamma_r; p_1, \gamma_1, \ldots, p_s, \gamma_s)$$

$$= \sqrt{r+1}(-1)^q \Phi^{(q,r+1,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p, \gamma, p_1, \gamma_1, \ldots, p_r, \gamma_r; p_1, \gamma_1, \ldots, p_s, \gamma_s)$$

$b^*_{\gamma,+}(p)$ is then given by:

$$\begin{gathered}(b^*_{\gamma,+}(p)\Phi)^{(q,r+1,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_{r+1}, \gamma_{r+1}; p_1, \gamma_1, \ldots, p_s, \gamma_s) = \\ \frac{(-1)^q}{\sqrt{r+1}} \sum_{i=1}^{r+1} (-1)^{i+1} \delta_{\gamma_i \gamma} \delta(p - p_i) \\ \Phi^{(q,r,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, \widehat{p_i, \gamma_i}, \ldots, p_{r+1}, \gamma_{r+1}; p_1, \gamma_1, \ldots, p_s, \gamma_s)\end{gathered}$$

where $\hat{\cdot}$ denotes that the i-th variable has to be omitted.

Similarly $b_{\gamma,-}(p)$ maps $\mathcal{F}_a^{(q,r,s+1)} \cap \mathcal{D}_D$ into $\mathcal{F}_a^{(q,r,s)} \cap \mathcal{D}_D$ such that

$$\begin{gathered}(b_{\gamma,-}(p)\Phi)^{(q,r,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_r, \gamma_r; p_1, \gamma_1, \ldots, p_s, \gamma_s) = \\ \sqrt{s+1}(-1)^{q+r} \Phi^{(q,r,s+1)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_r, \gamma_r; p, \gamma, p_1, \gamma_1, \ldots, p_s, \gamma_s)\end{gathered}$$

$b^*_{\gamma,-}(p)$ is then given by

$$\begin{gathered}(b^*_{\gamma,-}(p)\Phi)^{(q,r,s+1)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_s, \gamma_s; p_1, \gamma_1, \ldots, p_{s+1}, \gamma_{s+1}) = \\ \frac{1}{\sqrt{s+1}}(-1)^{q+r} \sum_{i=1}^{s+1} (-1)^{i+1} \delta_{\gamma,\gamma_i} \delta(p - p_i) \\ \Phi^{(q,r,s)}(p_1, \gamma_1, \ldots, p_q, \gamma_q; p_1, \gamma_1, \ldots, p_r, \gamma_r; p_1, \gamma_1, \ldots, \widehat{p_i, \gamma_i}, \ldots, p_{s+1}, \gamma_{s+1})\end{gathered}$$

Let us recall that $\Phi^{(q,r,s)}$ is antisymmetric in the bound states, the electron in the continuum and the positron variables separately. We have

$$\{b_{\gamma,+}(p), b^*_{\gamma',+}(p')\} = \{b_{\gamma,-}(p), b^*_{\gamma',-}(p')\} = \delta_{\gamma,\gamma'} \delta(p - p') \tag{12}$$

Any other anti-commutation relation is equal to zero. In particular we have

$$\{b_{\gamma,\pm}(p), b^{\#}_{\gamma',n}\} = \{b^*_{\gamma,\pm}(p), b^{\#}_{\gamma',n}\} = 0 \tag{13}$$

where $b^{\#}$ is b or b^*.

1.2.4 The Hamiltonian for the quantized Dirac-Coulomb field

The quantization of the Dirac-Coulomb Hamiltonian H_D, denoted by $d\Gamma(H_D)$, is now given by

$$\begin{aligned} d\Gamma(H_D) \;&=\; \sum_{\gamma,n} E_{\gamma,n}\, b^*_{\gamma,n}\, b_{\gamma,n} + \sum_{\gamma} \int dp\ \omega(p)\, b^*_{\gamma,+}(p)\, b_{\gamma,+}(p) \\ &\quad + \sum_{\gamma} \int dp\ \omega(p)\, b^*_{\gamma,-}(p)\, b_{\gamma,-}(p) \end{aligned}$$

The operator $d\Gamma(H_D)$ is well defined on the dense subset $\mathcal{D}_D$ and it is essentially self-adjoint on $\mathcal{D}_D$. The self-adjoint extension will be also denoted by $d\Gamma(H_D)$ with domain $D(d\Gamma(H_D))$. $d\Gamma(H_D)$ is the Hamiltonian of the quantized Dirac-Coulomb field. Moreover the operator number of electrons and positrons, denoted by N_D, is given by

$$N_D = \sum_{\gamma,n} b^*_{\gamma,n} b_{\gamma,n} + \sum_{\gamma} \int dp\ b^*_{\gamma,+}(p) b_{\gamma,+}(p) + \sum_{\gamma} \int dp\ b^*_{\gamma,-}(p) b_{\gamma,-}(p)$$

N_D is essentially self-adjoint on $\mathcal{D}_D$. The self-adjoint extension will be also denoted by N_D with domain $D(N_D)$.

Let Δ denote the set of all eigenvalues of the operator

$$d\Gamma(H_D)_d = \sum_{\gamma,n} E_{\gamma,n} b^*_{\gamma,n} b_{\gamma,n}$$

Each $E_{\gamma,n}$ is in Δ and we have

$$0 < E_0 = \inf_{\gamma,n} E_{\gamma,n}, \quad \text{and} \quad \inf_{E\in\Delta} E = 0. \tag{14}$$

0 is the eigenvalue associated with Ω_D. Futhermore

$$\sigma(d\Gamma(H_D)) = \Delta \cup [m_0 c^2, \infty)$$

$[m_0c^2, \infty)$ is the absolutely continuous spectrum of $d\Gamma(H_D)$. Let us remark that, for $E \in \Delta$ with $E > m_0c^2$, E is an eigenvalue embedded in the continuous spectrum.

1.3 The Fock space for transversal photons

The one photon Hilbert space is given by $L^2(\mathbb{R}^3, \mathbb{C}^2)$. For every f in $L^2(\mathbb{R}^3, \mathbb{C}^2)$, we shall write $f(k, \mu)$ where $k \in \mathbb{R}^3$ is the momentum variable of the photon and $\mu = 1, 2$ is its polarization index associated with two given independent transversal polarizations of the photon in the Coulomb gauge.

Let $\mathcal{F}_{ph}$ denote the Fock space for transversal photons :

$$\mathcal{F}_{ph} = \oplus_{n=0}^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$$

where $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^0} = \mathbb{C}$. Here $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$ is the symmetric n-tensor power of $L^2(\mathbb{R}^3, \mathbb{C}^2)$ appropriate for Bose-Einstein statistics.

For $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, the annihilation and creation operators, denoted by $a(f)$ and $a^*(f)$ respectively, are now given by:

$$\begin{aligned} a(f) &= \sum_{\mu=1,2} a_\mu(f(\cdot, \mu)) \\ a^*(f) &= \sum_{\mu=1,2} a_\mu^*(f(\cdot, \mu)) \end{aligned}$$

where

$$(a_\mu(f(\cdot, \mu))\psi)^{(n)}(k_1, \mu_1, k_2, \mu_2, \ldots, k_n, \mu_n) =$$

$$\sqrt{n+1} \int d^3k \, \overline{f(k, \mu)} \, \psi^{(n+1)}(k, \mu, k_1, \mu_1, k_2, \mu_2, \ldots, k_n, \mu_n)$$

$$(a_\mu^*(f(\cdot, \mu))\psi)^{(n)}(k_1, \mu_1, \ldots, k_n, \mu_n) =$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(k_i, \mu_i) \, \psi^{(n-1)}(k_1, \mu_1, \ldots, \widehat{k_i, \mu_i}, \ldots, k_n, \mu_n)$$

where $\widehat{\cdot}$ denotes that the i-th variable has to be omitted. Note that $a_\mu^*(f(\cdot, \mu))$ and $a_\mu(f(\cdot, \mu))$ are linear and anti-linear with respect to f respectively, so that we can introduce operator valued distributions, i.e, fields $a_\mu(k)$ and $a^*_\mu(k)$ such that

$$a_\mu(f(\cdot,\mu)) = \int d^3k \; \overline{f(k,\mu)}\, a_\mu(k)$$

and

$$a^*_\mu(f(\cdot,\mu)) = \int d^3k \; f(k,\mu)\, a^*_\mu(k)$$

where $f(\cdot,\mu) \in L^2(\mathbb{R}^3)$.

Let $\mathcal{D}_{ph}$ denote the set of smooth vectors $\psi \in \mathcal{F}_{ph}$ for which $\psi^{(n)}$ has a compact support and $\psi^{(n)} = 0$ for all but finitely many n. Then, for any $\psi \in \mathcal{D}_{ph}$, the action of $a_\mu(k)$ and $a^*_\mu(k)$ is given by

$$\begin{aligned}&(a_\mu(k)\psi)^{(n)}(k_1,\mu_1,k_2,\mu_2,\dots,k_n,\mu_n)\\ &\quad= \sqrt{n+1}\psi^{(n+1)}(k,\mu,k_1,\mu_1,\dots,k_n,\mu_n)\end{aligned}$$

$$\begin{aligned}&(a^*_\mu(k)\psi)^{(n+1)}(k_1,\mu_1;k_2,\mu_2,\dots,k_{n+1},\mu_{n+1})\\ &\quad= \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1} \delta_{\mu_i,\mu}\,\delta(k_i-k)\,\psi^{(n)}(k_1,\mu_1,\dots,\widehat{k_i,\mu_i},\dots,k_{n+1},\mu_{n+1})\end{aligned}$$

We have the canonical commutation relations

$$[a_\mu(k), a^*_{\mu'}(k')] = \delta_{\mu\mu'}\delta(k-k') \tag{15}$$

Any other commutation relation is equal to zero.

