LOCATING THE SPECTRUM FOR MAGNETIC SCHRÖDINGER AND DIRAC OPERATORS

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Abstract

Some spectral properties of magnetic Schrödinger and Dirac operators perturbed by long range magnetic fields are investigated. If the intensity of the field is small enough, a better location of the perturbed spectrum is given. In particular, if the unperturbed spectrum is discrete, we show that the perturbed eigenvalues are given in terms of an absolutely convergent series with respect to a magnetic parameter, from which the usual asymptotic expansion can be derived.

1 Introduction and the Results

This paper is concerned with the study of the spectrum for Schrödinger and Dirac operators perturbed by long range magnetic fields of small intensity. By long range, we mean that the field may not decay at infinity. This question has been addressed by many authors (see [8], [2],[16], [17], [10], [15], [11], [12], [1] or [13]). One of the common topics of these works is the spectral stability of the "band/gap" structure under a magnetic perturbation, in various situations of Schrödinger operators. For a review of these results and connections with physical aspects, see [16], [17] and references therein.

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In this paper, we want to revisit this problem through some mathematical technics borrowed from [7], [6], [13] and also from the multiple-well analysis [5],[12]. Together with some Combes-Thomas like arguments [3], our method enables us to sharpen the location of the spectrum for slightly perturbed magnetic Schrödinger and Dirac operators. Our results are given in terms of physical relevant quantities, i.e. magnetic fields.

Let us now give some more details and the results. We first consider the Schrödinger case. Let $H_0 = (\mathbf{p}-\mathbf{a}_0)^2+V$ be the unperturbed operator, densely defined in $L^2(\mathbb{R}^n)$, $n = 2, 3$, where $\mathbf{p} = -\imath\nabla$ is the momentum, $\mathbf{a}_0 \in [C^2(\mathbb{R}^n)]^n$ is the vector potential and V is the scalar electric potential supposed to be relatively bounded to $\mathbf{p}^2$ with relative bound smaller than one (which implies the same property with respect to $(\mathbf{p}-\mathbf{a}_0)^2$ via the diamagnetic inequality), so H_0 is an essentially self-adjoint operator on $C_0^\infty(\mathbb{R}^n)$ (see [8]). We will denote by $\mathbf{B}_0 = \mathrm{rot}(\mathbf{a}_0)$ the corresponding magnetic field. Here we restrict ourselves to dimensions $n = 2, 3$, but most of our results can be extended to higher dimensional settings, i.e. for N-body problems.

Now if $n = 3$, define an additional magnetic field, $\mathbf{B} \in [C^1(\mathbb{R}^3)]^3$, $\nabla\mathbf{B} = 0$, satisfying

$$\max_{j\in\{1,2,3\}} \max_{|\alpha|\leq 1} \sup_{\mathbf{x}\in\mathbb{R}^3} \{|D^\alpha B_j|(\mathbf{x})\} \leq 1, \tag{1.1}$$

i.e. $\mathbf{B}$ is "long-range", which means that all its components together with their first order derivatives are uniformly bounded.

An important remark for the case when $n = 2$: the magnetic field has only one component which is orthogonal to the plane in which the particle is moving; as a three dimensional vector, it may be represented as $(0, 0, B(x_1, x_2))$, where now $\mathbf{x} = (x_1, x_2, 0)$. With this identification, the formulae below can be used for both dimensions.

The transversal gauge (see [18]) generated by $\mathbf{B}$ is defined as

$$\mathbf{a}(\mathbf{x}) = \int_0^1 ds\ s\ \mathbf{B}(s\mathbf{x}) \wedge \mathbf{x}\ . \tag{1.2}$$

Our goal is to study the spectrum location of the perturbed operator (essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$)

$$H(b) = (\mathbf{p} - \mathbf{a}_0 - b\mathbf{a})^2 + V, \tag{1.3}$$

for small values of the real parameter b.

For the Dirac case, let $H_0 = \alpha\cdot(\mathbf{p}-\mathbf{a}_0)+\beta+V$ be the unperturbed operator in $[L^2(\mathbb{R}^3)]^4$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the usual Dirac matrices. The vector potential $\mathbf{a}_0$ belongs to $[C^2(\mathbb{R}^3)]^3$ and for simplicity, V is supposed to be bounded. We will consider the perturbed operator

$$H(b) = \alpha \cdot (\mathbf{p} - \mathbf{a}_0 - b\mathbf{a}) + \beta + V, \quad b \in \mathbb{R} \tag{1.4}$$

where the vector potential $\mathbf{a}$ is defined by (1.2); then $H(b)$ and H_0 are essentially self-adjoint operators on $[C_0^\infty(\mathbb{R}^3)]^4$ [18].

Our first result is stated in the following theorem:

Theorem 1.1. *Assume that* $\Delta = (\Delta_-, \Delta_+) \subset \mathbb{R}$, $\Delta_- < \Delta_+$ *is a spectral gap for* H_0 *i.e.* Δ *is contained in the resolvent set of* H_0, $\rho(H_0)$. *We also assume that* $\Delta_\pm \in \sigma(H_0) \cup \{-\infty\}$. *Let* $2/3 > \epsilon > 0$. *Then there exists a strictly positive constant* $b_0(\epsilon)$ *such that for any* $\lambda \in \Delta$ *with* $dist(\lambda, \sigma(H_0)) \geq b^{2/3-\epsilon}$, $0 \leq b \leq b_0(\epsilon)$, *one has that* $\lambda \in \rho(H(b))$.

The constant $b_0(\epsilon)$ strongly depends on the energy Δ_-, in particular $b_0 \to 0$ when $\Delta_- \to +\infty$. Notice that by using the technics developed by Helffer-Sjöstrand [12] and Nenciu [16], similar results can be obtained but with an "1/2- exponent". The authors believe that even the "2/3- exponent" is not optimal, and that the "true" behavior should be with exponent 1, as is the case for bounded magnetic potentials.

The above result can be improved when we study the nature and location for $\sigma(H(b))$ near a discrete eigenvalue E_0 of H_0. Let us assume that the multiplicity of E_0 is $M < \infty$. It is well known that generally, analytic perturbation theory fails when "long range" magnetic perturbations are considered, i.e. the Rayleigh-Schrödinger series does not converge near $b = 0$ (see [4]).

The asymptotic perturbation theory tells us that for sufficiently small b, $H(b)$ will have exactly M eigenvalues near E_0, which in the first order are given by $E_j(b) = E_0 + be_j + o(b)$, $j \in \{1, \ldots, M\}$ (see Theorem VIII 2.6 in [14]).

In that case, we have the following theorem (for simple eigenvalues, this result was announced in [17]):

Theorem 1.2. *Assume that the first order corrections* $\{e_j\}_{j=1}^M$ *are different one from each other. Then there exist two strictly positive constants* b_0 *and* C, *such that for any* $b_0 > b > 0$, *each eigenvalue* $E_j(b)$ *is given in terms of an absolutely convergent power series,*

$$E_j(b) = E_0 + \sum_{k \geq 1} b^k e_{j,k}(b) \tag{1.5}$$

where $|e_{j,k}(b)| \leq C^k$ *and* $\lim_{b \searrow 0} e_{j,1}(b) = e_j$.

All the coefficients which appear in (1.5) admit an asymptotic expansion in b and if we introduce them in (1.5), this gives the usual asymptotic expansion for $E_j(b)$.

Hydrogen type systems perturbed by small constant magnetic fields (the Zeeman effect), provide an interesting case in which the condition on $\{e_j\}_{j=1}^M$ is satisfied. Indeed, consider $n = 3$, $H_0 = -\Delta + V(\mathbf{x})$, $H(b) = (\mathbf{p} - b\mathbf{a})^2 + V(\mathbf{x})$ with V a radial potential (only depending on $|\mathbf{x}|$), $\mathbf{a}(\mathbf{x}) = 1/2(-x_2, x_1, 0)$ and $b > 0$. It is well known that the discrete eigenvalues of H_0 are given by three quantum numbers: the "radial" one $N = 1, 2, \dots$ and the "angular" one $l = 0, 1, \dots, N-1$ and the "magnetic" one $-l \leq m \leq l$; let us label them as $E_{N,l,m}$. The terms angular and magnetic come from the fact that H_0 commutes with the square of the angular momentum together with all its components. The generic behaviour of these eigenvalues is that they are always degenerate with respect to m; hence they will be located in groups of $2l+1$ eigenvalues having a common value denoted by $E_{N,l}$.

