DYNAMICAL DETERMINANTS
VIA DYNAMICAL CONJUGACIES
FOR POSTCRITICALLY FINITE POLYNOMIALS

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Dedicated to David Ruelle and Yasha Sinai on the occasion of their 130th birthday.

Abstract. We give an analogue of Levin-Sodin-Yuditskii's study of the dynamical Ruelle determinants of hyperbolic rational maps in the case of subhyperbolic quadratic polynomials. Our main tool is to reduce to an expanding situation. We do so by applying a dynamical change of coordinates on the domains of a Markov partition constructed from the landing ray at the postcritical repelling orbit. We express the dynamical determinants $d_\beta(z) = \exp - \sum_{k\ge 1} \frac{z^k}{k} \sum_{w \in \mathrm{Fix} f^k} \frac{1}{((f_c^k)'(w))^\beta} \frac{1}{1-\frac{1}{(f_c^k)'(w)}}$ ($\beta \in \mathbb{Z}_+$) as the product of an (entire) determinant with an explicit expression involving the postcritical repelling orbit, thus explaining the poles in $d_\beta(z)$.

§1. Introduction

Let f be a rational function on the Riemann sphere, with bounded Julia set J. We shall recall some classical results of Ruelle and more recent work of Levin-Sodin-Yuditskii which hold under a hyperbolicity assumption, before moving to our study of subhyperbolic polynomials.

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Assume then that J is real and that f is *hyperbolic* on J, i.e., there exist constants $K > 0$, $\rho > 1$ such that $|(f^k)'(x)| \geq K \cdot \rho^k$ for all x in J and all nonnegative integers k. The thermodynamical formalism of hyperbolic quadratic polynomials is well understood by now (see e.g. [Rue0, Zin, Rue2, Lev]), we shall only recall here results on Ruelle transfer operators which are connected to our study. Consider the transfer operator

$$\mathcal{L}\varphi(x) = \sum_{fy=x} \frac{\varphi(y)}{(f'(y))^2}\,, \tag{1.1}$$

acting on the space of functions φ analytic in a critical point-free complex neighbourhood of the Cantor set J. Ruelle [Rue0, Rue1] observed that the operator $\mathcal{L}$ acting on bounded holomorphic functions in a neighbourhood of J is nuclear in the sense of Grothendieck, in particular $\mathcal{L}$ admits a Fredholm determinant $\det(1-z\mathcal{L})$, that Ruelle expressed as the following dynamical function:

$$\det(1 - z\mathcal{L}) = \exp - \sum_{k\geq 1} \frac{z^k}{k} \sum_{w\in \mathrm{Fix} f^k} \frac{1}{((f^k)'(w))^2} \frac{1}{1 - \frac{1}{(f^k)'(w)}}\,. \tag{1.2}$$

Here, $\mathrm{Fix} f^k = \{w \in \mathbb{C} \mid f^k(w) = w\}$ (in particular we include complex fixed points – which will be present in most subhyperbolic examples introduced below – but do *not* include $w = \infty$).

More recently, Levin, Sodin, and Yuditskii [LSY1, (4.15), (5.4)] obtained remarkable explicit expressions for $\det(1 - z\mathcal{L})$, which only involve the iterates of the critical points of the expanding rational map f (with bounded real Julia set). In the special case of a quadratic polynomial, i.e., $f_c(z) = z^2 + c$ with $c < -2$, they find

$$\begin{aligned}\det(1 - z\mathcal{L}) &= \exp - \sum_{k\geq 1} \frac{z^k}{k} \sum_{w\in \mathrm{Fix} f_c^k} \frac{1}{((f_c^k)'(w))^2} \frac{1}{1 - \frac{1}{(f_c^k)'(w)}} \\ &= 1 + \sum_{k\geq 1} \frac{(z/2)^k}{f_c(0)\cdots f_c^k(0)}\,.\end{aligned} \tag{1.3}$$

Our aim here is to obtain analogous results for the *subhyperbolic (i.e., postcritically finite) case of quadratic polynomials.* Note that Levin et al. extended their results [LSY2] to more general hyperbolic rational maps f (than in [LSY1]), but considering only transfer operators $\mathcal{L}_Q\varphi(x) = \sum_{fy=x} \varphi(y)Q(y)$ associated to a rational weight Q with *no* pole on the Julia set. (In [LSY2, Section 5] the case $Q(y) = (f'(y))^2$ is considered, but the analysis does *not* cover subhyperbolic polynomials.)

Our first observation (see Section 4 for a proof) is that the second equality in (1.3) still holds formally for *all* polynomials:

Theorem A (Formal Levin-Sodin-Yuditskii identity). *Let f_c be a quadratic polynomial. Then we have (in the sense of formal power series with coefficients rational functions of c):*

$$\exp - \sum_{k\geq 1} \frac{z^k}{k} \sum_{w\in \mathrm{Fix} f_c^k} \frac{1}{((f_c^k)'(w))^2} \frac{1}{1-\frac{1}{(f_c^k)'(w)}} = 1 + \sum_{k\geq 1} \frac{(z/2)^k}{f_c(0)\cdots f_c^k(0)} \, .$$

Note that if f_c is a Collet-Eckmann quadratic polynomial, i.e.,

$$\liminf_{k\to\infty} \frac{1}{k} \log |(f_c^k)'(f_c(0))| > 0 \, ,$$

then the identity in Theorem A is an equality between convergent power series.

In the following we make the additional assumption that the critical point is preperiodic (many of our ideas could be extended to the Collet-Eckmann case). The thermodynamical formalism of preperiodic quadratic polynomials is not as straightforward as that of hyperbolic polynomials (see e.g. [Og] for a pedestrian construction of the absolutely continuous invariant measure and [MS, Sm] for a more recent analysis of the phase transitions).

The finiteness of the postcritical orbit implies that the right-hand-side in Theorem A defines a rational function. Our aim is now to interpret the left-hand side in Theorem A as a (necessarily modified, since it is not entire) dynamical determinant, similarly to the first equality of (1.3). For this, we shall conjugate our complex subhyperbolic quadratic polynomial with an expanding analytic dynamical system. This is related to the “Thurston orbifold” metric that Douady and Hubbard [DH] construct in a ramified covering space over a neighbourhood of the Julia set in order to exhibit the hyperbolic properties of subhyperbolic polynomials (see comments after the proof of Theorem B). Here we shall use a more direct construction (as was that of [Og], on the real axis) in order to analyse the dynamical determinant.

