C*–ALGEBRAS, DYNAMICAL SYSTEMS AT INFINITY AND THE ESSENTIAL SPECTRUM OF GENERALIZED SCHRÖDINGER OPERATORS

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Abstract. We introduce classes of C*-algebras to which anisotropic Schrödinger operators are affiliated. In an abstract framework, we study decompositions of the quotients of these C*-algebras by the ideal of compact operators. These are induced by invariant decompositions of a dynamical system "at infinity" defined by the class of potentials we consider. As a consequence, we get formulae for the essential spectrum of the affiliated anisotropic operators. The theory is illustrated by two types of examples which do not seem to be accessible to standard methods (as Weyl sequences).

0. Introduction

In this paper we shall be interested in the essential spectrum of *generalized Schrödinger operators* of the form

$$H := h(P) + V \tag{0-1}$$

acting in the Hilbert space $\mathcal{H} = L^2(X)$, $X = \mathbb{R}^n$. We denoted by $P := -i\nabla$ the momentum, $h : X \to \mathbb{R}$ is a continuous function diverging at infinity and $V : X \to \mathbb{R}$ is (for simplicity) continuous and bounded. In fact, most of the results will be proven in the more general context of abelian, locally compact groups X.

It is well-known that the behaviour of the potential V on compact sets has no relevance for the essential spectrum of H. More precisely, let us assume that V is the sum of two continuous terms V_0 and V_1 and that $V_0(x) \to 0$ when $\mid x \mid \to \infty$. Then the essential spectrum of the operators $h(P) + V$ and $h(P) + V_1$ coincide. It follows that what counts is the behaviour of V "at infinity", but until recently this vague assertion has been made precise only in some particular instances, the celebrated HWZ theorem for the case of N-body potentials being the best known (see [11]).

It is our intention in this article to give a unified treatement to this topic, working for large classes of anisotropic operators. See also [6] for a related approach; an early

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version of this paper partially inspired the present work. The leading idea is that a class of potentials V "having the same type of anisotropy" is associated to some compactification $\hat{X}$ of X by the property "all V's admit a continuous extension to $\hat{X}$". This leads to the study of the C*-algebra $C(\hat{X})$ of continuous, complex functions on $\hat{X}$, which may be identified with a C*-algebra of bounded, continuous functions on X. It is natural to ask that this C*-algebra be translational invariant, which ammounts to the fact that the action of X on itself by translations extends to an action on the entire $\hat{X}$. One can show that the informations about the essential spectrum are carried by the values taken by V on the boundary $Z := \hat{X} \setminus X$, which is the space of a dynamical system under the action of X. We shall prove that special kinds of decompositions of this dynamical system will lead to formulae for the essential spectrum.

At the technical level, to implement this ideas one needs two important ingredients:

(A) From the abelian C*-algebra assigned to the potentials V one has to go to some non-abelian C*-algebra connected in some way to the entire operator $h(P) + V$ (we shall say that the operator is *affiliated* to this C*-algebra). This is done essentially by using the notion of crossed product and here the invariance under translations is crucial.

(B) A strategy is needed to convert informations about the C*-algebra to which a self-adjoint operator is affiliated into spectral informations about the operator itself. This is the algebraic formalism of V. Georgescu (see [1], [3], [6]) and it has a wider range of applications than merely the study of the essential spectrum.

In the main body of the text we shall show how the vague discussion above can be made precise in a quite abstract framework, i.e. without asking a particular form for the C*-algebra of potentials. But to have some motivation we shall present here two applications in a very simplified form. Much more general examples will appear in Sections 2 and 3. We take for simplicity $X = \mathbb{R}^2$ and set $H_0 = h(P)$. All the functions will be assumed continuous.

Example 0.1. Set $V(x) = A(x)W[x + B(x)]$. The function W is $\mathbb{Z}^2$-periodic and the amplitude A and the phase B have radial limits at infinity:

$$\mid A(r\theta) - a(\theta) \mid + \mid B(r\theta) - b(\theta) \mid \to 0$$

when $r \to \infty$, uniformly in $\theta \in S_X$. Here a and b are two continuous functions on S_X, the unit sphere in X. Then the essential spectrum of $H = H_0 + V$ is given by $\boldsymbol{\sigma}_{\text{ess}}(H) = \cup_{\theta \in S_X} \boldsymbol{\sigma}(H_\theta)$, where $\boldsymbol{\sigma}(H_\theta)$ is the (band) spectrum of the Schrödinger operator $H_\theta := H_0 + V_\theta$ with periodic potential $V_\theta(x) = a(\theta)W[x + b(\theta)]$.

Example 0.2. Set $X := \mathbb{R}^2 \equiv \mathbb{R}_1 \times \mathbb{R}_2$ and $V(x_1, x_2) := V_1(x_1)V_2(x_2) + V_0(x_1, x_2)$ where V_0 decays at infinity and $V_j(x_j) \to c_j^{\pm} \in \mathbb{R}$ when $x_j \to \pm\infty$. Then $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(H_1^-) \cup \boldsymbol{\sigma}(H_1^+) \cup \boldsymbol{\sigma}(H_2^-) \cup \boldsymbol{\sigma}(H_2^+)$. We set $H_j^{\pm} := H_0 + V_j^{\pm}$ where, for example, $V_1^+(x_1, x_2) := c_2^+ V_1(x_1)$.

In both examples the essential spectrum of H is (the closure of) the union of the usual spectra of a family of simpler Schrödinger operators which we decided to call *asymptotic Hamiltonians*. In fact, the same is true for the HWZ theorem for which "the asymptotic Hamiltonians" are usually called "subhamiltonians" and are obtained from the total N-body Hamiltonian by throwing away some suitable interparticle potentials. The reason for our choice of terminology (which also can be justified for the N-body case) is the following: In both situations the potentials can be extended continuously to very simple compactifications of the space X and the asymptotic Hamiltonians are constructed with restrictions of this extension to suitable pieces of the boundary Z. In fact our main contribution will be to explain what "suitable" means. We describe the compactifications $\hat{X} = X \sqcup Z$ for our two cases briefly; a thorough discussion in much more general circumstances may be found in the main body of the article.

In Example 0.1. one can take $Z := S_X \times \mathbb{T}^2$, where $\mathbb{T}^2 := X/\mathbb{Z}^2$ is the 2-dimensional torus (a $\mathbb{Z}^2$-periodic function on X is identified with a function on $\mathbb{T}^2$). But Z may also be represented as the disjoint union $Z = \sqcup_{\theta \in S_X} \mathbb{T}^2_\theta$ with all $\mathbb{T}^2_\theta$ equal with $\mathbb{T}^2$ and the restrictions V_θ of V to the fibers $\mathbb{T}^2_\theta$ interpreted as functions on X give the asymptotic Hamiltonians.

In Example 0.2. the compactification $\hat{X}$ is the "square" $\overline{\mathbb{R}_1} \times \overline{\mathbb{R}_2}$. By $\overline{\mathbb{R}}$ we denote the two-point (radial) compactification $\mathbb{R} \sqcup \{-\infty\} \sqcup \{+\infty\}$. We obtain the four asymptotic Hamiltonians first by extending V to $\hat{X}$ and then restricting it respectively to the four closed edges, each one identical to a copy of $\overline{\mathbb{R}}$. Now the decomposition of Z is no longer a partition, as in the first case.

Obviously, there are many other ways of decomposing the bounderies and these would have led to wrong results. What makes the above decompositions "correct"? The formalism we shall develop shows that this is connected to the fact that the kinetic energy $H_0 = h(P)$ must also "live" on each member of the decomposition and this forces it to be closed and invariant under translations. If we further ask a natural minimality condition, the expression of the essential spectrum is completely determined. In particular, each piece giving an asymptotic Hamiltonian will be a quasi-orbit of the dynamical system defined on the boundary.

Let us say some words on the level of generality in which we place ourselves. Until now in this introductive discussion the space X has been $\mathbb{R}^n$, for reasons of simplicity. We feel that the natural context is to take X a locally compact, abelian group (see also [6]); this works well for almost all our assertions. Only a non-compact X is relevant. The particular case $X = \mathbb{Z}^n$ will be important for people interested in finite-difference operators. Many results extend easily to the non-abelian case. Because our lack of familiarity with this situation, we decided however to take X abelian. We hope to be able to treat more general situations in the future. The "free Hamiltonian" H_0 will have the form $h(P)$ for some continuous, proper, real function h defined on the dual group of X. We notice that the choice $h(p) =\mid p \mid^2$ doesn't always make sense. And, even for $X = \mathbb{R}^n$, there is no extra effort to go from the Laplace operator to a general $h(P)$. Even more general free Hamiltonians are available (including some variable coefficient differential operators on $\mathbb{R}^n$), but we stick to the case $H_0 = h(P)$ for simplicity. A more drastical simplification was performed at the level of potentials. We decided to treat only perturbations which are multiplication operators by bounded, continuous functions $V : X \to \mathbb{R}$. This already has full relevance for the formalism and the phenomena we want to put into evidence. The way one could treat more general (higher-order, locally singular) perturbations is very well illustrated in [1], [3] and [6].