The Hamiltonian of the quantized electromagnetic field, denoted by H_{ph}, is

$$H_{ph} = \sum_{\mu=1,2}\int d^3k\;\omega(k)\,a^*_\mu(k)\,a_\mu(k)$$

where $\omega(k) = c|k|$. H_{ph} is essentially self-adjoint on $\mathcal{D}_{\mathrm{ph}}$.

The state $(1,0,\dots) \in \mathcal{F}_{ph}$ is the vacuum of photons and will be denoted Ω_{ph}.

The spectrum of H_{ph} consists of an absoluetly continuous spectrum covering $[0,+\infty)$ and a simple eigenvalue, equal to zero, whose the corresponding eigenvector is the vacuum state $\Omega \in \mathcal{F}_{ph}$.

1.4 The Total Hamiltonian and The main results

The Fock space for electrons, positrons and photons is the following Hilbert space:

$$\mathcal{F} = \mathcal{F}_D \otimes \mathcal{F}_{ph}. \tag{16}$$

$\Omega = \Omega_D \otimes \Omega_{ph}$ is the vacuum of $\mathcal{F}$.

The free Hamiltonian for electrons, positrons and photons, denoted by H_0, is the following operator in $\mathcal{F}$:

$$H_0 = d\Gamma(H_D) \otimes \mathbb{1}_{ph} + \mathbb{1}_D \otimes H_{ph} \tag{17}$$

H_0 is a self-adjoint operator with domain

$$D(d\Gamma(H_D) \otimes \mathbb{1}_{ph}) \cap D(\mathbb{1}_D \otimes H_{ph})$$

H_0 has the same point spectrum as $d\Gamma(H_D)$ and its continuous spectrum covers the half axis $[0, \infty)$. Hence the point spectrum of $d\Gamma(H_D)$ is embedded in the continuous spectrum of H_0. The eigenfunctions of H_0 corresponding to the eigenvalues E have the form $\varphi \otimes \Omega_{ph}$, where φ are the eigenfunctions of $d\Gamma(H_D)$ corresponding to the eigenvalues E.

Let us now describe the interaction between electrons, positrons and photons in Coulomb gauge that we consider. We first recall the physical interaction in Q. E. D. with the Coulomb gauge.

The quantized Dirac-Coulomb field is given by

$$\begin{aligned} \psi(x) &= \sum_{\gamma,n} b_{\gamma,n} \psi_{\gamma,n}(x) \\ &\quad + \sum_{\gamma} \int_{\mathbb{R}^+} dp\ b_{\gamma,+}(p) \psi_{\gamma,+}(p,x) + \sum_{\gamma} \int_{\mathbb{R}^+} dp\ b^*_{\gamma,-}(p) \psi_{\gamma,-}(p,x) \end{aligned}$$

together with

$$\begin{aligned} \psi^\dagger(x) &= \sum_{\gamma,n} b^*_{\gamma,n} \psi^\dagger_{\gamma,n}(x) + \sum_{\gamma} \int_{\mathbb{R}^+} dp\ b^*_{\gamma,+}(p) \psi^\dagger_{\gamma,n}(p,x) \\ &\quad + \sum_{\gamma} \int_{\mathbb{R}^+} dp\ b_{\gamma,-}(p) \psi^\dagger_{\gamma,-}(p,x) \end{aligned}$$

for the quantized adjoint spinor.

Here $b_{\gamma,n}$, $b^*_{\gamma,n}$, $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ are the annihilation and creation operators defined above.

The density of charge is then given by

$$\rho(x) = e : \psi^\dagger(x)\, \psi(x) :$$

and the density of current by

$$j(x) = ec : \psi^\dagger(x)\, \alpha\, \psi(x) :$$

Here $: \cdot :$ is the normal order and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

The interaction of electrons and positrons with photons is given by two terms.

The first one :

$$\frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3x\, d^3x' \;\frac{\rho(x)\, \rho(x')}{|x - x'|}$$

describes the Coulomb interaction of densities of charges.

The second one

$$-e \int_{\mathbb{R}^3} d^3x \; j(x) \cdot A(x)$$

describes the interaction between the current and the transversal photons. Here $A(x)$ is the quantized electromagnetic field in Coulomb gauge :

$$A(x) = \sum_{\mu=1,2} \left(\frac{\hbar}{2\varepsilon_0 (2\pi)^3} \right)^{1/2} \int \frac{d^3k}{\sqrt{2\omega_{ph}(k)}} \left(\varepsilon_\mu(k) e^{ik\cdot x} a_\mu(k) + \varepsilon_\mu(k) e^{-ik\cdot x} a^*_\mu(k) \right)$$

(See [6] and [18]) where $\varepsilon_\mu(k)$ are real polarization vectors satisfying

$$\varepsilon_\mu(k) \cdot \varepsilon_{\mu'}(k) = \delta_{\mu\mu'} \quad \text{and} \quad k \cdot \varepsilon_\mu(k) = 0$$

$a^*_\mu(k)$ and $a_\mu(k)$ are the creation and annihilation operators on $\mathcal{F}_{ph}$.

In this first approach we neglect the Coulomb interaction of density of charge and we restrict ouselves to the interaction between the current and the transversal photons.

Then, by a formal computation, the interaction term can be expressed in terms of the annihilation and creation operators $b_{\gamma,n}$, $b^*_{\gamma,n}$, $b_{\gamma,\pm}(p)$, $b^*_{\gamma,\pm}(p)$, $a_\mu(k)$ and $a^*_\mu(k)$.

It is quite well-known that, in order to obtain a well defined Hamiltonian in the Fock space, we have to introduce several cutoff functions on the Dirac-Coulomb field and the electromagnetic vector potential.

Thus, from now on, the interaction between electrons, positrons and photons, denoted by H_I, has the following form :

$$
\begin{aligned}
&H_I = \\
&\sum_{\gamma,\gamma',n,\ell}\sum_{\mu=1,2}\int_{\mathbb{R}^3} d^3k\,(G^\mu_{d,\gamma,\gamma',n,\ell}(k)b^*_{\gamma,n}b_{\gamma',\ell}a^*_\mu(k)+h.c.)\\
&+\sum_{\gamma,\gamma',n}\sum_{\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^+} d^3k\,dp\,(G^\mu_{d,+,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n}b_{\gamma',+}(p)+b^*_{\gamma,+}(p)b_{\gamma',n})a^*_\mu(k)+h.c.)\\
&+\sum_{\gamma,\gamma',n}\sum_{\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^+} d^3k\,dp\,(G^\mu_{d,-,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n}b^*_{\gamma',-}(p)+b_{\gamma,-}(p)b_{\gamma',n})a^*_\mu(k)+h.c.)\\
&+\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^+\times\mathbb{R}^+} d^3k\,dp\,dp'(G^\mu_{+,-,\gamma,\gamma'}(p,p';k)(b^*_{\gamma,+}(p)b^*_{\gamma',-}(p')\\
&+b_{\gamma,-}(p)b_{\gamma',+}(p'))a^*_\mu(k)+h.c.)\\
&+\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^+\times\mathbb{R}^+} d^3k\,dp\,dp'(G^\mu_{+,+,\gamma,\gamma'}(p,p';k)b^*_{\gamma,+}(p)b_{\gamma',+}(p')a^*_\mu(k)+h.c.)\\
&+\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^+\times\mathbb{R}^+} d^3k\,dp\,dp'(G^\mu_{-,-,\gamma,\gamma'}(p,p';k)b^*_{\gamma,-}(p)b_{\gamma',-}(p')a^*_\mu(k)+h.c).
\end{aligned}
$$

In this note we consider the model of relativistic electrons and positrons in a Coulomb potential interacting with the quantized transversal radiation field associated with the following Hamiltonian :

$$H(g) = H_0 + g\,H_I \tag{18}$$

where g is the real coupling constant.

Let, for $\beta = 0, 1, 2$,

$$C_\beta = \sum_{\mu=1,2} \left(\sum_{\gamma,\gamma',n,\ell} \int_{\mathbb{R}^3} |G^\mu_{d,\gamma,\gamma',n,\ell}(k)|^2 \omega(k)^{-\beta} d^3k \right)^{1/2} \tag{19}$$

$$+ \sum_{\varepsilon=+,-} \sum_{\mu=1,2} \left(\sum_{\gamma,\gamma',n} \int_{\mathbb{R}^3\times\mathbb{R}^+} |G^\mu_{d,\varepsilon,\gamma,\gamma',n}(p;k)|^2 \omega(k)^{-\beta} dp\, d^3k \right)^{1/2}$$

$$+ \sum_{r} \sum_{\mu=1,2} \left(\sum_{\gamma,\gamma'} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+} |G^\mu_{r,\gamma,\gamma'}(p,p';k)|^2 \omega(k)^{-\beta} dp\, dp'\, d^3k \right)^{1/2}$$

where $r = \{+,+\}, \{+,-\}, \{-,-\}$.

We now state our main results :

Theorem 1. Suppose that $C_0 < \infty$ and $\frac{|g|}{E_0} C_1 < 1$. Then, $H(g)$ is self-adjoint on the domain $D(H_0)$.