Assume that there exists $N_0 \geq 2$ and $0 < l_0 \leq N_0 - 1$ such that E_{N_0,l_0} is isolated (unfortunately, this excludes pure Coulomb systems, since in that case $E_{N,l} = E_N$).

What every physicist does when asked about first order corrections, is to say that

$$H(b) = H_0 + bL_3 + \mathcal{O}(b^2),$$

where L_3 is the third component of the angular momentum.

It is also very well known that when restricted to the range of the orthogonal projector corresponding to E_{N_0,l_0}, L_3 has $M = 2l_0+1$ distinct eigenvalues given by the magnetic quantum number m, hence our condition is satisfied.

Let us now present the last result. Since under our assumption, V is not necessarily relatively compact to H_0, $\sigma_{ess}(H_0)$ could have nontrivial gaps and this is the interesting case. We want to study the "stability" of such gaps when the additional magnetic field $\mathbf{B}$ vanishes arbitrarily slow at infinity. For that, we introduce the quantity ($R \geq 1$):

$$b_R := \max_{j\in\{1,2,3\}} \max_{|\alpha|\leq 1} \sup_{|\mathbf{x}|\geq R} \{|D^\alpha B_j|(\mathbf{x})\}. \tag{1.6}$$

We shall say that the magnetic field goes to zero at infinity if $\lim_{R\to\infty} b_R = 0$. Define the vector potential $\mathbf{a}$ as in (1.2). The Schrödinger operator

$$H(\mathbf{a}) = (\mathbf{p} - \mathbf{a}_0 - \mathbf{a})^2 + V$$

on $L^2(\mathbb{R}^n)$ and the Dirac operator

$$H(\mathbf{a}) = \alpha \cdot (\mathbf{p} - \mathbf{a}_0 - \mathbf{a}) + \beta + V$$

on $[L^2(\mathbb{R}^3)]^4$ are essentially self adjoint operators on $[C_0^\infty(\mathbb{R}^n)]$ and $[C_0^\infty(\mathbb{R}^3)]^4$ respectively. Then the following result holds:

Theorem 1.3. *Assume that the perturbing magnetic field goes to zero at infinity; then the essential spectrum is stable:*

$$\sigma_{ess}(H(\mathbf{a})) = \sigma_{ess}(H_0) \ .$$

For Schrödinger operators, our theorem is just a particular case of a much more general result obtained by Iftimie [13]. Nevertheless, we decided to include it in this paper because its proof is almost completely contained in the proof of the other two theorems, and works equally well for both Schrödinger and Dirac operators.

The paper is organized as follows: section 2 is devoted to the description of the mathematical tools needed throughout the paper and contains the proof of Theorem 1.1. The sections 3 and 4 are concerned with the proofs of Theorems 1.2 and 1.3, respectively. In Appendix, the exponential decay of the Green functions involved in the proof of Theorem 1.1 is discussed.

2 The $\frac{2}{3}$ exponent localization

We shall only treat in detail the Schrödinger case, since most of our arguments can be immediately adapted for the Dirac problem. Let us begin with a technical result (for its proof, see [7]).

Define a new gauge (see also (1.1), (1.2) and the remark before (1.2))

$$\mathbf{a}(\mathbf{x}, \mathbf{x}') = \int_0^1 ds \; s\mathbf{B}(\mathbf{x}' + s(\mathbf{x} - \mathbf{x}')) \wedge (\mathbf{x} - \mathbf{x}') \ , \quad \forall \mathbf{x}' \in \mathbb{R}^n, \ n = 2, 3.$$

Proposition 2.1. *This gauge satisfies* $|\mathbf{a}(\mathbf{x}, \mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$ *and* $|\nabla_x \mathbf{a}(\mathbf{x}, \mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$*. Moreover, there exists* $\phi(\mathbf{x}, \mathbf{x}')$ *such that* $\phi(\mathbf{x}, \mathbf{x}) = 0$*,* $\phi(\mathbf{x}, 0) = \phi(0, \mathbf{x}) = 0$ *and* $\nabla_x \phi(\mathbf{x}, \mathbf{x}') = \mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \mathbf{x}')$*.* □

The proof of Theorem 1.1 is based on the construction of a pseudo-inverse $S_b(z)$ of the operator $H(b) - z$ on $L^2(\mathbb{R}^n)$, for all $z \in \triangle$. To do this, choose a function $g \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$, $0 \leq g \leq 1$, $g(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq 1/4$ and $g(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq 3/4$. If $\gamma \in \mathbf{Z}^n$, denote by $g_\gamma(\mathbf{x}) = g(\mathbf{x} - \gamma)$. The final requirement:

$$\sum_\gamma g_\gamma(\mathbf{x}) = 1 \; . \tag{2.1}$$

For $b > 0$, define $g_{b,\gamma}(\mathbf{x}) := g(b^{\frac{1}{3}}\mathbf{x} - \gamma)$. Then $g_{b,\gamma}(\mathbf{x}) = 1$ if $|\mathbf{x} - b^{-\frac{1}{3}}\gamma| \leq b^{-\frac{1}{3}}/4$, $g_{b,\gamma}(\mathbf{x}) = 0$ if $|\mathbf{x} - b^{-\frac{1}{3}}\gamma| \geq 3/4 \; b^{-\frac{1}{3}}$ and the derivatives of $g_{b,\gamma}$ behave like $b^{\frac{1}{3}}$ at least.
Let $\tilde{g} \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ such that $0 \leq \tilde{g} \leq 1$, $\tilde{g}(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq 1$ and $\tilde{g}(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq 2$ and let $\tilde{g}_{b,\gamma}(\mathbf{x}) := \tilde{g}(b^{\frac{1}{3}}\mathbf{x} - \gamma)$. Clearly, for every $b > 0$ and $\gamma \in \mathbf{Z}^n$, we have

$$g_{b,\gamma} \cdot \tilde{g}_{b,\gamma} = g_{b,\gamma} \; , \tag{2.2}$$

and there exists a strictly positive constant such that

$$\mathrm{dist}(\mathrm{supp}(g_{b,\gamma}), \mathrm{supp}(\nabla \tilde{g}_{b,\gamma})) \geq \mathbf{const} \cdot b^{-1/3} \; . \tag{2.3}$$

Our partition of unity is locally finite, i.e. given any γ, there exists (uniformly in γ) only a finite number of neighbors of g_γ whose supports are not disjoint from supp g_γ; this is true for $\{\tilde{g}_\gamma\}$, too. Moreover, this number remains the same when dilating the lattice.