In the first section we shall present the main abstract result, a general HWZ-type formula for the essential spectrum. In Section 2 we treat a class of potentials including those of the form $V = V_1 V_2$, where V_1 has vanishing oscillation at infinity and V_2 is almost periodic. This case is a significant extension of [8]. In Section 3 we take into account cartesian anisotropy. An appendix is devoted to graded C*-algebras, which play an important role in the N-body problem; it is shown how they fit in our general framework. Each section will start with a short abstract.

We mention that the formalism employed in this article may also be used for other purposes than the study of the essential spectrum. In [2] localization properties of anisotropic Schrödinger operators are proven by looking at ideals associated to invariant, closed subsets of a suitable dynamical system.

1. DECOMPOSITIONS IN THE CALKIN ALGEBRA

In this section we shall prove our main abstract result. To "calculate" the essential spectrum of the self-adjoint operator H acting in $\mathcal{H} = L^2(X)$, we need to show that its resolvents $\{(H-\zeta)^{-1}\}_{\zeta\notin\mathbb{R}}$ belong to some C*-subalgebra $\mathfrak{C}$ of $B(\mathcal{H})$ and to express its image in the Calkin algebra (i.e. the quotient of $\mathfrak{C}$ by the ideal $K(\mathcal{H})$ of compact operators) in an efficient way. The natural C*-algebra will be generated by the potentials and the resolvents of the kinetic energies. By using techniques involving crossed products we will be reduced to understanding the dynamical system $\mathcal{X} := (\hat{X} \setminus X, \alpha, X)$. Here $\hat{X}$ is a compactification of X which takes into account the behaviour at infinity of the potentials and on which the action by translations α of X on itself extends continuously. We introduce the central concept of invariant decomposition of a dynamical system and apply it to $\mathcal{X}$. It will lead straightforwardly to an embedding of $\mathfrak{C}/K(\mathcal{H})$ into a direct product of C*-algebras. These C*-algebras are canonically defined by the members of the invariant decomposition and they contain the resolvents of the asymptotic Hamiltonians. As a simple consequence, the essential spectrum of H will be equal to the closure of the union of the spectra of these asymptotic Hamiltonians.

[1.1] C*-algebras and the essential spectrum.

For any Hilbert space $\mathcal{H}$, we set $B(\mathcal{H})$ for the C*-algebra of all bounded, linear operators : $\mathcal{H} \to \mathcal{H}$. The ideal of compact operators in $\mathcal{H}$ will be denoted by $K(\mathcal{H})$. From now on, "ideal" will always mean "two-sided, closed ideal". It is known how to express the spectrum and the essential spectrum of a self-adjoint operator H in $\mathcal{H}$ in terms of its functional calculus with functions from $C_0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ continuous}, \ f(\lambda) \to 0 \text{ when } |\lambda| \to \infty\}$:

$$\boldsymbol{\sigma}(H) := \{\lambda \in \mathbb{R} \mid \text{ if } f \in C_0(\mathbb{R}) \text{ and } f(\lambda) \neq 0, \text{ then } f(H) \neq 0\},$$

$$\boldsymbol{\sigma}_{ess}(H) := \{\lambda \in \mathbb{R} \mid \text{ if } f \in C_0(\mathbb{R}) \text{ and } f(\lambda) \neq 0, \text{ then } f(H) \notin K(\mathcal{H})\}.$$

We shall extend these to an abstract setting, following [1]. We are given an ideal $\mathfrak{K}$ of a C*-algebra $\mathfrak{C}$ not necessarily represented in a Hilbert space.

Definition 1.1. *(a) A* (self-adjoint) observable affiliated to $\mathfrak{C}$ *is a functional calculus taking values in* $\mathfrak{C}$*, i.e. a *-morphism* $H : C_0(\mathbb{R}) \to \mathfrak{C}$*. Instead of* $H(f)$ *we shall write systematically* $f(H)$*.*

(b) The $\mathfrak{K}$-essential spectrum *of* H *is*

$$\boldsymbol{\sigma}^{\mathfrak{C}}_{\mathfrak{K}}(H) := \{\lambda \in \mathbb{R} \mid \ \textit{if } f \in C_0(\mathbb{R}) \textit{ and } f(\lambda) \neq 0, \ \textit{then } f(H) \notin \mathfrak{K}\}.$$

Obvious important cases are $\mathfrak{K} = \{0\}$ (we write simply $\boldsymbol{\sigma}^{\mathfrak{C}}_{\{0\}}(H) \equiv \boldsymbol{\sigma}^{\mathfrak{C}}(H)$) and $\mathfrak{K} = K(\mathcal{H})$, with $\boldsymbol{\sigma}^{\mathfrak{C}}_{K(\mathcal{H})}(H) = \boldsymbol{\sigma}_{\text{ess}}(H)$ for any C*-subalgebra $\mathfrak{C}$ of $B(\mathcal{H})$ containing $K(\mathcal{H})$.

One may transport observables by *-morphisms:

Definition 1.2. *Let* $\rho : \mathfrak{C} \to \mathfrak{C}'$ *a *-morphism and* H *an observable affiliated to* $\mathfrak{C}$. *Then* $\rho[H] : C_0(\mathbb{R}) \to \mathfrak{C}'$ *given by* $f(\rho[H]) := \rho[f(H)]$ *is an observable affiliated to* $\mathfrak{C}'$, *called* the image of H through ρ.

We remark that *-morphisms do not increase the spectrum:

$$\boldsymbol{\sigma}^{\mathfrak{C}'}(\rho[H]) \subset \boldsymbol{\sigma}^{\mathfrak{C}}(H)$$

and one even has equality if ρ is a monomorphism (=injective *-morphism).

When calculating the quotient of a C*-algebra $\mathfrak{C}$ by an ideal $\mathfrak{K}$, it may be useful to "surround" $\mathfrak{K}$ with a family of larger ideals $(\mathfrak{K}_a)_{a\in A}$. In practice we are especially interested in the case $\mathfrak{K} = K(\mathcal{H})$ and it will be desirable that the C*-algebras $\mathfrak{C}/\mathfrak{K}_a$ should be "small and intelligible". It is also useful to choose the family of ideals in an economical way.

Definition 1.3. *Let* $\mathfrak{C}$ *be a C*-algebra and* $\mathfrak{K}$ *an ideal of* $\mathfrak{C}$. *A family of ideals* $(\mathfrak{K}_a)_{a\in A}$ *will be called* $\mathfrak{K}$-sufficient *if*
(i) $\cap_{a\in A}\mathfrak{K}_a = \mathfrak{K}$,
(ii) $\mathfrak{K}_a$ *is not contained in* $\mathfrak{K}_b$ *if* $a \neq b$.
When $\mathfrak{K} = \{0\}$, *instead of* $\{0\}$*-sufficient we say simply* sufficient.

We can pass very simply from a general $\mathfrak{K}$ to $\mathfrak{K} = \{0\}$.

Lemma 1.4. *Let* $(\mathfrak{K}_a)_{a\in A}$ *a* $\mathfrak{K}$*-sufficient family of ideals in* $\mathfrak{C}$. *Then* $\mathfrak{J}_a := \mathfrak{K}_a/\mathfrak{K}$ *may be identified with an ideal of* $\mathfrak{C}/\mathfrak{K}$ *and* $(\mathfrak{J}_a)_{a\in A}$ *is a sufficient family of ideals in* $\mathfrak{C}/\mathfrak{K}$.

The basic technical result is

Proposition 1.5. *(a) Let* $(\mathfrak{K}_a)_{a\in A}$ *a* $\mathfrak{K}$*-sufficient family of ideals of the C*-algebra* $\mathfrak{C}$. *Then there is a canonical monomorphism*

$$\mathfrak{C}/\mathfrak{K} \hookrightarrow \prod_{a\in A} \mathfrak{C}/\mathfrak{K}_a.$$

(b) Let H *be an observable affiliated to* $\mathfrak{C}$. *For any* $a \in A$, *let us set* $H_a \equiv \pi_a[H]$ *the image of* H *by the canonical surjection* $\pi_a : \mathfrak{C} \to \mathfrak{C}/\mathfrak{K}_a$. *Then*

$$\boldsymbol{\sigma}^{\mathfrak{C}}_{\mathfrak{K}}(H) = \overline{\bigcup_{a\in A} \boldsymbol{\sigma}^{\mathfrak{C}/\mathfrak{K}_a}(H_a)}.$$

Proof. (a) The kernel of π_a is $\mathfrak{K}_a$. Thus, the kernel of $(\pi_a)_{a\in A}:\mathfrak{C}\rightarrow\prod_{a\in A}\mathfrak{C}/\mathfrak{K}_a$ is $\mathfrak{K}$.

(b) Follows from (a), from the invariance of the spectrum under monomorphisms and from the fact that the spectrum of an observable affiliated to a direct product is the closure of the union of the spectra of its components. □

It is known that any ideal $\mathfrak{K}$ is the intersection of the family $\mathfrak{M}(\mathfrak{K})$ of maximal ideals containing it and obviously $\mathfrak{M}(\mathfrak{K})$ form a family of $\mathfrak{K}$-sufficient ideals. If an ideal $\mathfrak{K}_a$ is maximal, then the quotient $\mathfrak{C}/\mathfrak{K}_a$ is simple. Unfortunatelly, usually we do not understand the set of all (maximal) ideals of a C*-algebra, so we have to work only with those which are known explicitly. In the sequel we shall show how this can be done systematically, in a configurational framework, for a rather general class of C*-algebras.

