Construction of Metastable States in Quantum Electrodynamics

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Abstract

In this paper, we construct metastable states of atoms interacting with the quantized radiation field, which emerge from the excited bound states of the non-interacting system, and we prove that these states obey a quasi-exponential time-decay law. The estimate of the error term improves a corresponding result in [BFS3]. This yields an enlargement of the time period during which *Fermi's Golden Rule* can be observed. A novel feature of our approach is the dependence of the complex dilatation parameter on the perturbation parameter (coupling constant).

1 Introduction and Survey of Results

1.1 The Mathematical Model of Nonrelativistic Electrons Coupled to the Quantized Radiation Field

This paper is concerned with the study of unstable, excited states of a nonrelativistic N-electron atom, coupled to the quantized electromagnetic field. We base our analysis on the *Standard Model of Nonrelativistic Quantum Electrodynamics* introduced in [BFS1, BFS2, BFS3]. We begin by briefly reviewing this model, and we refer the reader to [BFS1, BFS2, BFS3] for a detailed discussion of its derivation from first principles of quantum mechanics. Other contributions to various aspects of this model can be found in [CDG, HS1, HS2, Sk].

The Hilbert space $\mathcal{H}$, containing the state vectors of N electrons and the photon field, is given as the tensor product

$$\mathcal{H} := \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F} \cong \mathcal{A}_N L^2 \left[(\mathbb{R}^3 \times \{\uparrow, \downarrow\})^N ; \mathcal{F} \right] \tag{1}$$

of the *N-electron Hilbert space*

$$\mathcal{H}_{\mathrm{el}} := \mathcal{A}_N L^2 \left[(\mathbb{R}^3 \times \{\uparrow, \downarrow\})^N \right] \cong \bigwedge_{j=1}^{N} \left(L^2[\mathbb{R}^3] \otimes \mathbb{C}^2 \right), \tag{2}$$

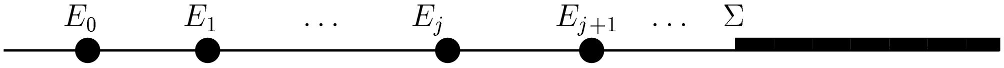

Figure 1: The spectrum of the electronic Hamiltonian $H_{\rm el}$.

of totally antisymmetric wave functions of N position and spin variables $\underline{x}_j \equiv (\vec{x}_j, \lambda_j) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\}$ and the *bosonic Fock space*

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \tag{3}$$

where $\mathcal{H}_n$ denotes the space of totally symmetric, square-integrable wave functions of n momentum and polarization variables $\underline{k}_j \equiv (\vec{k}_j, \lambda_j) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\}$. $\mathcal{F}$ is the Hilbert space of all sequences $(\psi_n \in \mathcal{H}_n)_{n \in \mathbb{N}_0} \equiv \bigoplus_{n=0}^{\infty} \psi_n$, for which $|\psi_0|^2 + \sum_{n=1}^{\infty} \langle \psi_n \,|\, \psi_n \rangle_{\mathcal{H}_n} < \infty$.

We assume that the electrons are bound in a Coulomb potential

$$V_C\,(\vec{x}_1, \ldots, \vec{x}_N) := \sum_{j=1}^{N} \frac{-Z}{|\vec{x}_j|} + \sum_{1 \le i < j \le N} \frac{1}{|\vec{x}_i - \vec{x}_j|} \tag{4}$$

of a single static nucleus located at the origin with nuclear charge $Z \in \mathbb{R}^+$. The electron dynamics is generated by the Schrödinger operator

$$H_{\rm el} := -\Delta_x + V_C(x), \tag{5}$$

$x \in \mathbb{R}^{3N}$, which is densely defined on the usual Sobolev space H^2 of antisymmetric wave functions. While it is well-known from the *HVZ-Theorem* (see, e.g., [CFKS]) that the essential spectrum of the *electron Hamiltonian* is given by $\sigma_{\rm ess}(H_{\rm el}) = [\Sigma, \infty)$, $\Sigma \le 0$, much less is known about the discrete spectrum of $H_{\rm el}$, and we have to require explicitly some additional properties of the atom to ensure stability of the electron system.

1.1 Hypothesis *The discrete spectrum of $H_{\rm el}$ consists of, at most, countably many, finitely degenerated eigenvalues $E_0 < E_1 < \cdots < E_j < E_{j+1} < \cdots < \Sigma = \inf(\sigma_{\rm ess}(H_{\rm el}))$. The spectrum, $\sigma(H_{\rm el})$, has no accumulation point in $\mathbb{R} \setminus \{\Sigma\}$ and*

$$\sigma(H_{\rm el}) = \{E_0, E_1, \ldots, E_j, E_{j+1}, \ldots\} \cup [\Sigma, \infty) \tag{6}$$

(see Fig. 1).

Hypothesis 1.1 is fulfilled for neutral atoms and positively charged ions (i.e., for $N \le Z$, see also [Zi]). It is known that there are no eigenvalues for highly charged negative ions with $N \ge 2Z + 1$ (see also [Li]).

The dynamics of the photons is generated by the Hamiltonian of the free, quantized electromagnetic field

$$H_{\rm f} := \int \omega(\vec{k}) a(\underline{k})^* a(\underline{k})\, d\underline{k}, \tag{7}$$

which is the second quantization of the dispersion relation $\omega(\vec{k}) := |\vec{k}|$ of massless bosons. The *creation and annihilation operators*, $a^*(\underline{k})$ and $a(\underline{k})$, resp., are the usual densely defined, operator-valued, tempered distributions representing the *canonical commutation relations* (CCR)

$$\begin{aligned} \left[a(\underline{k}), a(\underline{k}')^*\right] &= \delta(\underline{k} - \underline{k}')\, \mathbb{1}_{\mathcal{F}} \qquad \text{and} \\ \left[a(\underline{k})^*, a(\underline{k}')^*\right] &= \left[a(\underline{k}), a(\underline{k}')\right] = 0. \end{aligned} \tag{8}$$

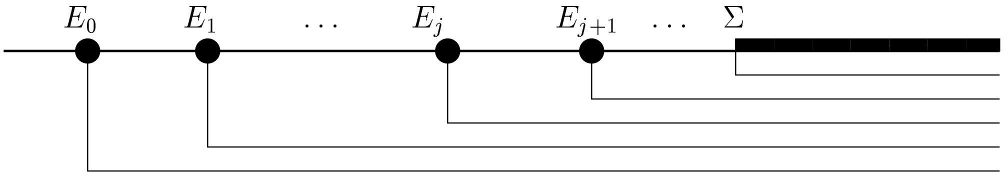

Figure 2: Spectrum of the unperturbed Hamiltonian H_0.

The normalized *vacuum state* $\Omega := (1, 0, 0, \dots) \in \mathcal{H}_0$ is uniquely determined (up to a phase factor) by

$$a(\underline{k})\Omega := 0, \tag{9}$$

for all $\underline{k} \in \mathbb{R}^3 \times \{\uparrow, \downarrow\}$.

It is easy to see that the spectrum of H_f is given by

$$\sigma(H_\mathrm{f}) = [0, \infty). \tag{10}$$

Furthermore, zero is the only eigenvalue of H_f. Its multiplicity is one, i.e.,

$$\sigma_\mathrm{pp}(H_\mathrm{f}) = \{0\}, \qquad \mathrm{Ker}(H_\mathrm{f}) = \mathcal{H}_0 = \mathbb{C}\Omega. \tag{11}$$

From the spectral properties of the electron Hamiltonian and the Hamiltonian of the free radiation field, the spectral properties of the unperturbed Hamilton operator,

$$H_0 := H_\mathrm{el} \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_{\mathcal{H}_\mathrm{el}} \otimes H_\mathrm{f}, \tag{12}$$

describing the non-interacting system of electrons and photons, are easily derived. The atomic eigenvalues $\{E_j \mid j \in \mathbb{N}_0\}$ persist as eigenvalues of H_0, but the energy levels are no longer isolated points of the spectrum. The eigenvalues E_k are the thresholds of half lines $E_k + [0, \infty)$ of absolutely continuous spectrum and are imbedded in the half lines of absolutely continuous spectrum, $E_l + [0, \infty)$ with $l < k$ (see Fig. 2).

We obtain the Hamiltonian H_g of the coupled system by replacing the momentum operator of the j^th electron: $\vec{p}_j \mapsto \vec{p}_j - c\vec{A}(\alpha\vec{x}_j)$ (*minimal coupling*), where $\vec{A}(\alpha\vec{x}_j)$ is the UV-regularized *vector potential* of the magnetic field. After normal ordering, the Hamiltonian reads

$$H_g := H_0 + W_g(\theta = 0) := H_0 + \sum_{m+n=1,2} g^{m+n} W_{m,n}(\theta = 0), \tag{13}$$

where $g > 0$ represents the strength of the electromagnetic coupling between electrons and photons. Further, the perturbation operators $W_{m,n}(\theta)$, which appear as coefficients in the terms of order $m + n = 1, 2$ in the coupling constant g, are defined by

$$\begin{aligned} W_{m,n}(\theta) := W_{n,m}(\overline{\theta})^* \quad := & \int d\underline{k}\, w_{m,n}(\underline{k}_1, \dots, \underline{k}_m, \underline{k}_{m+1}, \dots, \underline{k}_{m+n}, \theta) \cdot \\ & \cdot a(\underline{k}_1)^* \cdots a(\underline{k}_m)^* a(\underline{k}_{m+1}) \cdots a(\underline{k}_{m+n}). \end{aligned} \tag{14}$$

We require that the operators $w_{m,n}(\underline{k}_1, \dots, \underline{k}_m, \underline{k}_{m+1}, \dots, \underline{k}_{m+n}, \theta)$ on the Hilbert space $\mathcal{H}_\mathrm{el}$ fulfill the relative bounds

$$\begin{aligned} \left\| w_{m,n}(\underline{k}, \theta)(-\Delta_x + 1)^{-1/2} \right\|_{\mathcal{H}_\mathrm{el}} &\le J(\vec{k}) \qquad && (m + n = 1), \\ \| w_{m,n}(\underline{k}_1, \underline{k}_2, \theta) \|_{\mathcal{H}_\mathrm{el}} &\le J(\vec{k}_1) J(\vec{k}_2) \qquad && (m + n = 2) \end{aligned} \tag{15}$$

for a square-integrable function $J : \mathbb{R}^3 \to \mathbb{R}$ with $J(\vec{k}) \overset{|\tilde{k}|\to 0}{\sim} \omega(\vec{k})^{-1/2}$ and $J(\vec{k}) \overset{|\tilde{k}|\to\infty}{\sim} \exp(-|\vec{k}|^4)$. This guarantees the existence of

$$\Lambda_\beta := \left(\int_{\mathbb{R}^3} J(\vec{k})^2 \omega(\vec{k})^\beta \, d^3k \right)^{1/2}, \qquad \beta > -2. \tag{16}$$

Furthermore, we require analyticity of the maps $\theta \mapsto w_{m,n}(\underline{k}_1, \ldots, \underline{k}_m, \underline{k}_{m+1}, \ldots, \underline{k}_{m+n}, \theta)\,\psi$ in a complex neighbourhood of $\theta = 0$, for every vector ψ in the domain of $w_{m,n}(\ldots)$. This is ensured by a suitable UV-regularization of the vector potential as discussed in [BFS3]. θ is the dilatation parameter (see Sect. 1.3).

Under these assumptions, it is possible to deduce suitable relative bounds on the perturbation:

1.2 Lemma *For $m, n \in \mathbb{N}_0$, $1 \le m+n \le 2$, $\theta \in U_{\pi/4} := \left\{ \tilde{\theta} \in \mathbb{C} \,\middle|\, |\tilde{\theta}| < \pi/4 \right\}$, the operators $W_{m,n}(\theta)$ are defined on $\mathcal{D}(H_0) = \mathcal{D}(H_{\rm el}) \otimes \mathcal{D}(H_{\rm f})$ and satisfy the relative bound:*

$$\left\| W_{m,n}(\theta) |H_0 + i|^{-1} \right\|_{\mathcal{H}} \le \left[4(1 + \Lambda_0^2 + \Lambda_{-1}^2) \right]^{(m+n)/2}. \tag{17}$$

Proof: We refer the reader to [BFS3], Lemma I.1. ■

An immediate consequence is the following theorem which ensures that the perturbed Hamiltonian is semibounded and has the same domain as the unperturbed one.

1.3 Theorem *The perturbation W_g is defined on $\mathcal{D}(H_0) = \mathcal{D}(H_{\rm el}) \otimes \mathcal{D}(H_{\rm f})$. If $g < \frac{1}{10}(1 + \Lambda_0^2 + \Lambda_{-1}^2)^{-1/2}$, then $H_g = H_0 + W_g$ is self-adjoint on $\mathcal{D}(H_g) = \mathcal{D}(H_0)$ and bounded from below.*

Proof: (see [BFS3], Corollary I.2). ■

1.2 Survey of the Main Result

In [BFS3] it was proven that, for sufficiently small $g > 0$, the excited energy levels disappear (see also Thm. 1.10) when electrons and photons are minimally coupled. We study the fate of the corresponding eigenstates, stated in Sect. 2.

In Sect. 2.1 we construct metastable states close to the unperturbed eigenstates, i.e., we explicitely define a vector Φ in the Hilbert space $\mathcal{H}$ which approximates the unperturbed, excited eigenstate $\varphi_j \otimes \Omega$ with energy E_j of the decoupled system as the coupling constant g decreases.