Lemma 2.1. *Let $\{\mathcal{T}_b(\gamma)\}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$, where*

$$\sup_\gamma ||\mathcal{T}_b(\gamma)|| = f(b) < \infty \; .$$

Denote by $\{\chi_{b,\gamma}\}$ a family of measurable functions with the property that

$$\mathrm{supp}(\chi_{b,\gamma}) \subseteq \mathrm{supp}(\tilde{g}_{b,\gamma}) \quad \text{and} \quad |\chi_{b,\gamma}| \leq M < \infty \; .$$

Define on $C_0^\infty(\mathbb{R}^n)$:

$$\mathcal{S}_b := \sum_\gamma \exp\left(\imath b \phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\chi_{b,\gamma}\mathcal{T}_b(\gamma) g_{b,\gamma} \exp\left(-\imath b \phi(\cdot, b^{-\frac{1}{3}}\gamma)\right). \tag{2.4}$$

Let $v(\gamma)$ be the set of indices which correspond to those cut-off functions whose support are not disjoint from **supp** $\tilde{g}_{b,\gamma}$ *(we already know that $\#\{v(\gamma)\} \leq v < \infty$). Then $\mathcal{S}_b$ can be extended to a bounded operator on $L^2(\mathbb{R}^n)$ and*

$$||\mathcal{S}_b|| \leq M \; (1 + v)^{1/2} \; f(b). \tag{2.5}$$

Proof. Take $\psi \in C_0^\infty(\mathbb{R}^n)$. Define

$$\psi_{b,\gamma} := \exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\mathcal{T}_b(\gamma)\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)g_{b,\gamma}\psi\ . \tag{2.6}$$

Then:

$$||\psi_{b,\gamma}||^2 \leq f(b)^2||g_{b,\gamma}\psi||^2 = f(b)^2\langle\psi, g_{b,\gamma}^2\psi\rangle. \tag{2.7}$$

Since $\sum_\gamma g_{b,\gamma}^2 \leq \left(\sum_\gamma g_{b,\gamma}\right)^2 = 1$, it follows that

$$\sum_\gamma ||\psi_{b,\gamma}||^2 \leq f(b)^2\ ||\psi||^2.$$

Moreover, only a finite number of $\psi_{b,\gamma}$ will be nonzero. We have

$$||\mathcal{S}_b\psi||^2 = \sum_\gamma\sum_\alpha\langle\chi_{b,\alpha}\psi_{b,\alpha}, \chi_{b,\gamma}\psi_{b,\gamma}\rangle. \tag{2.8}$$

The nonzero terms from the above double sum are those for which $\alpha \in \{\gamma\} \cup v(\gamma)$, only. Then the Schwarz inequality implies

$$\begin{aligned} ||\mathcal{S}_b\psi||^2 &\leq (1/2)\sum_\gamma\sum_{\alpha\in\gamma\cup v(\gamma)}[||\chi_{b,\alpha}\psi_{b,\alpha}||^2 + ||\chi_{b,\gamma}\psi_{b,\gamma}||^2] \\ &\leq M^2(1+v)\sum_\gamma||\psi_{b,\gamma}||^2 \leq M^2(1+v)\ f(b)^2||\psi||^2. \end{aligned} \tag{2.9}$$

After a standard density argument, the lemma is proven. □

For $z \in \rho(H_0)$, the resolvent set of H_0, denote by $d_{0,z} := \mathrm{dist}(\Re(z), \sigma(H_0))$. Take $\mathcal{T}_b(\gamma) := (H_0 - z)^{-1}$; by the above lemma, the operator

$$S_b(z) := \sum_\gamma \exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\tilde{g}_{b,\gamma}(H_0 - z)^{-1}g_{b,\gamma}\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right), \tag{2.10}$$

is a bounded operator on $L^2(\mathbb{R}^n)$ and $\|S_b(z)\| \leq (1+v)^{1/2}d_{0,z}^{-1}$. Then Theorem 1.1 follows from,

Lemma 2.2. *i. Let* $2/3 > \epsilon > 0$. *Then there exists a strictly positive constant* b_ϵ *such that for any* $1 > b_\epsilon > b > 0$, *then* $\Delta(b) = (\Delta_- + b^{2/3-\epsilon}, \Delta_+ - b^{2/3-\epsilon}) \subset \rho(H(b))$.

ii. If $z = x + \imath y \in \mathbb{C}$ *with* $0 < |y| \leq 1$ *and* $x \in K$, *where* K *is a compact in* $\mathbb{R}$, *then there exists a positive constant* C *such that for any* $0 < b \leq 1$:

$$\sup_{x\in K}||(H(b) - z)^{-1} - S_b(z)|| \leq C \cdot \frac{b^{\frac{2}{3}}}{|y|^3}\ . \tag{2.11}$$

Proof. i. Let us remark the following commutation relation, valid in the form sense on $C_0^\infty(\mathbb{R}^n)$:

$$(-\imath\nabla_x - \mathbf{a}_0(\mathbf{x}) - b\mathbf{a}(\mathbf{x}))e^{\imath b\phi(\mathbf{x},\mathbf{x}')} = e^{\imath b\phi(\mathbf{x},\mathbf{x}')}(-\imath\nabla_x - \mathbf{a}_0(\mathbf{x}) - b\mathbf{a}(\mathbf{x},\mathbf{x}'))\ .$$

In particular, this implies:

$$\begin{aligned} & \exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)(H(b)-z)\exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right) = \\ = \ & \left[\mathbf{p} - \mathbf{a}_0 - b\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)\right]^2 + V - z = \\ = \ & H_0 - z - 2b\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)\cdot(\mathbf{p}-\mathbf{a}_0) + \imath\, b\ \mathrm{div}(\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)) + b^2\mathbf{a}^2(\cdot, b^{-\frac{1}{3}}\gamma) = \\ = \ & H_0 - z + bW_b(\gamma)\ . \end{aligned} \tag{2.12}$$

Hence, for $z \in \mathbb{C}$ and $\Re(z) \in \rho(H_0)$, we have in form sense on $C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned} & (H(b)-z)\exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\tilde{g}_{b,\gamma}(H_0-z)^{-1}g_{b,\gamma}\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right) = \\ = \ & e^{\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)}\left[H_0 - z + bW_b(\gamma)\right]\tilde{g}_{b,\gamma}(H_0-z)^{-1}g_{b,\gamma}e^{-\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)} = \tilde{g}_{b,\gamma}\cdot g_{b,\gamma} + \\ + \ & e^{\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)}\Big\{[H_0, \tilde{g}_{b,\gamma}] + 2\imath\ b\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)\cdot(\nabla\tilde{g}_{b,\gamma}) + b\tilde{g}_{b,\gamma}W_b(\gamma)\Big\}\times \\ \times \ & (H_0-z)^{-1}g_{b,\gamma}e^{-\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)}\ . \end{aligned} \tag{2.13}$$

Notice that $[H_0, \tilde{g}_{b,\gamma}] = -2\imath(\nabla\tilde{g}_{b,\gamma})\cdot(\mathbf{p}-a_0) - (\Delta\tilde{g}_{b,\gamma})$. We now want to estimate the r.h.s of 2.13. We suppose that the energy z lies in some compact subset of $\mathbb{C}$ containing Δ in its interior; by a straightforward computation, we have from Proposition 2.1

$$\sup_{\gamma}\ \sup_{\mathbf{x}\in\mathrm{supp}\tilde{g}_{b,\gamma}} \{|\mathbf{a}(\mathbf{x}, b^{-\frac{1}{3}}\gamma)| + |\mathrm{div}\mathbf{a}(\mathbf{x}, b^{-\frac{1}{3}}\gamma)|\} \le \mathbf{const}\cdot b^{-\frac{1}{3}}\ , \tag{2.14}$$

and by standard arguments [5] we get:

$$\max_j ||(\mathbf{p}-a_0)_j(H_0-z)^{-1}|| \le \mathbf{const}\ \max\left\{1, \frac{1}{d_{0,z}}\right\}. \tag{2.15}$$

Denote by $F_{b,\gamma}$ and $f_{b,\gamma}$ the characteristic functions of $\mathrm{supp}\tilde{g}_{b,\gamma}$ and $\mathrm{supp}g_{b,\gamma}$. Since by construction

$$\mathrm{dist}(\mathrm{supp}\nabla\tilde{g}_{b,\gamma}, \mathrm{supp}g_{b,\gamma}) = \mathcal{O}(b^{-\frac{1}{3}}),$$

the exponential estimates (see the Appendix) imply that for $\Re(z) \in \Delta$,

$$\max_j ||[H_0, \tilde{g}_{b,\gamma}](H_0-z)^{-1}f_{b,\gamma}|| \le \mathbf{const}\ b^{\frac{1}{3}}\ \max\left\{1, \frac{1}{d_{0,z}}\right\}\exp\{-\eta b^{-\epsilon/2}\} \tag{2.16}$$

for some constant $0 < \eta < 1$, since under our conditions (see the Appendix for its definition) $D_{0,z} \geq \eta b^{1/3-\epsilon/2}$. Notice that we also used here that

$$\sup|(\nabla\tilde{g}_{b,\gamma})| = \mathcal{O}(b^{\frac{1}{3}})\,,\ \sup|(\Delta\tilde{g}_{b,\gamma})| = \mathcal{O}(b^{\frac{2}{3}}).$$