[1.2] The abelian case; topological dynamical systems.

In the examples we have in mind, the relevant C*-algebra $\mathfrak{C}$ is generated by a translational invariant abelian C*-algebra $\mathcal{C}$ (related to the potentials) and a very simple C*-algebra to which the "free" Hamiltonians $h(P)$ are affiliated. As we shall see in [**1.4**], $\mathfrak{C}$ may be written as a crossed product $\mathcal{C}\rtimes X$ of the abelian part $\mathcal{C}$ by the group X. Algebraic operations as quotients and (restricted) direct products are distributive with respect to the crossed product, so it is useful to study first decomposition properties at the abelian level in the spirit of Definition 1.3. We will place ourselves in a context which is relevant for our purposes; most of the assertions have obvious more general counterparts.

Let us introduce first some notations. If $\mathfrak{B}$ is a C*-algebra and Y a locally compact space, the set of all continuous functions $\varphi:Y\rightarrow\mathfrak{B}$ which tend to 0 at infinity will be denoted by $C_0(Y;\mathfrak{B})$. It is a C*-algebra in a standard way. When $\mathfrak{B}=\mathbb{C}$, we write simply $C_0(Y)$. If Y is compact, we drop the index 0. $BC(Y)$ consists of continuous, bounded functions on Y. Those which are also uniformly continuous are elements of $BC_u(Y)$.

We describe now our framework. Let us denote by $(X,+)$ an abelian, locally compact group. We shall consider on X continuous potentials belonging to a C*-algebra $\mathcal{C}$. It will be necessary to make the following (rather natural) assumptions on $\mathcal{C}$: (a) $\mathcal{C}$ is a unital C*-subalgebra of $BC_u(X)$ containing $C_0(X)$, (b) $\mathcal{C}$ is stable under translations, i.e. if $\varphi(\cdot)$ is in $\mathcal{C}$, then $\varphi(\cdot+y)$ is also in $\mathcal{C}$ for all $y\in X$. By standard Gelfand theory, there is a topological dynamical system $\mathcal{X}:=(\hat{X},\alpha,X)$ which is canonically associated to $\mathcal{C}$. The space $\hat{X}$ is the spectrum of $\mathcal{C}$ and it is a *compactification* of X, i.e. a compact space in which X is dense. The elements of $\mathcal{C}$ are exactly the continuous functions on X which can be extended continuously on $\hat{X}$, so $\mathcal{C}$ and $C(\hat{X})$ are naturally isomorphic. The application $\alpha:X\times\hat{X}\rightarrow\hat{X}$ is continuous, extends the group law $X\times X\mapsto X$ and satisfies $\alpha_{x+y}=\alpha_x\circ\alpha_y$ for all $x,y\in X$; we set $\alpha_x(\hat{x}):=\alpha(x,\hat{x})$. One says that α is an action of X by homeomorphisms of $\hat{X}$ which extends the action of X on itself by translations.

It will be seen in [**1.4**] that the quotient $\mathfrak{C}/K(\mathcal{H})$ is easy to express if we have informations on the abelian quotient $\mathcal{C}/C_0(X)$. One identifies $\mathcal{C}$ with $C(\hat{X})$ as above and $C_0(X)$ with the ideal of the functions in $C(\hat{X})$ which are null on the boundary $Z:=\hat{X}\setminus X$. It follows that $\mathcal{C}/C_0(X)$ is canonically isomorphic to $C(Z)$.

In [**1.3**] we shall apply Proposition 1.5. to the case $\mathfrak{C} = \mathcal{C}$ and $\mathfrak{K} = C_0(X)$. By Lemma 1.4. and the discussion above, we are reduced to $\mathfrak{C} = C(Z)$ and $\mathfrak{K} = \{0\}$. Technical reasons connected with the properties of crossed products obliges us to look for sufficient families composed of *invariant* ideals in $C(Z)$. Before doing this, let us recall some general facts about topological dynamical systems.

So, let $\mathcal{Y} := (Y, \alpha, X)$ a topological dynamical system. We shall always assume that X is an abelian, locally compact group and that Y is a locally compact space. By a result of Ellis ([5]), the joint continuity of α follows from its separate continuity. We shall interpret the map $x \mapsto \alpha_x$ as an embedding of the group X into the group of all the homeomorphisms of Y.

Let us denote by $\mathfrak{Y}$ the topology of Y and by $\mathfrak{Y}_\alpha$ the family of all invariant sets belonging to $\mathfrak{Y}$. It is easy to show that $\mathfrak{Y}_\alpha$ is also a topology on Y, strictly coarser than $\mathfrak{Y}$ as soon as α is not trivial. We shall talk about *α-open* and *α-closed* sets.

Many familiar notions of topological dynamics may be reformulated in terms of this new topology. A singleton $\{y\}$ is α-closed iff y is a fixed point. The closure of an orbit is the α-closure of a point; this is called usually *a quasi-orbit.* A dynamical (sub)system for which the space is a quasi-orbit is called *topologically transitive.* An α-closed set is saturated with quasi-orbits. Remark that the set of all the quasi-orbits is *not* a partition of Y; two quasi-orbits may overlap or may even be contained strictly one into the other. We say that a quasi-orbit is *maximal* if it is not strictly contained into some other quasi-orbit. In terms of $\mathfrak{Y}_\alpha$, it is the closure of a point which is not contained strictly in the closure of a point. A closed, invariant subset Y_0 is *minimal* iff every point in it is α-dense. Closed, minimal invariant subsets are quasi-orbits which do not contain strictly some other quasi-orbit. If Y itself is minimal, we say that the dynamical system $\mathcal{Y}$ is minimal; of course, it will be topologically transitive. $\mathcal{Y}$ is minimal iff $\mathfrak{Y}_\alpha = \{\emptyset, Y\}$.

[1.3] Invariant decompositions.

Definition 1.6. *An* invariant decomposition *of the topological dynamical system $\mathcal{Y} := (Y, \alpha, X)$ is an approximate economical covering of Y by closed, invariant sets, i.e. a family $\mathcal{D} := (Y_a)_{a \in A}$ with $Y \setminus Y_a \in \mathfrak{Y}_\alpha$, such that*

(i) $\cup_{a \in A} Y_a$ is dense in Y,

(ii) If $a \neq b$ then Y_a is not a subset of Y_b.

The set of all invariant decompositions of $\mathcal{Y}$ will be denoted by $\mathbb{D}(\mathcal{Y})$.

If $\tilde{\mathcal{D}}$ satisfies (i), by dropping the sets Y_a which are properly contained in some other sets of $\tilde{\mathcal{D}}$, we get a true invariant decomposition. From now on we shall assume that Y is compact.

Proposition 1.7. *(a) There is a one-to-one correspondence between (i) invariant decompositions of $\mathcal{Y}$, (ii) sufficient families of invariant ideals of $C(Y)$.*

(b) Let $\mathcal{D} \equiv (Y_a)_{a \in A} \in \mathbb{D}(\mathcal{Y})$. Then there exist a monomorphism $i_{\mathcal{D}} : C(Y) \to \prod_{a \in A} C(Y_a)$.

(c) Elements $(\varphi_a)_{a \in A}$ of the image of $i_{\mathcal{D}}$ satisfy $\lim_{x \to 0} \sup_a \parallel \alpha_{a,x}[\varphi_a] - \varphi_a \parallel = 0$. We denetod by $\alpha_{a,x}$ the restriction of α_x to Y_a.

Proof. (a) follows from the well-known correspondence between ideals in C(Y) and closed sets in Y. Specifically, if Y_0 is a closed subset in Y, $C^{Y_0}(X) := \{\varphi \in C(Y) \mid$

$\varphi|_{Y_0} = 0\}$ is an ideal in $C(Y)$ and there are no others. If $Y_0 \subsetneqq Y_0'$, then $C^{Y_0'}(X) \subsetneqq C^{Y_0}(X)$, by Urysohn's lemma. We have $\cap_a C^{Y_a}(Y) = C^{\overline{\cup_a Y_a}}(Y)$; thus $\cap_a C^{Y_a}(Y) = \{0\}$ iff $\cup_a Y_a$ is dense in Y. It is easy to see that Y_0 is invariant iff $C^{Y_0}(Y)$ is invariant.

(b) We just set $i_{\mathcal{D}}(\varphi) := (\varphi|_{Y_a})_{a \in A}$. Obviously, it is an injective *-morphism from $C(Y)$ to $\prod_{a \in A} C(Y_a)$. Alternatively, one may combine (a) with Proposition 1.5.(a) by setting $\mathfrak{K}_a = C^{Y_a}(X)$ and $\mathfrak{K} = \{0\}$. The equivalence of the two approaches is clear, since $C(Y)/C^{Y_a}(Y)$ is isomorphic with $C(Y_a)$ and the canonical surjection $C(Y) \mapsto C(Y)/C^{Y_a}(Y)$ becomes the operation of restriction to Y_a.