In Sect. 2.2, we identify the time-evolution of this state Φ to be exponentially decaying, up to small errors. We compare this result with predictions from time-dependent perturbation theory to first order. Following standard expositions in physics textbooks (see, e.g., [Me], Chap. 20), the probability per unit time, $p_{\Phi\to\phi_0}$, of transition of a perturbed, excited state Φ into the ground state ϕ_0 is given by the matrix element of the perturbation:

$$p_{\Phi\to\phi_0} = 2\pi \left| \langle \phi_0 \mid W_g \, \Phi \rangle_{\mathcal{H}} \right|^2 = \mathcal{O}\left(g^2\right). \tag{18}$$

This constant transition rate implies an exponential decay of the state Φ under the time-evolution with a life-time $p_{\Phi\to\phi_0}^{-1}$, a result known as *Fermi's Golden Rule.*

The main result of this paper (Thm. 2.9) shows the weakness of the first order approximation of time dependent perturbation theory. It states that the time-decay law of the constructed vector Φ can be divided into a part of exponential decay, which is predicted by Fermi's Golden Rule, and an *error term* ("background") which dominates the decay for large times $t \gg 1$:

$$\Big| \langle\, \Phi \mid \exp(-iH_g t)\Phi\, \rangle_{\mathcal{H}} \Big| \leq \underbrace{C \exp(-S_g t)}_{\text{Fermi's Golden Rule}} + \underbrace{\frac{C_n}{t^n} g^{4+\beta}}_{\text{error term}}, \tag{19}$$

where $0 < \beta < 1/9$, $S_g = \mathcal{O}\left(g^2\right)$, $n = 0, 1$ and $C, C_n < \infty$ are suitable constants. We point out that this result is an improvement on [BFS3], where the time decay of the eigenstate $\varphi_j \otimes \Omega$ itself was identified to be of this kind, except for the error term; specifically, Bach, Fröhlich, and Sigal proved that the error term is $\mathcal{O}(t^{-n} g^4)$ (but here valid for all $n \in \mathbb{N}$). Some ideas for estimating metastable states used here are borrowed from this paper. Our aim is to improve the order of the error term (in [BFS3]) in g, $4 + \beta > 4$. Our improved estimate expands the time slot in which the exponential decay dominates the error term and, thus, in which Fermi's Golden Rule can be observed, i.e., where

$$C \exp(-S_g t) \gg \frac{C_n}{t^n} g^{4+\beta}. \tag{20}$$

This is fulfilled for

$$t \leq S_g^{-1} \left[\text{const} + (4 + \beta) \log(g^{-1})\right]. \tag{21}$$

Comparing to [BFS3], we note that the size of the time slot is enlarged by

$$\beta S_g^{-1} \log(g^{-1}) \overset{g \to 0}{\longrightarrow} \infty. \tag{22}$$

Increasing the order of the error term requires a refinement of the choice of Φ as compared to [BFS3]. The construction of the state Φ is complicated, and we devote an entire section to this task. In our paper, we borrow an idea of [Hu] for constructing metastable states. Hunziker proved for a quantum system with isolated eigenvalues of the unperturbed system that the error is of order $\mathcal{O}(t^{-n} g^M)$, for all $n \in \mathbb{N}$, assuming the existence of a *Rayleigh-Schrödinger expansion* to order M of the (nonexistent) perturbed eigenstate. Moreover, Hunziker showed how metastable states of a quantum system and *resonances* of the interacting system are related: The resonance energies are the complex eigenvalues of the *dilated Hamiltonian* $H_g(\theta)$ (defined in Sect. 1.3), and their imaginary parts determine the reciprocal life-time S_g of the metastable states. In our case, the Rayleigh-Schrödinger expansion to first order of the eigenstate corresponding to the resonance $\tilde{E}_j$ of the dilated operator $H_g(\theta)$, $\mathrm{Im}(\theta) > 0$, appears as

$$\psi_\theta^{(0)} = \left[P(\theta) - \overline{P}(\theta) \left(\overline{P}(\theta) H_g(\theta) \overline{P}(\theta) - \tilde{E}_j\right)^{-1} \overline{P}(\theta) W_g(\theta) P(\theta)\right] \varphi_j(\theta) \otimes \Omega \tag{23}$$

(as derived in [BFS1]), where $P(\theta)$ is a projection on the unperturbed eigenspace corresponding to E_j and on "small photon energies" and $\overline{P}(\theta) = \mathbb{1}_{\mathcal{H}} - P(\theta)$. It is the first prototype of a metastable state. But the transformation back to the undilated system is problematic because $\lim_{\mathrm{Im}(\theta) \to 0} \psi_\theta^{(0)}$ does not exist. In the limit $\mathrm{Im}(\theta) \to 0$, the spectrum of $H_g(\theta)$ is turned back from the lower half plane into the real axis. Thus, the resonance $\tilde{E}_j = E_j + \mathcal{O}\left(g^2\right) - i\mathcal{O}\left(g^2\right)$ is covered by a sector of absolutely continuous spectrum of $H_g(\theta)$, which results from perturbing the branches $E_k + e^{-\theta}[0, \infty)$, $k < j$, of the spectrum of $H_0(\theta)$ (this is illustrated in Fig. 3; for spectral properties of the dilated Hamiltonian see also Sect. 1.3 and Fig. 4). Note that this is the crucial difference to [Hu] in the construction of a metastable state because imbedded eigenvalues are excluded in the latter paper.

We briefly sketch our approach of constructing a metastable state. Inspired by [Hu], we modify the state $\psi_\theta^{(0)}$ so that it becomes analytic in θ (i.e., we require *dilatation*

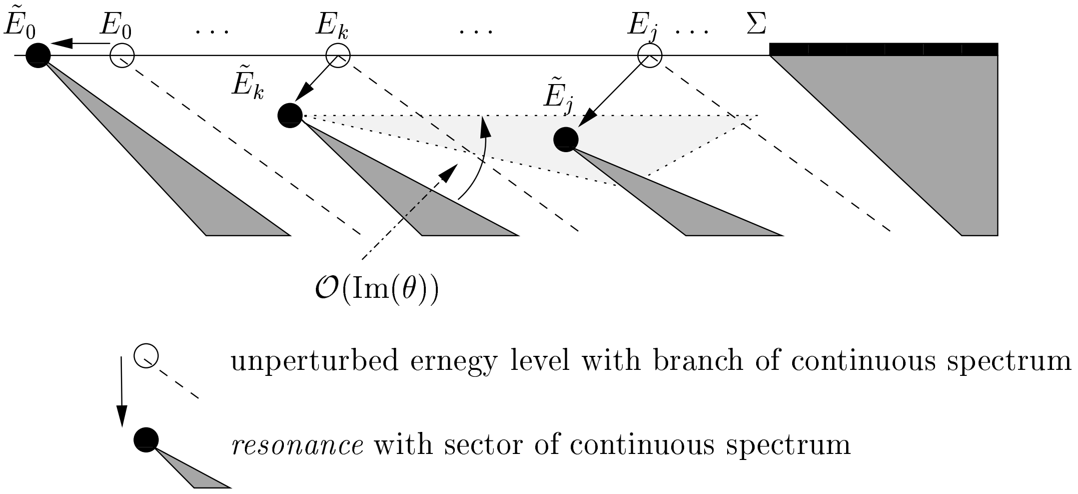

Figure 3: Spectrum of the perturbed, dilated Hamiltonian $H_g(\theta)$, $g > 0$, $\mathrm{Im}(\theta) > 0$, resulting from $H_0(\theta)$. The sectors of continuous spectrum fixed at the *resonances* $\tilde{E}_k$ are caused by perturbing the branches $E_k + e^{-\theta}[0, \infty)$. They cover the *resonance* $\tilde{E}_j$ as $\mathrm{Im}(\theta) \to 0$.

analyticity), and yet, the new state should approximate $\psi_\theta^{(0)}$ as $g \to 0$. We obtain this state by replacing, in a first step, the perturbed resolvent in (23) by its unperturbed analogue and then, in a second step, by introducing regularization terms to ensure the existence of the resolvents restricted to several eigenspaces:

$$\begin{aligned}\left(\overline{P}(\theta)H_g(\theta)\overline{P}(\theta) - \tilde{E}\right)^{-1} &\rightsquigarrow (H_0(\theta) - E_j)^{-1} \\ &\rightsquigarrow \sum_{k=0}^{j-1} \left(e^{-\theta}H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1} P_k(\theta) \\ &\quad + \left(e^{-\theta}H_{\mathrm{f}} + ig^{\gamma}\right)^{-1} P_j(\theta) + (H_0(\theta)P_{>j}(\theta) - E_j)^{-1} P_{>j}(\theta).\end{aligned}$$

Here, $P_k(\theta)$ is the projection on the eigenspace of $H_0(\theta)$ corresponding to E_k and $P_{>j}(\theta) := \mathbb{1}_{\mathcal{H}} - \sum_{k=0}^{j} P_k(\theta)$. In Sect. 2.1, we see that this modification of the resolvent enables us to define a state with the required properties. However, we note that the analyticity of the resolvent $\left(e^{-\theta}H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1}$ in θ is dependent on the spectral parameter $E_j - E_k + ig^{\alpha}$ and therefore on the coupling constant. Once more, we observe that the branches $E_k + e^{-\theta}[0, \infty)$ of absolutely continuous spectrum of $H_0(\theta)$, fixed at the lower eigenvalues E_k, $k < j$, hit the pole $E_j + ig^{\alpha}$ for $\tan(\mathrm{Im}(\theta)) = g^{\alpha}(E_k - E_j)^{-1}$. As a result, the variation of θ is restricted to a complex neighbourhood of zero which decreases with the coupling constant g. This is illustrated in detail in Remark 2.3, Fig. 5. Consequently, the complex deformation of the spectrum is coupled to the perturbation constant g. Since $g \to 0$ implies $\mathrm{Im}(\theta) \to 0$, more accurate considerations of the divergence of the resolvent $\lim_{g,\mathrm{Im}(\theta)\to 0} \left(e^{-\theta}H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1}$ are required. Corresponding estimates are derived in Sect. 2.1.

In [BFS3], Bach, Fröhlich, Sigal examine the time-decay of the zero-order Rayleigh-Schrödinger expansion of the perturbed eigenstates, i.e., of the unperturbed eigenstates themselves. As no resolvents enter their definition of mestable states, the dilatation parameter θ was unrestricted, while in our case, the consideration of the first-order Rayleigh-Schrödinger expansion allows us to find stronger estimates on the error term at the expense of g-dependence of the dilatation parameter. We note that the crucial difference between [BFS3] and this paper is the problem of handling the singularities of $\left(e^{-\theta}H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1}$ as $g, \mathrm{Im}(\theta) \to 0$. Following the above argument, this dependence seems to be inevitable whenever we work with Rayleigh-Schrödinger expansions in higher orders.

1.3 Dilatation Analyticity of the System

Complex deformations is a convenient tool for the analysis of resonances and corresponding metastable states, as well as, for the derivation of spectral properties of the perturbed Hamiltonian from the unperturbed (non-interacting) one. For estimates of the metastable states in Sect. 2.2 we need an analytic continuation of the matrix element $z \mapsto \left\langle \varphi \,\middle|\, (H_g - z)^{-1}\psi \right\rangle_{\mathcal{H}}$ from the upper half plane accross the real axis into the lower half plane for suitable *dilatation analytic* states $\varphi, \psi \in \mathcal{H}$. This enables us to deform integration contours into lower half plane by applying *Cauchy's integral formula* to approach the resonances. This analytic continuation is established by *dilatation analyticity.* In this section we describe the basic concepts. More details can be found in [AC, BC] on which these concepts are based.

1.4 Definition (Dilatation Group) *On the Hilbert space $\mathcal{H}$ we introduce a strongly continuous one parameter group of unitary operators $U(\theta)$, $\theta \in \mathbb{R}$, the* Dilatation Group, *which scales the position variables of the electrons and the photon momenta by $\vec{x} \mapsto e^{\theta}\vec{x}$ and $\vec{k} \mapsto e^{-\theta}\vec{k}$, resp.*

For $\psi \in \mathcal{H}$, $\theta \in \mathbb{R}$, we abbreviate

$$\psi(\theta) := U(\theta)\psi \tag{24}$$

and, for a closed operator A on $\mathcal{H}$, we write

$$A(\theta) := U(\theta)\, A\, U(\theta)^{-1}. \tag{25}$$

We collect the dilatation properties of the operators, involved in our analysis:

$$\begin{aligned} H_{\mathrm{el}}(\theta) &= e^{-2\theta}\,(-\Delta_x) + e^{-\theta} V_C(x), & (26)\\ H_{\mathrm{f}}(\theta) &= e^{-\theta} H_{\mathrm{f}}, & (27)\\ H_0(\theta) &= e^{-2\theta}\,(-\Delta_x \otimes \mathbb{1}_{\mathcal{F}}) + e^{-\theta}\,(V_C(x) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}_{\mathrm{el}}} \otimes H_{\mathrm{f}})\,. & (28) \end{aligned}$$

Note that the homogeneity of the Coulomb potential under dilatation is due to the fact that we consider an atom, i.e., we deal with a single nucleus at the origin.

The perturbation $W_g \equiv W_g(\theta = 0)$ transforms under dilatation as follows

$$W_g(\theta) = U(\theta)\, W_g\, U(\theta)^{-1} = \sum_{m+n \le 2} g^{m+n} W_{m,n}(\theta). \tag{29}$$

1.5 Definition (Dilatation Analyticity) *A vector $\psi \in \mathcal{H}$ is called* dilatation analytic, *iff the Hilbert space valued map $\mathbb{R} \ni \theta \mapsto \psi(\theta)$ has an analytic continuation in a neighbourhood of zero*

$$U_r := \{\theta \in \mathbb{C} \mid |\theta| < r\}$$

for some $0 < r < \pi/4$.

A closed operator A on $\mathcal{H}$ is called dilatation analytic *iff $U_r \ni \theta \mapsto A(\theta)$ defines a holomorphic family of type (A) (in the sense of [Ka], Chap. VII.2.1) on a neighbourhood U_r, $0 < r < \pi/4$, which conincides with the above definition (25) of $A(\theta)$, for $\theta \in \mathbb{R}$.*

The advantage of the concept of dilatation analytic operators and vectors lies in the following identities:

$$\left[A(\overline{\theta})\right]^* = A^*(\theta), \tag{30}$$

$$\langle \phi \,|\, A\psi \rangle_{\mathcal{H}} = \left\langle \phi(\overline{\theta}) \,\middle|\, A(\theta)\psi(\theta) \right\rangle_{\mathcal{H}}, \tag{31}$$

which hold for dilatation analytic operators A, A^* and vectors ϕ, ψ and $\theta \in \mathbb{C}$ sufficiently close to zero. By Eq. (31), a matrix element of an operator A can be transformed into a matrix element of the dilated operator $A(\theta)$, whose properties are more accessible in various applications. For instance, we transform

$$\left\langle \varphi \,\middle|\, (H_g - z)^{-1}\,\psi \right\rangle_{\mathcal{H}} = \left\langle \varphi(\overline{\theta}) \,\middle|\, (H_g(\theta) - z)^{-1}\,\psi(\theta) \right\rangle_{\mathcal{H}}$$

for dilatation analytic vectors φ, ψ, and use the spectral properties of $H_g(\theta)$ instead of H_g. Note that the dilatation of ψ with a dilatation parameter $\mathrm{Im}(\theta) > 0$ in the upper half plane requires the dilatation of φ with the conjugated parameter $\mathrm{Im}(\overline{\theta}) < 0$ in the lower half plane. For this reason, we couple the interaction constant g and the dilatation parameter θ to ensure analyticity of the metastable state which we define in Sect. 2.1 (see Remark 2.3).

The dilatation analyticity of the unperturbed Hamiltonian H_0 follows directly from Eq. (28). An immediate implication of Lemma 1.2, Eq. (29) and the analyticity of $\theta \mapsto w_{m,n}(\underline{k}_1, \ldots, \underline{k}_{m+n}, \theta)\,\psi$ is the dilatation analyticity of $W_g(\theta)$ and of $H_g(\theta)$:

1.6 Lemma *If $g < \frac{1}{10}(1 + \Lambda_0^2 + \Lambda_{-1}^2)^{-1/2}$ and $\theta \in U_{\pi/4}$, then the dilated perturbation $W_g(\theta)$ is relatively H_0-bounded with relative bound strictly smaller than one. Furthermore, the map*

$$W_g : U_{\pi/4} \to \mathcal{B}(\mathcal{D}(H_0); \mathcal{H}), \qquad \theta \mapsto W_g(\theta)$$

is analytic w.r.t. the graph norm on $\mathcal{D}(H_0)$. In particular, $W_g(\theta)$ and $H_g(\theta)$ are dilatation analytic.