More precisely, we first build a finite Markov partition in the complex plane reminiscent of (but not the same as) the Yoccoz puzzle (we consider external rays at the landing periodic point and iterate them backwards finitely many times). We then introduce conjugacies defined by iterates of the map itself, defining (Section 2) new coordinates for our dynamics. (This in fact is analogous to the conjugacy appearing in the towers of Young [Y], e.g.). We show in Section 3 that we thus obtain an expanding analytic system where the results of Ruelle [Rue1] apply and give rise to a Ruelle-Grothendieck-Fredholm dynamical determinant. One needs to relate the Fredholm determinants of the system in the old and the new coordinates, this is Lemma 6 in Section 3 where we prove that the only special periodic orbit is the postcritical one. Our main result is the following theorem, proved in Section 4:

Theorem B (Determinant for subhyperbolic quadratic polynomials). *Let f_c be a quadratic polynomial with a preperiodic critical point c, landing on a repelling periodic point of period $m \geq 1$, multiplier $\lambda_f > 1$, and admitting $q \geq 1$ external rays, each of period $d \cdot m$. For every nonnegative integer β, set*

$$d_\beta(z) = \exp - \sum_{k \geq 1} \frac{z^k}{k} \sum_{w \in \mathrm{Fix} f^k} \frac{1}{((f_c^k)'(w))^\beta} \frac{1}{1 - \frac{1}{(f_c^k)'(w)}} . \tag{1.4}$$

Then, there is a choice of $\mu_f = \sqrt{\lambda_f^d}$ with $\mu_f^2 = \lambda_f^d$ such that the product

$$\hat{d}_\beta(z) = d_\beta(z) \cdot \prod_{j \geq 0} \frac{(1 - z^{dm} \mu_f^{-j-\beta})^q}{1 - z^{dm} \lambda_f^{-j-\beta}} \tag{1.5}$$

extends to an entire function. The zeroes of this function are exactly the inverses of the nonzero eigenvalues of the transfer operator

$$\mathcal{L}_\beta \varphi(x) = \sum_{f_c(y) = x} \frac{\varphi(y)}{(f_c'(y))^\beta}$$

acting on a Banach space $\mathcal{B}_\beta$ of multivalued functions (with prescribed singularities along the postcritical orbit) in a neighbourhood of the Julia set of f_c.

The definition of $\sqrt{\lambda_f^d}$ is given in (3.11), and $\mathcal{B}_\beta$ is constructed in the proof of Theorem B in Section 4 (see (4.6), and our comments there linking our construction with the ramified covering in [DH]).

The simplest illustration of Theorem B is given by the quadratic map $f_{-2}(z) = z^2 - 2$, where $f_{-2}^2(0) = 2$ is a fixed point with derivative $\lambda_f = +4$. We may compute $d_\beta(z)$ for $\beta = 2$ from Theorem A, and find $1 - \sum_{k \geq 1} z^k / 4^k = (1 - z/2)/(1 - z/4)$. Theorem B (here $m = d = q = 1$, and (3.11) gives $\sqrt{4} = +2$) then says that $\hat{d}_2(z) = (1 - z/2) \prod_{j \geq 0} (1 - z/(8 \cdot 2^{2j}))$, an entire function which vanishes at $z = 2$ and $z = 8 \cdot 2^{2j}$ for all nonnegative integer j.

We mention that A. Epstein [Ep] and S. Ushiki [Us1, Us2] have independently studied related determinants with different approaches. The tower construction in Smirnov's Ph.D. [Sm, Section 4.1] is more closely related to our method, but was used towards different goals (in particular no determinants were considered).

Finally, we would like to point out that the assumption that all critical points are mapped to repelling periodic orbits is enough to extend our results to polynomials of higher degree, or even to a suitable class of rational maps. (The square root appearing in Theorem B should be replaced by a root of degree of the appropriate critical point, and the q, d-counting becomes more complicated.)

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§2. DYNAMICAL CHARTS FOR PREPERIODIC QUADRATIC POLYNOMIALS

Consider a quadratic polynomial $f(z) = z^2 + c$. Then ∞ is a super-attractive *critical fixed point* of f with $f(\infty) = f^{-1}(\infty) = \infty$. Thus there are an open neighbourhood V of ∞ and a unique holomorphic diffeomorphism ϕ defined on V, such that $\phi(\infty) = \infty$, $\phi'(\infty) = 1$ and $\phi(z^2) = f\big(\phi(z)\big)$. (This is the Boettcher conjugacy.) The other critical point of f is 0. The set $C = \{c_j = f^j(0)\}_{j=0}^{\infty}$ is called the *critical orbit* of f. If C does not intersect V, i.e., if it is bounded (the corresponding parameter value $c = c_1$ then lies in the Mandelbrot set, by definition), then ϕ can be holomorphically extended to the domain $\{z \in \overline{\mathbb{C}} \mid |z| > 1\}$.

Henceforth, we assume that C is bounded.

The image $B = \phi(\{z \in \overline{\mathbb{C}} \mid |z| > 1\})$ is the basin of ∞ under the iteration of f. Its complement $K = \overline{\mathbb{C}} \setminus B$ is called the filled-in Julia set of f, and the boundary $J = \partial K$ is called the *Julia set* of f. (It is well known that K is connected if and only if C is bounded.)

The image of a circle $\{z \in \mathbb{C} \mid |z| = r\}$ of radius $r > 1$ under ϕ is called an *equipotential curve* of f. The image γ_θ of a half-line $e_\theta = \{re^{2\pi i\theta} \mid r > 1\}$ is called an *external ray of angle* θ of f. An external ray γ_θ is said to *land at a point* w in K if the limit $\lim_{r\to 1+} \phi(re^{2\pi i\theta})$ exists and is w.

A point in K is called a *periodic point* of period $m > 0$ if $f^j(p) \neq p$ for $1 \leq j < m$ and $f^m(p) = p$. For a periodic point p of period m, its *multiplier* is $\lambda(p) = (f^m)'(p)$. The periodic point p is called *repelling* if $|\lambda(p)| > 1$, *attracting* if $|\lambda(p)| < 1$, *neutral* if $|\lambda(p)| = 1$. The following theorem is due to Douady (the polynomial need not be assumed quadratic):

Douady Landing Theorem. *Let f be a polynomial with a connected filled-in Julia set and let p be a repelling periodic point of f of period m. Then there is a nonempty finite set of (say $q \geq 1$) external rays landing at p. There is a divisor $d \geq 1$ of q such that each of these landing rays is periodic of period $m \cdot d$.*

A proof of the Douady landing theorem can be found e.g. in [Hu, Theorem I.A, Pe, Corollary B.1] (see [EL] for a statement without the assumption that the Julia set is connected).