Theorem 2. Suppose that $C_0 < \infty$, $\frac{|g|}{E_0} C_1 < 1$ and $C_2 < \infty$. Then, there exists $g_0 > 0$ such that, for every $g \in [-g_0, g_0]$, the self-adjoint operator $H(g)$ has a ground state.

2 Proofs of theorems 1 and 2

Let

$$\begin{aligned} I\!\!D &= \{\gamma = (j, m_j, k_j); j = \frac{1}{2}, \frac{3}{2}, \dots, m_j = -j+1, \dots, j, k_j = \pm 1, \pm 2, \dots, \} \\ I\!\!D_d &= \{(\gamma, n); \gamma \in I\!\!D \text{ and } n = 0, 1, 2, \dots\} \end{aligned}$$

We set

$$\begin{aligned} \Sigma_+ &= \Sigma_- = \mathbb{R}^+ \times I\!\!D \\ \Sigma_d &= \mathbb{R}^+ \times I\!\!D_d \end{aligned}$$

We will write $\xi = (p, \gamma)$ where (p, γ) belongs to Σ_+ and Σ_- respectively and $\xi_d = (p, \gamma, n)$ where (p, γ, n) belongs to Σ_d. Then

$$L^2(\Sigma_\pm) = \left\{ f : \Sigma_\pm \to \mathbb{C};\ \int_{\Sigma_\pm} |f(\xi)|^2 d\xi := \Sigma_{\gamma\in I\!\!D} \int_{\mathbb{R}^+} |f(p,\gamma)|^2 dp < \infty \right\}$$

Thus the map

$$(f_\gamma(\cdot))_{\gamma\in I\!\!D} \in \bigoplus_{\gamma\in I\!\!D} L^2(\mathbb{R}^+) \longrightarrow f(\xi) = f(p,\gamma) \in L^2(\Sigma_\pm)$$

is unitary and we will identify the space $\tilde{H}_{c_+}$ (resp. $\tilde{H}_{c_-}$) introduced in section 1 with $L^2(\Sigma_+)$(resp. $L^2(\Sigma_-)$).

Similarly we identify the space $\tilde{H}_d$ with a closed subspace, denoted by F, of $L^2(\Sigma_d)$. According to section 1, F is the closure of the following subspace

$$\left\{ \sum_{(\gamma,n)\in I\!\!D_d} c_{\gamma,n}\, 1\!\!1_{[n,n+1]}(p) \ ; \ \sum_{(\gamma,n)\in I\!\!D_d} |c_{\gamma,n}|^2 < \infty \right\}$$

Hence the Fermi-Fock space $\mathcal{F}_D$ will be identified with

$$\mathcal{F}_a(F) \otimes \mathcal{F}_a(L^2(\Sigma_+)) \otimes \mathcal{F}_a(L^2(\Sigma_-))$$

Let us now define the new annihilation and creation operators. Recall that (See [11])

$$\begin{aligned} \left\| \int_{\mathbb{R}^+} dp\, \overline{h_\pm(p,\gamma)}\, b_{\gamma\pm}(p) \right\| &= \left\| \int dp h_\pm(p,\gamma)\, b^*_{\gamma,\pm}(p) \right\| \\ &= ||h_\pm(\cdot,\gamma)||_{L^2(\mathbb{R}^+)} \end{aligned}$$

Thus the series $\sum_{\gamma\in I\!\!D} \int_{\mathbb{R}^+} dp\, \overline{h_\pm(p,\gamma)} b_{\gamma,\pm}(p)$ and $\sum_{\gamma\in I\!\!D} \int_{\mathbb{R}^+} dp\, h_\pm(p,\gamma) b^*_{\gamma,\pm}(p)$ are normally convergent in $\ell^2(I\!\!D,\ B(\mathcal{F}_a(\tilde{H}_{c\pm}))$ respectively, for every $h_\pm$ in $L^2(\Sigma_\pm)$.

Here $B(\mathcal{F}_a(\tilde{H}_{c\pm}))$ is the set of bounded operators in $\mathcal{F}_a(\tilde{H}_{c\pm})$ respectively.

We then set, for every $h_\pm$ in $L^2(\Sigma_\pm)$ respectively,

$$\begin{aligned} b_\pm(h_\pm) &:= \int d\xi\, \overline{h_\pm(\xi)}\, b_\pm(\xi) = \sum_{\gamma\in I\!\!D} \int_{\mathbb{R}^+} dp\, \overline{h_\pm(p,\gamma)}\, b_{\gamma,\pm}(p) \\ b^*_\pm(h_\pm) &:= \int d\xi\, h_\pm(\xi)\, b^*_\pm(\xi) = \sum_{\gamma\in I\!\!D} \int_{\mathbb{R}^+} dp\, h_\pm(p,\gamma)\, b^*_{\gamma,\pm}(p) \end{aligned}$$

where $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ are defined in section 1.

Similarly the series $\sum_{(\gamma,n)\in I\!\!D_d} \overline{c_{\gamma,n}}\, b_{\gamma,n}$ and $\sum_{(\gamma,n)\in I\!\!D_d} c_{\gamma,n}\, b^*_{\gamma,n}$, where $\sum_{(\gamma,n)\in I\!\!D_d} |c_{\gamma,n}|^2 < \infty$, are normally convergent in $\ell^2(I\!\!D_d, B(\mathcal{F}_a(\tilde{H}_d)))$.

Thus, for any $h_d(\xi_d) = \sum_{(\gamma,n)\in \mathbb{D}_d} c_{\gamma,n}\, 1\!\!1_{[n,n+1]}(p) \in F \subset L^2(\Sigma_d)$, we set

$$
\begin{aligned}
b_d(h_d) &= \sum_{(\gamma,b)_i n \mathbb{D}_d} \overline{c_{\gamma,n}}\, b_{\gamma,n} \\
b_d^*(h_d) &= \sum_{(\gamma,b)_i n \mathbb{D}_d} c_{\gamma,n}\, b_{\gamma,n}^*
\end{aligned}
$$

where $b_{\gamma,n}$ and $b_{\gamma,n}^*$ are defined in section 1.

The new canonical anticommutation relations are the following ones

$$
\{b_\pm(h_\pm^1),\ b_\pm^*(h_\pm^2)\} = (h_\pm^1, h_\pm^2)_{L^2(\Sigma_\pm)} \tag{20}
$$

$$
\{b_\pm^\#(h_\pm^1),\ b_\pm^\#(h_\pm^2)\} = 0 \tag{21}
$$

$$
\{b_\pm^\#(h_\pm),\ b_d^\#(h_d)\} = 0 \tag{22}
$$

$$
\{b_d(h_d^1),\ b_d^*(h_d^2)\} = (h_d^1, h_d^2)_{L^2(\Sigma_d)} \tag{23}
$$

Here $b^\#$ is b or b^*.

We now have the following lemmas whose proofs are well-known.

Lemma 2.1 *Let f be in $L^2(\mathbb{R}^3, \mathbb{C}^2; \omega(k)^{-\beta} d^3k)$ for $\beta = 0, 1$. We have*

$$
\begin{aligned}
||a(f)\psi||_{\mathcal{F}_{ph}}^2 &\leq \left(\sum_{\mu=1,2}\int d^3k \frac{|f(k,\mu)|^2}{\omega(k)}\, d^3k\right) ||H_{ph}^{1/2}\psi||_{\mathcal{F}_{ph}}^2 \\
||a^*(f)\psi||_{\mathcal{F}_{ph}}^2 &\leq \left(\sum_{\mu=1,2}\int \frac{|f(k,\mu)|^2}{\omega(k)}\, d^3k\right) ||H_{ph}^{1/2}\psi||_{\mathcal{F}_{ph}}^2 \\
&\quad + \left(\sum_{\mu=1,2}\int \frac{|f(k,\mu)|^2}{\omega(k)}\, d^3k\right) ||\psi||_{\mathcal{F}_{ph}}^2
\end{aligned}
$$

for every $\psi \in D(H_{ph})$ (see [3]).

Lemma 2.2 *For $h_\pm \in L^2(\Sigma_\pm)$, $b_\pm(h_\pm)$ and $b_\pm^*(h_\pm)$ are bounded operators and we have*

$$
||b_\pm(h)\psi||^2 + ||b_\pm^*(h)\psi||^2 = ||\psi||^2 ||h||_{L^2(\Sigma_\pm)}^2.
$$

In particular,

$$
||b_\pm(h)|| = ||b_\pm^*(h)|| = ||h||_{L^2(\Sigma_\pm)}
$$

for every $\psi \in \mathcal{F}_a(L^2(\Sigma_\pm))$ respectively.

Proof : (20) implies that

$$||b_\pm(h)\psi||^2 + ||b_\pm^*(h)\psi||^2 = (\{b_\pm(h), b_\pm^*(h)\}\psi, \psi) = ||h||^2\, ||\psi||^2,$$

which proves the lemma. ■

Lemma 2.3 *For every $h_d \in F$, $b_d^\#(h_d)$ is a bounded operator and we have*

$$||b_d(h_d)\psi||^2 + ||b_d^*(h_d)\psi||^2 = ||h_d||^2_{L^2(\Sigma_d)}||\psi||^2$$

for all $\psi \in \mathcal{F}_a(L^2(\Sigma_d))$.