Finally, we require the dilatation properties of eigenstates of the electronic Hamiltonian:

1.7 Lemma *Let $\varphi_k \in \mathcal{H}_{\mathrm{el}}$ be an eigenstate of H_{el} corresponding to the eigenvalue E_k. Then φ_k is dilatation analytic. The dilated state $\varphi_k(\theta)$ is an eigenstate of the dilated operator $H_{\mathrm{el}}(\theta)$ corresponding to E_k.*

Proof: We refer to [BC], Thm. 1. ■

1.8 Corollary *Let P_{el}^k be the orthogonal rank $N_k < \infty$ projection onto the eigenspace of H_{el} corresponding to the eigenvalue E_k. Then, P_{el}^k is dilatation analytic. In particular:*

$$\theta \mapsto P_{\mathrm{el}}^k(\theta)\,\psi = \sum_{l=1}^{N_k} \left\langle \varphi_k^l(\overline{\theta}) \,\middle|\, \psi \right\rangle_{\mathcal{H}_{\mathrm{el}}} \varphi_k^l(\theta) \tag{32}$$

is the analytic continuation on a disc U_r, where φ_k^l, $l = 1, \ldots, N_k$, are the normalized eigenstates of H_{el} corresponding to E_k which are dilatation analytic by Lemma 1.7. $P_{\mathrm{el}}^k(\theta)$ is the (not necessarily orthogonal) finite rank projection onto the eigenspace of $H_{\mathrm{el}}(\theta)$ corresponding to E_k.

To estimate the resolvent of $H_0(\theta)$ and $H_g(\theta)$ we collect properties of their spectra.

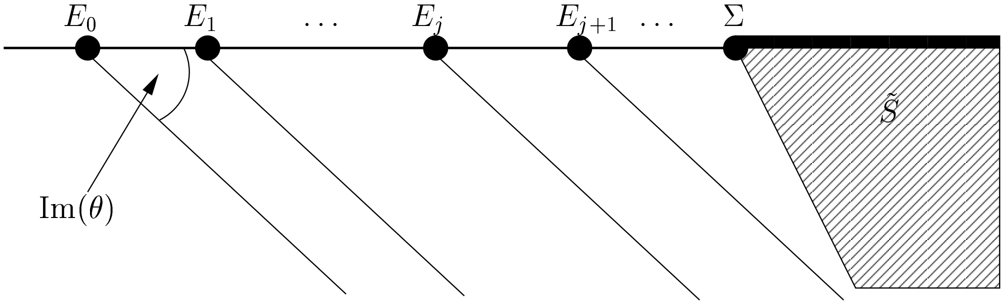

Figure 4: Spectrum of the unperturbed, dilated Hamiltonian $H_0(\theta)$, $\mathrm{Im}(\theta) > 0$.

1.9 Theorem *The spectrum of the dilated electronic Hamiltonian* $H_{\mathrm{el}}(\theta)$ *is given by*

$$\sigma(H_{\mathrm{el}}(\theta)) = \{E_k \mid k \in \mathbb{N}_0\} \cup S_{H_{\mathrm{el}}(\theta)}, \tag{33}$$

where

$$S_{H_{\mathrm{el}}(\theta)} \subseteq \tilde{S} := \left\{\Sigma + re^{-i\vartheta} \,\middle|\, r \geq 0, \vartheta \in [0, 2\mathrm{Im}(\theta)] \cup [2\mathrm{Im}(\theta), 0]\right\}. \tag{34}$$

Thus, the spectrum of $H_0(\theta) = H_{\mathrm{el}}(\theta) \otimes \mathbb{1}_{\mathcal{F}} + e^{-\theta}\mathbb{1}_{\mathcal{H}_{\mathrm{el}}} \otimes H_{\mathrm{f}}$ *is given by*

$$\sigma(H_0(\theta)) = \left\{E_k + re^{-i\mathrm{Im}(\theta)} \,\middle|\, k \in \mathbb{N}_0, r \geq 0\right\} \cup S_{H_0(\theta)}, \tag{35}$$

with $S_{H_0(\theta)} \subseteq \tilde{S}$ *(see Fig. 4). Furthermore, consider for* $P_k(\theta) := P_{\mathrm{el}}^k(\theta) \otimes \mathbb{1}_{\mathcal{F}}$ *the operator* $H_0(\theta)$ *restricted to the range of* $P_{>j}(\theta) := \mathbb{1}_{\mathcal{H}} - \sum_{k=0}^{j} P_k(\theta)$. *Its spectrum is given by*

$$\sigma(H_0(\theta)P_{>j}(\theta))_{\mathrm{Ran}(P_{>j}(\theta))} = \left\{E_k + re^{-i\mathrm{Im}(\theta)} \,\middle|\, k \geq j+1, r \geq 0\right\} \cup S_{H_0(\theta)}. \tag{36}$$

Proof: Eq. (33) is proven in [BC], Lemma 1 and Lemma 3. The remaining assertions are a simple consequence of $\sigma(e^{-\theta}H_{\mathrm{f}}) = e^{-i\mathrm{Im}(\theta)}[0, \infty)$ and the functional calculus. ■

1.10 Theorem *Fix* $0 < \varepsilon < 2/9$ *and define for* $j \in \mathbb{N}$

$$\mathcal{A}^{\varepsilon} := \mathcal{I}(\delta/2) + i\,[-S_g, \infty), \tag{37}$$

$$S_g := g^2\Gamma - g^{2+\varepsilon}C_S = \mathcal{O}\left(g^2\right) \tag{38}$$

with suitable constants $\Gamma, C_S > 0$, *defined as in [BFS3] (note that* $S_g > 0$ *for* g *sufficiently small). Let* $g > 0, \mathrm{Im}(\theta) > 0$ *and*

$$g^{\varepsilon} \leq \mathrm{const}\,\mathrm{Im}(\theta)^2 \qquad \text{and} \qquad g^{2-2\varepsilon} \leq \mathrm{const}\,\mathrm{Im}(\theta), \tag{39}$$

where the constants are suitably chosen. Then

$$\mathcal{A}^{\varepsilon} \cap \sigma(H_g(\theta)) = \emptyset \tag{40}$$

and $\mathbb{H}^+ \ni z \mapsto \left\langle \varphi \,\middle|\, (H_g - z)^{-1}\psi \right\rangle_{\mathcal{H}}$ *has an analytic continuation accross the real axis into the lower half plane for all dilatation analytic vectors* $\varphi, \psi \in \mathcal{H}$. *In particular, there are no eigenvalues of* H_g *in* $\mathcal{A}^{\varepsilon}$, *i.e., the electronic energy levels* $E_j, j \geq 1$, *are unstable against coupling to the photon field.*

Proof: (see [BFS3], Lemmata III.14, III.15). ■

The value S_g approximates the imaginary part of the resonance corresponding to E_j. It enters the estimates of the time decay of the metastable state as the reciprocal life-time (see also Fig. 6).

2 Time-Decay of Metastable States

The first part of this section deals with the construction of metastable states. We prove dilatation analyticity and some important estimates. In the second part, these estimates enter the proof of our main Theorem 2.9, which gives an estimate on time evolution of the metastable states.

2.1 Construction of a Metastable State

Before constructing a metastable state of H_g, we introduce some notation.

2.1 Notation *Let* $\sigma(H_{\rm el}) = \{E_0, E_1, \dots, E_j, E_{j+1}, \dots\} \cup [\Sigma, \infty)$ *be the spectrum of the electron Hamiltonian* $H_{\rm el}$ *as postulated in Hypothesis 1.1. For* $k \in \mathbb{N}_0$ *we consider the eigenvalue* E_k *(which is discrete and finitely degenerated by assumption) and denote by*

$$P_{\rm el}^k := \chi_{\left(E_k - \frac{\delta_k}{2}\, ,\, E_k + \frac{\delta_k}{2}\right)}(H_{\rm el}), \qquad \delta_k := \mathrm{dist}(E_k, \sigma(H_{\rm el}) \backslash \{E_k\}) > 0, \tag{41}$$

the orthogonal projection of rank $N_k < \infty$ *on the corresponding eigenspace. Furthermore, we define by*

$$P_k := P_{\rm el}^k \otimes \mathbb{1}_{\mathcal{F}}. \tag{42}$$

an orthogonal projection on $\mathcal{H}$. *We fix* $j \in \mathbb{N}$ *(note that* $j = 0$ *is excluded), set*

$$\delta := \delta_j, \tag{43}$$

and introduce

$$P_{>j} := \mathbb{1}_{\mathcal{H}} - \sum_{k=0}^{j} P_k = \chi\left[H_{\rm el} > E_j + \frac{\delta_j}{2}\right] \otimes \mathbb{1}_{\mathcal{F}}, \tag{44}$$

with $\chi[f(x) > a] := \chi_{\{y \in \mathbb{R} \,|\, f(y) > a\}}(x)$, *for a measurable function* f.

Now, we are in position to implement the strategy of constructing metastable states as outlined in Sect. 1.2.

2.2 Definition (Construction of a Metastable State) *Using Notation 2.1 we define, for* $g > 0$,

$$\begin{aligned} \phi \quad := \quad \Bigg[\mathbb{1}_{\mathcal{H}} - \Bigg\{ & \sum_{k=0}^{j-1} P_k \left(H_{\rm f} + E_k - E_j - i g^{\alpha}\right)^{-1} P_k \\ & + P_j \left(H_{\rm f} + g^{\gamma}\right)^{-1} P_j \\ & + P_{>j} \left(H_0 P_{>j} - E_j\right)^{-1} P_{>j} \Bigg\} W_g \Bigg] \varphi_j \otimes \Omega, \end{aligned} \tag{45}$$

where $\varphi_j \in \mathrm{Ran} P_{\rm el}^j$ *is an arbitrary, normalized eigenstate of* $H_{\rm el}$ *corresponding to the* j^{th} *exited energy level* E_j, *and* Ω *is the vacuum vector of the photon Fock space.* $\alpha, \gamma > 0$ *are positive constants fixed later and* $\left(H_0 P_{>j} - E_j\right)^{-1}$ *is the resolvent of the operator* H_0, *restricted to the range of* $P_{>j}$, *which is well defined because of* $\sigma(H_0 P_{>j})_{\mathrm{Ran}(P_{>j})} = \{E_k \,|\, k \geq j+1\} + \sigma(H_{\rm f}) = [E_{j+1}, \infty)$. *Furthermore, we define, for* $\omega > 0$, *the interval*

$$\mathcal{I}(\omega) := (E_j - \omega\, ,\, E_j + \omega)\,. \tag{46}$$

Let $F \in C_0^\infty(\mathcal{I}(\delta/2)\,;[0,1])$ *with* $F \equiv 1$ *on* $\mathcal{I}(\delta/4)$ *and let* F *be symmetric w.r.t.* E_j, *i.e.,* $F(E_j - \lambda) = F(E_j + \lambda)$. *The metastable state is defined as*

$$\Phi := F^{1/2}(H_g)\, \phi. \tag{47}$$

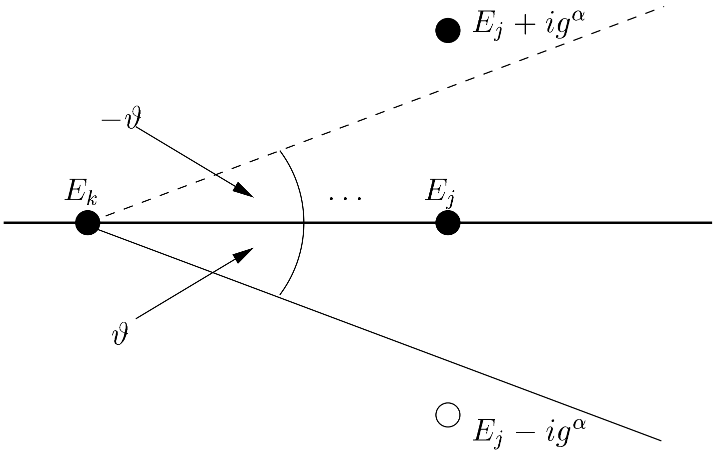

Figure 5: Illustration of the g-dependence of the analytic dilatation of ϕ.

We define two further states which are useful for our analysis,

$$\mathcal{F}\phi := (H_g - E_j)\phi \tag{48}$$

and

$$\xi_k := -ig^\alpha \left(H_\mathrm{f} + E_k - E_j - ig^\alpha\right)^{-1} P_k W_g \,\varphi_j \otimes \Omega, \qquad k = 0, \dots, j-1. \tag{49}$$

2.3 Remark *The complex region in which $\phi(\theta) = U(\theta)\phi$ has analytic continuation depends on the coupling constant g. The resolvents $(H_\mathrm{f} + E_k - E_j - ig^\alpha)^{-1}$ acting on the range of P_k $(k < j)$ are the terms that confine the dilatation parameter θ to a region that decreases in g. We observe that*

$$\left(e^{-\theta} H_\mathrm{f} + E_k - E_j - ig^\alpha\right)^{-1} = e^\theta \left(H_\mathrm{f} - e^\theta(-E_k + E_j + ig^\alpha)\right)^{-1}$$

diverges, for $\mathrm{Im}(\theta) \to \arctan\left(\frac{g^\alpha}{E_k - E_j}\right)$, because the spectral parameter $e^\theta(-E_k + E_j + ig^\alpha)$ becomes positive. Thus, for a fixed $g > 0$, we bound $\vartheta := \mathrm{Im}(\theta)$ by

$$\vartheta > \arctan\left(\frac{g^\alpha}{E_k - E_j}\right).$$

The boundary correponds to the critical angle ϑ in Fig. 5, so that the spectrum of $e^{-\theta} H_\mathrm{f} + E_k$ (the ray in the figure) touches the pole $E_j + ig^\alpha$.

Since we consider matrix elements of the type $\left\langle\, \phi(\overline{\theta}) \;\middle|\; A(\theta)\, \phi(\theta) \,\right\rangle_{\mathcal{H}}$, we get an equal bound on $-\vartheta = \mathrm{Im}(\overline{\theta})$, i.e., for the reflected ray in Fig. 5. Thus, the angle is confined to the sector

$$|\vartheta| < \arctan\left(\frac{g^\alpha}{E_j - E_k}\right) \overset{g\to 0}{\longrightarrow} 0.$$

Note that the bound on ϑ is due to the presence of resolvents in the definition of ϕ. This is in contrast to [BFS3], where metastable states were given by the electron eigenstates, such that dilatation analyticity holds true, uniformly in the coupling constant g.

Remark 2.3 suggests to choose the dilatation parameter ϑ dependend on the interaction strength g to ensure dilatation analyticity.