Since C is a finite set, then either 0 itself is periodic, or there is a nonzero periodic point in C. If 0 is not periodic but preperiodic (i.e., one of its forward iterates is periodic), then the polynomial f is called *subhyperbolic* (or preperiodic, or (post)critically finite). (The corresponding parameter value c is often called a *Misiurewicz point,* see e.g. [TL] for some of their nontrivial properties.) In this case, it is known that the filled-in Julia set K of f has no interior and coincides with the Julia set J. Moreover, the periodic orbit in C is repelling (see [Mi, CG]). We shall restrict henceforth to subhyperbolic quadratic polynomials.

For a preperiodic quadratic polynomial f we have

$$C = \{c_0 = 0, c_1 = c, \cdots, c_{n-1}, c_n, \cdots, c_{n+m-1}, c_{n+m} = c_n\}, \tag{2.1}$$

with $n = \min\{j \geq 0\,, c_j \text{ is periodic }\} \geq 1$, and $\{c_n = f(c_{n+m-1}), \cdots, c_{n+m-1}\}$ is a (repelling) periodic orbit, of multiplier denoted by

$$\lambda_f = (f^m)'(c_n)\,. \tag{2.2}$$

Finally, we write

$$f(C) = \{c_1, \cdots, c_{n-1}, c_n, \cdots, c_{n+m-1}\}$$

for the postcritical orbit of f.

The Douady landing Theorem says that there is a finite number q of external rays landing at c_n. Denote the union of the corresponding curves as Γ_n. Then Γ_n can be pulled back by f^{-1}, giving a union Γ_{n+m-1} of external rays landing at c_{n+m-1} and Γ_{n-1} of external rays landing at c_{n-1}. Through pull-back by appropriate branches of f^{-1} we obtain a collection of external rays Γ_j landing at c_j, for all $0 \leq j < n+m$. Note that $\#\Gamma_j = q$ for all $j \neq 0$ while $\#\Gamma_0 = 2q$ (the doubling comes from taking a square root). Let

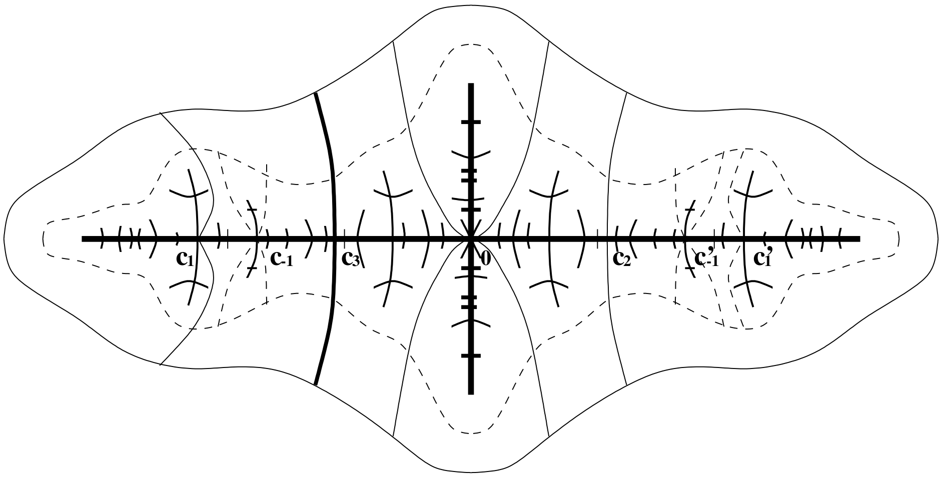
$$\Gamma = \cup_{j=0}^{n+m-1} \Gamma_j \tag{2.3}$$

be the finite set of external rays landing at the critical orbit, and let $U \supset J$ be an open domain bounded by an equipotential curve of f. For $0 \leq j < \infty$, set

$U_j = f^{-j}(U)$. Then $\overline{\Gamma}$ cuts $U = U_0$ into finitely many domains. This finite partition $\mathcal{P}_0 = \{\xi_i^0\}_{i=0}^{k_0}$ into open domains deserves to be called a *Markov partition* because it satisfies the following properties:

(1) The restriction of f on each ξ_i^0 is injective.
(2) For each $\xi_i^0 \in \mathcal{P}_0$, the image $f(\xi_i^0 \cap U_1)$ is a union of domains in $\mathcal{P}_0$ (neglecting boundaries).

In Figure 1, the Markov partition $\mathcal{P}_0$ is represented for the quadratic polynomial with $n = 3$ and $m = 1$, i.e., $f^3(0)$ is a repelling fixed point. (In this case, we have $q = 2$ external rays at $f^3(0) = c_3$.)



0 --> c_1 --> c_2 --> c_3 --> c_3
c'_1 --> c_2
c_{-1}, c'_{-1} --> 0
Bold solid curves are external rays landing at the fixed point c_3
Solid curves bound the initial Markov partition P_0
Dashed curves and solid curves bound our partition Q

Figure 1: Partitions $\mathcal{P}_0$ and $\mathcal{Q}$ in case $n = 3$, $m = 1$.

Pulling-back $\mathcal{P}_0$ via (both branches of) f^{-1}, we get a finer Markov partition $\mathcal{P}_1 = \mathcal{P}_0 \vee f^{-1}(\mathcal{P}_0)$, and construct inductively

$$\mathcal{P}_\ell = \mathcal{P}_{\ell-1} \vee f^{-\ell}(\mathcal{P}_0) = \{\xi_j^\ell\}_{j=1}^{k_\ell}\,, \tag{2.4}$$

where each ξ_j^ℓ is an open domain in U_ℓ such that $f^\ell : \xi_j^\ell \to \xi_j^0$ is a homeomorphism. Thus, we construct a sequence $\{\mathcal{P}_\ell\}_{\ell=0}^\infty$ of nested Markov partitions. Adapting to our slightly different partition the proof that the pieces of the Yoccoz puzzle containing a point z shrink to that point for nonrenormalizable quadratic polynomials (see [Hu, Theorem 5.7, Remarks 5.2 and 7.7(a)]), we get that the maximal diameter $\max_j\{\operatorname{diam}(\xi_j^\ell)\}$ tends to zero as ℓ goes to infinity. This gives the following lemma, defining our initial partition:

Lemma-Definition 2 (Partition $\mathcal{Q}$). *There is an integer $\ell_0 \geq 0$ such that $\overline{\xi} \cap f(C)$ contains at most one point for each $\xi = \xi_i^{\ell_0} \in \mathcal{P}_{\ell_0}$. Taking the smallest such integer ℓ_0, define the partition $\mathcal{Q} = \mathcal{P}_{\ell_0}$.*

In the case illustrated in Figure 1 (where $\mathcal{Q}$ is also drawn), we have $\ell_0 = 1$.