(c) We have

$$\sup_a \| \alpha_{a,x}[\varphi_a] - \varphi_a \| = \sup_a \| \alpha_{a,x}[\varphi|_{Y_a}] - \varphi|_{Y_a} \| = \| \alpha_x[\varphi] - \varphi \| \xrightarrow[x \to 0]{} 0.$$

and this finishes the proof. □

The monomorphism $i_{\mathcal{D}}$ is rarely surjective even if A is finite.

It is often useful to apply Proposition 1.7. for $\mathcal{D}_{\max} = (Y_a)_{a \in A}$, the most refined invariant decomposition, for which every component Y_a is a (maximal) quasi-orbit, i.e. $\mathcal{Y}_a := (Y_a, \alpha, X)$ is topologically transitive. Recall that we are mainly interested in the case when Y is the boundary $Z = \hat{X} \setminus X$ of X into some compactification $\hat{X}$ which is the spectrum of the C*-algebra of potentials $\mathcal{C}$. Then we may say that *we expressed the C*-algebra $\mathcal{C}/C_0(X)$ by means of an invariant decomposition of the dynamical system "at infinity" $\mathcal{Z} = (Z, \alpha, X)$ into topologically transitive components.* Note that when $\mathcal{Y}$ itself is topologically transitive, no decomposition is possible. This shows clearly why going from $\mathcal{C}$ to the quotient $\mathcal{C}/C_0(X)$ leads often to new opportunities. The initial dynamical system $\mathcal{X} = (\hat{X}, \alpha, X)$ is topologically transitive by definition, so, in order to have some chance to make some decomposition, one has first to eliminate the "large" (dense) orbit X. After performing the crossed product, similar assertions may be stated about the quotient $\mathfrak{C}/K(\mathcal{H})$.

[1.4] Crossed products.

Since it plays only the role of an intermediary tool, we decided to adopt a very pragmatic point of view upon the crossed product of a C*-algebra by a group. In fact we do not define it, but just give a convenient description and enumerate some of its properties. We shall take profit of all the specific facilities of our framework (abelian objects, Fourier transform, etc). For the general theory we send to [9]. A brief survey of the facts that are relevant for spectral analysis may be found in [6], Section 3; this includes the precise definition of the crossed product and the proofs of most of the assertions we shall make in the sequel. We hope that the present approach will be convenient for the untrained reader and not too loose for the expert.

We shall call *abelian C*-dynamical system* a triplet $(\mathcal{A}, \alpha, X)$, where $\mathcal{A}$ is an abelian C*-algebra, X is an abelian, locally compact group and α is a continuous action of X by *-automorphisms of $\mathcal{A}$. By Gelfand theory, there is a bijective correspondence between abelian C*-dynamical systems and topological dynamical systems as in [**1.2**]. As a rule, we shall denote by the same letter the action of X on the C*-algebra $\mathcal{A}$ and on its spectrum. We endow the dual $X^\sharp$ (the space of all continuous morphisms : $X \to \mathbb{T} := \{z \in \mathbb{C} \mid \mid z \mid = 1\})$ with the standard structure; it becomes an abelian, locally compact group. $X^\sharp$ is compact iff X is discrete. An important role will be played by the C*-algebra $C_0(X^\sharp)$.

(A) Let us fix an abelian C*-dynamical system $(\mathcal{A}, \alpha, X)$. We denote by $\mathcal{A} \odot C_0(X^\sharp)$ the algebraic tensor product of $\mathcal{A}$ and $C_0(X^\sharp)$. It consists of finite linear combinations of elements of the form $\varphi \otimes \psi$, with $\varphi \in \mathcal{A}$ and $\psi \in C_0(X^\sharp)$. *The crossed product of* $\mathcal{A}$ *by* X is a C*-algebra denoted by $\mathcal{A} \rtimes_\alpha X \equiv \mathcal{A} \rtimes X$, in which $\mathcal{A} \odot C_0(X^\sharp)$ is a dense subspace. The precise algebraic and topological structures on $\mathcal{C} \rtimes X$ will not be given here; but note that they depend on the action α of X on $\mathcal{A}$. One has $\mathbb{C} \rtimes X = C_0(X^\sharp)$. In fact, our $\mathcal{A} \rtimes X$ is a Fourier transformed version of the one which appears usually in the literature.

(B) Let us consider a second abelian C*-dynamical system $(\mathcal{A}', \alpha', X)$ and a *-morphism $\mu : \mathcal{A} \to \mathcal{A}'$ which is X-covariant (i.e. $\alpha'_x \circ \mu = \mu \circ \alpha_x$ for all $x \in X$). Then there is a unique *-morphism $\mu_\rtimes : \mathcal{A} \rtimes X \to \mathcal{A}' \rtimes X$ such that $\mu_\rtimes[\varphi \otimes \psi] = \mu[\varphi] \otimes \psi$. Let us denote by $\mathcal{J}$ the kernel of μ; it is invariant under the action α. Then $\mathcal{J} \rtimes X$ may be identified to an ideal of $\mathcal{A} \rtimes X$ which coincides with the kernel of $\mu_\rtimes$. One has a canonical isomorphism $[\mathcal{A} \rtimes X]/[\mathcal{J} \rtimes X] \cong [\mathcal{A}/\mathcal{J}] \rtimes X$ (on $\mathcal{A}/\mathcal{J}$ there is a natural action of X) which sends the class modulo $\mathcal{J} \rtimes X$ of $\varphi \otimes \psi$ into $(\varphi + \mathcal{J}) \otimes \psi$.

(C) We are given now a family $(\mathcal{A}_a, \alpha_a, X)_{a \in A}$ of abelian C*-dynamical systems. We can consider the natural product action of X in $\prod_{a \in A} \mathcal{A}_a$. Unfortunately, it is not always continuous. But we can form the largest C*-subalgebra $\prod'_{a \in A} \mathcal{A}_a$ of $\prod_{a \in A} \mathcal{A}_a$ on which continuity is restored:

$$\prod_{a \in A}{}' \mathcal{A}_a := \left\{ (\varphi_a)_{a \in A} \in \prod_{a \in A} \mathcal{A}_a \;\middle|\; \lim_{x \to 0} \sup_a \parallel \alpha_{a,x}[\varphi_a] - \varphi_a \parallel = 0 \right\}.$$

Then there exists a monomorphism $\left(\prod'_{a \in A} \mathcal{A}_a\right) \rtimes X \mapsto \prod_{a \in A}(\mathcal{A}_a \rtimes X)$ sending $(\varphi_a)_{a \in A} \otimes \psi$ into $(\varphi_a \otimes \psi)_{a \in A}$.

We remark that the image of $i_\mathcal{D}$ in Proposition 1.7. is in fact comprised in $\prod'_{a \in A} C(Y_a)$.

(D) Let $\mathcal{A}$ be a C*-subalgebra of $BC_u(X)$, stable under translations. We work in the Hilbert space $\mathcal{H} := L^2(X)$ (with a Haar measure) and identify $\mathcal{A}$ with its image in $B(\mathcal{H})$ formed by multiplication operators. In **[1.5]** operators $\psi(P)$ in $\mathcal{H}$ are assigned bijectively to a large class of functions ψ defined on the dual group $X^\sharp$. In particular, we get a canonical identification of $C_0(X^\sharp)$ with a C*-subalgebra of $B(\mathcal{H})$, denoted in the same way. One may form $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$, the closure of the vector subspace generated by $\mathcal{A}$ and $C_0(X^\sharp)$. It can be shown that it is a C*-subalgebra and that it coincides with $\langle C_0(X^\sharp) \cdot \mathcal{A} \rangle$ (obvious notation). The really useful fact is that $\mathcal{A} \rtimes X$ and $\langle \mathcal{A} \cdot C_0(X) \rangle$ are isomorphic, the element $\varphi \otimes \psi$ corresponding to $\varphi \cdot \psi(P)$.

(E) With the notations of **(D)**, one has $\langle C_0(X) \cdot C_0(X^\sharp) \rangle = K(\mathcal{H})$. Thus the crossed product $C_0(X) \rtimes X$ can be identified with the ideal of all compact operators in $L^2(X)$.

[1.5] Generalized Schrödinger operators and their essential spectrum.

In this paragraph we are finally in a position to define generalized Schrödinger operators and give a result on their essential spectrum. We shall fix the following

Framework. *(i) X will be an abelian, locally compact group, with dual $X^\sharp$.*

(ii) $\mathcal{C}$ will be an unital C-subalgebra of $BC_u(X)$ which is invariant under the action α of X by left translations.*

(iii) We denote by $\hat{X}$ the compact spectrum of $\mathcal{C}$ and by Z the boundary $\hat{X} \setminus X$. The extension to $\hat{X}$ of the action of X, as well as all its possible restrictions will also be denoted by α.

(iv) $h : X^\sharp \to \mathbb{R}$ will be a continuous function, satisfying $\lim_{p\to\infty}|h(p)| = \infty$.