2.4 Lemma (Dilatation Analyticity of ϕ) *The states $\phi, \mathcal{F}\phi, \xi_k$ are dilatation analytic in the sense of Definition 1.5. More precisely, let $\beta \geq \alpha$ and define*

$$\vartheta_g := C_\vartheta g^\beta, \tag{50}$$

where $C_\vartheta > 0$ is a sufficiently small positive constant so that

$$\frac{g^\alpha}{\tan(2\vartheta_g)} \geq \sup_{k=0,\ldots,j-1} (E_j - E_k + 2), \tag{51}$$

for $g > 0$ sufficiently small. Then the maps $\mathbb{R} \ni \theta \mapsto \phi(\theta), \mathcal{F}\phi(\theta), \xi_k(\theta)$ have Hilbert space valued analytic continuations to the disc $U_{2\vartheta_g} = \{\theta \in \mathbb{C} \mid |\theta| < 2\vartheta_g\}$, for $g > 0$ sufficiently small, namely

$$\begin{aligned} \phi(\theta) \;=\; & \Bigg[\mathbb{1}_{\mathcal{H}} - \Bigg\{ \sum_{k=0}^{j-1} P_k(\theta) \left(e^{-\theta} H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1} P_k(\theta) \\ & \quad + P_j(\theta) \left(e^{-\theta} H_{\mathrm{f}} + g^\gamma\right)^{-1} P_j(\theta) \\ & \quad + P_{>j}(\theta) \left(H_0(\theta) P_{>j}(\theta) - E_j\right)^{-1} P_{>j}(\theta) \Bigg\} W_g(\theta) \Bigg] \varphi_j(\theta) \otimes \Omega, \end{aligned} \tag{52}$$

$$\mathcal{F}\phi(\theta) = (H_g(\theta) - E_j)\phi(\theta), \tag{53}$$

$$\xi_k(\theta) = -ig^\alpha \left(e^{-\theta} H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1} P_k(\theta) W_g(\theta) \varphi_j(\theta) \otimes \Omega. \tag{54}$$

Proof: For $\theta \in \mathbb{R}$, the dilatation $\phi(\theta)$ looks like in (52). We already proved in Corollary 1.6, Lemma 1.7 and Corollary 1.8 that the eigenstate $\varphi_j \otimes \Omega$, the projections P_k (and therefore also the projection $P_{>j} = \mathbb{1}_{\mathcal{H}} - \sum_{k=0}^{j} P_k$) and the interaction W_g are dilatation analytic vectors and operators, resp. It remains to show the analyticity of the resolvents in (52).

For a fixed $g > 0$ and $|\theta| < \pi/4$, we have (using functional calculus for normal operators):

$$\left\| \partial_\theta \left(e^{-\theta} H_{\mathrm{f}} + g^\gamma\right)^{-1} \right\|_{\mathcal{F}} = \left\| e^{-\theta} H_{\mathrm{f}} \left(e^{-\theta} H_{\mathrm{f}} + g^\gamma\right)^{-2} \right\|_{\mathcal{F}} \leq 2g^{-\gamma}.$$

For $|\theta| < 2\vartheta_g$, we have (using the abbreviations $E_{jk} := E_j - E_k > 0$ and $\vartheta := \mathrm{Im}(\theta)$):

$$\begin{aligned} & \left\| \partial_\theta \left(e^{-\theta} H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1} \right\|_{\mathcal{F}} = \left\| e^{-\theta} H_{\mathrm{f}} \left(e^{-\theta} H_{\mathrm{f}} - E_{jk} - ig^\alpha\right)^{-2} \right\|_{\mathcal{F}} \\ = \;& \frac{1}{\cos(\vartheta)} \sup_{r \geq 0} \left\{ \frac{r}{(r - E_{jk})^2 + g^{2\alpha} \left(1 + r \frac{\tan(\vartheta)}{g^\alpha}\right)^2} \right\} \\ \leq \;& \frac{1}{\cos(\vartheta)} \left[1 + E_{jk} + \sup_{|r - E_{jk}| \leq 1} \left\{ \frac{r}{(r - E_{jk})^2 + g^{2\alpha} \left(1 + r \frac{\tan(\vartheta)}{g^\alpha}\right)^2} \right\} \right] \\ \leq \;& \frac{1 + E_{jk}}{\cos(\vartheta)} \left[1 + \left(\frac{E_{jk} + 2}{g^\alpha} \right)^2 \right]. \end{aligned} \tag{55}$$

Here, the last step is implied by (see (51)):

$$\left| r \frac{\tan(\vartheta)}{g^\alpha} \right| \leq \frac{|r|}{E_{jk} + 2} \leq \frac{E_{jk} + |r - E_{jk}|}{E_{jk} + 2} \leq 1 - \frac{1}{E_{jk} + 2},$$

for $|r - E_{jk}| \leq 1$, which implies

$$\left|1 + r\frac{\tan(\vartheta)}{g^\alpha}\right| \geq 1 - \left|r\frac{\tan(\vartheta)}{g^\alpha}\right| \geq \frac{1}{E_{jk}+2}.$$

To show the analyticity of $\left(H_0(\theta)P_{>j}(\theta) - E_j\right)^{-1} P_{>j}(\theta)$, we cite Thm. 1.9. Using Eq. (36) and the analyticity of $H_0(\theta), P_{>j}(\theta)$, we have:

$$\left\|\partial_\theta \left(H_0(\theta)P_{>j}(\theta) - E_j\right)^{-1} P_{>j}(\theta)\right\|_{\mathcal{H}_{\mathrm{el}}} \leq \mathrm{const}(E_{j+1} - E_j)^{-1}$$

(uniformly in $|\vartheta| < \pi/4$).

The proofs of the dilatation analyticity of $\mathcal{F}\phi$ and ξ_k are similar. ■

For the estimate of the time-decay of the metastable state Φ, we have to examine the behaviour of the dilated states $\phi(\theta), \mathcal{F}\phi(\theta), \xi_k(\theta)$ in the limit $g \to 0$. As mentioned in Remark 2.3 and Lemma 2.4, the dilatation parameter $\theta = \theta_g$ is assumed to approach 0, as well. The effects of both limits are studied in the following lemmata.

2.5 Lemma *Let $\theta_g := \pm i\vartheta_g = \mathcal{O}\left(g^\beta\right)$ and $0 < \alpha \leq \beta \leq 1$, $0 < \gamma \leq 1$. Then, the states $\phi, \mathcal{F}\phi, \tau_k$ defined in (45), (48), (49) obey the following estimates:*

$$\begin{aligned} \left(\phi(\theta_g) - \varphi_j(\theta_g) \otimes \Omega\right) &= \mathcal{O}\left(g^{1-\max\{\frac{\beta}{2},\frac{\gamma}{2}\}}\right), && (56)\\ \xi_k(\theta_g) &= \mathcal{O}\left(g^{1+\alpha-\frac{\beta}{2}}\right), && (57)\\ \mathcal{F}\phi(\theta_g) - \sum_{k=0}^{j-1}\xi_k(\theta_g) &= \mathcal{O}\left(g^{1+\min\{\frac{\gamma}{2},1-\frac{\beta}{2}\}}\right), && (58) \end{aligned}$$

and therefore

$$\mathcal{F}\phi(\theta_g) = \mathcal{O}\left(g^{1+\min\{\frac{\gamma}{2},\alpha-\frac{\beta}{2}\}}\right). \tag{59}$$

Proof: We first establish Eq. (59).

$$\begin{aligned} &\left(H_g(\theta_g) - E_j\right)\phi(\theta_g)\\ &= \Bigg[\left(H_0(\theta_g) + W_g(\theta_g) - E_j\right) - \Bigg\{\sum_{k=0}^{j-1}\left(E_k + e^{-\theta_g}H_{\mathrm{f}} + W_g(\theta_g) - E_j\right)P_k(\theta_g)\\ &\qquad + \left(e^{-\theta_g}H_{\mathrm{f}} + W_g(\theta_g)\right)P_j(\theta_g) + \left(H_0(\theta_g) + W_g(\theta_g) - E_j\right)P_{>j}(\theta_g)\Bigg\}\cdot\\ &\qquad \cdot\Bigg\{\sum_{k=0}^{j-1}\left(e^{-\theta_g}H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1}P_k(\theta_g) + \left(e^{-\theta_g}H_{\mathrm{f}} + g^\gamma\right)^{-1}P_j(\theta_g)\\ &\qquad\quad + \left(H_0(\theta_g)P_{>j}(\theta_g) - E_j\right)^{-1}P_{>j}(\theta_g)\Bigg\}W_g(\theta_g)\Bigg]\varphi_j(\theta_g)\otimes\Omega\\ &= \sum_{k=0}^{j-1}\Big[\xi_k(\theta_g) - \underbrace{W_g(\theta_g)\left(e^{-\theta_g}H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1}P_k(\theta_g)W_g(\theta_g)\,\varphi_j(\theta_g)\otimes\Omega}_{=:T_1^k(\theta_g)}\Big]\\ &\quad - \underbrace{\left(W_g(\theta_g) - g^\gamma\right)\left(e^{-\theta_g}H_{\mathrm{f}} + g^\gamma\right)^{-1}P_j(\theta_g)W_g(\theta_g)\,\varphi_j(\theta_g)\otimes\Omega}_{=:T_2(\theta_g)}\\ &\quad - \underbrace{W_g(\theta_g)\left(H_0(\theta_g)P_{>j}(\theta_g) - E_j\right)^{-1}P_{>j}(\theta_g)W_g(\theta_g)\,\varphi_j(\theta_g)\otimes\Omega}_{=:T_3(\theta_g)} \end{aligned}$$

$$= \sum_{k=0}^{j-1} \left[\xi_k(\theta_g) - T_1^k(\theta_g)\right] - T_2(\theta_g) - T_3(\theta_g).$$

We estimate the four terms $\xi_k(\theta_g), T_1^k(\theta_g), T_2(\theta_g), T_3(\theta_g)$ separately. It is obvious that $T_3(\theta_g) = \mathcal{O}\left(g^2\right)$, because $\left\|W_g(\theta)\left(H_0+i\right)^{-1}\right\|_{\mathcal{H}} = \mathcal{O}\left(g^1\right)$. Next, we concentrate on the computation of $\xi_k(\theta_g)$:

$$\begin{aligned} \|\xi_k(\theta_g)\|_{\mathcal{H}} &= g^\alpha \left\|\left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1} P_k(\theta_g) W_g(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega\right\|_{\mathcal{H}} \\ &= \mathcal{O}\left(g^{1+\alpha-\frac{\beta}{2}}\right), \end{aligned}$$

where we used Lemma 2.7 to estimate the norm. Analogously, we see that

$$\begin{aligned} \left\|T_1^k(\theta_g)\right\|_{\mathcal{H}} &\leq \|W_g(\theta_g) P_k(\theta_g)\|_{\mathcal{H}} \cdot \\ &\quad \cdot \left\|\left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j - ig^\alpha\right)^{-1} P_k(\theta_g) W_g(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega\right\|_{\mathcal{H}} \\ &= \mathcal{O}\left(g^{2-\frac{\beta}{2}}\right). \end{aligned}$$

To estimate $T_2(\theta_g)$, we introduce the polar decomposition of the normal operator

$$\left(e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right)^{-1} = \left|e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right|^{-1/2} U \left|e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right|^{-1/2}, \tag{60}$$

where U is a unitary operator. Using this equation, we have

$$\begin{aligned} \|T_2(\theta_g)\|_{\mathcal{H}} &= \left\|\left(W_g(\theta_g) - g^\gamma\right) P_j(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right)^{-1} P_j(\theta_g) W_g(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega\right\|_{\mathcal{H}} \\ &\leq \left\|\left(W_g(\theta_g) - g^\gamma\right) P_j(\theta_g)\right\|_{\mathcal{H}} \left\|\left|e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right|^{-1/2}\right\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \cdot \\ &\quad \cdot \left\|\left|e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right|^{-1/2} P_j(\theta_g) W_g(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega\right\|_{\mathcal{H}}. \end{aligned}$$

To obtain

$$\left\|\left|e^{-\theta_g} H_{\mathrm{f}} + g^\gamma\right|^{-1/2} P_j(\theta_g) W_g(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega\right\|_{\mathcal{H}} = \mathcal{O}\left(g^1\right),$$

we refer to Lemma 2.8. This estimate implies that $\|T_2(\theta_g)\|_{\mathcal{H}} = \mathcal{O}\left(g^{1+\frac{\gamma}{2}}\right)$, and finally

$$\begin{aligned} \mathcal{F}\phi(\theta_g) - \sum_{k=0}^{j-1} \xi_k(\theta_g) &= -\sum_{k=0}^{j-1} T_1^k(\theta_g) - T_2(\theta_g) - T_3(\theta_g) = \mathcal{O}\left(g^{1+\min\{\frac{\gamma}{2}, 1-\frac{\beta}{2}\}}\right), \\ \mathcal{F}\phi(\theta_g) &= \sum_{k=0}^{j-1} \xi_k(\theta_g) + \mathcal{O}\left(g^{1+\min\{\frac{\gamma}{2}, 1-\frac{\beta}{2}\}}\right) = \mathcal{O}\left(g^{1+\min\{\frac{\gamma}{2}, \alpha-\frac{\beta}{2}\}}\right). \end{aligned}$$

The estimate of (56) is similar. ■

2.6 Remark *Later, we choose β smaller than $\varepsilon/2 < 1/9$ in order to use properties of the dilated spectrum of H_g. To get the largest possible exponent in (57) and (59), we choose α as large as possible, i.e. $\alpha = \beta$, in accordance with Lemma 2.4. Furthermore,* *we choose* $\gamma \geq \alpha = \beta$. *Then, the Eqs. (56) – (59) read:*

$$\begin{aligned} (\phi(\theta_g) - \varphi_j(\theta_g) \otimes \Omega) &= \mathcal{O}\left(g^{1-\frac{\gamma}{2}}\right), & (61) \\ \xi_k(\theta_g) &= \mathcal{O}\left(g^{1+\frac{\beta}{2}}\right), & (62) \\ \mathcal{F}\phi(\theta_g) - \sum_{k=0}^{j-1} \xi_k(\theta_g) &= \mathcal{O}\left(g^{1+\frac{\gamma}{2}}\right), & (63) \\ \mathcal{F}\phi(\theta_g) &= \mathcal{O}\left(g^{1+\frac{\beta}{2}}\right). & (64) \end{aligned}$$

2.7 Lemma *Let* $0 < \alpha \leq \beta < 1/9$, ϑ_g *as defined in (50), and* $\theta_g := \pm i\vartheta_g = \mathcal{O}\left(g^{\beta}\right)$. *Abbreviate* $E_{jk} := E_j - E_k > 0$. *Then, we have the estimates:*

$$\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - i g^{\alpha}\right)^{-1} W_g(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} = \mathcal{O}\left(g^{1-\frac{\beta}{2}}\right) \tag{65}$$

and

$$\begin{aligned} &\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} \mp \omega + is\right)^{-1} \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - i g^{\alpha}\right)^{-1} W_g(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} \\ &\qquad = \mathcal{O}\left(g^{1-\frac{\beta}{2}}\right), \end{aligned} \tag{66}$$

uniformly in $\delta/4 \leq \omega \leq \delta/2$ *and* $-\mathrm{const}\, g^2 \leq s \leq \mathrm{const}\, g^2$.