Therefore the family of operators

$$T_b(\gamma,z) := \left\{[H_0,\tilde{g}_{b,\gamma}] + 2\imath\, b\mathbf{a}(\cdot,b^{-\frac{1}{3}}\gamma)\cdot(\nabla\tilde{g}_{b,\gamma}) + b\tilde{g}_{b,\gamma}W_b(\gamma)\right\}(H_0-z)^{-1}f_{b,\gamma} \tag{2.17}$$

satisfies $(0 < b \leq 1)$:

$$\sup_{\gamma}||T_b(\gamma,z)|| \leq \mathbf{const}\ b^{\frac{2}{3}}(1+b^{-\frac{1}{3}}\exp\{-\eta b^{-\epsilon/2}\})\max\left\{1,\frac{1}{d_{0,z}}\right\}.$$

Using Lemma 2.1, the operator

$$T_b(z) := \sum_{\gamma}\exp\left(\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)\right)F_{b,\gamma}T_b(\gamma,z)g_{b,\gamma}\exp\left(-\imath b\phi(\cdot,b^{-\frac{1}{3}}\gamma)\right)$$

is a bounded operator on $L^2(\mathbb{R}^3)$ and

$$||T_b(z)|| \leq \mathbf{const}\ b^{\frac{2}{3}}(1+b^{-\frac{1}{3}}\exp\{-\eta b^{-\epsilon/2}\})\max\left\{1,\frac{1}{d_{0,z}}\right\}. \tag{2.18}$$

Define $0 < b_{1,\epsilon} < 1$ such that for any $0 < b \leq b_{1,\epsilon}$, the interval $\Delta(b)$ is open, i.e. there exists at least one point $\lambda \in \Delta$ such that $d_{0,\lambda} \geq b^{\frac{2}{3}-\epsilon}$. If $\Re(z) \in \Delta$ and $d_{0,z} \geq b^{\frac{2}{3}-\epsilon}$, then there exists a constant $C(\epsilon) > 0$ such that

$$b^{-\frac{1}{3}}\exp\{-\eta b^{-\epsilon/2}\} \leq C(\epsilon).$$

Inserting the above inequality in (2.18), one obtains:

$$||T_b(z)|| \leq \mathbf{const}\ b_{1,\epsilon}^{\epsilon}(1+C(\epsilon))\ ,\ (\forall)\ 0 < b \leq b_{1,\epsilon}\ . \tag{2.19}$$

Let ψ and φ belong to $C_0^\infty(\mathbb{R}^n)$, $z \in \mathbb{C}$, $\Re(z) \in \Delta$ and $\Im(z) > 0$. Then ($\overline{z}$ means the conjugate of z and we employ (2.1) and (2.2))

$$\langle(H(b)-\overline{z})\psi, S_b(z)\varphi\rangle = \langle\psi,\varphi\rangle + \langle\psi, T_b(z)\varphi\rangle\ ,$$

and since $H(b)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, (2.18) implies that $(H(b)-z)S_b(z)$ is a bounded operator on $C_0^\infty(\mathbb{R}^n)$ and then on $L^2(\mathbb{R}^n)$. Moreover,

$$(H(b)-z)S_b(z) = 1 + T_b(z). \tag{2.20}$$

Choose $0 < b_\epsilon \leq b_{1,\epsilon}$ small enough such that $||T_b(z)|| < 1/2$ if $0 < b \leq b_\epsilon$ (see (2.19)); then (2.20) implies:

$$(H(b) - z)^{-1} = S_b(z)[1 + T_b(z)]^{-1} \qquad \text{and} \qquad ||(H(b) - z)^{-1}|| \leq \frac{\mathbf{const}}{d_{0,z}} \,, \tag{2.21}$$

where the norm estimate is uniform in $\Im(z)$. The final result is obtained through analytic continuation.

For Dirac operators, after a similar reasoning one obtains $W_b(\gamma) = -\alpha \cdot \mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)$ and

$$T_b(\gamma, z) = [-\imath\alpha \cdot (\nabla\tilde{g}_{b,\gamma}) + b\tilde{g}_{b,\gamma}W_b(\gamma)]\,(H_0 - z)^{-1}f_{b,\gamma} \,,$$

therefore everything works as before.

ii. Fix the compact $K \subset \mathbb{R}$ and take $z = x + \imath y$, $x \in K$, $0 < |y| < 1$. A norm estimate for the operators in (2.17) is ($0 < b \leq 1$):

$$\sup_\gamma ||T_b(\gamma, z)|| \leq \mathbf{const}\, \frac{1}{|y|} b^{\frac{2}{3}}(1 + b^{-\frac{1}{3}} \exp\{-\eta|y|b^{-\frac{1}{3}}\}) \leq \mathbf{const}\, \frac{b^{\frac{2}{3}}}{|y|^2} \,, \tag{2.22}$$

where the exponential estimate is obtained via a standard Combes-Thomas argument, that is (for $\eta > 0$ sufficiently small and $\langle \mathbf{x} \rangle = \sqrt{1 + \mathbf{x}^2}$)

$$|| \exp(\eta|y|\langle\cdot\rangle)(H_0 - x - \imath y)^{-1} \exp(-\eta|y|\langle\cdot\rangle)|| \leq \frac{\mathbf{const}}{|y|}.$$

Now fix b and increase $|y|$; then $||T_b(z)|| < 1/2$ and (2.20) implies

$$\begin{aligned} (H(b) - z)^{-1} &= S_b(z)[1 + T_b(z)]^{-1} = S_b(z) - S_b(z)[1 + T_b(z)]^{-1}T_b(z) \\ &= S_b(z) - (H(b) - z)^{-1}T_b(z). \end{aligned} \tag{2.23}$$

Through analytic continuation, the equality between the first and the last term should hold for any $y \neq 0$ and (2.11) follows. □

The last proposition also has a nice consequence:

Corollary 2.1. *Let* $f \in C_0^\infty(\mathbb{R}; \mathbb{R})$ *and define* $\mathcal{G}_b$ *as in (2.4) with* $\chi_{b,\gamma} = \tilde{g}_{b,\gamma}$ *and* $\mathcal{T}_b(\gamma) = f(H_0)$. *Then*

$$||f(H(b)) - \mathcal{G}_b|| = \mathcal{O}(b^{2/3}). \tag{2.24}$$

Proof. Take $\tilde{f}$ a compactly supported, quasi-analytic extension of f (in the sense of Helffer and Sjöstrand; see [9]) such that

$$\sup_{x\in\mathbb{R}} |\frac{\partial \tilde{f}}{\partial \overline{z}}(x+\imath y)| \leq \mathbf{const}\cdot |y|^4\ ,\ |y|\leq 1\ . \tag{2.25}$$

Then

$$f(H(b)) = \frac{1}{\pi}\int_{z\in \text{supp}\ \tilde{f}} dx\ dy\ \frac{\partial \tilde{f}}{\partial \overline{z}}(z)\ (H(b)-z)^{-1}$$

and

$$\mathcal{G}_b = \frac{1}{\pi}\int_{z\in \text{supp}\ \tilde{f}} dx\ dy\ \frac{\partial \tilde{f}}{\partial \overline{z}}(z)\ S_b(z)\ ,$$

where the last equality comes from the fact that the sum with respect to γ in $S_b(z)$ commutes with the normly convergent integral. The use of (2.11) finishes the proof. □

3 Perturbation of eigenvalues

3.1 Proof of Theorem 1.2

In this section, we only treat the Schrödinger operators since the case of Dirac operators follows from the same type of argument. Because we are studying the behavior of a discrete eigenvalue under a small magnetic perturbation, we need a few additional well known properties, summarized in the next proposition.

Denote by $\langle \mathbf{x}\rangle = (1+\mathbf{x}^2)^{1/2}$.