We give a sense to the free operator $H_0 := h(P)$ in $\mathcal{H} := L^2(X)$ by using the unitary Fourier transform $\mathcal{F} : L^2(X) \to L^2(X^\sharp)$. In $L^2(X^\sharp)$ we consider the maximal operator M_h of multiplication by h and we set $h(P) := \mathcal{F}^{-1}M_h\mathcal{F}$. Writting H_0 as a function of P (ment to be the momentum) is just a reminiscence of the situations in which this can be soundly justified, $X = \mathbb{R}^n$ for instance.

We shall call *generalized Schrödinger operator* the self-adjoint operator $H := H_0 + V$, where V is the operator of multiplication with the real-valued function (also denoted by) V, which we choose in $\mathcal{C}$. This function will be fixed from now on and its continuous extension to $\hat{X}$ will be denoted by $\hat{V}$.

The point (b) of the next result is our main abstract contribution.

Theorem 1.8. *(a) The self-adjoint operator H is affiliated to the C*-subalgebra $\mathfrak{C} := \langle \mathcal{C} \cdot C_0(X^\sharp) \rangle$ of $B(\mathcal{H})$.*

(b) Let $\mathcal{D} = (Z_a)_{a\in A}$ be an invariant decomposition of the dynamical system $\mathcal{Z} = (Z, \alpha, X)$. For any $a \in A$, let $\rho_a : C(Z_a) \to BC_u(X)$ be a monomorphism. We set $V_a := \rho_a \left[\hat{V}|_{Z_a}\right]$ and $H_a := H_0 + V_a$. Then

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \overline{\bigcup_{a\in A} \boldsymbol{\sigma}[H_a]}. \tag{1-1}$$

Remark. We assumed that the abelian C*-algebras $C(Z_a)$ are faithfully represented in $\mathcal{H}$ by multiplication operators because this really happens in the examples we have in mind. If this is not the case, there is a less concrete version of Theorem 1.8., in which the asymptotic Hamiltonian H_a is defined as observables affiliated to the abstract C*-algebra $C(Z_a) \rtimes X$.

Remark. Much larger classes of Hamiltonians may be treated by using more sofisticated affiliation criteria, as those in [1] or [3].

Proof of Theorem 1.8. (a) It is proven in [1] that H is affiliated to $\mathfrak{C}$ if $(H-\zeta)^{-1}$ is in $\mathfrak{C}$ for some number $\zeta \notin \mathbb{R}$. We choose ζ such that the Neumann series

$$(H-\zeta)^{-1} = \sum_{m\in\mathbb{N}} (-1)^m (H_0-\zeta)^{-1} \left[V(H_0-\zeta)^{-1}\right]^m \tag{1-2}$$

converges. We have to prove that each term in (1-2) is in $\mathfrak{C}$. By taking into account the definition of $\mathfrak{C}$ and the fact that V is in $\mathcal{C}$, we see that it is enough to show that $(H_0-\zeta)^{-1} \in C_0(X^\sharp) \subset B(\mathcal{H})$. The function $\psi(\cdot) := [h(\cdot)-\zeta]^{-1}$ belongs to $C_0(X^\sharp)$, since h is continuous and diverges at infinity on $X^\sharp$. Thus we finish the proof of (a) by remarking that $(H_0-\zeta)^{-1} = \psi(P)$.

(b) Let us denote by $\cong$ a canonical isomorphism and by $\hookrightarrow$ a monomorphism of C*-algebras. By applying successively **(E)**, **(D)**, **(B)**, Proposition 1.7.(b), **(C)** and **(D)** again, we have:

$$\begin{aligned}\mathfrak{C}/K(\mathcal{H}) &= \left\langle \mathcal{C}\cdot C_0(X^\sharp)\right\rangle / \left\langle C_0(X)\cdot C_0(X^\sharp)\right\rangle \cong [\mathcal{C}\rtimes X]/[C_0(X)\rtimes X]\cong\\ &\cong [\mathcal{C}/C_0(X)]\rtimes X\hookrightarrow \left[\prod_{a\in A}{}' C(Z_a)\right]\rtimes X\hookrightarrow\\ &\hookrightarrow \prod_{a\in A}[C(Z_a)\rtimes X]\cong \prod_{a\in A}\left\langle \rho_a[C(Z_a)]\cdot C_0(X^\sharp)\right\rangle .\end{aligned}$$

We shall denote by $I_{\mathcal{D}}$ the resulting monomorphism. By keeping track of the explicit form of all the *-morphisms we see that $I_{\mathcal{D}}$ transforms the class modulo $K(\mathcal{H})$ of $V\psi(P)$ into $\left(\rho_a\left[\hat{V}|_{Z_a}\right]\psi(P)\right)_{a\in A}$, which is equal to $\left(V_a(H_0-\zeta)^{-1}\right)_{a\in A}$.

Let us denote by $\pi:\mathfrak{C}\to\mathfrak{C}/K(\mathcal{H})$ the canonical surjection. Then $\pi(H)$ is the image of H in the Calkin algebra. Using (1-2), we see that $I_{\mathcal{D}}[\pi(H)]=(H_a)_{a\in A}$, in the sense of Definition 1.2., where $H_a=H_0+V_a$ is affiliated to $\mathfrak{C}_a:=\left\langle\rho_a[C(Z_a)]\cdot C_0(X^\sharp)\right\rangle$.

Now we need only to observe that the spectrum of $\pi(H)$ is the essential spectrum of H and that

$$\boldsymbol{\sigma}^{\prod_a\mathfrak{C}_a}[(H_a)_{a\in A}]=\overline{\bigcup_{a\in A}\boldsymbol{\sigma}^{\mathfrak{C}_a}(H_a)}.$$

This finishes the proof. □

Remark. Let us denote by $C^{Z_a}(\hat{X})$ the invariant ideal consisting of the elements in $C(\hat{X})$ which are null on Z_a. Another way to express the core of the proof of the point (b) is to say that in the C*-algebra $\left\langle C(\hat{X})\cdot C_0(X^\sharp)\right\rangle$, the family $\left(\left\langle C^{Z_a}(\hat{X})\cdot C_0(X^\sharp)\right\rangle\right)_{a\in A}$ is a $\langle C_0(X)\cdot C_0(X^\sharp)\rangle$-sufficient family of ideals in the sense of Definition 1.3. Then one applies Proposition 1.5.(b); but, of course, the explicit form of the observables $H_a=\pi_a[H]$ has to be calculated.

2. Potentials of class $\langle VO(X)\cdot AP(X)\rangle$

We shall introduce here two important C*-algebras: (a) $VO(X)$ consisting of continuous functions : $X\to\mathbb{C}$ having oscillation which is small at infinity and (b) $AP(X)$, the C*-algebra of continuous, almost periodic functions on X. The spectrum of $VO(X)$ is the largest compactification $\tilde{X}$ of X such that the action of translations in $R:=\tilde{X}\setminus X$ is trivial. The spectrum of $AP(X)$ is the Bohr group T. The abelian C*-algebra generated by $VO(X)$ and $AP(X)$ is a very nice one; its spectrum is the union between X and the cartesian product of the two compact spaces R and T. The finest invariant decomposition of $Z:=R\times T$ is $(T)_{r\in R}$. By applying Theorem 1.8. one gets results on the essential spectra of Schrödinger operators with potentials in the C*-algebra generated by $VO(X)$ and $AP(X)$. In this case, the asymptotic Hamiltonians are all almost periodic.

[2.1] The C*-algebra VO(X).

An abelian, locally compact, non-compact group X will be fixed from now on. We denote by α the action of X on itself by translations $(\alpha_y(x) = x + y)$ and also the induced action on functions $(\alpha_y(\varphi) = \varphi \circ \alpha_{-y} = \varphi(\cdot - y))$.

Definition 2.1. *We say that a function* $\varphi \in BC_u(X)$ has asymptotic vanishing oscillation *and we write* $\varphi \in VO(X)$ *if for any* $y \in X$, $\alpha_y(\varphi) - \varphi \in C_0(X)$.

Obviously, $VO(X)$ is a unital C*-algebra placed between $C_0(X)$ and $BC_u(X)$, which is stable by translations. To get a link with a more traditional definition, let us set for any function $\psi : X \to \mathbb{C}$ and any $K \subset X$ compact neighbourhood of the origin $\mathrm{osc}_K(\psi) : X \to \mathbb{R}_+$, $\mathrm{osc}_K(\psi)(x) := \sup_{y \in K} \mid \psi(x - y) - \psi(x) \mid$. Then it is easy to see that osc_K is a well-defined map : $BC_u(X) \to BC_u(X)$ and that a function $\varphi \in BC_u(X)$ is in $VO(X)$ iff $\mathrm{osc}_K(\varphi) \in C_0(X)$. This shows in particular that if one defines $VO(X)$ by using osc_K, the choice of K is irrelevant.