Starting from $\mathcal{Q}$, we now construct a system of charts and dynamical conjugacies. It is convenient to introduce some further notation. We will say that a ray γ is a *boundary ray* of a domain $\xi \in \mathcal{Q}$ if $\gamma \cap \overline{\xi}$ is non-empty. The landing point for each such ray is called a *vertex* point of ξ. By construction a vertex point v is either in the critical orbit or a pre-image (up to ℓ_0 iterates backwards) of the critical point. The closure of ξ is the disjoint union of ξ, its boundary rays and its vertex points (plus a piece of an equipotential curve which is, however, not of interest here). An element $\xi \in \mathcal{Q}$ is called *non-singular* if its closure does not intersect the postcritical orbit $f(C)$, otherwise (i.e., if at least one of its vertices belongs to $f(C)$) it is called *singular.* In fact, when ξ is singular, by definition of ℓ_0, the intersection $\overline{\xi} \cap f(C)$ is a singleton $\{c_j\}$ for some $1 \leq j < n+m$; we then say that ξ is *singular of index j,* and write $\operatorname{ind}(\xi) = j$. (For consistency we denote $\operatorname{ind}(\xi') = 0$ when ξ' is not singular.) If $\xi, \eta \in \mathcal{Q}$ and $f(\eta) \supseteq \xi$, we write

$$f_{\eta\xi}^{-1} : \xi \to \eta$$

for the corresponding inverse map.

From now on, let us fix an external ray $\gamma \in \Gamma_n$. As in the construction of Γ we may pull-back $\gamma = \gamma_n$ along the postcritical orbit taking suitable pre-images by f. All pull-backs are local diffeomorphisms, except for the last which produces two external rays $\gamma_{0\pm}$ at c_0 which cut the Riemann sphere into two topological disks $\mathcal{H}_\pm$. Choosing one of these, say $\mathcal{H}_+$ (we shall see below that this choice is immaterial, see e.g. Lemma 6), and denoting by γ_1 the image ray at c_1, we may define an inverse branch of f_c,

$$r(z) = \sqrt{z - c_1}\,, \tag{2.5}$$

which maps $\mathbb{C} \setminus \gamma_1$ conformally onto $\mathcal{H}_+$.

We shall now recursively define our dynamical coordinates, constructing a collection of (conformal) maps and open charts $\{(\psi_\xi, V_\xi) \mid \xi \in \mathcal{Q}\}$, with

$$\psi_\xi : \xi \to V_\xi \subset \mathbb{C}, \qquad \xi \in \mathcal{Q}\,.$$

Definition 3 (Dynamical charts).

(1) *If $\xi \in \mathcal{Q}$ is non-singular, then $V_\xi = \xi$ and $\psi_\xi(z) = z$ is the identity map.*

(2) *If ξ is singular of index one, then $V_\xi = r(\xi)$, and $\psi_\xi(z) = r(z)$, i.e., ψ is the restriction of the fixed inverse branch r.*

(3) *If ξ is singular of index $j > 1$, there is a unique singular $\eta \in \mathcal{Q}$ which has index $j - 1$ and such that $f(\eta) \supseteq \xi$. We define inductively ψ_ξ (and V_ξ) by $\psi_\xi(z) := \psi_\eta \circ f_{\eta\xi}^{-1}(z)$ (this recursion is finite, with at most $n + m - 1$ steps).*

We may view the conjugacies ψ_ξ either as holomorphic charts of our partition $\mathcal{Q}$ or as a map ψ from a subset of the complex plane to a tower consisting in the disjoint union of finitely many open complex coordinate charts:

$$\psi : \bigcup_{\xi \in \mathcal{Q}} \xi \to \bigsqcup_{0 \le j < n+m} \bigsqcup_{\eta' \in \mathcal{Q}, \mathrm{ind}\,(\eta') = j} V_{\eta'}\,, \quad \psi|_\xi = \psi_\xi\,.$$

Note that when $\xi \in \mathcal{Q}$ has index $j > 0$, we may also caracterise V_ξ as the unique connected component in $\mathcal{H}_+$ with $0 = c_0 \in \overline{V_\xi}$ such that $f^j : V_\xi \to \xi$ is a conformal bijection (and ψ_ξ is then the inverse map of f^j).

Finally, we construct a dynamical system on the tower, based on the inverse branches of f in the new coordinates. For domains ξ and η in $\mathcal{Q}$ such that $f(\eta) \supset \xi$, we define a *conjugated inverse branch:*

$$g_{\eta\xi} = \psi_\eta \circ f_{\eta\xi}^{-1} \circ \psi_\xi^{-1} : V_\xi \to V_\eta\,.$$

§3. PROPERTIES OF THE DYNAMICS IN THE NEW CHARTS

In this section, we list the properties of our preperiodic quadratic polynomial viewed in the charts constructed in Section 2. They will suffice to prove Theorem B in the next section. The first step is to extend the local inverse branches in the charts:

Lemma 4 (Holomorphic extension in dynamical charts). *There are open neighbourhoods $\Omega_\xi \supset \overline{V}_\xi$, for $\xi \in \mathcal{Q}$, such that each conjugated inverse branch $g_{\eta\xi} : V_\xi \to V_\eta$ extends holomorphically to $g_{\eta\xi} : \Omega_\xi \to \mathbb{C}$.*

Proof of Lemma 4. When $\overline{\xi}$ does not intersect the postcritical orbit the conjugating map ψ_ξ is the identity map and thus extends as a conformal bijection to a larger domain. Any inverse branch of f also extends to a larger domain because the critical value is not on the boundary. The image of such an enlarged domain does not intersect the postcritical orbit either (if this domain is not taken too large) and ψ_η extends holomorphically as well.

If ξ is singular of index $j \geq 1$ but η is not, then ψ_η is the identity map and

$$g_{\eta\xi} = f_{\eta\xi}^{-1} \circ \psi_\xi^{-1} = f_{\eta\xi}^{-1} \circ f^j \,. \tag{3.1}$$

When $j > 1$, $f_{\eta\xi}^{-1}$ can be extended to a small neighbourhood of $\bar{\xi}$ (because $c_1 \notin \bar{\xi}$), and the map g can be extended to a small neighbourhood of $\overline{V}_\xi$. (Note that in this case the origin is on the boundary of V_ξ but that the conjugated map $g_{\eta\xi}(w) = \text{const} + w^2 + O(w^4)$ is not injective in a neighborhood of the origin). If $j = 1$, then $f_{\eta\xi}^{-1} = \pm r|\xi$. So $g_{\eta\xi}(z) = \pm z$.