Proposition 3.1. *Let P_0 be the unperturbed projector corresponding to E_0. Then there exists $\alpha>0$ such that*

i. $\exp(\alpha\langle\cdot\rangle)P_0\exp(\alpha\langle\cdot\rangle)$, $\exp(\alpha\langle\cdot\rangle)(\mathbf{p}-\mathbf{a}_0)P_0\exp(\alpha\langle\cdot\rangle)$ *are in* $B(L^2(\mathbb{R}^n))$*;*

ii. The free resolvent $(H_0-z)^{-1}$ is meromorphic near E_0, and if $\mathcal{C}$ denotes a positively oriented circle centered in E_0, with no other eigenvalues of H_0 in its interior and $\text{dist}(\mathcal{C},\sigma(H_0)) = d > 0$ *then:*

$$\begin{aligned}(H_0-z)^{-1} &= \frac{P_0}{E_0-z} + \frac{\imath}{2\pi}\int_{\mathcal{C}}\frac{1}{\mu-z}(H_0-\mu)^{-1}\,d\mu \\ &:= \frac{P_0}{E_0-z} + R_1(z), \end{aligned}\tag{3.1}$$

where $R_1(z)$ *is holomorphic near* E_0. *Moreover, for any compact* $K \subset \mathbb{C}$ *contained in the interior of* $\mathcal{C}$, *one has*

$$\sup_{z\in K} \| \exp\left(\alpha\langle\cdot\rangle\right) R_1(z) \exp\left(-\alpha\langle\cdot\rangle\right)\| < \infty. \tag{3.2}$$

Then the operator $S_b(z)$ in (2.10) reads as

$$\begin{aligned} S_b(z) &= \frac{1}{E_0 - z}\sum_\gamma \exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\tilde{g}_{b,\gamma}P_0 g_{b,\gamma}\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right) + \\ &+ \sum_\gamma \exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)\tilde{g}_{b,\gamma}R_1(z) g_{b,\gamma}\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right) \\ &:= \frac{1}{E_0 - z}\mathcal{P}_b + \mathcal{R}_b(z), \end{aligned} \tag{3.3}$$

where the last two notations are obvious. Notice that Proposition 2.1 implies that $\mathcal{R}_b(z)$ is holomorphic in the interior of $\mathcal{C}$, hence

$$\frac{\imath}{2\pi}\int_{\mathcal{C}} S_b(z)\ dz = \mathcal{P}_b\ . \tag{3.4}$$

A similar treatment can be carried out for $T_b(z)$. Inserting (3.1) in (2.17), one obtains the decomposition $T_b(\gamma, z) := \frac{1}{E_0 - z}A_b(\gamma) + D_b(\gamma, z)$ where

$$A_b(\gamma) = \left\{[H_0, \tilde{g}_{b,\gamma}] + 2\imath\, b\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)\cdot(\nabla\tilde{g}_{b,\gamma}) + b\tilde{g}_{b,\gamma}W_b(\gamma)\right\} P_0 f_{b,\gamma}, \tag{3.5}$$

$$D_b(\gamma, z) = \left\{[H_0, \tilde{g}_{b,\gamma}] + 2\imath\, b\mathbf{a}(\cdot, b^{-\frac{1}{3}}\gamma)\cdot(\nabla\tilde{g}_{b,\gamma}) + b\tilde{g}_{b,\gamma}W_b(\gamma)\right\} R_1(z) f_{b,\gamma} \tag{3.6}$$

With obvious notations,

$$T_b(z) = \frac{1}{E_0 - z}A_b + D_b(z), \tag{3.7}$$

and $D_b(z)$ is holomorphic near E_0.

Denote with $w(b) := [\mathbf{a}\cdot(\mathbf{p} - \mathbf{a}_0) + (\mathbf{p} - \mathbf{a}_0)\cdot\mathbf{a} - b\mathbf{a}^2]$. Then, we have

Lemma 3.1. *There exist* $0 < \alpha' < \alpha$ *and* $b_0 > 0$ *such that for* $0 < b < b_0$*:*

$$||\mathcal{P}_b - P_0|| = \mathcal{O}(e^{-\alpha' b^{-1/3}}) \text{ and } ||A_b/b + w(b)P_0|| = \mathcal{O}(e^{-\alpha' b^{-1/3}})\ . \tag{3.8}$$

Proof. We only prove the second estimate, the first one can be handled with the same technique. We start by rewriting A_b as ($\phi(\mathbf{x}, 0) = 0$)

$$A_b = A_b(0)g_{b,0} + \sum_{\gamma\neq 0} \exp\left(\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right)F_{b,\gamma}A_b(\gamma)g_{b,\gamma}\exp\left(-\imath b\phi(\cdot, b^{-\frac{1}{3}}\gamma)\right). \quad (3.9)$$

By using the exponential estimates of Proposition 3.1 i., the second part of the r.h.s of (3.9) is of order $\mathcal{O}(e^{-\alpha' b^{-1/3}})$ for some $0 < \alpha' < \alpha$ since the support of $g_{b,\gamma}$, $\gamma \neq 0$ is located outside of a ball centered at the origin and with a radius of order $b^{-1/3}$. Hence we are left with the first term only:

$$A_b(0)g_{b,0} = \left\{[H_0, \tilde{g}_{b,0}] + 2\imath\, b\mathbf{a}(\cdot)\cdot(\nabla\tilde{g}_{b,0}) + b\tilde{g}_{b,0}W_b(0)\right\} P_0 g_{b,0}.$$

The first two terms are also exponentially small because of the properties of the supports of $\nabla\tilde{g}_{b,0}$ and $g_{b,0}$. Notice also that $W_b(0) = -w(b)$, then by using the decay of P_0,

$$A_b = -b\tilde{g}_{b,0}w(b)P_0 f_{b,0} + \mathcal{O}(e^{-\alpha' b^{-1/3}}) = -bw(b)P_0 + \mathcal{O}(e^{-\alpha' b^{-1/3}}). \quad \square$$

We then are ready to finish the proof of the theorem. Choose b_0 small enough and $z \in \mathcal{C}$. Then (2.18) yields $||T_b(z)|| \leq C\ b^{2/3} < 1/2$, uniformly in $0 < b < b_0$.

The perturbed resolvent can be expressed as before:

$$(H(b) - z)^{-1} = S_b(z)(1 + T_b(z))^{-1} = S_b(z) - S_b(z)T_b(z)(1 + T_b(z))^{-1}\ . \quad (3.10)$$

Then the projector $P(b)$ which corresponds to the part of $\sigma(H(b))$ which is in the interior of $\mathcal{C}$, reads as:

$$P(b) = \frac{\imath}{2\pi}\int_{\mathcal{C}}(H(b) - z)^{-1}\, dz = \mathcal{P}_b - \frac{\imath}{2\pi}\int_{\mathcal{C}} S_b(z)T_b(z)(1 + T_b(z))^{-1}\ . \quad (3.11)$$

From (3.7), since $||D_b(z)|| = o(1)$ for $z \in \mathcal{C}$,

$$(1 + T_b(z))^{-1} = (1 + D_b(z))^{-1}\left[1 + \frac{1}{E_0 - z}A_b(1 + D_b(z))^{-1}\right]^{-1}. \quad (3.12)$$

Notice that Lemma 3.1 implies that $\frac{1}{b}||A_b|| = \mathcal{O}(1)$. Now since $(z - E_0)S_b(z)$ is holomorphic near E_0,

$$\begin{aligned} P(b)(H(b) - E_0)P(b) &= \frac{\imath}{2\pi}\int_{\mathcal{C}}(z - E_0)(H(b) - z)^{-1}\, dz \qquad (3.13)\\ &= \sum_{j\geq 0}\frac{\imath}{2\pi}\int_{\mathcal{C}}\frac{1}{(z - E_0)^{j-1}}S_b(z)T_b(z)(1 + D_b(z))^{-1}[A_b(1 + D_b(z))^{-1}]^j\, dz\ . \end{aligned}$$

Equations (3.11) and (3.8) state in particular that $P(b)$ converges in norm to P_0. Therefore, consider the Nagy unitary operator which intertwines the projectors, i.e. $U_b P(b) = P_0 U_b$:

$$U_b := \left[1 - (P(b) - P_0)^2\right]^{-1/2} \left[P_0 P(b) + (1 - P_0)(1 - P(b))\right] . \tag{3.14}$$