Proof: Note that

$$W_g(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega = \left(g W_{1,0}(\theta_g) + \sum_{m+n=2} g^2 W_{m,n}(\theta_g) \right) \varphi_j(\theta_g) \otimes \Omega,$$

because of

$$W_{0,1}(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega = \int d\underline{k} \, w_{0,1}(\underline{k}, \theta_g) \, \varphi_j(\theta_g) \otimes a(\underline{k}) \, \Omega = 0.$$

We start with Eq. (65). First, we consider for $m + n = 2$:

$$\begin{aligned} &\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - i g^{\alpha}\right)^{-1} g^2 W_{m,n}(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} \\ \leq \;& g^2 \left\| \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - i g^{\alpha}\right)^{-1} \right\|_{\mathcal{F}} \left\| W_{m,n}(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} = \mathcal{O}\left(g^{2-\alpha}\right). \end{aligned}$$

Thus, it remains to consider the case $(m, n) = (1, 0)$. To this end, we write the projection $P_k(\theta_g) = P_{\mathrm{el}}^k(\theta_g) \otimes \mathbb{1}_{\mathcal{F}}$ as a finite sum

$$P_{\mathrm{el}}^k(\theta_g) \, \psi = \sum_{l=1}^{N_k} \left\langle \varphi_k^l(\overline{\theta}_g) \,\middle|\, \psi \right\rangle_{\mathcal{H}_{\mathrm{el}}} \varphi_k^l(\theta_g), \qquad \psi \in \mathcal{H}_{\mathrm{el}}, \tag{67}$$

where φ_k^l, $l = 1, \ldots, N_k$, are the (dilatation analytic) normalized eigenstates of H_{el} corresponding to the eigenvalue E_k (Corollary 1.8). Furthermore, we make use of the

Pull-Through formula (Lemma A.1) and derive the following estimate,

$$\begin{aligned}
&\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - ig^\alpha\right)^{-1} g W_{1,0}(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}}^2 \\
&= g^2 \left\| \int d\underline{k}\, P_k(\theta_g) w_{1,0}(\underline{k},\theta_g)\, \varphi_j(\theta_g) \otimes a(\underline{k})^* \left(e^{-\theta_g}(H_{\mathrm{f}} + \omega(\vec{k})) - E_{jk} - ig^\alpha\right)^{-1} \Omega \right\|_{\mathcal{H}}^2 \\
&= g^2 \left\| \int d\underline{k} \sum_{l=1}^{N_k} \varphi_k^l(\theta_g) \frac{\langle\, \varphi_k^l(\overline{\theta}_g) \,|\, w_{1,0}(\underline{k},\theta_g)\, \varphi_j(\theta_g)\, \rangle_{\mathcal{H}_{\mathrm{el}}}}{e^{-\theta_g}\omega(\vec{k}) - E_{jk} - ig^\alpha} \otimes a(\underline{k})^*\, \Omega \right\|_{\mathcal{H}}^2 \\
&\leq g^2 \left[\sum_{l=1}^{N_k} \|\varphi_k^l(\theta_g)\|_{\mathcal{H}_{\mathrm{el}}} \left\| \int d\underline{k}\, \frac{\langle\, \varphi_k^l(\overline{\theta}_g) \,|\, w_{1,0}(\underline{k},\theta_g)\, \varphi_j(\theta_g)\, \rangle_{\mathcal{H}_{\mathrm{el}}}}{e^{-\theta_g}\omega(\vec{k}) - E_{jk} - ig^\alpha} a(\underline{k})^*\, \Omega \right\|_{\mathcal{F}} \right]^2 \\
&\leq g^2 N_k \max_{l=1,\dots,N_k} \|\varphi_k^l(\theta_g)\|_{\mathcal{H}_{\mathrm{el}}}^2 \int d\underline{k} \left| \frac{\langle\, \varphi_k^l(\overline{\theta}_g) \,|\, w_{1,0}(\underline{k},\theta_g)\, \varphi_j(\theta_g)\, \rangle_{\mathcal{H}_{\mathrm{el}}}}{e^{-\theta_g}\omega(\vec{k}) - E_{jk} - ig^\alpha} \right|^2 \\
&\leq g^2 C \int d\underline{k}\, \frac{J(\vec{k})^2}{\left|e^{-\theta_g}\omega(\vec{k}) - E_{jk} - ig^\alpha\right|^2} \leq g^2 C' \int_0^\infty \frac{r^{-1} + \exp(-2r^4)}{|e^{\mp i\vartheta_g} r - E_{jk} - ig^\alpha|^2}\, r^2\, dr
\end{aligned}$$

for some suitable constants $C, C' < \infty$, where C can be choosen as

$$C := N_k \max_{l=1,\dots,N_k} \|\varphi_k^l(\theta_g)\|_{\mathcal{H}_{\mathrm{el}}}^2 \|\varphi_k^l(\overline{\theta}_g)\|_{\mathcal{H}_{\mathrm{el}}}^2 \left\|(-\Delta_x + 1)^{1/2} \varphi_j(\theta_g)\right\|_{\mathcal{H}_{\mathrm{el}}}^2 .$$

The constant C' takes the transformation to polar coordinates and the asymptotic behaviour of $J(\vec{k})$ into account. Now, we prove that the last integral is of order $\mathcal{O}\left(g^{-\beta}\right)$. We apply the bound (51) on ϑ_g again in a similar fashion as in (55):

$$\begin{aligned}
&\int_0^\infty \frac{r\left(1 + r\exp(-2r^4)\right)}{(\cos(\vartheta_g)r - E_{jk})^2 + (\sin(\pm\vartheta_g)r + g^\alpha)^2}\, dr \\
&\leq \cos(\vartheta_g)^{-3} \Bigg[\int_{|r-E_{jk}|>1} \frac{r\left(1 + r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r - E_{jk})^2}\, dr \\
&\qquad + \int_{|r-E_{jk}|\leq 1} \frac{r\left(1 + r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r - E_{jk})^2 + \tan(\vartheta_g)^2 \left(r + \frac{g^\alpha}{\tan(\pm\vartheta_g)}\right)^2}\, dr \Bigg] \\
&\leq \cos(\vartheta_g)^{-3} \left[C'' + C''' \frac{1}{\tan(\vartheta_g)^2} \int_{-\infty}^{\infty} \frac{\tan(\vartheta_g)}{1 + \left(\frac{r}{\tan(\vartheta_g)}\right)^2} \frac{dr}{\tan(\vartheta_g)} \right] \\
&= \mathcal{O}(\tan(\vartheta_g)^{-1}) = \mathcal{O}\left(g^{-\beta}\right),
\end{aligned} \tag{68}$$

with $C'', C''' < \infty 0$. Taking square roots, we have:

$$\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - ig^\alpha\right)^{-1} g W_{1,0}(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} = \mathcal{O}\left(g^{\frac{2-\beta}{2}}\right) = \mathcal{O}\left(g^{1-\frac{\beta}{2}}\right),$$

which proves (65).

Now, we turn to the proof of (66). For $m + n = 2$, we estimate as above:

$$\begin{aligned}
&\left\| P_k(\theta_g) \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} \mp \omega + is\right)^{-1} \left(e^{-\theta_g} H_{\mathrm{f}} - E_{jk} - ig^\alpha\right)^{-1} g^2 W_{m,n}(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega \right\|_{\mathcal{H}} \\
&\quad = \mathcal{O}\left(g^{2-2\beta}\right) = \mathcal{O}\left(g^{1-\frac{\beta}{2}}\right).
\end{aligned}$$

In the case of $(m,n) = (1,0)$, we proceed similarly to (65). Using the *Pull-Through formula*, twice we get (as above), for a suitable constant $C < \infty$:

$$\begin{aligned} &\left\| P_k(\theta_g)\left(e^{-\theta_g}H_{\mathrm{f}} - E_{jk} \mp \omega + is\right)^{-1}\left(e^{-\theta_g}H_{\mathrm{f}} - E_{jk} - ig^{\alpha}\right)^{-1} gW_{1,0}(\theta_g)\varphi_j(\theta_g)\otimes\Omega\right\|_{\mathcal{H}}^2 \\ &\quad \le \frac{g^2C}{\cos(\vartheta_g)^3}\int_0^\infty \Bigg[\frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk}\mp\omega)^2+\tan(\vartheta_g)^2\left(r-\frac{s}{\tan(\pm\vartheta_g)}\right)^2} \\ &\qquad\qquad \cdot \frac{1}{(r-E_{jk})^2+\tan(\vartheta_g)^2\left(r+\frac{g^{\alpha}}{\tan(\pm\vartheta_g)}\right)^2}\Bigg]\, dr. \end{aligned} \tag{69}$$

The integral in (69) can be split as

$$\begin{aligned} \int_0^\infty \Big[\dots\Big]\, dr \quad &\le \frac{64}{\delta^2}\int_{|r-E_{jk}|>\frac{\delta}{8}} \frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk}\mp\omega)^2+\tan(\vartheta_g)^2\left(r-\frac{s}{\tan(\pm\vartheta_g)}\right)^2}\, dr \\ &\quad + \frac{64}{\delta^2}\int_{|r-E_{jk}|\le\frac{\delta}{8}} \frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk})^2+\tan(\vartheta_g)^2\left(r+\frac{g^{\alpha}}{\tan(\pm\vartheta_g)}\right)^2}\, dr \\ &\le \frac{64}{\delta^2}\int_0^\infty \frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk}\mp\omega)^2+\tan(\vartheta_g)^2\left(r-\frac{s}{\tan(\pm\vartheta_g)}\right)^2}\, dr + \mathcal{O}\left(g^{-\beta}\right), \end{aligned}$$

since $(r-E_{jk})^2 > \frac{\delta^2}{64}$ for $|r-E_{jk}| > \frac{\delta}{8}$ and $(r-E_{jk}\mp\omega)^2 \ge (|\omega| - |r-E_{jk}|)^2 \ge (\frac{\delta}{4}-\frac{\delta}{8})^2 = \frac{\delta^2}{64}$ for $|r-E_{jk}| \le \frac{\delta}{8}$. The second integral was already estimated in (68) to be a term of order $\mathcal{O}\left(g^{-\beta}\right)$. We consider the remaining integral. First, we note that

$$E_{jk} \pm \omega \ge E_j - E_{j-1} - \frac{\delta}{2} \ge \frac{\delta}{2} > 0.$$

Since $|s| \le \mathrm{const}\, g^2$ and $\tan(\pm\vartheta_g) = \mathcal{O}\left(g^{\beta}\right)$ with $\beta < 1/9$ we have

$$\left|\frac{s}{\tan(\pm\vartheta_g)}\right| < \frac{\delta}{8}$$

for $g > 0$ sufficiently small. Thus

$$\begin{aligned} &\int_0^\infty \frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk}\mp\omega)^2+\tan(\vartheta_g)^2\left(r-\frac{s}{\tan(\pm\vartheta_g)}\right)^2}\, dr \\ &\quad = \mathcal{O}\left(g^0\right) + \int_{\delta/4}^\infty \frac{r\left(1+r\exp(-2r^4\cos(\vartheta_g)^{-4})\right)}{(r-E_{jk}\mp\omega)^2+\tan(\vartheta_g)^2\left(r-\frac{s}{\tan(\pm\vartheta_g)}\right)^2}\, dr = \mathcal{O}\left(g^{-\beta}\right), \end{aligned}$$

as in (68). Inserting these estimates in (69), we arrive at

$$\begin{aligned} &\left\| P_k(\theta_g)\left(e^{-\theta_g}H_{\mathrm{f}} - E_{jk} \mp \omega + is\right)^{-1}\left(e^{-\theta_g}H_{\mathrm{f}} - E_{jk} - ig^{\alpha}\right)^{-1} gW_{1,0}(\theta_g)\varphi_j(\theta_g)\otimes\Omega\right\|_{\mathcal{H}}^2 \\ &\quad = \mathcal{O}\left(g^{2-\beta}\right) \end{aligned}$$

which proves (66). ■

2.8 Lemma *Let $\theta = i\vartheta \in U_{\vartheta_g}$, i.e., $|\vartheta| \le \vartheta_g$ and $\gamma \le 2$. Then, we have*

$$\left\| P_j(\theta)\left|e^{-\theta}H_{\mathrm{f}} + g^{\gamma}\right|^{-1/2} W_g(\theta)\left|e^{-\theta}H_{\mathrm{f}} + g^{\gamma}\right|^{-1/2} P_j(\theta)\right\|_{\mathcal{H}} = \mathcal{O}\left(g^{1-\frac{\gamma}{2}}\right), \tag{70}$$

$$\left\| P_j(\theta)\left|e^{-\theta}H_{\mathrm{f}} + g^{\gamma}\right|^{-1/2} W_g(\theta)\varphi_j(\theta)\otimes\Omega\right\|_{\mathcal{H}} = \mathcal{O}\left(g^{1}\right), \tag{71}$$

uniformly in $\vartheta \in (-\vartheta_g, +\vartheta_g)$.

Proof: Recall that $W_g(\theta) = \sum_{m+n=1,2} g^{m+n} W_{m,n}(\theta)$. First, we derive a lemma:

Proposition *There is a constant $C > 0$ so that for $m + n = 1$:*

$$\begin{aligned} &\left\| P_j(\theta)\left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} W_{m,n}(\theta)\left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} P_j(\theta)\right\|_{\mathcal{H}} \\ &\qquad \le C\sqrt{\frac{2\left(\frac{1+\mu}{\cos(\vartheta)}\Lambda_{-1}^2 + \Lambda_0^2\right)}{\mu}} \end{aligned} \tag{72}$$

for all $\mu > 0$.

Proof of (72): Let $\psi \in \mathcal{H}$. Using the *Pull-Through formula* (Lemma A.1), we obtain:

$$\begin{aligned} &\left\| P_j(\theta)\left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} W_{0,1}(\theta)\left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} P_j(\theta)\,\psi\right\|_{\mathcal{H}} \\ &= \left\| \int \left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} P_j(\theta) w_{0,1}(\underline{k},\theta) P_j(\theta)\left|e^{-\theta}(H_{\mathrm{f}} + \omega(\vec{k})) + \mu\right|^{-1/2} a(\underline{k})\,\psi\, d\underline{k}\right\|_{\mathcal{H}} \\ &\le \left[\int \left\| \left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} P_j(\theta) w_{0,1}(\underline{k},\theta) P_j(\theta)\left|e^{-\theta}(H_{\mathrm{f}} + \omega(\vec{k})) + \mu\right|^{-1/2}\right.\right. \\ &\qquad \left.\left. \cdot\,(H_{\mathrm{f}} + \omega(\vec{k}))^{1/2}\right\|_{\mathcal{H}}^2 \frac{d\underline{k}}{\omega(\vec{k})}\right]^{1/2} \left[\int \left\|(H_{\mathrm{f}} + \omega(\vec{k}))^{-1/2} a(\underline{k})\,\psi\right\|_{\mathcal{H}}^2 \omega(\vec{k})\, d\underline{k}\right]^{1/2}. \end{aligned}$$