When both ξ and η are singular there are two possibilities: First, $c_j \in \bar{\xi}$ and $c_{j-1} \in \bar{\eta}$ for some $1 < j < n+m$. From the definition of the ψ's,

$$\psi_\xi = \psi_\eta \circ f_{\eta\xi}^{-1} \,.$$

Hence, $g_{\eta\xi}(z) = z$ is the identity map and can be extended holomorphically.

The other possibility (the most interesting one) is that $c_n \in \bar{\xi}$ and $c_{n+m-1} \in \bar{\eta}$. We need to map both ξ and η to domains containing the critical point on the boundary. We shall do so in two steps: First we conjugate to domains with the *critical value* on the boundary and thereafter to domains with the critical point on the boundary:

Let ξ' be the (unique) singular domain of index n such that $f^{m-1}(\xi') \supset \eta$, i.e., $f^m(\xi') \supset \xi$. The domains ξ and ξ' may be different (in fact, this happens precisely when $d > 1$ where d was the divisor of q in the Douady landing Theorem). Let $f_{\xi\xi'}^{-m}$ be the inverse of $f^m : \xi' \to \xi$. Then, by definition,

$$g_{\eta\xi} = \psi_\eta f_{\eta\xi}^{-1} \psi_\xi^{-1} = \psi_{\xi'} f_{\xi'\xi}^{-m} \psi_\xi^{-1} \,. \tag{3.2}$$

Let ξ_1 be the singular domain of index 1 such that $f^{n-1}(\xi_1) \supset \xi$. The corresponding inverse is $f_{\xi_1\xi}^{1-n} : \xi \to \xi_1$. We define similarly ξ_1' and the associated maps (relative to ξ').

Then for $z \in V_\xi$ we may express the conjugated map as follows:

$$g_{\eta\xi}(z) = r \circ f_{\xi_1'\xi'}^{1-n} f_{\xi'\xi}^{-m} f_{\xi\xi_1}^{n-1} \circ f(z) = r \circ h \circ f(z) \,. \tag{3.3}$$

It is clear that $f_{\xi'\xi}^{-m}$ can be analytically extended to a small neighbourhood of $\bar{\xi}$ (since its linearisation around the fixed point c_n is of the form $z \mapsto z/\lambda_f$). When starting from the critical value the associated iterates do not meet the critical point and therefore $f_{\xi_1\xi}^{1-n}$ can also be analytically extended to a small neighbourhood of

$\bar{\xi}$. These maps are all local diffeomorphisms. Hence, the middle composition h has c_1 as a fixed point and in a neighborhood of this fixed point a local power series expansion,

$$h(w) = f^{1-n}_{\xi_1'\xi'} f^{-m}_{\xi'\xi} f^{n-1}_{\xi\xi_1}(w) = c_1 + \lambda_f^{-1}(w - c_1) + O((w - c_1)^2)\,. \tag{3.4}$$

By post- and precomposing with r and f, respectively, we see that $g_{\eta\xi}$ can be extended to a small neighbourhood of $\overline{V}_\xi$ with the local power series expansion

$$g_{\eta\xi}(z) = \pm(\sqrt{\lambda}_f)^{-1} z + O(z^2)\,. \tag{3.5}$$

Note that when $d > 1$ the sign of the square root may depend on the choice of the ray γ which was used to define the map r. This ambiguity will disappear in the dynamical determinant (where a dth iterate occur, see Lemma 6). $\square$

Our next step is to obtain contraction in the sense of the Schwartz lemma (we will thus be in a position to apply standard results of Ruelle on Fredholm determinants for holomorphic contractions). The basic idea in the proof is fairly standard in complex dynamics (using "thickened" puzzle pieces), but some complications arise from the conjugacies.

Lemma 5 (Contraction in dynamical charts). *Up to taking slightly smaller domains* $\Omega_\xi \supset \overline{V}_\xi$ *than in Lemma 4, we have*

$$g_{\eta\xi}(\overline{\Omega}_\xi) \subset \Omega_\eta$$

for each conjugated inverse branch $g_{\eta\xi} : \Omega_\xi \to \Omega_\eta$*, and* $\xi, \eta \in \mathcal{Q}$*.*

Proof of Lemma 5. There are two steps. First we will cover the postcritical orbit and up to ℓ_0 pre-images thereof by a suitable collection of small balls. These will provide domains for local contractions. We will then use slightly "thickened" external rays to fill in and obtain global contractions (see [Ji, pp. 213–214] for a similar construction).

Consider a small open ball $B(c_n, \delta)$ of radius $\delta > 0$ centered at the point, c_n. Since $c_n = c_{n+m}$ is a repelling periodic point of f^m, there are $\delta^* > 0$ and $\rho > 1$ such that for $0 < \delta < \delta^*$, we have $f^m(B(c_n, \delta)) \supset \overline{B(c_n, \delta\rho^m)}$. When $v \in \overline{\xi}$ is a vertex point there is $j \in \{-\ell_0, \dots, n+m-1\}$ (called the index of the vertex point) such that $f^{j-n}(c_n) = v$. Let B_v be the connected component of $f^{j-n}(B(c_n, \delta\rho^{n-j}))$ which contains the vertex point, v (when $j \geq n$ we consider the image, while for $j < n$ we take the pre-image). Since ℓ_0 is finite we may take δ small enough so that the collection of balls, B_v, v a vertex, is disjoint.

Consider a transition $\eta \to \xi$ between two domains. Let $v \in \overline{\xi}$ be a vertex point of ξ and let $w = f_{\eta\xi}^{-1}(v) \in \overline{\eta}$. There are three possibilities: either, case I, w is itself a vertex point of η of index one smaller than the index of v, case II, $v = c_n$ and $w = c_{n+m-1}$, or, case III, w is not a vertex point but belongs to the open domain η. In case I, the fact that $\rho > 1$ ensures that $f_{\eta\xi}^{-1}(\overline{B_v}) \subset B_w$. In case II, the choice of ρ and δ was such that, $f_{\eta\xi}^{-1}(\overline{B_v}) \subset f_{\eta\xi}^{m-1}(\overline{B(c_n, \delta\rho^{-m})}) \subset B_w$. Finally, in case III, possibly by making δ smaller, we may ensure that $f_{\eta\xi}^{-1}(\overline{B_v}) \subset \eta$.

In summary, with suitable choices of δ and ρ we have that $f_{\eta\xi}^{-1}(\overline{B_v}) \subset \eta \cup \bigcup_w B_w$, the union being taken over vertices of η. This ends the first step in the construction.