Then $P(b)(H(b) - E_0)P(b)$ is unitarily equivalent with $h(b) : \mathrm{Ran}(P_0) \mapsto \mathrm{Ran}(P_0)$ given by

$$\begin{aligned} h(b) &:= \frac{\imath}{2\pi}\int_{\mathcal{C}} (z - E_0) P_0 U_b S_b(z) T_b(z) (1 + D_b(z))^{-1} U_b^* P_0 \, dz + \\ &+ \sum_{j\geq 1} b^j \frac{\imath}{2\pi}\int_{\mathcal{C}} \frac{1}{(z-E_0)^{j-1}} P_0 U_b S_b(z) T_b(z)(1 + D_b(z))^{-1} \times \\ &\times \left[\frac{1}{b} A_b (1 + D_b(z))^{-1}\right]^j U_b^* P_0 \, dz := h_0(b) + \sum_{j\geq 1} h_j(b) b^j . \end{aligned} \tag{3.15}$$

Notice that due to the presence of $T_b(z)$, the norm of $h_j(b)$ behaves like $C^j b^{2/3}$ for some constant $C > 0$.

From the asymptotic expansion for $h(b)$, the following limit should hold:

$$\lim_{b\searrow 0} \left\| \frac{1}{b}\{h(b) - P_0[\mathbf{a}\cdot(\mathbf{p} - \mathbf{a}_0) + (\mathbf{p} - \mathbf{a}_0)\cdot \mathbf{a}]P_0\}\right\| = 0 .$$

For completeness, we shall prove that this follows from (3.15). First, the series in $h(b)$ is $o(b)$, hence only the first term should contribute to the limit. Inserting (3.3) in the first term, and knowing that $(z - E_0)T_b(z)$ is holomorphic near E_0, one obtains:

$$\begin{aligned} h_0(b) &= \frac{\imath}{2\pi}\int_{\mathcal{C}} P_0 U_b \mathcal{P}_b T_b(z) (1 + D_b(z))^{-1} U_b^* P_0 \, dz \\ &= \frac{\imath}{2\pi}\int_{\mathcal{C}} \frac{1}{(z - E_0)} P_0 U_b \mathcal{P}_b A_b (1 + D_b(z))^{-1} U_b^* P_0 \, dz \\ &= -P_0 U_b \mathcal{P}_b A_b (1 + D_b(E_0))^{-1} U_b^* P_0 . \end{aligned} \tag{3.16}$$

Using $\lim_{b\searrow 0} U_b = 1$ and (3.8), the limit is proved.

Let us now rewrite (3.15) as (we know that $h(b)$ is self adjoint):

$$h(b)/b = \tilde{h}_0(b) + \sum_{j\geq 1} b^j \tilde{h}_j(b), \tag{3.17}$$

where

$$\tilde{h}_0(b) = \frac{1}{2}\{[h_0(b) + h_0^*(b)]/b + h_1(b) + h_1^*(b)\}$$

and

$$\tilde{h}_j(b) = \frac{1}{2}\{h_{j+1}(b) + h_{j+1}^*(b)\} \quad \text{if} \quad j \geq 1 \ .$$

Clearly, $\tilde{h}_0(b)$ converges in norm to $P_0[\mathbf{a} \cdot (\mathbf{p} - \mathbf{a}_0) + (\mathbf{p} - \mathbf{a}_0) \cdot \mathbf{a}]P_0$, whose eigenvalues are in fact the first order corrections given by the asymptotic perturbation theory and we assumed that they are all different one from each other. Denote by $\{e_{j,1}(b)\}_{j=1}^M$ the eigenvalues of $\tilde{h}_0(b)$, where the labeling agrees with $\lim_{b\searrow 0} e_{j,1}(b) = e_j$. Therefore, for b_0 sufficiently small and for $0 < b < b_0$, the eigenvalues $\{e_{j,1}(b)\}_{j=1}^M$ will be different one from each other, too.

Now fix $\lambda \in (0, b_0)$ and define

$$\tilde{h}(\lambda, b) = \tilde{h}_0(\lambda) + \sum_{j\geq 1} b^j \tilde{h}_j(\lambda) := \tilde{h}_0(\lambda) + b\tilde{h}_1(\lambda, b)$$

which is an analytic family of operators in a neighborhood of $b = 0$. This can be seen from the estimate (see (3.15))

$$||h_j(\lambda)|| \leq C^j,$$

which is valid if $0 < \lambda < b_0$ and b_0 is sufficiently small.

We know that $\{e_{j,1}(\lambda)\}_{j=1}^M$ are separated for $0 < \lambda < b_0$. Then, applying the Rellich theorem, there exist M eigenvalues $\{\tilde{e}_{j,1}(\lambda, b)\}_{j=1}^M$ for the operator $\tilde{h}(\lambda, b)$ which are analytic in a neighborhood of $b = 0$. Now since $\|\tilde{h}_1(\lambda, b)\| \leq \tilde{C}$, then $\{\tilde{e}_{j,1}(\lambda, b)\}_{j=1}^M$ remain separated hence analytic as long as

$$|b| \leq \frac{1}{2\,\tilde{C}} \inf_{i\neq j}\{|e_{i,1} - e_{j,1}|\} \ .$$

Finally, one has to replace λ by b and (1.5) is proved.

Remark. We would like to motivate the condition we asked for the first order corrections to be separated. We are not claiming that this condition is necessary, but we simply want to warn the reader about the possible difficulties one might encounter in its absence.

Consider the following family of 2×2 matrices acting on $\mathbb{C}^2$, where $b_0 > \lambda > 0$ is fixed:

$$\tilde{h}(\lambda, b) = \left[\begin{array}{ll} 1 - \lambda^{3/2} & 0 \\ 0 & 1 + \lambda^{3/2} \end{array} \right] + b \left[\begin{array}{ll} 0 & 1 \\ 1 & 0 \end{array} \right] . \tag{3.18}$$

Although we explicitly compute its eigenvalues below, let us describe their generic behavior.

The operator family is analytic in b near zero; since the zero-th order has distinct eigenvalues, Relich theorem says that $\tilde{h}(\lambda, b)$ has two eigenvalues centered at $e_{1,1}(\lambda) = 1 - \lambda^{3/2}$ and $e_{2,1}(\lambda) = 1 + \lambda^{3/2}$. Those two eigenvalues will also be analytic in b near zero, having a radius of convergence of the order $\lambda^{3/2}$, i.e. the same as the splitting. But then we cannot put $\lambda = b$ because $b^{3/2}$ is much smaller than b near zero therefore we would be outside the convergence disk.

In the particular case we have chosen, an explicit computation of the eigenvalues yields:

$$1 \pm \lambda^{3/2} \sqrt{1 + \frac{b^2}{\lambda^3}} \, .$$

3.2 Asymptotic Expansions

From our formulae (3.11) and (3.15) one can derive the usual asymptotic expansions for the projector $P(b)$, the reduced operator $h(b)$ and therefore for the eigenvalues $E_j(b)$. We only do this for the projector.

