Strong Dynamical Localization for the almost Mathieu model

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Abstract

In this note we prove Strong Dynamical Localization for the almost Mathieu operator $H_{\theta,\lambda,\omega} = -\Delta + \lambda\cos(2\pi(\theta + x\omega))$ for all $\lambda > 2$ and Diophantine frequencies ω. This improves the previous known result [22, 13] which established Dynamical Localization for a.e. θ and for $\lambda \geq 15$.

1 Introduction and main result

In this paper we prove the strong version of Dynamical Localization for the almost Mathieu operator $H_{\theta,\lambda,\omega}$, acting on $\ell^2(\mathbb{Z})$,

$$(H_{\theta,\lambda,\omega}u)(x) = u(x-1) + u(x+1) + \lambda\cos(2\pi(\theta + x\omega))u(x),\ x \in \mathbb{Z}, \tag{1}$$

with $\lambda > 2$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$, and $\theta \in \mathbb{T} = [0,1[$. We shall also use the shorter notation H_θ instead of $H_{\theta,\lambda,\omega}$. More precisely our result holds for Diophantine frequencies $\omega \in \mathbb{R}\backslash\mathbb{Q}$, meaning that there exist $c(\omega)$ and $1 < r(\omega) < \infty$ such that

$$\forall\, j \neq 0, \quad |\sin 2\pi j\omega| > \frac{c(\omega)}{|j|^{r(\omega)}}. \tag{2}$$

Pure point spectrum with exponentially decaying eigenfunctions for a.e. θ, $\lambda > 2$ and Diophantine ω was proved in [21]. While the latter property is often referred to as Anderson localization, the physical understanding of localization depends on the dynamical properties, particularly the nonspreading of initially localized wave-packets under the Schrödinger time-evolution. One, generally accepted, formulation of it is, for any $q > 0$, and ψ with fast decay at ∞,

$$\sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H,t} \leq C_{\psi,q}, \tag{3}$$

where the moments of the position operator X are defined by

$$\langle\langle |X|^q \rangle\rangle_{\psi,H,t} = \langle e^{-iHt}\psi,\ |X|^q e^{-iHt}\psi\rangle,$$

or if necessary using the initial state $E_H(I)\psi$ rather than ψ, where $E_H(I)$ is the spectral projector of H onto the interval I. One refers to the bound (3) as Dynamical Localization.

For ergodic families of operators H_θ, $\theta \in \Theta$ (denoting by $\mathbb{P}$ the probability measure), the constant $C_{\psi,q}$ may also depend on the parameter θ: $\sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H_\theta,t} \le C_{\psi,\theta,q}$. The following stronger form of (3) then makes sense.

Definition 1.1. *We shall say that the family $(H_\theta)_{\theta\in\Theta}$ acting on a Hilbert space $\mathcal{H}$ is strongly dynamically localized, if, for any $q > 0$ and initial state $\psi \in \mathcal{H}$ that decays faster than any polynomial, there exists a constant $C(q,\psi) < \infty$ such that*

$$\int_\Theta \mathrm{d}\mathbb{P}(\theta) \sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H_\theta,t} \le C(q,\psi). \tag{4}$$

If necessary we can restrict ourselves to an interval of energies I, and then work with the initial state $E_{H_\theta}(I)\psi$ instead of ψ. This will define Strong Dynamical Localization on the interval I.