Proof : Lemma 2.3 follows from (23) directly. ■

Let $\{f_j, j = 1, 2, \dots\}$ (resp. $\{g_k, k = 1, 2, \dots\}$, $\{h_i, i = 1, 2, \dots\}$) be an orthonormal basis of $L^2(\Sigma_+)$ (resp. $L^2(\Sigma_-), F$).

We will now consider vectors in $\mathcal{F}_D$ of the following form :

$$\Phi^{(l,m,n)} := b_d^*(h_{i_1}) \dots b_d^*(h_{i_l}) b_+^*(f_{j_1}) \dots b_+^*(f_{j_m}) b_-^*(g_{k_1}) \dots b_-^*(g_{k_n})\, \Omega \qquad (24)$$

where l, m, n are positive integers and $\Omega = \Omega_d \otimes \Omega_{c+} \otimes \Omega_{c-}$.

The indexes will be assumed ordered such that $i_1 < \dots < i_l$, $j_1 < \dots < j_m$ and $k_1 < \dots < k_n$.

It is known that the set $\{\Phi^{(l,m,n)}; l, m, n = 0, 1, 2, \dots\}$ is an orthonormal basis of $\mathcal{F}_D$ (See [11]).

In the following propositions we investigate several operators in $\mathcal{F}_D$ buildt from product of creation and annihilation operators.

For $G \in L^2(\Sigma_\pm \times \Sigma_\pm)$, we set

$$A_\pm = \int\int_{\Sigma_\pm \times \Sigma_\pm} d\xi\, d\xi'\, \overline{G(\xi, \xi')}\, b_\pm(\xi)\, b_\pm(\xi')$$

Futhermore, for $G \in L^2(\Sigma_+ \times \Sigma_-)$, we set

$$A_{+-} = \int\int_{\Sigma_+ \times \Sigma_-} d\xi\, d\xi'\, \overline{G(\xi, \xi')}\, b_+(\xi)\, b_-(\xi')$$

Similarly, for $G \in L^2(\Sigma_- \times \Sigma_+)$, we set

$$A_{-+} = \int\int_{\Sigma_- \times \Sigma_+} d\xi\, d\xi'\, \overline{G(\xi, \xi')}\, b_-(\xi)\, b_+(\xi')$$

We then have

Proposition 2.4 *For $G \in L^2(\Sigma_\pm \times \Sigma_\pm)$, we set*

$$G_a(\xi, \xi') = G(\xi, \xi') - G(\xi', \xi)$$

We then have

$$||A_\pm|| = ||A^*_\pm|| \leq ||G_a||_{L^2(\Sigma_\pm \times \Sigma_\pm)} \tag{25}$$

Similarly, for $G \in L^2(\Sigma_+ \times \Sigma_-)$, we have

$$||A_{+-}|| = ||A^*_{+-}|| \leq ||G_a||_{L^2(\Sigma_+ \times \Sigma_-)} \tag{26}$$

and, for $G \in L^2(\Sigma_- \times \Sigma_+)$, we have

$$||A_{-+}|| = ||A^*_{-+}|| \leq ||G_a||_{L^2(\Sigma_- \times \Sigma_+)}$$

Proof : We first prove (25) . We only consider A_+ and A_{+-}. The cases of A_- and A_{-+} are similar.

Let

$$\Phi^{(\ell,m,n)} = \prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{j=1}^{m} b^*_+(g_j) \otimes \prod_{k=1}^{n} b^*_-(h_k)\Omega$$

be a vector of the form (24).

We claim that

$$A_+\Phi^{(\ell,m,n)} = \sum_{1 \leq \alpha \leq \beta \leq m} (-1)^{1+\beta}(G_a, g_\alpha \otimes g_\beta)_{L^2(\Sigma_+ \times \Sigma_+)}$$

$$\prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{\substack{j=1 \\ j\neq\alpha, j\neq\beta}}^{m} b^*_+(g_j) \otimes \prod_{k=1}^{n} b^*_-(h_k)\Omega \tag{27}$$

where

$$(G_a, g_\alpha \otimes g_\beta)_{L^2(\Sigma_+ \times \Sigma_+)} = \int\int_{\Sigma_+ \times \Sigma_+} \overline{G_a(\xi, \xi')}\, g_\alpha(\xi) g_\beta(\xi') d\xi\, d\xi'$$

Indeed the canonical anticommutation relations (CAR) (17)-(20) yield

$$\int d\xi' \overline{G(\xi, \xi')}\, b_+(\xi')\, \Phi^{(\ell,m,n)}$$

$$= (-1)^\ell \sum_{\alpha=1}^{m} (-1)^{\alpha+1} (G(\xi, \cdot), g_\alpha)_{L^2(\Sigma_+)}$$

$$\cdot \prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{\substack{j=1 \\ j\neq\alpha}}^{m} b^*_+(g_j) \otimes \prod_{k=1}^{n} b^*_-(h_k)\Omega \tag{28}$$

Applying $b_+(\xi)$ to the right hand side of (28), using the CAR as above and integrating with respect to ξ, we get (27).

Since

$$\left((I \otimes \prod_{\substack{j=1\\ j\neq\alpha, j\neq\beta}}^{m} b_+^*(g_j) \otimes I)\Omega\right)_{1\leq\alpha<\beta\leq n}$$

is an orthonormal family, it follows from (25) that

$$\begin{aligned} ||A_+\Phi^{(\ell,m,n)}||^2 &= \sum_{1\leq\alpha<\beta\leq n} |(G_a, g_\alpha \otimes g_\beta)_{L^2(\Sigma_+\times\Sigma_+)}|^2 \\ &\leq ||G_a||^2_{L^2(\Sigma_+\times\Sigma_+)} ||\Phi^{(\ell,m,n)}||^2 \end{aligned} \tag{29}$$

We recall that $(g_j)_j$ is an orthonormal family and that $||\Phi^{(\ell,m,n)}|| = 1$.

Using (27) we show, in the same way, that we have

$$||A_+\psi||^2 \leq ||G_a||^2_{L^2(\Sigma_+\times\Sigma_+)} ||\psi||^2 \tag{30}$$

for every $\psi = \sum_{\ell,m,n=0}^{N} \alpha_{\ell,m,n} \Phi^{(\ell,m,n)}$.

Since the set of the vectors ψ is dense in $\mathcal{F}_D$, (25) follows from (30).

Similarly (26) follows from

$$\begin{aligned} &\int_{\Sigma_+\times\Sigma_-} d\xi\, d\xi' \overline{G(\xi,\xi')}\, b_+(\xi)\, b_-(\xi')\Phi^{(\ell,m,n)} \\ &\quad = (-1)^m \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} (-1)^{\alpha+\beta} (G_a, g_\alpha \otimes h_\beta)_{L^2(\Sigma_+\times\Sigma_-)} \\ &\quad \cdot \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j=1\\ j\neq\alpha}}^{m} b_+^*(g_j) \otimes \prod_{\substack{k=1\\ k\neq\beta}}^{n} b_-^*(h_k)\Omega \end{aligned}$$

Thus the proposition 2.4 is proved. ■

Proposition 2.5 *For $G \in L^2(\Sigma_\pm \times \Sigma_\pm)$ we set*

$$B_\pm = \int\int_{\Sigma_\pm\times\Sigma_\pm} d\xi\, d\xi' G(\xi,\xi') b_\pm^*(\xi) b_\pm(\xi')$$

We have

$$||B_\pm\, (N_D + 1)^{-1/2}|| \leq ||G||_{L^2(\Sigma_\pm\times\Sigma_\pm)} \tag{31}$$

$$||(N_D+1)^{-1/2}\, B_\pm|| \leq ||G||_{L^2(\Sigma_\pm\times\Sigma_\pm)} \tag{32}$$

Proof : Since

$$B_\pm^* = \int\int_{\Sigma_\pm\times\Sigma_\pm} d\xi\, d\xi'\, \overline{G(\xi',\xi)}\, b_\pm^*(\xi)\, b_\pm(\xi')$$

and $N_D = N_D^*$, (32) follows from (31).

Let us prove (31). By (28) we obtain

$$\begin{aligned} b_+(G(\xi,\cdot))\Phi^{(\ell,m,n)} = (-1)^l \sum_{\alpha=1}^{m}(-1)^{\alpha+1}\prod_{i=1}^{\ell} b_d^*(f_i) \quad &\otimes \quad \left[b_+^*((G(\xi,\cdot),g_\alpha(\cdot)) \prod_{\substack{j=1\\ j\neq\alpha}}^{m} b_+^*(g_j)\right] \\ &\otimes \quad \prod_{k=1}^{n} b_-^*(h_k)\Omega \end{aligned} \tag{33}$$

Since $(g_j)_j$ is an orthonormal basis of $L^2(\Sigma_+)$ we have

$$(G(\xi,\cdot),g_\alpha(\cdot)) = \sum_\beta (G, g_\beta\otimes g_\alpha)\, g_\beta \tag{34}$$