If $X = \mathbb{R}^n$ or $X = \mathbb{Z}^n$, $VO(X)$ contains the C*-algebra $C^{\mathrm{rad}}(X)$ of all continuous functions admitting radial limits at infinity, i.e. those which can be extended continuously on the radial compactification $\check{X}$ of X (obtained by adding a sphere at infinity). Anisotropic Schrödinger operators with potentials in $\left\langle C^{\mathrm{rad}}(X) \cdot AP(X) \right\rangle$ were treated in [8] by a somewhat different technique, which cannot be extended to $\langle VO(X) \cdot AP(X) \rangle$. In fact $VO(X)$ is much larger than $C^{\mathrm{rad}}(X)$, so Theorem 2.5. will extend considerably the result in [8]. If $X = \mathbb{R}^n$ for example, all the bounded C^1-functions φ such that $\nabla\varphi$ is of class C_0 are in $VO(X)$ and they need not have radial limits. A simple typical example is $\varphi(x) := f\left(|x|^p\right)$ for $p \in [-\infty, 1)$ and f a C^1 periodic function of one variable. In addition, the fact that $VO(X)$ makes sense for any abelian, locally compact group X may be considered very satisfying.

Let us denote by $\tilde{X} = X \sqcup R$ the spectrum of $VO(X)$.

Lemma 2.2. *(a) The extension to* $\tilde{X}$ *of the action of* X *(also denoted by* α*) is trivial on* R*: all the elements of* R *are fixed points.*

(b) $VO(X)$ *is the largest* X*-invariant C*-subalgebra of* $BC_u(X)$ *satisfying (a).*

Proof. (a) Let us consider an element r of R, i.e. a non-null morphism $r : VO(X) \to \mathbb{C}$ whose restriction to $C_0(X)$ is null. For any $y \in X$ and $\varphi \in VO(X)$ we have $[\alpha_y(r) - r][\varphi] = r[\varphi(\cdot + y) - \varphi] = 0$, by the definition of $VO(X)$.

(b) follows by a slight rearrangement of the proof of the point (a).

To apply Theorem 1.8. to generalized Schrödinger operators with potentials of class $VO(X)$, we begin by noting that for any dense subset R_0 of R, $\mathcal{D} := (\{r\})_{r \in R_0}$ is an invariant decomposition of the dynamical system $\mathcal{R} := (R, \alpha, X)$. The C*-algebra $C(\{r\})$ will be, of course, identified to $\mathbb{C}$, interpreted as a C*-subalgebra either of $BC_u(X)$ or of $B(\mathcal{H})$, $\mathcal{H} = L^2(X)$. Thus we have the structure result

$$\left\langle VO(X) \cdot C_0(X^\sharp) \right\rangle / K(\mathcal{H}) \hookrightarrow \prod_{r \in R_0} C_0(X^\sharp).$$

For $V \in VO(X)$, the image $\tilde{V}(R)$ of R through the extension $\tilde{V}$ of V to $\tilde{X} := X \sqcup R$ is just *the asymptotic range* of V, i.e. the set $V(X)_{\mathrm{asy}} := \cap_{K \in \kappa(X)} \overline{V(X \setminus K)}$, where $\kappa(X)$ is the set of all compact neighbourhoods of the origin in X. Then Theorem 1.8. gives:

Corollary 2.3. *Let $h : X^\sharp \to \mathbb{R}$ a continuous function diverging at infinity by positive values. Let $V \in VO(X)$. Then the self-adjoint operator acting in $\mathcal{H}$ $H := h(P) + V$ is affiliated to the C*-algebra $\langle VO(X) \cdot C_0(X^\sharp)\rangle$ and for any dense subset R_0 of R we have*

$$\boldsymbol{\sigma}_{\rm ess}\,[h(P)+V] = \overline{\bigcup_{r\in R_0} \boldsymbol{\sigma}[h(P)+\tilde{V}(r)]} = [\,\min h + \min V(X)_{\rm asy}, \infty)\,. \tag{2-1}$$

It seems that this simple result does not appear in the literature. It will be a rather trivial particular case of our Theorem 2.5. We put it into evidence here because it has a feature that is specific to $VO(X)$: the essential spectrum is determined by *the values* taken by the potential at infinity. Assume for simplicity that $X = \mathbb{R}^n$ and $h(p) = |p|^2$. The bottom of the essential spectrum of $|P|^2 + V$ coincides with the least value taken by V at infinity. $VO(X)$ is the largest C*-algebra of potentials for which such kind of result is true "non-accidentally".

[2.2] The C*-algebra of almost periodic functions and the Bohr group.

Definition 2.4. *A bounded, continuous function $\psi : X \to \mathbb{C}$ satisfying one of the next equivalent conditions is called* almost periodic*:*

(a) The set $\{\psi(\cdot + y) \mid y \in X\}$ is relatively compact in $BC(X)$.

(b) For any $\varepsilon > 0$, there exists a trigonometric polynomial p on X such that $\| \psi - p \| < \varepsilon$.

We recall that a trigonometric polynomial is just a finite linear combination of characters of X. The equivalence of (a) and (b) is a particular case of [4], 16.2.1. The set of all almost periodic functions on X is a translational invariant C*-subalgebra of $BC_u(X)$, denoted $AP(X)$. All the continuous functions : $X \to \mathbb{C}$ which are periodic with respect to some closed subgroup Γ of X with compact quotient X/Γ lie in $AP(X)$, but there are many others.

The spectrum of $AP(X)$ is a compact group T, often called the Bohr group; hence AP(X) is canonically isomorphic with C(T). We recall briefly the construction of T; for further details we send to [4] and [7].

On the dual $X^\sharp$ of X we consider the discrete topology. One gets a discrete, abelian group, denoted by $X^\sharp_d$. The dual group $T := (X^\sharp_d)^\sharp$ with the usual (dual) topology is then a compact group. By standard duality arguments one can identify X with a dense subgroup of T. More precisely, there exists a continuous morphism τ from X to T with dense image. We avoid to call T the Bohr compactification of X (as it is done occasionally), since the topology induced by T on X (called *the Bohr topology*) is usually much weaker than the initial one (i.e. the morphism $\tau : X \to T$ is not open). In particular, if X is not compact, no neighbourhood of the origin in the Bohr topology is relatively compact with respect to the initial topology on X.

So X will be identified with a dense subgroup of T. We denote by γ the action of X on T obtained by restricting to $X \times T$ the group law of T; it extends the action α of X on itself. It can be shown that (T, γ, X) is a minimal dynamical system, i.e. the orbit $\{\gamma_x(t) \mid x \in X\}$ is dense in T for all $t \in T$. In fact, in a certain sense, it is the largest minimal dynamical system with group X for which the space is also a group; see [7].

[2.3] Schrödinger operators with potentials os class $\langle VO(X)\cdot AP(X)\rangle$.

Let us consider now Schrödinger operators with potentials in the C*-subalgebra $\mathcal{A}$ of $BC_u(X)$ generated by $VO(X)$ and $AP(X)$. We shall need some notations. Let us fix an element V of $\mathcal{A}$. It may be writen as $V=\lim_{n\to\infty}\sum_{j=1}^{m_n}U_j^nW_j^n$, with $m_n\in\mathbb{N}$, $U_j^n\in VO(X)$ and $W_j^n\in AP(X)$. For any $r\in R$ and $x\in X$ we set

$$V_r(x):=\lim_{n\to\infty}\sum_{j=1}^{m_n}\tilde{U}_j^n(r)W_j^n(x), \tag{2-2}$$

where $\tilde{U}_j^n$ is the continuous extension of U_j^n to $\tilde{X}=X\sqcup R$. It will become clear in the proof of Theorem 2.5. that the almost periodic function V_r does not depend on the representation choosen for V.

Theorem 2.5. *Let $h:X^\sharp\to\mathbb{R}$ continuous, such that $\lim_{p\to\infty}h(p)=\infty$. Let $V\in\mathcal{A}$.*

(a) The self-adjoint operator $H:=h(P)+V$ is affiliated to the C-subalgebra $\big\langle\mathcal{A}\cdot C_0(X^\sharp)\big\rangle$ of $B(\mathcal{H})$, $\mathcal{H}=L^2(X)$.*

(b) We denote by H_r the almost periodic operator $h(P)+V_r$. Then

$$\boldsymbol{\sigma}_{\rm ess}(H)=\overline{\bigcup_{r\in R_0}\boldsymbol{\sigma}(H_r)}$$

for any dense subset R_0 of R.

Example. Let us consider the very simple example $V=UW$, where U has vanishing oscillation at infinity and W is almost periodic. By using the asymptotic range of U (introduced in [**2.1**]) one may write

$$\boldsymbol{\sigma}_{\rm ess}\left[h(P)+V\right]=\overline{\bigcup_{\mu\in U(X)_{\rm asy}}\boldsymbol{\sigma}\left[h(P)+\mu W\right]}.$$

The values taken by U at infinity serve as "coupling constants" for the asymptotic Hamiltonians assigned to H.

Proof of Theorem 2.5. $\mathcal{A}$ is indeed a X-invariant, unital C*-subalgebra of $BC_u(\mathcal{H})$. (a) follows directly from the point (a) of Theorem 1.8. We shall verify that Theorem 1.8.(b) implies our point (b).