Now, let γ_θ be an external ray bounding a domain $\xi \in \mathcal{Q}$, where θ is its angle, and let v be its landing point (which is a vertex of ξ). We may choose $\epsilon > 0$ small enough so that every external ray (independently of whether or not it lands) with angle in $(\theta - \epsilon, \theta + \epsilon)$ will intersect the ball $\overline{B_v}$. As the number of vertices is finite, ϵ can be chosen uniformy in v. Let γ'_ζ denote the part of the ray of angle ζ between its entry in the equipotential neighbourhood U_{ℓ_0} and its first intersection with $\overline{B_v}$. Finally, let $\gamma'_{\theta\pm\epsilon}$ denote the "thickened ray" which is the union of all γ'_ζ for $\zeta \in (\theta - \epsilon, \theta + \epsilon)$.

Consider now a pre-image γ_ζ of γ_θ, and let w be its landing point. The pre-image is either bounding some domain $\eta \in \mathcal{Q}$, or it is inside such a domain. In the first case, the pre-image possesses a neighbourhood $\gamma'_{\zeta\pm\epsilon}$ of rays, the point w is a vertex of η, and by construction one has (chosing for the inverse the connected component intersecting γ'_ζ) $f_{\eta\xi}^{-1}(\overline{\gamma'_{\theta\pm\epsilon}}) \subset \gamma'_{\zeta\pm\epsilon} \cup B_w$, essentially because a square root halves angle-differences. For the second case, we may choose a smaller ϵ so that (uniformly in these rays), $f_{\eta\xi}^{-1}(\overline{\gamma'_{\theta\pm\epsilon}}) \subset \eta$.

When the domain ξ is non-singular, we simply define $\Omega_\xi = V_\xi \bigcup_v B_v \bigcup_i \gamma'_{\theta_i\pm\epsilon}$ the unions being over vertices and bounding rays of the domain. If the domain is singular, say of index $j > 0$, then ψ_ξ maps ξ to the domain V_ξ containing the origin at the boundary. If γ_θ is a ray which bounds ξ then it has one or two pre-image(s) which bound(s) V_ξ (i.e., intersect(s) the closure of V_ξ). To each such ray pre-image we associate a corresponding pre-image (the connected component containing the ray), $\gamma'_{\theta'\pm\epsilon}$, of the thickened ray $\gamma'_{\theta\pm\epsilon}$. Finally, to each vertex v, we associate a connected component B'_v of the corresponding ball B_v. We let $\Omega_\xi = V_\xi \bigcup_{\theta'} \gamma'_{\theta'\pm\epsilon} \bigcup_v B'_v$ and one verifies that by construction each $g_{\eta\xi}(\overline{\Omega}_\xi) \subset \Omega_\eta$.

Some remarks may clarify the procedure in the singular case: First note that if only one ray, say γ_θ, lands at c_n, then both pre-images of $\gamma'_{\theta\pm\epsilon}$ will bound the domain V_ξ and hence both should be included in Ω_ξ. Second, each V_ξ is an open subset of the "half-plane" $\mathcal{H}_+$ (since it is in the image of r). The extended domain

Ω_ξ, however, is a neighbourhood of the closure of V_ξ, in particular it contains a neighbourhood of the origin. Recall that if η was non-singular then $g_{\eta\xi}$ behaves as a quadratic map around the origin. This illustrates a rather peculiar effect of our dynamical conjugation, namely that on the extended domains, the (noninjective) map $g_{\eta\xi}$ is a strict contraction but not the inverse of an expanding map. (This reflects the fact that our construction is just another description of the orbifold metric, see comments after the proof of Theorem B.) □

It remains to understand the relation between the periodic orbits and multipliers of $f : \mathbb{C} \to \mathbb{C}$ and those of the system of contractions

$$g = \{g_{\xi\eta}\} : \Omega \to \Omega \,, \quad \Omega = \{\Omega_\xi \mid \xi \in \mathcal{Q}\} \,.$$

(Slightly abusing terminology, we continue to call vertex of an Ω_ξ a vertex of ξ.) In Lemma 6 we shall see that, except for the postcritical repelling point, nothing changes. For this repelling periodic point, however, the period is multiplied by $d \geq 1$, and the multiplier must be replaced by an appropriate square root of λ_f^d. Let us first define this square root:

Dynamical choice of $\sqrt{\lambda_f^d}$.

The external ray γ landing at c_n that we used to construct the partition $\mathcal{Q}$ may be parametrised as a real-analytic map $t \in \mathbb{R} \mapsto \gamma(t) \in \mathbb{C} - \{c_n\}$, where $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to-\infty} \gamma(t) = c_n$. The ray is invariant under the mdth iterate of the map f, and it is convenient to choose the parametrisation (through a simple expression involving the Boettcher conjugacy ϕ) so that for all $t \in \mathbb{R}$,

$$f^{md}(\gamma(t)) = \gamma(t + md) \,. \tag{3.6}$$

Since γ is a continuous curve and does not contain c_n, we may write

$$\gamma(t) = c_n + \exp(\alpha(t)) \,, \tag{3.7}$$

where α is unique up to multiples of $2\pi i$. By linearizing the invariance relation (3.6) for γ, we see that (note here the extra dth power)

$$\lim_{t\to-\infty} \exp(\alpha(t) - \alpha(t - md)) = \lambda_f^d \,. \tag{3.8}$$

Recall that the last two external rays $\gamma_{0\pm}$ at c_0 obtained by pulling back γ in the construction of our partition arose from taking a square root. We may thus write

$$\gamma_{0\pm}(t) = f^{-n}(\gamma(t)) = c_0 \pm \exp(\alpha_0(t)) \,. \tag{3.9}$$

Here, α_0 is unique up to multiples of πi, whence the plus-minus. Through linearisation we see that (the constant is a logarithm of the derivative along the postcritical orbit)

$$\lim_{t\to-\infty} \alpha_0(t) - \alpha(t)/2 = \text{const}\,. \tag{3.10}$$

This relation allows us finally to define a square root of the multiplier:

$$\sqrt{\lambda_f^d} \equiv \lim_{t\to-\infty} e^{\alpha_0(t)-\alpha_0(t-md)} = \lim_{t\to-\infty} e^{(\alpha(t)-\alpha(t-md))/2}\,. \tag{3.11}$$

It is interesting to note that this definition does not depend on the choice of the parametrisations α and α_0 (which is clear, including the fact that $+\gamma_0$ and $-\gamma_0$ lead to the same choice), nor of the initial ray γ landing at c_n. To see the latter we will use a simple homotopy argument: Let $\tilde{\gamma}$ be another ray landing at c_n. It does not intersect γ and a logarithm of the parametrisation, $\tilde{\alpha}$, can thus not intersect the family of curves $\alpha + i\mathbb{Z}$. The same must be true for the pulled-back rays and this forces the limit $\lim_{t\to-\infty}\lim_{k\to\infty}(\tilde{\alpha}_0(t) - \tilde{\alpha}_0(t-mdk))/k$ to be independent of the various choices. But this limit yields the unique (relative to the family of rays) logarithm of $\sqrt{\lambda_f^d}$. Taking exponentials we obtain the wanted conclusion.