The bound of the second factor in (73) follows from

$$\begin{aligned} &\int \left\|(H_{\mathrm{f}} + \omega(\vec{k}))^{-1/2} a(\underline{k})\psi\right\|_{\mathcal{H}}^2 \omega(\vec{k})\, d\underline{k} \\ &= \int \left\langle (H_{\mathrm{f}} + \omega(\vec{k}))^{-1/2} a(\underline{k})\psi \,\middle|\, (H_{\mathrm{f}} + \omega(\vec{k}))^{-1/2} a(\underline{k})\psi \right\rangle_{\mathcal{H}} \omega(\vec{k})\, d\underline{k} \\ &= \int \left\langle a(\underline{k})(H_{\mathrm{f}} P_{\Omega}^{\perp})^{-1/2} P_{\Omega}^{\perp}\,\psi \,\middle|\, a(\underline{k})(H_{\mathrm{f}} P_{\Omega}^{\perp})^{-1/2} P_{\Omega}^{\perp}\,\psi \right\rangle_{\mathcal{H}} \omega(\vec{k})\, d\underline{k} = \left\|P_{\Omega}^{\perp}\,\psi\right\|_{\mathcal{H}}^2, \end{aligned}$$

where $P_{\Omega}^{\perp} = \mathbb{1}_{\mathcal{H}} - P_{\Omega}$ and $P_{\Omega} \equiv \mathbb{1}_{\mathcal{H}_{\mathrm{el}}} \otimes P_{\Omega}$ denotes the orthogonal projection on the kernel $\mathcal{H}_{\mathrm{el}} \otimes \mathbb{C}\Omega$ of $H_{\mathrm{f}} \equiv \mathbb{1}_{\mathcal{H}_{\mathrm{el}}} \otimes H_{\mathrm{f}}$. The integrand of the first factor in (73) is estimated as follows,

$$\begin{aligned} &\left\| \left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} P_j(\theta) w_{0,1}(\underline{k},\theta) P_j(\theta)\left|e^{-\theta}(H_{\mathrm{f}} + \omega(\vec{k})) + \mu\right|^{-1/2} (H_{\mathrm{f}} + \omega(\vec{k}))^{1/2}\right\|_{\mathcal{H}} \\ &\quad \le \left\| \left|e^{-\theta}H_{\mathrm{f}} + \mu\right|^{-1/2} (H_{\mathrm{f}} + \omega(\vec{k}))^{1/2}\right\|_{\mathcal{F}} \cdot \left\|P_j(\theta) w_{0,1}(\underline{k},\theta) P_j(\theta)\right\|_{\mathcal{H}_{\mathrm{el}}} \\ &\qquad \cdot \left\| \left|e^{-\theta}(H_{\mathrm{f}} + \omega(\vec{k})) + \mu\right|^{-1/2}\right\|_{\mathcal{F}} \\ &\quad \le C\sqrt{\frac{1}{\cos(\vartheta)} + \frac{\omega(\vec{k})}{\mu}}\, J(\vec{k})\sqrt{\frac{1}{\omega(\vec{k})\cos(\vartheta) + \mu}} \end{aligned}$$

$$
\begin{aligned}
&\leq \; CJ(\vec{k}) \sqrt{\left(\frac{1}{\cos(\vartheta)} + \frac{\omega(\vec{k})}{\mu}\right)\left(1 + \frac{1}{\omega(\vec{k})\cos(\vartheta) + \mu}\right)} \\
&= \; \frac{CJ(\vec{k})}{\sqrt{\mu \cos(\vartheta)}} \sqrt{1 + \mu + \omega(\vec{k})\cos(\vartheta)}. \qquad (73)
\end{aligned}
$$

Inserting the estimates (73) and (73) into (73) we finally get:

$$
\begin{aligned}
&\left\| P_j(\theta) \left| e^{-\theta} H_{\mathrm{f}} + \mu \right|^{-1/2} W_{0,1}(\theta) \left| e^{-\theta} H_{\mathrm{f}} + \mu \right|^{-1/2} P_j(\theta) \right\|_{\mathcal{H}} \\
&\leq \; \frac{C}{\sqrt{\mu \cos(\vartheta)}} \left[\int \frac{1 + \mu + \omega(\vec{k})\cos(\vartheta)}{\omega(\vec{k})} J(\vec{k})^2 \, d\underline{k} \right]^{1/2} = C \sqrt{\frac{2\left(\frac{1+\mu}{\cos(\vartheta)}\Lambda_{-1}^2 + \Lambda_0^2\right)}{\mu}}.
\end{aligned}
$$

Replacing $W_{0,1}(\theta)$ by $W_{1,0}(\theta) = W_{0,1}(\overline{\theta})^*$ and similar estimates finally prove (72). □

We return to the proof of Eq. (70). Since, obviously,

$$
\left\| P_j(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} g^2 W_{m,n}(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} P_j(\theta) \right\|_{\mathcal{H}} = \mathcal{O}\left(g^{2-\gamma}\right),
$$

for $m + n = 2$, it is sufficient to consider the case $m + n = 1$. But this case easily handled using (72) and replacing μ by g^γ,

$$
\begin{aligned}
&\left\| P_j(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} g W_{m,n}(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} P_j(\theta) \right\|_{\mathcal{H}} \\
&\leq \; g^{1-\frac{\gamma}{2}} C \sqrt{2\left(\frac{1 + g^\gamma}{\cos(\vartheta)}\Lambda_{-1}^2 + \Lambda_0^2\right)} = \mathcal{O}\left(g^{1-\frac{\gamma}{2}}\right),
\end{aligned}
$$

uniformly in $\vartheta \in (-\vartheta_g, +\vartheta_g)$.

Finally, Eq. (71) follows from:

$$
\begin{aligned}
&\left\| P_j(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} W_g(\theta) \varphi_j(\theta) \otimes \Omega \right\|_{\mathcal{H}} \\
&\leq \; \left\| P_j(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} W_g(\theta) \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{-1/2} P_j(\theta) \right\|_{\mathcal{H}} \cdot \\
&\quad \cdot \left\| \left| e^{-\theta} H_{\mathrm{f}} + g^\gamma \right|^{1/2} \varphi_j(\theta) \otimes \Omega \right\|_{\mathcal{H}} = \mathcal{O}\left(g^1\right),
\end{aligned}
$$

uniformly in $\vartheta \in (-\vartheta_g, +\vartheta_g)$. ■

2.2 Quasi-Exponential Time-Decay Law

2.9 Theorem (Quasi-Exponential Time-Decay of Metastable States) *Let $\Phi = F^{1/2}(H_g)\,\phi \in \mathcal{D}(H_{\mathrm{el}}) \otimes \mathcal{D}(H_{\mathrm{f}}) = \mathcal{D}(H_g)$ be the state defined in (45) and (47). Then there exist constants $C, C_n < \infty 0$, for $n = 0, 1$ such that, for any $0 < \beta < 1/9$,*

$$
\left| \langle\, \Phi \mid \exp(-iH_g t)\Phi \,\rangle_{\mathcal{H}} \right| \leq C \exp(-S_g t) + \frac{C_n}{t^n} g^{4+\beta}, \qquad (74)
$$

for $t > 1$ and sufficiently small $g > 0$.

2.10 Remark *The constants α, γ appearing in the Definition 2.2 of ϕ and the constant β in Corollary 2.4 are fixed by $\alpha := \beta$, where $0 < \beta < 1/9$ is arbitrary, and $\gamma := 1$. $0 < S_g = \mathcal{O}\left(g^2\right)$ is the reciprocal of the* life-time *of the metastable state defined in (38).*

Proof: We roughly follow the strategy of the proof of Thm. III.5 in [BFS3]. We first use the fact that

$$F(\lambda) = -\int_{\lambda-E_j}^{\delta/2} F'(E_j+\omega)\,d\omega = -\int_{\delta/4}^{\delta/2} F'(E_j+\omega)\chi_{\mathcal{I}(\omega)}(\lambda)\,d\omega \tag{75}$$

and *Stones formula* (see e.g. [RS2]), to rewrite

$$\begin{aligned}\langle\, \Phi \mid \exp(-iH_g t)\Phi\,\rangle_{\mathcal{H}} &= \langle\, \phi \mid \exp(-iH_g t)F(H_g)\phi\,\rangle_{\mathcal{H}}\\ &= -\int_{\delta/4}^{\delta/2} F'(E_j+\omega)\,\langle\, \phi \mid \exp(-iH_g t)\chi_{\mathcal{I}(\omega)}(H_g)\phi\,\rangle_{\mathcal{H}}\; d\omega\\ &= -\frac{1}{2\pi i}\lim_{\eta\searrow 0}\int_{\delta/4}^{\delta/2} d\omega\, F'(E_j+\omega)\int_{\mathcal{I}(\omega)} d\lambda\, \exp(-i\lambda t)\\ &\qquad \cdot\left[\left\langle \phi \,\middle|\, (H_g-\lambda-i\eta)^{-1}\,\phi\right\rangle_{\mathcal{H}} - \left\langle \phi \,\middle|\, (H_g-\lambda+i\eta)^{-1}\,\phi\right\rangle_{\mathcal{H}}\right].\end{aligned} \tag{76}$$

Now, we use our results of the dilatation analyticity derived in Sect. 1.3. Fix $0<\varepsilon<2/9$, and let $\theta_g := i\vartheta_g = iC_\vartheta g^\beta$, as defined in (50). We note that the Conditions (39) are satisfied, for $\beta<\varepsilon/2<1/9$ and sufficiently small $g>0$. Thus, appealing to Thm. 1.10, we transform the matrix elements in (76) by complex dilatation, using the fact that ϕ is dilatation analytic on U_{ϑ_g} (Lemma 2.4),

$$\begin{aligned}\left\langle \phi \,\middle|\, (H_g-\lambda-i\eta)^{-1}\,\phi\right\rangle_{\mathcal{H}} &= \left\langle \phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g)-\lambda-i\eta)^{-1}\,\phi(\theta_g)\right\rangle_{\mathcal{H}},\\ \left\langle \phi \,\middle|\, (H_g-\lambda+i\eta)^{-1}\,\phi\right\rangle_{\mathcal{H}} &= \left\langle \phi(\theta_g) \,\middle|\, \left(H_g(\overline{\theta}_g)-\lambda+i\eta\right)^{-1}\,\phi(\overline{\theta}_g)\right\rangle_{\mathcal{H}},\end{aligned}$$

because $H_g(\theta_g)^* = H_g(\overline{\theta}_g)$. Inserting this into (76), we have:

$$\begin{aligned}&\langle\, \Phi \mid \exp(-iH_g t)\Phi\,\rangle_{\mathcal{H}}\\ &= -\frac{1}{2\pi i}\lim_{\eta\searrow 0}\int_{\delta/4}^{\delta/2} d\omega\, F'(E_j+\omega)\int_{\mathcal{I}(\omega)} d\lambda\, \exp(-i\lambda t)\,\cdot\\ &\qquad\cdot\Big[\left\langle \phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g)-\lambda-i\eta)^{-1}\,\phi(\theta_g)\right\rangle_{\mathcal{H}}\\ &\qquad\qquad - \left\langle \phi(\theta_g) \,\middle|\, \left(H_g(\overline{\theta}_g)-\lambda+i\eta\right)^{-1}\,\phi(\overline{\theta}_g)\right\rangle_{\mathcal{H}}\Big]\\ &= -\frac{1}{\pi}\int_{\delta/4}^{\delta/2} d\omega\, F'(E_j+\omega)\int_{\mathcal{I}(\omega)} d\lambda\, e^{-i\lambda t}\,\mathrm{Im}\left\langle \phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g)-\lambda)^{-1}\,\phi(\theta_g)\right\rangle_{\mathcal{H}},\end{aligned} \tag{77}$$

where the limit $\eta\searrow 0$ exists due to the analyticity of $\mathcal{A}^\varepsilon \ni z \mapsto (H_g(\theta_g)-z)^{-1}$. The analyticity also allows us to deform the integration contour along $\mathcal{I}(\omega)$ in (77) into the lower half plane (see Fig. 6). Applying *Cauchy's integral formula*, we get:

$$\langle\, \Phi \mid \exp(-iH_g t)\Phi\,\rangle_{\mathcal{H}} = M_- - M_+ + M_{\|}(\theta_g) - M_{\|}(\overline{\theta}_g), \tag{78}$$

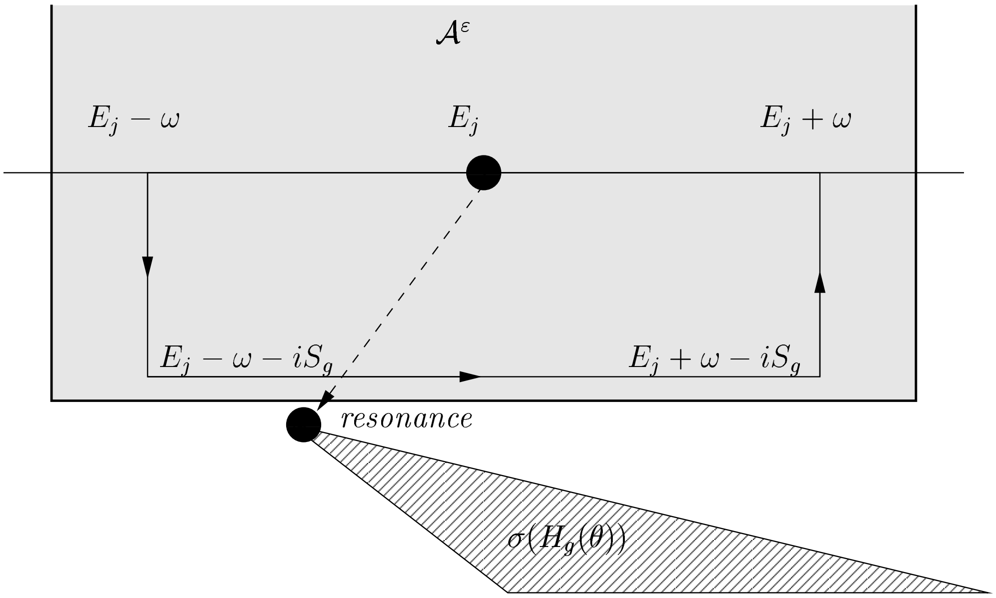

Figure 6: Deformation of the integration contour into the lower half plane.

where

$$\begin{aligned} M_\pm \quad &:= \quad -\int_{\delta/4}^{\delta/2} d\omega \, \frac{F'(E_j+\omega)}{2\pi i} \int_0^{S_g} ds \, \exp(-it(E_j \pm \omega - is)) \\ &\quad \cdot \Big[\Big\langle \phi(\overline{\theta}_g) \Big| (H_g(\theta_g) - E_j \mp \omega + is)^{-1} \, \phi(\theta_g) \Big\rangle_{\mathcal{H}} \\ &\qquad - \Big\langle \phi(\theta_g) \Big| \big(H_g(\overline{\theta}_g) - E_j \mp \omega + is\big)^{-1} \, \phi(\overline{\theta}_g) \Big\rangle_{\mathcal{H}} \Big], \qquad (79) \\ M_\|(\theta_g) \quad &:= \quad -\int_{\delta/4}^{\delta/2} d\omega \, \frac{F'(E_j+\omega)}{2\pi i} \\ &\quad \cdot \int_{\mathcal{I}(\omega)} d\lambda \, e^{-it(\lambda - iS_g)} \Big\langle \phi(\overline{\theta}_g) \Big| (H_g(\theta_g) - \lambda + iS_g)^{-1} \, \phi(\theta_g) \Big\rangle_{\mathcal{H}} . \qquad (80) \end{aligned}$$

Thm. 2.9 follows directly from Lemmata 2.11 and 2.12 below which state that the integration along the horizonzal contour $-iS_g + \mathcal{I}(\omega)$ (see Fig. 6) gives the exponential decay in (74), and the integration along the vertical contour yields the error term $\mathcal{O}(t^{-n} g^{4+\beta})$. ■

2.11 Lemma *Under the conditions of Thm. 2.9 and for $n = 0, 1$, there is a constant $C_n < \infty$, such that*

$$|M_\pm| \le \frac{C_n}{2t^n} g^{4+\beta}, \qquad (81)$$

provided $t > 1$, and $g > 0$ is sufficiently small.