Thus, from (33) and (34), we get

$$\begin{aligned} B_+\Phi^{(\ell,m,n)} &= \sum_{\alpha=1}^{m}(-1)^{\alpha+1}\left(\sum_\beta (G, g_\beta\otimes g_\alpha)\prod_{k=1}^{\ell} b_d^*(f_i)\right) \\ &\quad \otimes \left[b_+^*(g_\beta) \prod_{\substack{j=1\\ j\neq\alpha}}^{m} b_+^*(g_j)\right] \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \\ &= \sum_{\alpha=1}^{m}(-1)^{\alpha+1}(G, g_\alpha\otimes g_\alpha)\prod_{i=1}^{\ell} b_d^*(f_i) \\ &\quad \otimes \left[b_+^*(g_\alpha) \prod_{\substack{j=1\\ j\neq\alpha}}^{m} b_+^*(g_j)\right] \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \\ &+ \sum_{\alpha=1}^{m}(-1)^{\alpha+1}\sum_{\beta=m+1} (G, g_\beta\otimes g_\alpha)\prod_{i=1}^{\ell} b_d^*(f_i) \\ &\quad \otimes \left[b_+^*(g_\beta) \prod_{\substack{j=1\\ j\neq\alpha}}^{m} b_+^*(g_j)\right] \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \end{aligned} \tag{35}$$

where we have used the fact that

$$\prod_{j=1}^{m} b_+^*(f_j)\Omega_{c_+} = 0 \quad \text{if} \quad g_{j_0} = g_{j_1} \quad \text{with} \quad j_0 = j_1.$$

We have

$$(-1)^{\alpha+1} b_+^*(g_\alpha) \prod_{\substack{j=1 \\ j\neq\alpha}}^{m} b_+^*(g_j)\Omega_{c_+} = \prod_{j=1}^{m} b_+^*(g_j)\Omega_{c_+} \tag{36}$$

Hence, from (35) and (36), we get

$$\begin{aligned} B_+\Phi^{(\ell,m,n)} &= \left(\sum_{\alpha=1}^{m}(G, g_\alpha \otimes g_\alpha)\right) \Phi^{(\ell,n,m)} \\ &\quad + \sum_{\alpha=1}^{m}(-1)^{\alpha+1} \sum_{\beta=m+1}^{\infty} (G, g_\beta \otimes g_\alpha) \prod_{i=1}^{\ell} b_d^*(f_i) \\ &\quad \otimes \left[b_+^*(g_\beta) \prod_{j=1}^{m} b_+^*(g_j) \right] \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \end{aligned} \tag{37}$$

Two different vectors in the right hand side of (37) are orthogonal. Therefore we get

$$\begin{aligned} ||B_+\Phi^{(\ell,m,n)}||^2 &= \left|\sum_{\alpha=1}^{m}(G, g_\alpha \otimes g_\alpha)\right|^2 \\ &\quad + \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{\infty} |(G, g_\beta \otimes g_\alpha)|^2 \end{aligned} \tag{38}$$

from which we deduce

$$||B_+\Phi^{(\ell,m,n)}||^2 \leq m||G||^2 \leq ||G||^2 \; ||N_D^{1/2}\, \Phi^{(\ell,m,n)}||^2$$

The proof is now completed by density as in proposition 2.4. ■

We then have the following lemmas

Lemma 2.6 *Let A (resp. B) a positive self-adjoint operator in the Hilbert space H_1 (resp. H_2) with domain $D(A)$ (resp. $D(B)$).*

Then the operator $H_0 = A \otimes I + I \otimes B$ *with domain* $D(A \otimes I) \cap D(I \otimes B)$ *is self-adjoint in* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *and we have*

$$\begin{aligned} ||(A \otimes I)\, \psi|| &\leq ||H_0\, \psi|| \\ ||(I \otimes B)\, \psi|| &\leq ||H_0\, \psi|| \end{aligned}$$

for every $\psi \in D(A \otimes I) \cap D(I \otimes B)$.

Proof : Let $D(A)$ (resp. $D(B)$) any domain on which A (resp. B) is essentially self-adjoint. Then $A \otimes I$, $I \otimes B$ and $H_0 = A \otimes I + I \otimes B$ are essentially self-adjoint on $D(A) \otimes D(B)$, the algebraic tensor product of $D(A)$ and $D(B)$. (See [10, chap.VIII]).

The lemma is proved if we show that $D(H_0)$ is contained in $D(A \otimes I) \cap D(B \otimes I)$. Let $(u_n)_{n \geq 1}$ be a sequence such that $u_n \in D(A) \otimes D(B)$, $u_n \underset{n \to \infty}{\longrightarrow} u$ and $H_0\, u_n \underset{n \to \infty}{\longrightarrow} w$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. We have

$$\begin{aligned} ||H_0(u_n - u_m)||^2 &= ||(A \otimes I + I \otimes B)\, (u_n - u_m)||^2 \\ &= ||(A \otimes I)\, (u_n - u_m)||^2 + ||(I \otimes B)\, (u_n - u_m)||^2 \\ &+ 2||A^{1/2} \otimes B^{1/2}(u_n - u_m)||^2 \end{aligned}$$

Therefore

$$\begin{aligned} ||H_0(u_n - u_m)|| &\geq ||(A \otimes I)\, (u_n - u_m)|| \\ ||H_0(u_n - u_m)|| &\geq ||(I \otimes B)\, (u_n - u_m)|| \end{aligned}$$

Thus the sequences $((A \otimes I)u_n)_{n \geq 1}$ and $((A \otimes B)u_n)_{n \geq 1}$ are convergent in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Finally it follows from the self-adjointness of $A \otimes I$ and $I \otimes B$ that $u \in D(A \otimes I) \cap D(I \otimes B)$, $(A \otimes I)u_n \underset{n \to \infty}{\longrightarrow} (A \otimes I)u$, $(I \otimes B)u_n \underset{n \to \infty}{\longrightarrow} (I \otimes B)u$ and $H_0\, u_n \underset{n \to \infty}{\longrightarrow} H_0 u = w$.

Hence the lemma is proved. ■

Lemma 2.7 *Let* $E_0 = \inf_{(\gamma,n)\in D_d} E_{\gamma,n} > 0$. *We have*

$$E_0 ||N_D\, \psi|| \leq ||d\Gamma(H_D)\, \psi||$$

for any $\psi \in D(d\Gamma(H_D))$.

Proof : The inequality is satisfied both on $(P_{\Omega_D} \otimes I)\mathcal{F}$ and $(P_{\Omega_D}^{\perp} \otimes I)\mathcal{F}$ since $\omega(p) \geq E_0$ for any $p \in \mathbb{R}^+$. ■

Applying Lemma 2.6 to $A = d\Gamma(H_D)$ and $B = H_{ph}$ we get from lemma 2.7 the following proposition :

Proposition 2.8 *For any* $\psi \in D(d\Gamma(H_D) \otimes I) \cap D(I \otimes H_{ph})$, *we have*

$$\begin{aligned} E_0 ||N_D \otimes I)\psi|| &\leq ||H_0\psi|| \\ ||(I \otimes H_{ph})\psi|| &\leq ||H_0\psi|| \end{aligned}$$

where $H_0 = d\Gamma(H_D) \otimes I + I \otimes H_{ph}$

2.1 Proof of Theorem 1

The interaction between electrons, positrons and photons can be written (see section 1.4) :

$$H_I = \sum_{i=1}^{6} \sum_{\mu=1,2} \int d^3k (v_i^{\mu}(k) \otimes a_{\mu}^*(k) + v_i^{\mu *}(k) \otimes a_{\mu}(k)) \tag{39}$$

where

$$
\begin{aligned}
v_1^\mu(k) &= \sum_{\gamma,\gamma',n,\ell} G^\mu_{d,\gamma,\gamma',n,\ell}(k) b^*_{\gamma,n} b_{\gamma',l} \\
v_2^\mu(k) &= \sum_{\gamma,\gamma',n} \int dp G^\mu_{d,+,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n} b_{\gamma',+}(p) + b^*_{\gamma,+}(p) b_{\gamma',n}) \\
v_3^\mu(k) &= \sum_{\gamma,\gamma',n} \int dp G^\mu_{d,-,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n} b^*_{\gamma',-}(p) + b_{\gamma,-}(p) b_{\gamma',n}) \\
v_4^\mu(k) &= \sum_{\gamma,\gamma'} \int\int dp\, dp' G^\mu_{+,-,\gamma,\gamma'}(p,p';k)(b^*_{\gamma,+}(p) b^*_{\gamma',-}(p') + b_{\gamma,-}(p) b_{\gamma',+}(p')) \\
v_5^\mu(k) &= \sum_{\gamma,\gamma'} \int\int dp\, dp' G^\mu_{+,+,\gamma,\gamma'}(p,p';k) b^*_{\gamma,+}(p) b_{\gamma',+}(p') \\
v_6^\mu(k) &= \sum_{\gamma,\gamma'} \int\int dp\, dp' G^\mu_{-,-,\gamma,\gamma'}(p,p';k) b^*_{\gamma,-}(p) b_{\gamma',-}(p')
\end{aligned}
\tag{40}
$$

It follows from Proposition 2.4, Proposition 2.5 and Lemma 2.7 that $v_i^\mu(k)$, $i \neq 5,6$, $v_j^\mu(k)^*(N_D+1)^{-1/2}$, $(N_D+1)^{-1/2} v_j^\mu(k)$, $j = 5,6$ are bounded from $\mathcal{F}_D$ into $\mathcal{F}_D$.