Let us denote by $\hat{X}=X\sqcup Z$ the spectrum of $\mathcal{A}$; it is a compactification of X and the space of a dynamical system with group X. It will be useful to know that the compact space Z is naturally homeomorphic to the cartesian product $R\times T$. This is by no means a general result. It expresses the fact that $VO(X)$ and $AP(X)$ are "asymptotically independent", a concept introduced and studied by Power in a series of articles even for non-commuting C*-algebras. The natural candidate for the homeomorphism is $\theta:Z\to R\times T$ given by $\theta(z):=\big(z|_{VO(X)},z|_{AP(X)}\big)$. It is clear that it is a well-defined continuous, open injection. The non-trivial part of the proof is to check surjectivity. But this can be done by performing obvious modifications in the proof of Lemma 1. from [10].

It is clear that θ is even an isomorphism of X-dynamical systems. On $R \times T$ we consider the invariant decomposition $\mathcal{D}' := (\{r\} \times T)_{r \in R_0}$; we transport it through θ^{-1} and obtain an invariant decomposition $\mathcal{D}$ of Z consisting of minimal components Z_r, $r \in R_0$. The C*-algebras $C(Z_r)$ and $C(\{r\} \times T)$ are isomorphic. By composing with the natural identification between $C(\{r\} \times T)$ and $AP(X)$, we get an isomorphism $\rho_r : C(Z_r) \to AP(X)$. It is obvious that $\rho_r[\hat{V}|_{Z_r}]$ (notation as in Theorem 1.8.) coincides with V_r given by (2-2); this also shows the independence of V_r of the representation chosen for V.

Now the point (b) of Theorem 1.8. can be applied straightforwardly. □

Remark. The basic structure result which allows deducing Theorem 2.5. from Theorem 1.8. is the fact that the C*-algebra $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$ can be embedded into $\prod_{r \in R_0} \langle AP(X) \cdot C_0(X^\sharp) \rangle$. In turn, this result follows from properties of the crossed product, from the identification of the spectrum of $\mathcal{A}/C_0(X)$ with $R \times T$ and from the way the group X acts on R and T (trivially on R and minimally on T). The points of the space $V(X)_{\text{asy}}$ *do not* serve to define asymptotic Hamiltonians for H.

3. Cartesian products

We shall study here products of dynamical systems. For notational convenience, only two factors will be considered; the case of an arbitrary, finite number of factors is essentially the same.

So, let $\mathcal{X}^j := (\hat{X}^j, \alpha^j, X^j)$, $j = 1, 2$ be two topological dynamical systems, where $\hat{X}^j$ is some compactification of the locally compact group X^j and α^j is an extension of the usual action of X^j on itself by translations. We set $X := X^1 \times X^2$ (direct product of topological groups), $\hat{X} := \hat{X}^1 \times \hat{X}^2$ (cartesian product of compact spaces) and $\alpha := \alpha^1 \times \alpha^2$ (the product action). We will be concerned with the product dynamical system $\mathcal{X} := (\hat{X}, \alpha, X)$, so the abelian C*-algebra of potentials will be a tensor product. The boundary $Z := (\hat{X}^1 \times \hat{X}^2) \setminus (X^1 \times X^2)$ admits the obvious partition $Z = (X^1 \times Z^2) \sqcup (Z^1 \times X^2) \sqcup (Z^1 \times Z^2)$, where $Z^j := \hat{X}^j \setminus X^j$. But note that among the three invariant components of the right-hand side only the last one is closed. This time, invariant decompositions are somewhat more complicated than in Section 2; they do not form partitions of Z. We shall study some of them and write corresponding formulae for the essential spectrum of the affiliated Hamiltonians in an interesting particular case.

[3.1] Cartesian invariant decompositions.

We are interested only in certain invariant decompositions of $\mathcal{Z} := (Z, \alpha, X)$, whose elements are cartesian products.

Definition 3.1. *The invariant decomposition $\mathcal{D} := (Z_a)_{a \in A} \in \mathbb{D}(\mathcal{Z})$ is called* cartesian *if there are $\mathcal{D}^j := (Z^j_{a^j})_{a^j \in A^j} \in \mathbb{D}(\mathcal{Z}^j)$, $j = 1, 2$ such that for any $a \in A$ either $Z_a = \hat{X}^1 \times Z^2_{a^2}$ for some $a^2 \in A^2$ or $Z_a = Z^1_{a^1} \times \hat{X}^2$ for some $a^1 \in A^1$. The set of all cartesian invariant decompositions of $\mathcal{Z}$ will be denoted by $\mathbb{D}_c(\mathcal{Z})$.*

It is obvious that there is a bijection between $\mathbb{D}_c(\mathcal{Z})$ and $\mathbb{D}(\mathcal{Z}^1) \times \mathbb{D}(\mathcal{Z}^2)$. Indeed, the most general cartesian invariant decomposition of $\mathcal{Z}$ has the form

$$\mathcal{D} := \left(\hat{X}^1 \times Z^2_{a^2} \ , \ Z^1_{a^1} \times \hat{X}^2 \right)_{(a^1, a^2) \in A^1 \times A^2} \tag{3-1}$$

for some $\mathcal{D}^j = (Z^j_{a^j})_{a^j \in A^j} \in \mathbb{D}(\mathcal{Z}^j)$, $j = 1, 2$.

So let us fix now $\mathcal{D}^1 \in \mathbb{D}(\mathcal{Z}^1)$ and $\mathcal{D}^2 \in \mathbb{D}(\mathcal{Z}^2)$ and denote by $\mathcal{D} \in \mathbb{D}_c(\mathcal{Z})$ the corresponding cartesian invariant decomposition of $\mathcal{Z}$, as in (3-1). A direct application of Proposition 1.7. gives

$$\text{(3-2)} \quad C(\hat{X})/C_0(X) \hookrightarrow \left(\prod_{a^2 \in A^2}{}' C(\hat{X}^1) \otimes C(Z^2_{a^2}) \right) \times \left(\prod_{a^1 \in A^1}{}' C(Z^1_{a^1}) \otimes C(\hat{X}^2) \right).$$

Let us set $\mathfrak{C} := C(\hat{X}) \rtimes X$, $\mathfrak{K} := C_0(X) \rtimes X$ and $\mathfrak{C}^j := C(\hat{X}^j) \rtimes X^j$; one has $\mathfrak{C} = \mathfrak{C}^1 \otimes \mathfrak{C}^2$. Then by using (3-2), [**1.4**] and the formula $(\mathcal{A}^1 \otimes \mathcal{A}^2) \rtimes (X^1 \times X^2) \cong (\mathcal{A}^1 \rtimes X^1) \otimes (\mathcal{A}^2 \rtimes X^2)$ (cf. [12]) one obtains easily the monomorphism

$$\text{(3-3)} \qquad \mathfrak{C}/\mathfrak{K} \hookrightarrow \left(\prod_{a^2 \in A^2} \mathfrak{C}^1 \otimes [C(Z^2_{a^2}) \rtimes X^2] \right) \times \left(\prod_{a^1 \in A^1} [C(Z^1_{a^1}) \rtimes X^1] \otimes \mathfrak{C}^2 \right).$$

This is supposed to be a progress, since $C(Z^j_{a^j}) \rtimes X^j$ is usually simpler than $\mathfrak{C}^j$. If one knows explicitly the most refined invariant decompositions $\mathcal{D}^j_{max} \in \mathbb{D}(\mathcal{Z}^j)$, $j = 1, 2$, a good idea is to use them in the relations above.

In fact, (3-3) is the main structure result. We shall not give the formula for the essential spectrum in this generality. The next paragraph presents in detail a relevant particular case.

[3.2] A simple case.

Many special instances of cartesian anisotropy may be studied. In particular, the two compactifications $\hat{X}^1$ and $\hat{X}^2$ may have "different natures". Let us restrict ourselves here to a simple, though not trivial case, for which the formula for the essential spectrum is quite explicit.

For $j = 1, 2$, let $\hat{X}^j := \tilde{X}^j$ be the spectrum of the C*-algebra $VO(X^j)$. Note that $\tilde{X}^1 \times \tilde{X}^2$ and $\widetilde{X^1 \times X^2}$ differs a lot. While the boundary R of $\widetilde{X^1 \times X^2}$ consists only of fixed points for the translations, in $Z := (\tilde{X}^1 \times \tilde{X}^2) \setminus X^1 \times X^2 = (X^1 \times R^2) \sqcup (R^1 \times X^2) \sqcup (R^1 \times R^2)$ only the points of $R^1 \times R^2$ are fixed. Another way to formulate this is to say that $VO(X^1) \otimes VO(X^2)$ and $VO(X^1 \times X^2)$ are quite different.