We can now state the last ingredient to prove Theorem B:

Lemma 6 (Periodic orbits of f and g). *There is an isomorphism between the set of periodic points for f which are not in the postcritical periodic orbit and the set of periodic points of $g : \Omega\to\Omega$ which are not vertices of domains. The isomorphism preserves the period and the multiplier of these periodic points.*

Let $m\cdot d$ be as in the Douady landing Theorem. To each c_j, with $n \le j < n+m$ in the postcritical periodic orbit, there corresponds q periodic points of g which are vertices of domains. Each such point z has period $m\cdot d$ and multiplier $1/\sqrt{\lambda_f^d}$ (as defined in (3.11)*). There are no other periodic points for g.*

Proof of Lemma 6. Consider a point $z \in \Omega_\xi$ for some $\xi \in \mathcal{Q}$. This point is periodic of period $k \ge 1$ for g precisely when there is a symbol sequence $\xi_0, \ldots, \xi_{k-1}$, with $\xi = \xi_0$ and such that there are transitions $\xi_j\to\xi_{j+1 \bmod k}$ for each $j = 0, \ldots, k-1$ (we say such a cyclic symbol sequence is admissible). In that case we have $g_{\xi_0\xi_1} \circ \cdots g_{\xi_{k-1}\xi_0}(z) = z$ and furthermore, either $z \in V_\xi$ (i.e., in the chart of the open domain of the original partition) or z is (a lift of a point) in the postcritical periodic orbit. To see this note first that contraction of the family g implies that each admissible cyclic symbol sequence yields a unique fixed point $z \in \Omega_{\xi_0} = \Omega_\xi$ following that sequence. Now, $g_{\eta\xi} : \Omega_\xi\to\Omega_\eta$ in fact maps V_ξ to V_η, and it follows that the fixed point must be in the closure $\overline{V}_\xi$ of the domain in question. If z is not in the interior it must be on the boundary. Clearly it can not be on the piece of equipotential curve and no ray γ_θ which bounds the domain can contain any periodic points. Hence z must be a vertex of the domain and only vertices which are lifts of points in the postcritical periodic orbit can be periodic.

Since on the open domains f and g are conjugated by diffeomorphisms, we have already proved the first claim.

We now prove the remaining statements. Since there are q rays landing at c_n, there are q domains in $\mathcal{Q}$ having c_n as a vertex. The same holds for all points in the postcritical orbit, whence the (over)counting of the corresponding periodic vertices for g. The (minimal) period $m \cdot d$ of the rays gives the same minimal period of domains under the mapping g, proving our claim on the period of the vertices. Finally to get the multiplier of the return map of a periodic vertex z, let ξ be a domain having $z = c_n$ as a vertex. Take any $w \in \Omega_\xi - \{c_0 = 0\}$. Then the multiplier will be the limit of $(g^{mdk}(w) - c_0)/(g^{md(k-1)}(w) - c_0)$ as $k \to \infty$. If here we let w be a point on the landing ray γ_0^+ then it is easy to check that this limit coincides with the one over the limit in (3.11) for $\sqrt{\lambda_f^d}$. (Recall that if we took γ_0^- or another ray we would get the same limit, by the observations made above. Note also that the choice between $\mathcal{H}^+$ and $\mathcal{H}^-$ does not intervene here.) $\square$

§4. DYNAMICAL DETERMINANTS - PROOFS OF THE THEOREMS

We first give the simple proof of the generalised Levin-Sodin-Yuditskii [LSY1] formula:

Proof of Theorem A. One views the second equality in (1.3) as an identity between formal power series, i.e., between the critical orbit power series and the exponential of the sum over fixed points. The coefficient of z^k is thus the same on both sides when $-c$ is real and large enough. We will show that each such coefficient $a_k(c)$ of the left-hand-side (involving the periodic points) extends as a meromorphic function in c to the whole plane. Uniqueness of meromorphic extensions will then give Theorem A.

Since the coefficient $a_k(c)$ of z^k on the left-hand-side is a sum and product of terms which only involve sums over fixed points of order not greater than k, it suffices to show that each term of the type

$$\sum_{w \in \mathrm{Fix} f_c^k} \frac{1}{(f_c^k)'(w)((f_c^k)'(w) - 1)} \tag{4.1}$$

defines a (single-valued) meromorphic function of c. Assume now that for each k, the zeroes $w(c)$ of

$$F(w, c) = f_c^k(w) - w$$

are isolated as a function of c. (This will be proved in Sublemma 7 below.) Then the sum over $\mathrm{Fix} f^k$ has only finitely many singular points, solutions to algebraic equations. Take a loop in the complex plane which avoids these singularities. Traversing the loop yields a permutation of the fixed points. Hence the sum is unchanged, and the function is single-valued as we wanted to show. $\square$

Sublemma 7. *Let $F(w,c)$, $G(w,c)$, and $H(w,c)$ be polynomials in w and c such that:*

1. *$F(w,c) - w^N$ is a polynomial in w and c of degree at most $N-1$ in w.*
2. *There is $\hat{c}$ such that $H(w,\hat{c}) \neq 0$ at all w for which $F(w,\hat{c}) = 0$ (this guarantees that the function $R(c)$ below is well-defined on a small disc).*

Then the function

$$R(c) = \sum_{w:F(w,c)=0} \frac{G(w,c)}{H(w,c)} \tag{4.2}$$

is rational.

In our application, we take $F(w,c) = f_c^k(w) - w$, $G(w,c) = 1$, $H(w,c) = (f_c^j)'(w)((f_c^j)'(w)-1)$, and $N = 2k$. We note (although we shall not need this) that one can extend the above result to more general holomorphic or even meromorphic functions.

Proof of Sublemma 7. Let $w_1, ..., w_N$ be N complex variables, and let $\sigma_1, ..., \sigma_N$ denote the elementary symmetric functions

$$\sigma_1 = w_1 + \cdots + w_N\,, \quad \sigma_2 = \sum_{i<j} w_i w_j\,, \quad \ldots\,, \quad \sigma_N = w_1 \cdots w_N\,.$$

For fixed c, let $\{w_1(c), ... w_N(c)\}$ denote the zero-set $\{w : F(w,c) = 0\}$ (counted with multiplicity, we use the degree claim in (1) here). We have

$$F(w,c) = (w - w_1(c)) \cdots (w - w_N(c)) = w^N - \sigma_1(c) w^{N-1} \pm \cdots \sigma_N(c)\,, \tag{4.3}$$

where $\sigma_1(c), ..., \sigma_N(c)$ are the elementary polynomials evaluated in the roots $w_i(c)$. Each $\sigma_i(c)$ is a polynomial function of c because we assumed $F(w,c)$ to be polynomial in c.