For $\beta = 0,1$, $\mu = 1,2$ and $i \in \{1,2,\dots,6\}$, we set

$$
\begin{aligned}
a^\mu_{\beta,i} &= \left(\int_{\mathbb{R}^3} \omega(k)^{-\beta} ||v_i^\mu(k)||^2 d^3k \right)^{1/2}, \qquad i \neq 5,6 \\
a^\mu_{\beta,i} &= \left(\int_{\mathbb{R}^3} \omega(k)^{-\beta} ||v_i^\mu(k)(N_D+1)^{-1/2}||^2 d^3k \right)^{1/2}, \qquad i = 5,6.
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
b_i^\mu &= \left(\int \omega(k)^{-1} ||v_i^\mu(k)||^2 d^3k \right)^{1/2}, \qquad i \neq 5,6 \\
b_i^\mu &= \left(\int \omega(k)^{-1} ||(N_D+1)^{-1/2}\, v_i^\mu(k)||^2 d^3k \right)^{1/2}, \qquad i = 5,6.
\end{aligned}
\tag{42}
$$

Lemma 2.9 *For every $\psi \in D(H_0)$, we have*

$$
\left\| \int d^3k\, v_i^\mu(k)^* \otimes a_\mu(k)\, \psi \right\| \leq \frac{b_i^\mu}{E_0} ||(H_0 + m_0c^2)\psi||, \tag{43}
$$

$$
\begin{aligned}
\left\| \int d^3k\, v_i^\mu(k) \otimes a_\mu^*(k)\, \psi \right\|^2 \leq\ & \frac{(a^\mu_{1,i})^2 + \varepsilon (a^\mu_{0,1})^2}{E_0^2} ||(H_0 + m_0c^2)\psi||^2 \\
& + \frac{(a^\mu_{0,i})^2}{4\varepsilon} ||\psi||^2
\end{aligned}
\tag{44}
$$

for every $\varepsilon > 0$.

Proof : In the appendix, we prove that, for every $\psi \in D(H_0)$ and every $\varepsilon > 0$, we have

$$\left\| \int d^3k\, v_i^\mu(k)^* \otimes a_\mu(k)\, \psi \right\| \leq b_i^\mu\, ||(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\psi||, \tag{45}$$

$$\begin{aligned}\left\| \int d^3k\, v_i^\mu(k) \otimes a_\mu^*(k)\, \psi \right\|^2 &\leq (a_{1,i}^\mu)^2 ||(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\psi||^2 \\ &+ (a_{0,i}^\mu)^2 \left[\varepsilon||(N_D+1)\otimes\psi||^2 + \frac{1}{4\varepsilon}||\psi||^2\right],\end{aligned} \tag{46}$$

which together with Lemma 2.6 and Lemma 2.7 give (43) and (44) by noting that $E_0 < m_0c^2$. ■

Finally, from (39), (43) and (44), we get

Corollary 2.10 *The operator H_I is $H_0 + m_0c^2$-bounded with $H_0 + m_0c^2$-bound*

$$C_1' = \sum_{\mu=1,2} \sum_{i=1}^{6} \frac{a_{1,i}^\mu + b_{1,i}^\mu}{E_0}.$$

Choosing g such that $|g|C_1' < 1$, the theorem 1.1 follows from the Proposition 2.4, the proposition 2.5, the corollary 2.11 and theorem V.4.3 of [9].

3 Proof of Theorem 2

Throughout this section we assume that the assumptions of Theorem 2 are satisfied. Let us introduce an infrared cutoff in the interaction

Let $H_{I,m}$be the operator obtained from (39) by substituting $v_{i,m}^\mu(k) = 1_{\{k;\omega(k)\geq m\}}v_i^\mu(k)$, $m > 0$, for $v_i^\mu(k)$. We then define

$$H_m = H_0 + g\ H_{I,m}. \tag{47}$$

Theorem 2 will be a simple consequence of the following result.

Theorem 3.1 *There exists $g_0 > 0$ such that, for every $g \in [-g_0, g_0]$, the following properties hold.*

(i) $H_m\,\psi \underset{m\mapsto 0}{\longrightarrow} H\psi$ *for every* $\psi \in D(H_0)$.

(ii) For every $m > 0$, H_m *has a normalized ground state* ϕ_m.

(iii) Fix λ *in* $(E_0, m_0\, c^2)$. *For every* $m > 0$, *we have*

$$(\phi_m, P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}\phi_m) \geq 1 - \delta_g \tag{48}$$

where δ_g *tends to zero when* g *tends to zero.*

Theorem 2 is easely deduced from Theorem 3.1 as follows:

Let H_m and ϕ_m be the same as in Theorem 3.1. Since $||\phi_m|| = 1$, there exists a subsequence (ϕ_{m_k}) of (ϕ_m) such that $w - \lim \phi_{m_k} = \phi$.

On the other hand, since $P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}$ is of finite rank for $\lambda \in (E_0, m_0 c^2)$, it follows from (48) that

$$(\phi, P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}\phi) \geq 1 - \delta_g,$$

which implies that $\phi \neq 0$.

Now, in order to conclude the proof of Theorem 2, it suffices to apply the following well known result :

Lemma 3.2 *Let* T_n, $n \geq 1$, *and* T *be bounded below self-adjoint operators on a Hilbert space.*

We assume that, for every $n \geq 1$, T_n *has a normalized ground state* ϕ_n *with ground state energy* E_n *such that* $\lim_{n\to\infty} E_n = E$ *and* $\omega - \lim_{n\mapsto+\infty} \phi_n = \phi \neq 0$. *Then* ϕ *is a ground state of* T *with ground state energy* E.

Theorem 3.1 will be proved in several steps. We first prove (i). We then prove (ii). Finally, by the periodic approximation method, we will prove the existence of a ground state for H_m, $m > 0$.

3.1 Proof of (i)

For $m > 0$, set

$$\tilde{v}_i^{\mu,m}(k) = 1_{\{k;\omega(k)\leq m\}}(k)\, v_i^{\mu}(k).$$

We denote $a_{\beta,j}^{\mu,m}$ (resp. $b_j^{\mu,m}$) the expression (41) and (42) that we obtain by substituting $\tilde{v}_j^{\mu,m}(k)$ for $v_j^{\mu}(k)$.

Since $H - H_m = g(H_I - H_{I,m})$, it follows from Lemma 2.9 that

$$||(H-H_m)\phi|| \leq \sum_{\mu=1,2}\sum_{j=0}^{6} \frac{a_{1,j}^{\mu,m} + a_{0,j}^{\mu,m} + b_j^{\mu,m}}{E_0}\left[||(H_0 + m_0 c^2)\phi|| + ||\phi||\right]$$

for every $\phi \in D(H_0)$.

By Lebesgue theorem $(a_{\beta,j}^{\mu,m} + b_j^{\mu,m})$ tends to zero when m tends to zero. We then get

$$\lim_{n\mapsto 0} ||(H-H_m)\phi|| = 0, \text{ for every } \phi \in D(H_0).$$

Thus the first statement of Theorem 3.1 is proved.

■

3.2 Proof of (iii)

The proof of (48) will follow from the following lemmas.

Lemma 3.3 *For every $m > 0$, we have*

$$E_m := \inf_{\substack{\phi\in D(H_0)\\ ||\phi||=1}} (H_m\phi,\phi) \leq 0.$$

Proof :

Since $(a_\mu(k)\Omega_{ph}, \Omega_{ph}) = (\Omega_{ph}, a_\mu^*(k)\Omega_{ph}) = 0$, we have

$$(H_{I,m}\Omega_D\otimes\Omega_{ph}, \Omega_D\otimes\Omega_{ph}) = 0$$

and

$$((I\otimes H_{ph})\,\Omega_D\otimes\Omega_{ph}, \Omega_D\otimes\Omega_{ph}) = 0$$

Hence we get

$$
\begin{aligned}
(H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}) &= ((d\Gamma(H_D) \otimes I)\, \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \\
&= 0
\end{aligned}
$$

$$E_m = \inf_{\substack{\phi \in D(H_0) \\ ||\phi||=1}} (H_m \phi, \phi) \leq (H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}) = E_0$$

■

In Theorem 3.10 below we show that, for g sufficiently small, H_m has a ground state ϕ_m, i.e. there exists, for every $m > 0$, a normalized solution to $H_m \phi_m(g) = E_m \phi_m(g)$ such that $||\phi_m|| = 1$. We then have :

Lemma 3.4 *There exist* $g_1 > 0$ *and* $\delta(g_1) > 0$ *independent of* m *such that*

$$||(H_0 + m_0 c^2) \phi_m(g)|| \leq \delta(g_1)$$

for every $m > 0$ *and for every* $g \in [-g_1, g_1]$.

Proof : We have :

$$H_0 \phi_m(g) = H_m \phi_m(g) - g\, H_{I,m} \phi_m(g) = E_m \phi_m(g) - g\, H_{I,m} \phi_m(g) \tag{49}$$

By lemme 2.9, we get

$$||H_{I,m} \phi_m(g)|| \leq \frac{C_1 + C_0}{E_0} [||(H_0 + m_0 c^2) \phi_m(g)|| + ||\phi_m(g)||]. \tag{50}$$

¿From (49) and (50) we obtain

$$||H_0 \phi_m(g)|| \leq E_m + |g| \, \frac{C_1 + C_0}{E_0} \, [||(H_0 + m_0 c^2) \phi_m(g)|| + 1],$$

which, together with Lemma 3.3, implies

$$||(H_0 + m_0 c^2) \phi_m(g)|| \leq \left[m_0 c^2 + \frac{|g|}{E_0} (C_0 + C_1) \right] \left[1 - \frac{|g|}{E_0} (C_0 + C_1) \right]^{-1}.$$

To conclude, we only need to choose $|g| \leq g_1$ where $g_1 < \dfrac{E_0}{C_0 + C_1}$. ■

Lemma 3.5 *Assume $C_0 < \infty$ and $C_1 < \infty$. Then we have*

$$\sum_{\mu=1,2} ||[1 \otimes a_\mu(k), H_{I,m}]\,\phi||^2 \leq \frac{G(k)^2}{E_0^2}[||H_0\phi||^2 + ||\phi||^2]$$

for every $\phi \in D(H_0)$ and for a.e. $k \in \mathbb{R}^3$.