Proof: Recall the definition (79) of $M_\pm$. Since

$$\exp(-it(E_j \pm \omega - is)) = (\mp it)^{-n} \frac{d^n}{d\omega^n} \exp(-it(E_j \pm \omega - is)),$$

for $n = 0, 1$, $t > 0$, we can transform $M_\pm$ into

$$\begin{aligned} M_\pm &= \frac{-i^n}{2\pi i\, t^n} \int_0^{S_g} ds \int_{\delta/4}^{\delta/2} d\omega \, \exp(-it(E_j \pm \omega - is)) \sum_{\nu=0}^{n} \frac{(\pm 1)^\nu n!}{(n-\nu)!} F^{(n-\nu+1)}(E_j + \omega) \cdot \\ &\quad \cdot \Big[\Big\langle \phi(\overline{\theta}_g) \,\Big|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \, \phi(\theta_g) \Big\rangle_{\mathcal{H}} \\ &\quad - \Big\langle \phi(\theta_g) \,\Big|\, \big(H_g(\overline{\theta}_g) - E_j \mp \omega + is\big)^{-\nu-1} \, \phi(\overline{\theta}_g) \Big\rangle_{\mathcal{H}} \Big] \end{aligned} \tag{82}$$

using integration by parts (w.r.t. $d\omega$, boundary terms do not appear, because $F'(E_j + \delta/4) = F'(E_j + \delta/2) = 0$). It is important to consider the difference

$$\begin{aligned} D_\nu(s,\omega) &:= \Big\langle \phi(\overline{\theta}_g) \,\Big|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \, \phi(\theta_g) \Big\rangle_{\mathcal{H}} \\ &\quad - \Big\langle \phi(\theta_g) \,\Big|\, \big(H_g(\overline{\theta}_g) - E_j \mp \omega + is\big)^{-\nu-1} \, \phi(\overline{\theta}_g) \Big\rangle_{\mathcal{H}} . \end{aligned} \tag{83}$$

To compute $D_\nu(s,\omega)$, we observe an operator equality,

$$\begin{aligned} &(H_g(\theta_g) - E_j \mp \omega + is)^{-1} \\ &= \frac{1}{\mp\omega + is} \Big[\mathbb{1}_{\mathcal{H}} - (H_g(\theta_g) - E_j \mp \omega + is)^{-1} (H_g(\theta_g) - E_j) \Big] , \end{aligned} \tag{84}$$

on the domain $\mathcal{D}(H_g) = \mathcal{D}(H_{\mathrm{el}}) \otimes \mathcal{D}(H_{\mathrm{f}})$. Introducing two further definitions,

$$\begin{aligned} D_\nu^{\mathcal{F}}(s,\omega) &:= \Big\langle \phi(\overline{\theta}_g) \,\Big|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \, \mathcal{F}\phi(\theta_g) \Big\rangle_{\mathcal{H}} \\ &\quad - \Big\langle \phi(\theta_g) \,\Big|\, \big(H_g(\overline{\theta}_g) - E_j \mp \omega + is\big)^{-\nu-1} \, \mathcal{F}\phi(\overline{\theta}_g) \Big\rangle_{\mathcal{H}} , \end{aligned} \tag{85}$$

$$\begin{aligned} D_\nu^{\mathcal{F}^2}(s,\omega) &:= \Big\langle \mathcal{F}\phi(\overline{\theta}_g) \,\Big|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \, \mathcal{F}\phi(\theta_g) \Big\rangle_{\mathcal{H}} \\ &\quad - \Big\langle \mathcal{F}\phi(\theta_g) \,\Big|\, \big(H_g(\overline{\theta}_g) - E_j \mp \omega + is\big)^{-\nu-1} \, \mathcal{F}\phi(\overline{\theta}_g) \Big\rangle_{\mathcal{H}} , \end{aligned} \tag{86}$$

and repeatedly replacing the resolvent in (83) by (84), we obtain the recursion relations

$$D_\nu(s,\omega) = \frac{1}{\mp\omega + is} \left[D_{\nu-1}(s,\omega) - D_\nu^{\mathcal{F}}(s,\omega) \right] \tag{87}$$

and

$$D_\nu^{\mathcal{F}}(s,\omega) = \frac{1}{\mp\omega + is} \Big[D_{\nu-1}^{\mathcal{F}}(s,\omega) - D_\nu^{\mathcal{F}^2}(s,\omega) \Big] . \tag{88}$$

By induction in ν, we prove the following

Proposition *For $\nu \in \{-1, 0, \ldots, n\}$, the estimates*

$$\begin{aligned} \sup_{\substack{0 \le s \le S_g \\ \delta/4 \le \omega \le \delta/2}} |D_\nu(s,\omega)| &= \mathcal{O}\left(g^{2+\beta}\right) , \\ \sup_{\substack{0 \le s \le S_g \\ \delta/4 \le \omega \le \delta/2}} |D_\nu^{\mathcal{F}}(s,\omega)| &= \mathcal{O}\left(g^{2+\beta}\right) . \end{aligned} \tag{89}$$

hold true.

Proof of (89): We start the induction with $\nu = -1$ (for technical reasons) and note that

$$\begin{aligned} D_{-1}(s,\omega) &= \left\langle \phi(\overline{\theta}_g) \,\middle|\, \phi(\theta_g) \right\rangle_{\mathcal{H}} - \left\langle \phi(\theta_g) \,\middle|\, \phi(\overline{\theta}_g) \right\rangle_{\mathcal{H}} = 0, \\ D^{\mathcal{F}}_{-1}(s,\omega) &= \left\langle \phi(\overline{\theta}_g) \,\middle|\, \mathcal{F}\phi(\theta_g) \right\rangle_{\mathcal{H}} - \left\langle \phi(\theta_g) \,\middle|\, \mathcal{F}\phi(\overline{\theta}_g) \right\rangle_{\mathcal{H}} = 0 \\ D^{\mathcal{F}^2}_{-1}(s,\omega) &= \left\langle \mathcal{F}\phi(\overline{\theta}_g) \,\middle|\, \mathcal{F}\phi(\theta_g) \right\rangle_{\mathcal{H}} - \left\langle \mathcal{F}\phi(\theta_g) \,\middle|\, \mathcal{F}\phi(\overline{\theta}_g) \right\rangle_{\mathcal{H}} = 0 \end{aligned}$$

because of the dilatation analyticity of ϕ and $\mathcal{F}\phi = (H_g - E_j)\phi$. Let us assume now that (89) is already proven for $\nu - 1 < n = 0, 1$. For the induction, we recall the recursion relations (87) and (88). It is obvious that the proposition is implied by the estimate

$$\sup_{\substack{0 \le s \le S_g \\ \delta/4 \le \omega \le \delta/2}} |D^{\mathcal{F}^2}_{\nu}(s,\omega)| = \mathcal{O}\left(g^{2+\beta}\right), \tag{90}$$

for $\nu = -1, 0, 1$. This is already done for $\nu = -1$. To derive (90) in the cases $\nu = 0, 1$ we note that

$$\left\| (H_g(\theta_g) - E_j \mp \omega + is)^{-1} \right\|_{\mathcal{H}} \le \frac{\text{const}}{\tan(\vartheta_g)\left(|\omega + \mathcal{O}(g^2)| + \mathcal{O}(g^2)\right)} = \mathcal{O}\left(g^{-\beta}\right), \tag{91}$$

uniformly in $\delta/4 \le \omega \le \delta/2$ and $|s| \le S_g$. This result is also derived in [BFS3], Eqn. (III.33) (which is based on Thm. III.2, (III.17), including (III.14)). Furthermore, we have an estimate for the unperturbed resolvent:

$$\left\| (H_0(\theta_g) - E_j \mp \omega + is)^{-1} \right\|_{\mathcal{H}} = \mathcal{O}\left(g^{-\beta}\right). \tag{92}$$

To establish this estimate consider Thm. 1.9, Eq. (35) and conclude that for $\delta/4 \le \omega \le \delta/2$:

$$|\mathrm{dist}\,(E_j \pm \omega - is, \sigma(H_0(\theta)))| \ge 2\omega |\sin(\mathrm{Im}(\theta))|$$

for $g > 0$ sufficiently small and uniformly in $|s| \le S_g$ which implies (92). Combining (91) with (92), we get:

$$\begin{aligned} &\left\| (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu} - (H_0(\theta_g) - E_j \mp \omega + is)^{-\nu} \right\|_{\mathcal{H}} \\ \le\; &\left\| (H_g(\theta_g) - E_j \mp \omega + is)^{-1} \right\|^{\nu}_{\mathcal{H}} \\ &\cdot \Big\| \left[(H_g(\theta_g) - E_j \mp \omega + is)^{\nu} - (H_0(\theta_g) - E_j \mp \omega + is)^{\nu} \right] \\ &\quad \cdot (H_0(\theta_g) - E_j \mp \omega + is)^{-\nu} \Big\|_{\mathcal{H}} = \mathcal{O}\left(g^{1-2\nu\beta}\right). \end{aligned}$$

We are now in position to estimate $D^{\mathcal{F}^2}_{\nu}(s,\omega)$. Abbreviating

$$\xi(\theta_g) := \sum_{k=0}^{j-1} \xi_k(\theta_g), \tag{93}$$

we have (because of Remark 2.6 and (93))

$$\begin{aligned}
&\left\langle \mathcal{F}\phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \mathcal{F}\phi(\theta_g) \right\rangle_{\mathcal{H}} \\
&\quad = \left\langle \xi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \xi(\theta_g) \right\rangle_{\mathcal{H}} \\
&\qquad + \left\langle \xi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} (\mathcal{F}\phi(\theta_g) - \xi(\theta_g)) \right\rangle_{\mathcal{H}} \\
&\qquad + \left\langle \mathcal{F}\phi(\overline{\theta}_g) - \xi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \xi(\theta_g) \right\rangle_{\mathcal{H}} \\
&\qquad + \left\langle \mathcal{F}\phi(\overline{\theta}_g) - \xi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} (\mathcal{F}\phi(\theta_g) - \xi(\theta_g)) \right\rangle_{\mathcal{H}} \\
&\quad = \left\langle \xi(\overline{\theta}_g) \,\middle|\, (H_0(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \xi(\theta_g) \right\rangle_{\mathcal{H}} \\
&\qquad + \left\langle \xi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} - (H_0(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \xi(\theta_g) \right\rangle_{\mathcal{H}} \\
&\qquad + \mathcal{O}\left(g^{2+\frac{\gamma-(2\nu+1)\beta}{2}}\right) + \mathcal{O}\left(g^{2+\frac{\gamma-(2\nu+1)\beta}{2}}\right) + \mathcal{O}\left(g^{2+\gamma-(\nu+1)\beta}\right) \\
&\quad = \sum_{k=0}^{j-1} \left\langle \xi_k(\overline{\theta}_g) \,\middle|\, \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j \mp \omega + is\right)^{-\nu-1} \xi_k(\theta_g) \right\rangle_{\mathcal{H}} \\
&\qquad + \mathcal{O}\left(g^{3-(2\nu+1)\beta}\right) + \mathcal{O}\left(g^{2+\frac{\gamma-(2\nu+1)\beta}{2}}\right) + \mathcal{O}\left(g^{2+\gamma-(\nu+1)\beta}\right),
\end{aligned}$$

where we choose $\gamma = 1$. It remains to consider the scalar products. For $\nu = 0$, we have

$$\begin{aligned}
&\left| \left\langle \xi_k(\overline{\theta}_g) \,\middle|\, \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j \mp \omega + is\right)^{-\nu-1} \xi_k(\theta_g) \right\rangle_{\mathcal{H}} \right| \\
&\quad \leq \left\| \xi_k(\overline{\theta}_g) \right\|_{\mathcal{H}} g^{\alpha} \Big\| \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j \mp \omega + is\right)^{-1} \\
&\qquad\qquad \cdot \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1} P_k(\theta_g) W_g(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \Big\|_{\mathcal{H}} \\
&\quad = \mathcal{O}\left(g^{2+\beta}\right)
\end{aligned}$$

(in the case $\alpha = \beta$), uniformly in $\delta/4 \leq \omega \leq \delta/2$ and $|s| \leq \mathrm{const}\, g^2$. Here we made use of the results of Remark 2.6 and Lemma 2.7, Eq. (66). For $\nu = 1$, the estimate is similar:

$$\begin{aligned}
&\left| \left\langle \xi_k(\overline{\theta}_g) \,\middle|\, \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j \mp \omega + is\right)^{-\nu-1} \xi_k(\theta_g) \right\rangle_{\mathcal{H}} \right| \\
&\quad \leq g^{\alpha} \Big\| \left(e^{-\overline{\theta}_g} H_{\mathrm{f}} + E_k - E_j \mp \omega - is\right)^{-1} \\
&\qquad\qquad \cdot \left(e^{-\overline{\theta}_g} H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1} P_k(\overline{\theta}_g) W_g(\overline{\theta}_g) \, \varphi_j(\overline{\theta}_g) \otimes \Omega \Big\|_{\mathcal{H}} \\
&\qquad \cdot g^{\alpha} \Big\| \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j \mp \omega + is\right)^{-1} \\
&\qquad\qquad \cdot \left(e^{-\theta_g} H_{\mathrm{f}} + E_k - E_j - ig^{\alpha}\right)^{-1} P_k(\theta_g) W_g(\theta_g) \, \varphi_j(\theta_g) \otimes \Omega \Big\|_{\mathcal{H}} = \mathcal{O}\left(g^{2+\beta}\right)
\end{aligned}$$

($\alpha = \beta$), uniformly in ω and s, where we used (66) once more. Finally, we have

$$\left\langle \mathcal{F}\phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega + is)^{-\nu-1} \mathcal{F}\phi(\theta_g) \right\rangle_{\mathcal{H}} = \mathcal{O}\left(g^{2+\beta}\right)$$

for $\gamma = 1$ and $\beta < 1/9$. We omit the estimate of

$$\begin{aligned}
&\left\langle \mathcal{F}\phi(\theta_g) \,\middle|\, \left(H_g(\overline{\theta}_g) - E_j \mp \omega + is\right)^{-\nu-1} \mathcal{F}\phi(\overline{\theta}_g) \right\rangle_{\mathcal{H}} \\
&\qquad = \overline{\left\langle \mathcal{F}\phi(\overline{\theta}_g) \,\middle|\, (H_g(\theta_g) - E_j \mp \omega - is)^{-\nu-1} \mathcal{F}\phi(\theta_g) \right\rangle_{\mathcal{H}}}
\end{aligned}$$

which is analogous. This proves (90) and therefore the proposition (89). □

We return to the proof of Lemma 2.11. By inserting (83) into (82), we estimate:

$$\begin{aligned} |M_\pm| &\le \frac{S_g}{2\pi\, t^n} \\ &\quad \cdot \sup_{0\le s\le S_g} \left| \int_{\delta/4}^{\delta/2} d\omega\, e^{-it(E_j\pm\omega-is)} \sum_{\nu=0}^{n} \frac{(\pm1)^\nu n!}{(n-\nu)!} F^{(n-\nu+1)}(E_j+\omega)\cdot D_\nu(s,\omega) \right| \\ &\le \frac{\delta S_g}{8\pi\, t^n} \sum_{\nu=0}^{n} \frac{n!}{(n-\nu)!} \sup_{\substack{0\le s\le S_g \\ \delta/4\le\omega\le\delta/2}} \left| F^{(n-\nu+1)}(E_j+\omega)\cdot D_\nu(s,\omega)\right| \\ &\le \frac{C_n}{2t^n} g^{4+\beta} \end{aligned} \tag{94}$$

for some positive constant $C_n < \infty$ and sufficiently small $g > 0$ because of (89), $S_g = \mathcal{O}\left(g^2\right)$, and since all derivatives of F are bounded. ■

2.12 Lemma *Under the conditions of Thm. 2.9, there is a constant $C < \infty$, such that*

$$|M_\|(\theta_g)| + |M_\|(\overline{\theta}_g)| \le C\, \exp(-S_g t), \tag{95}$$

for $t > 1$ and sufficiently small $g > 0$.