We can now conclude: The function $H^*(c) = \prod_{w=w_i(c)} H(w,c)$ is a symmetric polynomial in the roots $w_i(c)$. The fundamental theorem of symmetric functions (see e.g. [vW]) asserts that it is a polynomial in the elementary functions evaluated at the roots, i.e., $H^*(c) = \pi_H(\sigma_1(c), \ldots, \sigma_N(c))$, for some polynomial π_H. By the observation in the previous paragraph, $H^*(c)$ is thus a polynomial in c.

Finally,

$$R(c)\, H^*(c) = \sum_{u=w_i(c)} G(u,c) \prod_{v=w_j(c):j\neq i} H(v,c) \tag{4.4}$$

is also a symmetric polynomial in the roots hence a polynomial in c (not identically zero by the assumptions on F and H). It follows that $R(c)$ is rational. $\square$

Proof of Theorem B. Using Lemmas 4 and 5, we can apply Theorem 1 of [Rue1], and its proof, to the system of holomorphic contractions $g : \Omega \to \Omega$ and the holomorphic weight $(g')^\beta$ for any fixed nonnegative integer β. The corresponding transfer operator $\widehat{\mathcal{L}}_\beta$ is thus nuclear when acting on the Banach space $\widehat{\mathcal{B}}$ of bounded holomorphic functions on the disjoint union of the open complex domains Ω_ξ (with the supremum norm) through

$$\widehat{\mathcal{L}}_\beta \hat{\varphi}(\hat{w}) = \sum_{\text{admissible } \eta} (g'_{\eta\xi}(\hat{w}))^\beta \cdot \hat{\varphi}(g_{\eta\xi}(\hat{w}))\,, \quad \hat{w} \in \Omega_\xi\,. \tag{4.5}$$

The trace of its kth power is given by [Rue1, p. 235]

$$\sum_{\hat{w}\in \mathrm{Fix} g^k} ((g^k)'(\hat{w}))^\beta \frac{1}{1-(g^k)'(\hat{w})}$$

and produces the announced (entire) Fredholm-Grothendieck determinant $\hat{d}_\beta(z)$.

By Lemma 6, the only difference between the determinant $\hat{d}_\beta(z)$ of $\widehat{\mathcal{L}}_\beta$ and the formal determinant $d_\beta(z)$ is the fact that the postcritical point of period m is (over)counted $q \geq 1$ times in the tower, its period is multiplied by d, and its multiplier is $\sqrt{\lambda_f^d}$ instead of λ_f^d. Since

$$\begin{aligned} \exp\Big[-\sum_{k\geq 1} \frac{z^{kmd}}{km} \frac{qm}{(\sqrt{\lambda_f^d})^{k\beta}} \frac{1}{1-\frac{1}{(\sqrt{\lambda_f^d})^k}} + \sum_{k\geq 1} \frac{z^{km}}{km} \frac{m}{\lambda_f^{k\beta}} \frac{1}{1-\frac{1}{\lambda_f^k}}\Big] \\ = \prod_{j\geq 0} \Big(1-\frac{z^{dm}}{\sqrt{\lambda_f^d}^{\beta}} \frac{1}{\sqrt{\lambda_f^d}^{j}}\Big)^q \cdot \prod_{j\geq 0}\Big(1-\frac{z^{dm}}{\lambda_f^\beta}\frac{1}{\lambda_f^j}\Big)^{-1}\,, \end{aligned}$$

the resulting formula is as described in Theorem B.

Recall that the conjugacies $\psi_\xi : \xi \to V_\xi$ are holomorphic, and that $g_{\eta\xi} \circ \psi_\xi = \psi_\eta \circ f_{\eta\xi}^{-1}$ on ξ, so that $[(g'_{\eta\xi} \circ \psi_\xi) \cdot \psi'_\xi]^\beta = [(\psi'_\eta \circ f_{\eta\xi}^{-1}) \cdot (f_{\eta\xi}^{-1})']^\beta$ there. We may use this conjugacy to define a Banach subspace $\mathcal{B} = \mathcal{B}_\beta$ of the set of functions φ which are holomorphic on the (disjoint) union of the open domains $\xi \in \mathcal{Q}$. For this, we consider the continuous embedding of $\widehat{\mathcal{B}}$ through

$$\varphi|_\xi = (\psi'_\xi)^\beta \cdot \hat{\varphi}_{|V_\xi} \circ \psi_\xi . \quad \xi \in \mathcal{Q} , \tag{4.6}$$

Uniqueness of analytic continuations implies that this map is injective (hence we get an embedding), and we may therefore define the norm of φ to be the corresponding (sup-)norm of $\hat{\varphi} \in \widehat{\mathcal{B}}$ (see [Rug] for a similar construction). Note that $\varphi \in \mathcal{B}_\beta$ is such that $((\psi_\xi^{-1})')^\beta \cdot (\varphi_{|\xi} \circ \psi_\xi^{-1})$ extends to a bounded holomorphic function on Ω_ξ, thus imposing a specific type of singularities of ϕ along the postcritical orbit. The weights in the above conjugacy have been chosen so that $\widehat{\mathcal{L}}_\beta$ is conjugated to $\mathcal{L}_\beta$ through the above isometry, proving our last claim. $\square$

To analyse subhyperbolic quadratic polynomials, Douady-Hubbard [DH] construct a covering $\widetilde{\Omega}$ over a neighbourhood of the Julia set, ramified above the postcritical orbit and endow it with a metric which makes the lifted polynomial $\tilde{f}$ uniformly expanding (see also [McM, §3.2 and Appendix A] for a description of this orbifold metric). Constructing a Markov partition $\widetilde{\mathcal{Q}}$ of $\widetilde{\Omega}$ with ideas similar to those of the present paper, one could define a Banach space $\widetilde{\mathcal{B}}$ of holomorphic bounded functions on $\widetilde{\Omega}$ (which would be just another way to see $\widehat{\mathcal{B}}$) such that the transfer operator associated to $\tilde{f}$ and the weight $(\tilde{f}')^{-\beta}$ (with the modifications inherent to the partition $\widetilde{\mathcal{Q}}$) is nuclear. We leave out the details in this approach.

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