Here

$$\begin{aligned} G(k)^2 &= \sum_{\mu=1,2} \sum_{\gamma,\gamma',n,\ell} |G^\mu_{d,\gamma,\gamma',n,l}(k)|^2 + 4 \sum_{\mu=1,2} \sum_{r=+,-} \sum_{\gamma,n} \int |G^\mu_{d,r,\gamma}(p;k)|^2 dp \\ &+ 4 \sum_{\mu=1,2} \sum_{\gamma,\gamma'} \int\int |G_{+,-,\gamma,\gamma'}(p,p';k)|^2 dp dp' \\ &+ \sum_{\mu=1,2} \sum_{r=+,-} \sum_{\gamma,\gamma'} \int\int |G_{r,r,\gamma,\gamma'}(p,p';k)|^2 dp dp' \end{aligned}$$

Proof : By using the commutation relations, a simple computation shows that

$$[1 \otimes a_\mu(k), H_{I,m}] = \sum_{i=1}^{6} v^\mu_{i,m}(k)$$

The lemma now follows from Proposition 2.4 and Proposition 2.5. ∎

Lemma 3.6 *There exists $C > 0$, independent of g such that*

$$(1 \otimes N_{ph}\phi_m, \phi_m) \leq C\ g^2\ \left(\int_{\mathbb{R}^3} \frac{G(k)^2}{\omega(k)^2}\, d^3k\right),$$

for all $m > 0$. Here

$$N_{ph} := \sum_{\mu=1,2} \int_{\mathbb{R}^3} d^3k\, a^*_\mu(k)\, a_\mu(k)$$

is the operator number of photons.

Proof : One easily checks that we have

$$[1 \otimes a_\mu(k), d\Gamma(H_D) \otimes 1\!\!1]\,\psi = 0$$

and

$$[1 \otimes a_\mu(k), 1\!\!1 \otimes H_{ph}]\,\psi = \omega(k)\,(1\!\!1 \otimes a_\mu(k))\,\psi$$

for every $\psi \in D(H_0)$ and for a.e. $k \in \mathbb{R}^3$.

We then get :

$$(H_m + \omega(k))\, 1 \otimes a_\mu(k) - 1 \otimes a_\mu(k)\, H_m = g[H_{I,m}, 1 \otimes a_\mu(k)]$$

for a.e. $k \in \mathbb{R}^3$.

Applying this equality to ϕ_m and taking the scalar product with $1 \otimes a_\mu(k)\, \phi_m$, we get

$$||1 \otimes a_\mu(k)\phi_m|| \le \frac{|g|}{\omega(k)}\; ||[1 \otimes a_\mu(k), H_{I,m}]\, \phi_m|| \tag{51}$$

for a.e. $k \in \mathbb{R}^3$.

Since

$$(1 \otimes N_{ph}\, \phi_m, \phi_m) = \sum_{\mu=1,2} \int ||1 \otimes a_\mu(k)\, \phi_m||^2 d^3k$$

Lemma 3.6 follows from (51), Lemma 3.4 and Lemma 3.5. ∎

Lemma 3.7 *Fix λ in $(0, m_0 c_0^2)$. There exists $\delta_g(\lambda) > 0$ such that $\delta_g(\lambda)$ tends to zero when $|g|$ tends zero and*

$$(P_D^\perp(\lambda) \otimes P_{\Omega_{ph}}\phi_m, \phi_m) \le \delta_g(\lambda)$$

for every $m > 0$. Here $P_D^\perp(\lambda) = 1 - P_D(\lambda) = \mathbb{1}_{[\lambda,\ \infty)}(d\Gamma(H_D))$.

Proof : Using $P_{\Omega_{ph}} H_{ph} = 0$, we get

$$\begin{aligned}&(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}})\,(H_m - E_m) = \\ &\qquad P_D(\lambda)^\perp (d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}} + g(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}})\, H_{I,m}.\end{aligned} \tag{52}$$

From (52) we obtain

$$\begin{aligned} 0 &= (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}})\,(H_m - E_m)\,\phi_m \qquad (53)\\ &= P_D^\perp(\lambda)\,(d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}}\phi_m + g\; P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} H_{I,m}\phi_m \end{aligned}$$

We then have

$$d\Gamma(H_D) P_D(\lambda)^\perp \ge \lambda\; P_D^\perp(\lambda).$$

By (53) and Lemma 3.3, we obtain

$$(P_D^{\perp}(\lambda) \otimes P_{\Omega_{ph}} \phi_m, \phi_m) \leq -\lambda^{-1} g((P_D^{\perp}(\lambda) \otimes P_{\Omega_{ph}}) H_{I,m} \phi_m, \phi_m)$$

for every $m > 0$.

Lemma 3.8 then follows by remarking that Corollary 2.10 also holds for $H_{I,m}$ and from Lemma 3.4. ■

We are now able to prove the inequality (48).

The left hand side of (48) is equivalent to

$$((P_D^{\perp}(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes P_{\Omega_{ph}}^{\perp})\, \phi_m, \phi_m) \leq \delta_g,$$

Since $P_{\Omega_{ph}}^{\perp} \leq N_{ph}$, it suffices to show that

$$((P_D^{\perp}(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes N_{ph}) \phi_m, \phi_m) \leq \delta_g,$$

which follows from Lemma 3.6 and Lemma 3.7. ■

3.3 Existence of a ground state for H_m

¿From now on we mimick the proof of [1] and [3] (See also [7], [8] and [19]) and we use the same notations as in [3].

For $m > 0$, we set

$$\mathcal{A}_\ell = \{k \in \mathbb{R}^3 \ ; \ \omega(k) \geq m\}, \quad \mathcal{A}_s = \mathbb{R}^3 \setminus \mathcal{A}_\ell$$

$$h_{s/\ell} = L^2(\mathcal{A}_{s/\ell}).$$

Let $\mathcal{F}(h_{\ell/s})$ be the bosonic Fock space associated with $h_{\ell/s}$.

Using the following lemma (See [3]),

Lemma 3.8 *Let h_i, $i = 1, 2$ be two Hilbert spaces. There exists a unitary operator from the Fermi-Dirac Fock space (resp. the photonic space) over $h_1 \oplus h_2$ to*

$$\mathcal{F}_D(h_1) \otimes \mathcal{F}_D(h_2) \qquad (resp.\ \mathcal{F}_{ph}(h_1) \otimes \mathcal{F}_{ph}(h_2)).$$

we may identify H_{ph} (resp. $g\ H_I$) with

$$H_{ph,\ell} \otimes 1\!\!1_s + 1\!\!1_\ell \otimes H_{ph,s}, \qquad (resp. g\ H_{I,m} \otimes I_s).$$

Here

$$H_{ph,\ell/s} = \sum_{\mu=1,2} \int_{\mathcal{A}_{\ell/s}} d^3k\, \omega(k)\, a_\mu^*(k)\, a_\mu(k).$$

We set

$$H_{m,\ell} = d\Gamma(H_D) \otimes 1\!\!1_\ell + 1\!\!1_D \otimes H_{ph,\ell} + g\, H_{I,m}$$

In this representation, H_m appears as

$$H_m = H_{m,\ell} \otimes 1\!\!1_s + 1\!\!1_D \otimes 1\!\!1_\ell \otimes H_{ph,s} \tag{54}$$

We then have

Lemma 3.9 *$H_{m,\ell}$ has a ground state $\phi_{m,\ell}$ if and only if H_m has a ground state $\phi_m = \phi_{m,\ell} \otimes \Omega_s$.*

Proof : In view of (54), one has

$$H_m(\phi_{m,\ell} \otimes \Omega_s) = H_{m,\ell}\, \phi_{m,\ell} \otimes \Omega_s \tag{55}$$

Futhermore, it results from (54) that

$$\begin{aligned} \sigma(H_m) &= \overline{\sigma(H_{m,\ell}) + \sigma(H_{ph,s})} \\ &= [\inf \sigma(H_{m,\ell}), +\infty) \end{aligned}$$

which, together with (55), gives the lemma. ■

By Lemma 3.9, Theorem 3.1 then follows from

Theorem 3.10 *There exists $g_0 > 0$ such that, for every $g \in (-g_0, g_0)$ and every $m > 0$, $H_{m,\ell}$ has a ground state $\phi_{m,\ell}$ at $E_m = \inf \sigma(H_m) = \inf \sigma(H_{m,\ell})$.*

Let $\varepsilon > 0$ be a parameter and $\gamma \in \mathbb{Z}^3$. We set

$$C_\varepsilon(\gamma) = \left\{ k \in \mathbb{R}^3 \; ; \; k - \varepsilon\gamma \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right)^3 \right\}$$

For $F \in L^1_{loc}(\mathbb{R}^3)$, we define its ε-avereage by

$$\langle F \rangle_\varepsilon(k) = \varepsilon^{-3} \int_{C_\varepsilon(\gamma)} F(k')\, d^3k' \quad \text{for} \quad k \in C_\varepsilon(\gamma).$$

More generally, when $G \in L^2(\Sigma \times \mathbb{R}^3_k)$, we define its ε-avereage with respect to k by

$$\langle G \rangle_\varepsilon(k) := G_\varepsilon(\cdot, k) = \varepsilon^{-3} \int_{C_\varepsilon(\gamma)} G(\cdot, k')\, d^3k' \quad \text{for} \quad k \in C_\varepsilon(\gamma).$$

Let $H^\varepsilon_{I,m}$ be the operator obtained from $H_{I,m}$ by substituting $\langle v^\mu_{i,m} \rangle_\varepsilon(k)$ for $v^\mu_{i,m}(k)$.