For each j, we choose the greatest invariant decomposition $\mathcal{D}^j_{\max} := (\{r^j\})_{r^j \in R^j}$ of $\mathcal{R}^j := \big(R^j, \alpha^j, X^j\big)$. Then the C*-algebras $C(Z^j_{a^j})$ in (3-3) coincide all with $\mathbb{C}$ and one gets immediately

$$\text{(3-4)} \qquad \mathfrak{C}/\mathfrak{K} \hookrightarrow \prod_{r^2 \in R^2} C\left[(X^2)^\sharp; \mathfrak{C}^1\right] \times \prod_{r^1 \in R^1} C\left[(X^1)^\sharp; \mathfrak{C}^2\right].$$

To apply Theorem 1.8. we need some preparations. We shall work in the Hilbert space $\mathcal{H} := L^2(X)$, which we identify with $\mathcal{H}^1 \otimes \mathcal{H}^2$, $\mathcal{H}^j := L^2(X^j)$. Let $h : X^\sharp \to \mathbb{R}$ be a continuous, proper function and V a real element of $VO(X^1) \otimes VO(X^2)$. Let us consider an element Z_a of the cartesian invariant decomposition, for example $\{r^1\} \times \tilde{X}^2$ for some $r^1 \in R^1$. We identify in an obvious way $C(\{r^1\} \times \tilde{X}^2)$ with the C*-subalgebra $\mathcal{C}^2$ of $BC_u(X)$ consisting of functions of the form $\varphi(x^1, x^2) = \varphi^2(x^2)$

with $\varphi^2 \in VO(X^2)$. Then the function $V^2_{r^1}(x^2) := \hat{V}(r^1, x^2)$ ($\hat{V}$ is the continuous extension of V to $\hat{X} = \tilde{X}^1 \times \tilde{X}^2$) is an element of $\mathcal{C}^2$. In a similar way we get elements $V^1_{r^2} \in \mathcal{C}^1 :=$ the C*-subalgebra of $BC_u(X)$ consisting of functions of the form $\varphi(x^1, x^2) = \varphi^1(x^1)$ with $\varphi^1 \in VO(X^1)$.

Then by a straightforward application of Theorem 1.8. we get

Corollary 3.2. *The self-adjoint operator $H := h(P) + V$ is affiliated to the C*-subalgebra $\langle [VO(X^1) \otimes VO(X^2)] \cdot C_0(X^\sharp) \rangle$ of $B(\mathcal{H})$ and its essential spectrum is given by*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \overline{\left(\bigcup_{r^1 \in R^1_0} \sigma[h(P) + V^2_{r^1}] \right)} \cup \overline{\left(\bigcup_{r^2 \in R^2_0} \sigma[h(P) + V^1_{r^2}] \right)}, \tag{3-5}$$

where R^j_0 is some arbitrary dense subset of R^j.

This result is quite different in nature from (2-1).

One can give a more detailed version of (3-5), by taking profit of the fact that the functions $V^2_{r^1}$ and $V^1_{r^2}$ do not depend of all the variables. We shall do this only for an interesting particular case. Let us consider Schrödinger operators in $X := \mathbb{R}^{n_1+n_2} = X^1 \times X^2$, i.e. $h(P) = \Delta = P^2 = -\nabla^2 = \Delta^1 \otimes 1 + 1 \otimes \Delta^2$. We interpret now $V^k_{r^j}$ as a function in $X^k = \mathbb{R}^{n_k}$ and set $H^k_{r^j} := \Delta^k + V^k_{r^j}$ (self-adjoint operator in $\mathcal{H}^k$) and $b^k_{r^j} := \inf(H^k_{r^j})$, $j, k = 1, 2$, $j \neq k$. Then we have

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \left[\min \left\{ \inf_{r^1} b^2_{r^1}, \inf_{r^2} b^1_{r^2} \right\}, \infty \right)$$

as a consequence of (3-5) and well-known facts about the spectrum of factorized operators.

By obvious particularizations one gets Example 0.2. of the Introduction. One has to take into account the specificity of the case $n_1 = n_2 = 1$ and the fact that functions with radial limits have the vanishing oscillation property.

Appendix A. Graded C*-algebras

The importance of graded C*-algebras in the generalized N-body problem was put into evidence in [1] and [3]. We do not intend to discuss this concept in detail. Our purpose will be just to show how the basic technical result (A-2) follows from Proposition 1.5.(a). We take profit of the occasion to introduce some new objects in graded C*-algebras.

Let $(\mathcal{L}, \leq)$ a finite lattice, with $\mathsf{o} := \min\mathcal{L} \neq \mathsf{1} := \max\mathcal{L}$. For $b \in \mathcal{L}$, $\mathcal{M} \subset \mathcal{L}$ we say that b *is under* $\mathcal{M}$ and we write $b \preccurlyeq \mathcal{M}$ if there is some $a \in \mathcal{M}$ such that $b \leq a$. Obviously $\mathsf{o} \preccurlyeq \mathcal{M}$ for any $\mathcal{M}$. If $b_1 \leq b_2 \preccurlyeq \mathcal{M}$ then $b_1 \preccurlyeq \mathcal{M}$ and if $b \preccurlyeq \mathcal{M} \subset \mathcal{M}'$ then $b \preccurlyeq \mathcal{M}'$.

We say that $\mathcal{M} \in 2^{\mathcal{L}}$ *is down-stable* and we write $\mathcal{M} \in DS(\mathcal{L})$ if $b \preccurlyeq \mathcal{M}$ implies $b \in \mathcal{M}$. This is equivalent to $\mathcal{M} = \mathcal{M}^{\text{down}}$, with $\mathcal{M}^{\text{down}} := \{b \in \mathcal{L} \mid b \preccurlyeq \mathcal{M}\}$. Note that $(\cup_j \mathcal{M}_j)^{\text{down}} = \cup_j \mathcal{M}^{\text{down}}_j$ and $(\cap_j \mathcal{M}_j)^{\text{down}} \subset \cap_j \mathcal{M}^{\text{down}}_j$ (strictly in general: take $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$). The sets $\{a\}^{\text{down}}$ are sublattices and were denoted in [1] by $\mathcal{L}_a$. We have:

Lemma A.1. *$DS(\mathcal{L})$ is a Boole subalgebra of $(2^{\mathcal{L}}, \subset)$ which includes canonically $\mathcal{L}$. The application $d : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$, $d(\mathcal{M}) := \mathcal{M}^{\mathrm{down}}$ is an increasing, idempotent map with range $DS(\mathcal{L})$.*

Let now $\mathfrak{C} = \sum_{a\in\mathcal{L}} \mathfrak{C}(a)$ be a *$\mathcal{L}$-graded C*-algebra*. This means that the sum is direct and that $\mathfrak{C}(a)\mathfrak{C}(b) \subset \mathfrak{C}(a \vee b)$. For any $\mathcal{M} \in 2^{\mathcal{L}}$ let us set $\mathfrak{K}_{\mathcal{M}} := \sum_{b\notin\mathcal{M}} \mathfrak{C}(b)$. This is a closed subspace of $\mathfrak{C}$ having $\mathfrak{C}_{\mathcal{M}} := \sum_{b\in\mathcal{M}} \mathfrak{C}(b)$ as a supplement, but in order that $\mathfrak{K}_{\mathcal{M}}$ be an ideal we have to impose the restriction $\mathcal{M} \in DS(\mathcal{L})$. The critical relation, very easy to obtain, is $\cap_j \mathfrak{K}_{\mathcal{M}_j} = \mathfrak{K}_{\cup_j \mathcal{M}_j}$ if every $\mathcal{M}_j$ is in $DS(\mathcal{L})$. By applying Proposition 1.5.(a) we get

Corollary A.2. *Set $\mathcal{L}' := \mathcal{L} \setminus \{\imath\}$ and $\mathfrak{K} := \mathfrak{K}_{\mathcal{L}'} = \mathfrak{C}(\imath)$ the minimal ideal. Then*

$$\mathfrak{C}/\mathfrak{K} \hookrightarrow \prod_{j\in J} \mathfrak{C}/\mathfrak{K}_{\mathcal{M}_j} \tag{A-1}$$

for any family $(\mathcal{M}_j)_{j\in J} \subset DS(\mathcal{L})$ such that $\cup_{j\in J}\mathcal{M}_j = \mathcal{L}'$ and $\mathcal{M}_j$ is not a subset of $\mathcal{M}_{j'}$ if $j \neq j'$.

Of course, in applications $\mathfrak{K}$ will be (isomorphic to) the ideal of compact operators. There is still a reason of discontentement in the way the Corollary is formulated, since the quotients in the right-hand side are not identified to C*-subalgebras of $\mathfrak{C}$. Let us observe that $\mathfrak{C}_{\mathcal{M}}$ is a C*-subalgebra if $\mathcal{M}$ is $\vee$-stable and in particular this is verified for $\mathcal{M} = \{a\}^{\mathrm{down}} = d(\{a\})$. We note also that, setting A for the set of all coatoms of $\mathcal{L}$ (= maximal elements in $\mathcal{L}'$), $\left(\mathfrak{K}_{\{a\}^{\mathrm{down}}}\right)_{a\in A}$ is a $\mathfrak{K}$-sufficient family of ideals. Therefore we can write instead of (A-1)

$$\mathfrak{C}/\mathfrak{K} \hookrightarrow \prod_{a\in A} \mathfrak{C}_{\{a\}^{\mathrm{down}}} \equiv \prod_{a\in A} \mathfrak{C}_a. \tag{A-2}$$

This formula is basic for the spectral analysis of self-adjoint operators affiliated to graded C*-algebras, the N-body Hamiltonians being the best-known example. It gives immediately the HWZ formula for the essential spectrum and it is at the base of a proof of the Mourre estimate.

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