Following Nenciu [17, Lemma 4.2], define

$$P_j(b) := (-1)^j \frac{\imath}{2\pi} \int_{\mathcal{C}} (H_0 - z)^{-1} [w(b)(H_0 - z)^{-1}]^j \, dz \, ,$$

which are bounded and

$$P(b) = \sum_{j=0}^{N} b^j P_j(b) + \mathcal{O}(b^{N+1}) \, . \tag{3.19}$$

We now want to prove that (3.19) can be derived from (3.11):

Lemma 3.2. *Fix an integer $N \geq 1$.*

i. There exists another integer $j(N) \geq 3N/2$ such that for $0 < b < 1$ we have:

$$P(b) = \mathcal{P}_b + \sum_{j=1}^{j(N)} (-1)^j b^j \frac{\imath}{2\pi} \int_{\mathcal{C}} S_b(z) [b^{-1} T_b(z)]^j \, dz + \mathcal{O}(b^{N+1}) \, ; \tag{3.20}$$

ii. For $j \geq 1$, we have:

$$\frac{\imath}{2\pi}\int_{\mathcal{C}} S_b(z)[b^{-1}T_b(z)]^j \, dz = P_j(b) + \mathcal{O}(\exp(-\alpha' b^{-1/3})) \ . \tag{3.21}$$

Proof. i. Since $||T_b(z)|| = \mathcal{O}(b^{2/3})$, there exists an integer $j(N)$ such that (3.20) holds.

ii. We start by decomposing $T_b(z)$ (see (2.17)) as

$$T_b(0,z)g_{b,0} + \tilde{T}_b(z),$$

where $\tilde{T}_b(z)$ contains all the terms with $\gamma \neq 0$. Then

$$T_b^j(z) = [T_b(0,z)g_{b,0}]^j + \text{terms containing at least one } \tilde{T}_b(z) \ .$$

Using Proposition 3.1 and the same arguments as those employed in the proof of Lemma 3.1, we get

$$\frac{\imath}{2\pi}\int_{\mathcal{C}} \{S_b(z)[b^{-1}T_b(z)]^j - S_b(z)[b^{-1}T_b(0,z)g_{b,0}]^j\} \, dz = \mathcal{O}(\exp(-\alpha' b^{-1/3})) \ , \tag{3.22}$$

because after the integration, we only obtain terms which are exponentially localized due to the presence of P_0 (see also the arguments employed by Nenciu in the proof of Lemma 4.2 [17]).

We also decompose $S_b(z)$ as

$$S_b(z) = \tilde{g}_{b,0}(H_0 - z)^{-1}g_{b,0} + \tilde{S}_b(z) \ ,$$

where $\tilde{S}_b(z)$ contains all the terms with $\gamma \neq 0$. By the same argument as above, the second term in the l.h.s. of (3.22) becomes:

$$\frac{\imath}{2\pi}\int_{\mathcal{C}} \tilde{g}_{b,0}(H_0 - z)^{-1}g_{b,0}[b^{-1}T_b(0,z)g_{b,0}]^j \, dz + \mathcal{O}(\exp(-\alpha' b^{-1/3})) \ .$$

By (2.17), $T_b(0,z) = b\tilde{g}_{b,0}w(b)(H_0 - z)^{-1}f_{b,0} + \mathcal{O}(\exp(-\alpha' b^{-1/3}))$, hence the last formula reads as:

$$\frac{\imath}{2\pi}\int_{\mathcal{C}} \tilde{g}_{b,0}(H_0 - z)^{-1}g_{b,0}[\tilde{g}_{b,0}w(b)(H_0 - z)^{-1}g_{b,0}]^j \, dz + \mathcal{O}(\exp(-\alpha' b^{-1/3})) \ .$$

We now can get rid of the cut-off functions using the exponential localization of P_0, and we are done. □

4 The stability of the essential spectrum

Since the Schrödinger case has been extensively studied in the literature, we choose to give the proof for the Dirac case.

As we have already said, the proof of this theorem is almost completely contained in the previous two sections. First, let us remark that only the tail of the magnetic field counts; a rigorous statement of this fact is given by the next lemma. Let us start with a few notations: take $\eta \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$, $0 \leq \eta \leq 1$ such that $\eta(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq 1$ and $\eta(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq 2$. Then $\eta_R(\mathbf{x}) = \eta(\mathbf{x}/R)$ is a dilated cut-off function. Define $\mathbf{a}_R := (1-\eta_R)\mathbf{a}$ and $H_R := \alpha \cdot (\mathbf{p} - \mathbf{a}_0 - \mathbf{a}_R) + \beta + V$.

Lemma 4.1. *$\sigma_{ess}(H(\mathbf{a})) = \sigma_{ess}(H_R)$, $(\forall)R > 1$.*

Proof. From the fact that $\eta_R\mathbf{a}$ has compact support, together with the local compactness of the operators involved, (see [8], [18]) it follows that $[H(\mathbf{a}) - i]^{-1} - [H_R - i]^{-1}$ is compact. The use of Weyl's theorem finishes the proof. □

The main property of $\mathbf{a}_R$ is that the corresponding magnetic field $\mathbf{B}_R := \nabla \wedge \mathbf{a}_R = (1-\eta_R)\mathbf{B} - (\nabla\eta_R) \wedge \mathbf{a}$ is *everywhere small.* It is easy to see that for $(1-\eta_R)\mathbf{B}$ and we shall only treat here the second term. Define $\mathbf{a}_{1,R} := \int_0^{R^{-1/2}} ds\ s\ \mathbf{B}(s\mathbf{x}) \wedge \mathbf{x}$ and $\mathbf{a}_{2,R} := \int_{R^{-1/2}}^1 ds\ s\ \mathbf{B}(s\mathbf{x}) \wedge \mathbf{x}$; then $\mathbf{a} = \mathbf{a}_{1,R} + \mathbf{a}_{2,R}$. Now let us prove that $\sup_{\mathbf{x}\in\mathbb{R}^3} |(\nabla\eta_R) \wedge \mathbf{a}| \leq \mathbf{const}\ \max\{1/R, b_{R^{1/2}}\}$; the same kind of bound can be obtained for its first order derivatives, but we shall skip that. Because $\mathbf{supp}(\nabla\eta_R) \subseteq \{R \leq |\mathbf{x}| \leq 2R\}$, then in the estimate of $|\mathbf{a}_{1,R}|$ one can consider $R \leq |\mathbf{x}| \leq 2R$; using (1.1):

$$|\mathbf{a}_{1,R}| \leq 2R \int_0^{R^{-1/2}} ds\ s = 1\ . \tag{4.1}$$

But $|\nabla\eta_R| \leq \mathbf{const}\ 1/R$, so we are done with $|(\nabla\eta_R) \wedge \mathbf{a}_{1,R}|$. For $(\nabla\eta_R) \wedge \mathbf{a}_{2,R}$, we see that $R^{-1/2} \leq s \leq 1$ implies $s|\mathbf{x}| \in [R^{1/2}, 2R]$; therefore,

$$|\mathbf{a}_{2,R}| \leq 2R \sup_{|\mathbf{x}'| \geq R^{1/2}} (|\mathbf{B}(\mathbf{x}')|) \int_0^1 ds\ s \leq \mathbf{const}\ R\ b_{R^{1/2}}. \tag{4.2}$$

Use again $|\nabla\eta_R| \leq \mathbf{const}\ 1/R$ and the needed estimate follows. We have seen that $\mathbf{B}_R$ is everywhere small. Denote:

$$\tilde{b}_R := \max_{j\in\{1,2,3\}} \max_{|\alpha|\leq 1} \sup_{\mathbf{x}\in\mathbb{R}^3} \{|D^\alpha(\mathbf{B}_R)_j|(\mathbf{x})\}. \tag{4.3}$$

Clearly, $\lim_{R\to\infty} \tilde{b}_R = 0$.

Remarks 1. If $\tilde{b}_R = 0$ for some R, then the theorem (1.3) is proved. That is why we assume $\tilde{b}_R > 0$, $\forall R > 1$.
2. Another observation is that $\mathbf{B}_R$ was defined as the curl of $\mathbf{a}_R = (1 - \eta_R)\mathbf{a}$; up to a gauge transformation, H_R is unitarily equivalent with $\tilde{H}_R := \alpha \cdot (\mathbf{p} - \tilde{\mathbf{a}}_R) + \beta + V$, where $\tilde{\mathbf{a}}_R$ is the transversal gauge derived from $\mathbf{B}_R$. From now on, we shall drop the tilde, and by $\mathbf{a}_R$ we understand the transversal gauge corresponding to $\mathbf{B}_R$ (the same thing for H_R).
3. Notice that there are examples of magnetic fields with decay at infinity i.e. $\mathbf{B}(\mathbf{x}) \sim |\mathbf{x}|^{-\alpha}$, $0 < \alpha < 1$ for which it is impossible to find out a bounded vector potential (see [18] for a more detailed discussion). For the transversal gauge, one can see that in those cases the formal perturbation "$H_R - H_0$" is not even relatively bounded to H_0.