Similarly, we define $H^{m,\varepsilon}_{ph,\ell/s}$ by substituting $\langle \omega(k) \rangle_\epsilon$ for $\omega(k)$ in $H^m_{ph,\ell/s}$.

A simple calculus shows that

$$|\omega(k) - \langle \omega \rangle_\varepsilon(k)| \leq \tilde{C}\, \varepsilon\, \omega(k),$$

which implies

$$\pm(H^m_{ph,\ell} - H^{m,\varepsilon}_{ph,\ell}) \leq \tilde{C}\, \varepsilon\, H^m_{ph,\ell}.$$

Combining this with Corollary 2.10 and using the fact that

$$H^\varepsilon_{m,\ell} - H_{m,\ell} = H^{m,\varepsilon}_{ph,\ell} - H^m_{ph,\ell} + g(H^\varepsilon_{I,m} - H_{I,m})$$

We get

$$||(H^\varepsilon_{m,\ell} - H_{m,\ell})\, (H_{m,\ell} - E_m + 1)^{-1}|| \leq \text{const.} \tag{56}$$

$$\sum_{\beta=0,1} \sum_{j=1}^{6} (a^{\mu,m,\varepsilon}_{\beta,j} + b^{\mu,m,\varepsilon}_j),$$

where $a^{\mu,m,\varepsilon}_{\beta,j}$ (resp. $b^{\mu,m,\varepsilon}_j$) denotes the expression (41) (resp. (42)) that we obtain when we substitute $v^\mu_{j,m}(k) - \langle v^\mu_{j,m} \rangle_\varepsilon(k)$ for $v^\mu_j(k)$.

By Lebesgue dominated convergence theorem, the right hand side of (56) converges to 0 as $\varepsilon \mapsto 0$.

On the other hand, a well known proof (see [3]) shows that the finite volume approximation $H^{\varepsilon}_{m,\ell}$ has discrete spectrum in $(-\infty, E_m + m_2)$, for $m_2 < m$. Therefore, Theorem 3.10 follows from (56) and from the following lemma (see [1, Lemma 5.3]).

Lemma 3.11 *Let $(T_n)_{n\geq 1}$ and T be bounded below self-adjoint operators on a Hilbert space.*

Suppose that

i) $T_n \to T$ as $n \mapsto +\infty$ in the norm resolvent sense.

ii) T_n has purely discrete spectrum in $[\inf \sigma(T_n), \inf \sigma(T_n) + C)$, where C is a constant independent of n.

Then T has purely discrete spectrum in $[\inf \sigma(T), \inf \sigma(T) + C)$.

■

Appendix A

In this appendix, we prove (43) and (44) of the section 2. We keep the same notations.

Lemma A *Under the assumptions of Theorem 1, we have :*

$$||\int d^3k\, v_i^\mu(k)^* \otimes a_\mu(k)\,\psi|| \leq b_i^\mu ||(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\psi||, \tag{A.1}$$

$$\begin{aligned} ||\int d^3k\, v_i^\mu(k) \otimes a_\mu^*(k)\,\psi||^2 &\leq (a_{1,i}^\mu)^2 ||(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\psi||^2 \\ &+ (a_{0,i}^\mu)^2[\varepsilon||(N_D+1)\otimes I\,\psi||^2 + \frac{1}{4\varepsilon}||\psi||^2], \end{aligned} \tag{A.2}$$

for every $\psi \in D(H_0)$ *and every* $\varepsilon > 0$.

Proof : We only give the proof of (A.2) for $i=5$. The case of $i=6$ is quite similar.

The other cases, i.e. $i \neq 5,6$ are simpler since $v_i^\mu(k)$ is bounded on $\mathcal{F}_D$ for $i \neq 5,6$.

In the following we omit the indexes μ and 5.

Set

$$\begin{aligned} a(v) &= \int d^3k\, v^*(k) \otimes a(k) \\ a(v)^* &= \int d^3k\, v(k) \otimes a^*(k) \end{aligned} \tag{A.3}$$

Let $u \in D(H_0)$. For $k \in \mathbb{R}^3$, set

$$\phi(k) = \omega(k)^{1/2}((N_D+1)^{1/2} \otimes a(k))\,u.$$

One has

$$\begin{aligned} \int ||\phi(k)||^2 d^3k &= \int \omega(k)\,(N_D+1)^{1/2} \otimes a(k)u, (N_D+1)^{1/2} \otimes a(k)u) d^3k \\ &= \left((N_D+1)^{1/2} \otimes \int \omega(k)a^*(k)a(k)d^3k u, (N_D+1)^{1/2} \otimes I\ u\right) \\ &= (((N_D+1)^{1/2} \otimes H_{ph})\,u, ((N_D+1)^{1/2} \otimes I)\,u) \\ &= ||((N_D+1)^{1/2} \otimes H_{ph}^{1/2})u||^2 \qquad \text{(A.4} \end{aligned}$$

Using (A.4) we get

$$||a(v)u||^2 = \int (v^*(k) \otimes a(k)u, v^*(k') \otimes a(k')u) d^3k\, d^3k'$$

$$= \int \omega(k)^{-1/2}\omega(k')^{-1/2}(v^*(k)((N_D{+}1)^{-1/2}{\otimes}I)\phi(k), v^*(k')((N_D{+}1)^{-1/2}{\otimes}I)\phi(k'))d^3k\, d^3k'$$

$$\leq \left[\int \omega(k)^{-1/2}||v^*(k)(N_D+1)^{-1/2}||_{\mathcal{F}_D}||\phi(k)||\, d^3k\right]^2.$$

The Cauchy-Schwarz inequality now implies that

$$||a(v)u||^2 \leq \left(\int \omega(k)^{-1}||v^*(k)\,(N_D+1)^{-1/2}||^2_{\mathcal{F}_D} d^3k\right)\left(\int ||\phi(k)||^2 d^3k\right),$$

which, together with (A.4), gives (A.1).

Let us prove (A.2). We have

$$||a^*(v)u||^2 = \int (v(k) \otimes a^*(k)u, v(k') \otimes a^*(k')u)\, d^3k\, d^3k'.$$

Using the commutation relations for a and a^* we obtain

$$||a^*(v)u||^2 = \int (v(k) \otimes a^*(k)a(k')u, v(k') \otimes I\, u)\, d^3k\, d^3k'$$

$$+\int ||(v(k) \otimes I)\, u||^2 d^3k = (1) + (2) \qquad (A.5)$$

The first term in the right hand side of (A.5) can be written in the following way

$$\int (v(k) \otimes a(k')u, v(k') \otimes a(k)u)\, d^3k\, d^3k' = \int \omega(k)^{-1/2}\omega(k')^{-1/2}$$
$$(v(k)\,((N_D+1)^{-1/2} \otimes I)\phi(k'), v(k')((N_D+1)^{-1/2} \otimes I)\phi(k))\, d^3k\, d^3k'$$

Hence

$$\begin{aligned}|(2)| &\leq \left[\int \omega(k)^{-1/2}||v(k)(N_D+1)^{-1/2}||_{\mathcal{F}_D}||\phi(k)||d^3k\right]^2\\ &\leq \left(\int \omega(k)^{-1}||v(k)(N_D+1)^{-1/2}||^2_{\mathcal{F}_D}d^3k\right)\left(\int ||\phi(k)||^2 d^3k\right)\end{aligned}$$

which, together with (A.4), yields

$$\begin{aligned}|(2)| &\leq \Big(\int \omega(k)^{-1/2}||v(k)(N_D+1)^{-1/2}||^2_{\mathcal{F}_D}d^3k\Big]^2\ ||((N_D+1)^{1/2} \otimes H_{ph}^{1/2}||^2\\ &= \left(\int \omega(k)^{-1}||(N_D+1)^{-1/2}v(k)^*||^2_{\mathcal{F}_D}d^3k\right)\ ||((N_D+1)^{1/2} \otimes H_{ph}^{1/2})v||^2.\end{aligned}$$

The second term of the R.H.S. of (A.5) is bounded by

$$\left(\int ||v(k)(N_D+1)^{-1/2}||^2_{\mathcal{F}_D} d^3k\right) ||(N_D+1)\otimes I)u|| \quad = \\ \left(\int ||v(k)(N_D+1)^{-1/2}||^2_{\mathcal{F}_D} d^3k\right) ((N_D+1)u,u)$$

Using

$$ab \leq \frac{\varepsilon}{2}\,a^2 + \frac{1}{2\varepsilon}\,b^2, \quad a>0, b>0$$

we get

$$|(2)| \leq \left(\int ||v(k)(N_D+1)^{-1/2}||^2 d^3k\right) \left[\varepsilon||(N_D+1)u||^2 + \frac{1}{4\varepsilon}\,||u||^2\right] \quad \text{(A.7)}$$

Finally, (A.5), (A.6) and (A.7) give (A.2). ■

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