Proof: Recall the definition (80) of $M_\|(\theta_g)$. We introduce a quadratic form

$$\begin{aligned} \widetilde{M}^\theta\,(\varphi\,,\,\psi) &:= -\int_{\delta/4}^{\delta/2} d\omega\, \frac{F'(E_j+\omega)}{2\pi i} \\ &\quad \cdot \int_{\mathcal{I}(\omega)} d\lambda\, e^{-it(\lambda-iS_g)} \left\langle \varphi \,\middle|\, (H_g(\theta)-\lambda+iS_g)^{-1}\,\psi \right\rangle_{\mathcal{H}} \end{aligned} \tag{96}$$

to rewrite

$$\begin{aligned} M_\|(\theta_g) &= \widetilde{M}^{\theta_g}\left(\varphi_j(\overline{\theta}_g)\otimes\Omega\,,\,\varphi_j(\theta_g)\otimes\Omega\right) \\ &\quad + \widetilde{M}^{\theta_g}\left(\varphi_j(\overline{\theta}_g)\otimes\Omega\,,\,\phi(\theta_g)-\varphi_j(\theta_g)\otimes\Omega\right) \\ &\quad + \widetilde{M}^{\theta_g}\left(\phi(\overline{\theta}_g)-\varphi_j(\overline{\theta}_g)\otimes\Omega\,,\,\varphi_j(\theta_g)\otimes\Omega\right) \\ &\quad + \widetilde{M}^{\theta_g}\left(\phi(\overline{\theta}_g)-\varphi_j(\overline{\theta}_g)\otimes\Omega\,,\,\phi(\theta_g)-\varphi_j(\theta_g)\otimes\Omega\right). \end{aligned}$$

By [BFS3], Eq. (III.34), we have

$$\left\|(H_g(\pm i\vartheta_g)-\lambda+iS_g)^{-1}\right\|_{\mathcal{H}} \le \frac{\text{const}}{\tan(\vartheta_g)\,\left(|E_j-\lambda+\mathcal{O}\left(g^2\right)|+|C_S g^{2+\varepsilon}|\right)} \tag{97}$$

which implies

$$\begin{aligned} \left|\widetilde{M}^{\pm i\vartheta_g}\,(\varphi\,,\,\psi)\right| &\le \frac{C\,\|\varphi\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}}}{\tan(\vartheta_g)} e^{-S_g t} \int_{\mathcal{I}(\delta/2)} \frac{1}{|E_j-\lambda+\mathcal{O}\left(g^2\right)|+|C_S g^{2+\varepsilon}|}\, d\lambda \\ &\le \frac{2C\,\|\varphi\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}}}{\tan(\vartheta_g)} \exp(-S_g t) \\ &\quad \cdot \left[\ln\left(\frac{\delta}{2}+C_S g^{2+\varepsilon}+\mathcal{O}\left(g^2\right)\right) - \ln\left(C_S g^{2+\varepsilon}+\mathcal{O}\left(g^2\right)\right)\right], \end{aligned}$$

for a suitable constant $C < \infty$, because F' is bounded. By (61), it follows that

$$\begin{aligned} & \left| \widetilde{M}^{\pm i\vartheta_g} \left(\varphi_j(\mp i\vartheta_g) \otimes \Omega \,,\, \phi(\pm i\vartheta_g) - \varphi_j(\pm i\vartheta_g) \otimes \Omega \right) \right| \\ \leq \quad & \text{const} \frac{\| \phi(\pm i\vartheta_g) - \varphi_j(\pm i\vartheta_g) \otimes \Omega \|_{\mathcal{H}}}{\tan(\vartheta_g)} \ln(\mathcal{O}\left(g^2\right)) \exp(-S_g t) \\ = \quad & \mathcal{O}\left(g^{1-\frac{\gamma}{2}-\beta} \ln(g) \right) \exp(-S_g t) \leq \text{const} \cdot \exp(-S_g t). \end{aligned}$$

The estimates

$$\begin{aligned} \left| \widetilde{M}^{\pm i\vartheta_g} \left(\phi(\mp i\vartheta_g) - \varphi_j(\mp i\vartheta_g) \otimes \Omega \,,\, \varphi_j(\pm i\vartheta_g) \otimes \Omega \right) \right| \quad & \leq \quad \text{const} \cdot \exp(-S_g t), \\ \left| \widetilde{M}^{\pm i\vartheta_g} \left(\phi(\mp i\vartheta_g) - \varphi_j(\mp i\vartheta_g) \otimes \Omega \,,\, \phi(\pm i\vartheta_g) - \varphi_j(\pm i\vartheta_g) \otimes \Omega \right) \right| \quad & \leq \quad \text{const} \cdot \exp(-S_g t), \end{aligned}$$

follow similarly. It remains to consider the term $\widetilde{M}^{\pm i\vartheta_g} \left(\varphi_j(\mp i\vartheta_g) \otimes \Omega \,,\, \varphi_j(\pm i\vartheta_g) \otimes \Omega \right)$. To this end, we define a projection

$$P(\theta) := P_{\text{el}}^{j}(\theta) \otimes \chi \left[H_{\text{f}} \leq g^{2-2\zeta} \right] \tag{98}$$

on small photon energies, where $\frac{3}{2}\varepsilon < \zeta < \frac{1}{3}$. We observe that $P(\theta_g)\, \varphi_j(\theta_g) \otimes \Omega = \varphi_j(\theta_g) \otimes \Omega$ and hence

$$\begin{aligned} & \left| \left\langle \varphi_j(\overline{\theta}_g) \otimes \Omega \, \middle| \, \left(H_g(\theta_g) - \lambda + iS_g \right)^{-1} \varphi_j(\theta_g) \otimes \Omega \right\rangle_{\mathcal{H}} \right| \\ \leq \quad & \left| \left\langle \varphi_j(\overline{\theta}_g) \otimes \Omega \, \middle| \, \left(e^{-\theta_g} H_{\text{f}} + E_j - \lambda + iS_g + \operatorname{Re}(\mathcal{O}\left(g^2\right)) - ig^2 \Gamma \right)^{-1} \varphi_j(\theta_g) \otimes \Omega \right\rangle_{\mathcal{H}} \right| \\ & + \frac{\text{const} \cdot g^{2+\zeta}}{\tan(\vartheta_g) \left[\left(E_j - \lambda + \mathcal{O}\left(g^2\right) \right)^2 + \left(C_S g^{2+\varepsilon} \right)^2 \right]}, \end{aligned} \tag{99}$$

where we used (97) and [BFS3], Thm. III.4, which states that

$$\begin{aligned} & \left\| \left[\mathcal{F}_{P(\theta_g)} \left(H_g(\theta) - z \right) - \left(e^{-\theta} H_{\text{f}} + E_j - z + \operatorname{Re}(\mathcal{O}\left(g^2\right)) - ig^2 \Gamma \right) \right] P(\theta) \right\|_{\mathcal{H}} \\ & \leq \text{const} \cdot g^{2+\zeta}, \end{aligned} \tag{100}$$

(uniformly in $z \in \mathcal{A}^{\varepsilon}$) where the *Feshbach map* is defined by

$$\begin{aligned} \mathcal{F}_{P(\theta)} \left(H_g(\theta) - z \right) \quad := \quad & P(\theta) \left(H_g(\theta) - z \right) P(\theta) \\ & - P(\theta) W_g(\theta) \overline{P}(\theta) \left(H_g(\theta) - z \right)^{-1} \overline{P}(\theta) W_g(\theta) P(\theta) \end{aligned}$$

on $\operatorname{Ran}(P(\theta))$ (for details see [BFS1], Chap. II). Furthermore, we compute:

$$\begin{aligned} & \int\limits_{\mathcal{I}(\omega)} \frac{g^{2+\zeta}}{\tan(\vartheta_g) \left[\left(E_j - \lambda + \mathcal{O}\left(g^2\right) \right)^2 + \left(C_S g^{2+\varepsilon} \right)^2 \right]} \, d\lambda \\ \leq \quad & \frac{2g^{2+\zeta}}{\tan(\vartheta_g)} \int\limits_{0}^{\delta/2+\mathcal{O}\left(g^2\right)} \frac{1}{\lambda^2 + \left(C_S g^{2+\varepsilon} \right)^2} \, d\lambda \\ \leq \quad & \frac{2g^{2+\zeta}}{\tan(\vartheta_g) C_S g^{2+\varepsilon}} \int\limits_{0}^{\infty} \frac{1}{\lambda^2 + 1} \, d\lambda = \mathcal{O}\left(g^{\zeta-\varepsilon-\beta} \right) = \mathcal{O}\left(g^0 \right), \end{aligned}$$

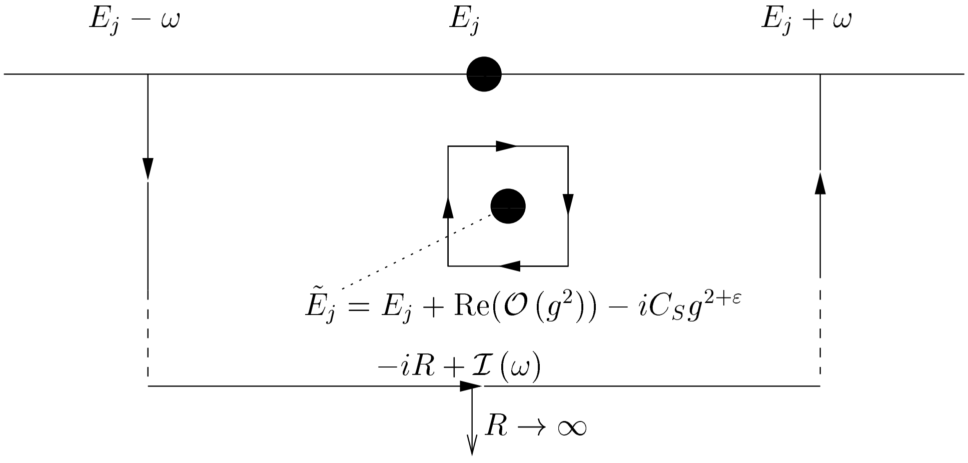

Figure 7: Applying *Cauchy's integral formula* and *residue theorem.*

because $\zeta \geq \frac{3}{2}\varepsilon$ and $\beta < \varepsilon/2$. Inserting this and (99) in (96), we can estimate (since F' is bounded):

$$\left|\widetilde{M}^{\theta_g}\left(\varphi_j(\overline{\theta}_g)\otimes\Omega\,,\,\varphi_j(\theta_g)\otimes\Omega\right)\right| \leq C\exp(-S_g t)$$
$$+C' \sup_{\delta/4\leq\omega\leq\delta/2}\left|\int\limits_{\mathcal{I}(\omega)} d\lambda\,\frac{\exp(-it(\lambda-iS_g))}{E_j-\lambda+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,\|\varphi_j\otimes\Omega\|^2_{\mathcal{H}}\right|.$$

Finally, we get the bound of the integral by applying *Cauchy's integral formula* and the *residue theorem* (see Fig. 7):

$$\begin{aligned}
&\int\limits_{\mathcal{I}(\omega)}\frac{\exp(-it(\lambda-iS_g))}{E_j-\lambda+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,d\lambda\\
&= \int\limits_{S_g}^{\infty}\frac{\exp(-it(E_j-\omega-is))}{\omega+is-iS_g+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,ds\\
&\quad-\int\limits_{S_g}^{\infty}\frac{\exp(-it(E_j+\omega-is))}{-\omega+is-iS_g+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,ds\\
&\quad+\lim_{R\to\infty}\int\limits_{\mathcal{I}(\omega)}\frac{\exp(-it(\lambda-iR))}{E_j-\lambda-iS_g+iR+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,d\lambda\\
&\quad-\frac{1}{2\pi i}\exp(-it(\lambda-iS_g))\Big|_{\lambda=E_j+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}},
\end{aligned}$$

where

$$\begin{aligned}
\left|\int\limits_{S_g}^{\infty}\frac{\exp(-it(E_j\mp\omega-is))}{\omega+is-iS_g+\mathrm{Re}(\mathcal{O}(g^2))-iC_S g^{2+\varepsilon}}\,ds\right| &\leq \frac{4}{\delta+\mathcal{O}(g^2)}\int\limits_{S_g}^{\infty}\exp(-st)\,ds\\
&< \frac{4}{\delta+\mathcal{O}(g^2)}\exp(-S_g t),
\end{aligned}$$

because $t > 1$,

$$\lim_{R\to\infty} \left| \int_{\mathcal{I}(\omega)} \frac{\exp(-it(\lambda - iR))}{E_j - \lambda - iS_g + iR + \mathrm{Re}(\mathcal{O}\left(g^2\right)) - iC_S g^{2+\varepsilon}}\, d\lambda \right| \;\leq\; \lim_{R\to\infty} \int_{\mathcal{I}(\omega)} \frac{e^{-Rt}}{R}\, d\lambda = 0$$

and

$$\left| \exp(-it(\lambda - iS_g)) \Big|_{\lambda = E_j + \mathrm{Re}(\mathcal{O}(g^2)) - iC_S g^{2+\varepsilon}} \right| = \exp(-t(S_g + C_S g^{2+\varepsilon})) \leq \exp(-S_g t).$$

Thus, we conclude

$$\left| \widetilde{M}^{\pm i\vartheta_g} \left(\varphi_j(\overline{\theta}_g) \otimes \Omega \,,\, \varphi_j(\theta_g) \otimes \Omega \right) \right| \leq \mathrm{const} \cdot \exp(-S_g t)$$

(note that we did not make use of the sign of $\mathrm{Im}(\theta_g)$). This proves the lemma. ■

A The Pull-Through Formula

A useful tool for calculations with the free field Hamiltonian and creation and annihilation operators is the so called *Pull-Through Formula*, which can easily be derived from functional calculus (see, e.g., [BFS1], Lemma A.1):

A.1 Lemma (Pull-Through Formula) *Let $F : \mathbb{R}^+ \to \mathbb{C}$ be a measurable function, obeying $F(r) = \mathcal{O}(r)$. Then $F(H_\mathrm{f})$ is defined on $\mathcal{D}(H_\mathrm{f})$ and*

$$\begin{aligned} a(\underline{k})\, F(H_\mathrm{f}) &= F(H_\mathrm{f} + \omega(\vec{k}))\, a(\underline{k}), \\ F(H_\mathrm{f})\, a(\underline{k})^* &= a(\underline{k})^*\, F(H_\mathrm{f} + \omega(\vec{k})). \end{aligned} \tag{101}$$

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