Let us now begin the proof of the theorem. We first show that if $\lambda \notin \sigma_{ess}(H_0)$, then $\lambda \notin \sigma_{ess}(H(\mathbf{a}))$; in other words, we prove the inclusion

$$\sigma_{ess}(H(\mathbf{a})) \subseteq \sigma_{ess}(H_0). \tag{4.4}$$

Since $\lambda \notin \sigma_{ess}(H_0)$, we can assume without loss that λ is an isolated discrete eigenvalue of H_0. Applying the results in Section 2, one gets that for $\tilde{b}_R$ sufficiently small (i.e. R sufficiently large), the operator H_R has purely discrete spectrum near λ. It means that $\lambda \notin \sigma_{ess}(H_R)$ if $R \geq R(\lambda)$, and then Lemma 4.1 states that $\lambda \notin \sigma_{ess}(H(\mathbf{a}))$.

For the reversed inclusion, denote by $\tilde{\mathbf{a}}_0 = \mathbf{a}_0 + \mathbf{a}$, $\tilde{\mathbf{a}} = -\mathbf{a}$, $\tilde{H}_0 = \alpha \cdot (\mathbf{p} - \tilde{\mathbf{a}}_0) + \beta + V$ and $\tilde{H}(\tilde{\mathbf{a}}) = \alpha \cdot (\mathbf{p} - \tilde{\mathbf{a}}_0 - \tilde{\mathbf{a}}) + \beta + V$. Clearly, $\tilde{H}_0 = H(\mathbf{a})$ and $\tilde{H}(\tilde{\mathbf{a}}) = H_0$. We next apply (4.4) for the couple $(\tilde{H}_0, \tilde{H}(\tilde{\mathbf{a}}))$ and we are done.

5 Appendix: Exponential Estimates

In this section we want to give some exponential estimates which we have used in the previous sections. For the Schrödinger case, this result has been obtained in [3]. Consider the Schrödinger operator:

$$H_0 = (\mathbf{p} - \mathbf{a}_0)^2 + V(\mathbf{x}); \mathbf{x} \in \mathbb{R}^n,$$

$\mathbf{a}_0 \in [C^1(\mathbb{R}^n)]^n$ and V is supposed to be relatively bounded to $(\mathbf{p} - \mathbf{a}_0)^2$ with relative bound smaller than one. Suppose that H_0 has a gap $\Delta = (\Delta_-, \Delta_+)$ and let K be a compact subset of $\mathbb{C}$ containing Δ in its interior. For $z \in K$

and $\Re(z) = \lambda \in \Delta$, denote by $d_- := \mathrm{dist}(\lambda, \Delta_-)$ and $d_+ := \mathrm{dist}(\lambda, \Delta_+)$. For such z, define $D_{0,z} := (1/2)\min\{1, (d_+d_-)^{1/2}, (d_+)^{1/2}\}$.

Let χ_{Ω_1} and χ_{Ω_2} be two smoothed out characteristic functions of Ω_1, $\Omega_2 \subset \mathbb{R}^n$. Take $\mathbf{x}_0 \in \mathbb{R}^n$ and define

$$D_2 = \inf_{\mathbf{x}\in \mathrm{supp}\chi_{\Omega_2}} \{|\mathbf{x} - \mathbf{x}_0|\} > D_1 = \sup_{\mathbf{x}\in \mathrm{supp}\chi_{\Omega_1}} \{|\mathbf{x} - \mathbf{x}_0|\}.$$

Proposition 5.1. *For $z \in K$, $\Re(z) = \lambda \in \Delta$, then*

i. $$||\chi_{\Omega_2}(H_0 - z)^{-1}\chi_{\Omega_1}|| \le \mathbf{const}\ \exp\{-D_{0,z}(D_2 - D_1)\}\frac{1}{d_{0,z}}\ ,$$

ii. $$||\chi_{\Omega_2}(\mathbf{p} - \mathbf{a}_0)(H_0 - z)^{-1}\chi_{\Omega_1}|| \le \mathbf{const}\ \exp\{-D_{0,z}(D_2 - D_1)\}\frac{1}{d_{0,z}}\ . \quad \square$$

A similar result can be obtained for Dirac operators, too. Let $H_0 = \alpha \cdot (\mathbf{p} - \mathbf{a}_0) + \beta + V(\mathbf{x}); \mathbf{x} \in \mathbb{R}^3$, $\mathbf{a}_0 \in [C^1(\mathbb{R}^3)]^3$ and V is bounded. Define in the form sense on $[C_0^\infty(\mathbb{R}^3)]^4$,

$$H(\rho) := e^\rho H_0 e^{-\rho} = H_0 - i\alpha \cdot \nabla\rho,$$

where $\rho(\mathbf{x}) := \delta(1 + |\mathbf{x} - \mathbf{x}_0)|^2)^{1/2}$ and $\delta > 0$ will be fixed later.

$H(\rho)$ is a closed operator in $[L^2(\mathbb{R}^3)]^4$. Suppose that $\Delta = (\Delta_-, \Delta_+)$ is a gap for H_0 and let $\psi \in [C_0^\infty(\mathbb{R}^3)]^4$, $\|\psi\| = 1$. Denote by $\psi_- := P_{(-\infty,\Delta_-]}\psi$, $\psi_+ := P_{[\Delta_+,+\infty)}\psi$ where $P_.$ are the spectral projectors of H_0; for $\Re(z) \in \Delta$, one has

$$\begin{aligned} \|(H(\rho) - z)\psi\| &\ge \mathrm{Re}\Big((\psi_+ - \psi_-), (H(\rho) - z)(\psi_+ + \psi_-)\Big) \\ &\ge d_+\|\psi_+\|^2 + d_-\|\psi_-\|^2 - 2\|\alpha \cdot \nabla\rho\|\|\psi_+\|\|\psi_-\|. \quad (5.1) \end{aligned}$$

Choose $\delta = (1/2)(d_+d_-)^{1/2}$; then $0 < \|\alpha \cdot \nabla\rho\| < (1/2)(d_+d_-)^{1/2}$ and

$$\|(H(\rho) - z)\psi\| \ge (1/2)\min(d_+, d_-).$$

At least for $\Re(z) \in \Delta$, $(H(\rho) - z)$ is an invertible operator and

$$\|(H(\rho) - z)^{-1}\psi\| \le \frac{2}{d_{0,z}}.$$

Notice that by standard arguments, we have

$$\|e^\rho(H_0 - z)^{-1}e^{-\rho}\| = \|(H(\rho) - z)^{-1}\|.$$

On the other hand, since

$$||\chi_{\Omega_2}(H_0 - z)^{-1}\chi_{\Omega_1}|| \leq ||\chi_{\Omega_2}e^{-\rho}|| \cdot ||e^{\rho}(H_0 - z)^{-1}e^{-\rho}|| \cdot ||\chi_{\Omega_1}e^{\rho}||,$$

we have proven:

Proposition 5.2. *In the same conditions as above,*

$$||\chi_{\Omega_2}(H_0 - z)^{-1}\chi_{\Omega_1}|| \leq \exp\{-D_{0,z}(D_2 - D_1)\})\frac{2}{d_{0,z}}, \tag{5.2}$$

with $D_{0,z} := (1/2)(d_+d_-)^{1/2}$. ☐

Remarks. 1. Notice that here the function $D_{0,z}$ is different from the one used for the Schrödinger case.
2. Suppose that the potential V is such that $P_{(-\infty,\Delta_-)} = 0$. Then for any $\delta > 0$,

$$||(H(\rho) - z)^{-1}\psi|| \leq \frac{1}{d_{0,z}},$$

because the term which contains δ is purely imaginary; hence for any $\delta > 0$,

$$||\chi_{\Omega_2}(H_0 - z)^{-1}\chi_{\Omega_1}|| \leq \exp\{-\delta(D_2 - D_1)\}\frac{1}{d_{0,z}}. \tag{5.3}$$

This finally implies that $(H_0 - z)^{-1} = 0$ which is a contradiction. We conclude that a bounded potential cannot create an infinite gap.

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