While the pure point spectrum follows from the Dynamical Localization (see, e.g. [4]), an example in [8, 9] shows that Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) may coexist with almost ballistic dynamics (which is the worst possibility, according to [27]). Therefore, proving Dynamical Localization requires going further in the proof of pure point spectrum. The Aizenman-Molchanov technique ([1] and ref. therein), wherever it works, leads naturally to a strong bound of the form (4). However, other treatments are needed for continuous or deterministic models. In [9], a property called SULE has been introduced. This property establishes a control on the size of the support of the eigenfunctions in terms of the centers of localization, $x_{E,\theta}$, namely $|\varphi_E^\theta(x)| \le C_\varepsilon e^{\varepsilon |x_{E,\theta}|} e^{-\gamma_0 |x - x_{E,\theta}|}$, for any $\varepsilon > 0$ and it is shown to entail Dynamical Localization [9]. In [14] SULE has been derived for a large class of random Schrödinger operators to which the multi-scale analysis [12] of the form [10] applies. Thereby, Dynamical Localization was proved for those models. Very recently, in [5], this result has been strenghened to *Strong* Dynamical Localization, namely it is shown that for random Hamiltonians the property SULE in the form obtained in [14] actually implies (4). In [15], an even more straightforward argument is given. We mention that these results also apply to the random dimer operator [7], a discrete 1-D model where the Aizenman-Molchanov approach fails.

Besides random, another important class of ergodic operators for which results on the pure point spectrum have been obtained is the class of quasi-periodic operators. As well as with the point spectrum itself, establishing Dynamical Localization in this case requires approaches that are quite different from that of the random case. There have been several new important localization results for quasiperiodic models recently: Anderson localization for the almost Mathieu operator with Diophantine frequencies [21], which is the operator studied in this paper, and even more recently, Bourgain-Goldstein [3] proved Anderson localization for a large class of one and two-frequencies potentials in the regime of positive Lyapunov exponents. Also, pure point spectrum was proved by Eliasson [11] for an even larger class of one-frequency potentials at high disorder. These models are natural candidates for dynamical strengthening.

From the dynamical point of view, for quasi-periodic models, the motion is knwon to be quasi-ballistic for a generic set of frequencies ω [6] (it was proved for Liouville frequencies and for the almost Mathieu model in [24]). Dynamical Localization (the *a.e.* θ version) for the almost Mathieu operator with $\lambda \geq 15$, Diophantine frequencies, has been obtained in [22] by establishing SULE. A somewhat weaker version of Dynamical Localization was independently obtained in [13] by establishing some weaker property on the decay of the eigenfunctions. Both these proofs were, in a certain sense, "second iterations" of the proof in [19], and neither extends automatically to the general $\lambda > 2$ case, nor to the strong form (4). In this paper we achieve both these goals for the almost Mathieu model. Our main result is the following:

Theorem 1.1. *Let* $\lambda > 2$ *and let* ω *be Diophantine. Then the family* $(H_{\theta,\lambda,\omega})_{\theta\in\mathbb{T}}$ *is strongly dynamically localized.*

We note that our proof can be adjusted to establish SULE. One can then adapt the approach of [5] and use our Lemma 2.1 in order to get the strong form of Dynamical Localization stated in Theorem 1.1. We, however, bypass the SULE condition and provide a more straightforward proof which is closer to the spirit of [13]. The idea is that the Strong Dynamical Localization of the family $(H_{\theta,\lambda,\omega})_{\theta\in\mathbb{T}}$ can be obtained in a very natural way using the (usually rather unexploited) properties of the BKG eigenfunctions expansion [2] [26] [23]. The latter is developed for some families of random operators like Schrödinger and Classical Waves in [15]. In the particular setting of the present model (a discrete one-dimensional case with simple pure point spectrum) the arguments become even more direct, since one can skip the explicit use of that expansion in eigenfunctions by roughly reconstructing it "by hand". See Section 2 for further comments.

Finally we make the additional remark that in the setting of quasi-periodic potential the result (4) - in comparison to (3) - has another physical interest. It has been argued by Piechon [25], for the Fibonacci chain, that after averaging the quantity $\langle\langle|X|^2\rangle\rangle_{\psi,H_\theta,t}$ over the parameter θ one could get some ballistic motion. The argument relies on the idea that averaging over θ of the spectral measure μ_θ of H_θ yields the density of states, which is continuous (at least for the almost Mathieu model with Diophantine frequency: see [17] together with [21]). Then by Guarneri's arguments [18] [24] $\int_{\mathbb{T}}\mathrm{d}\theta\left(\frac{1}{T}\int_0^T \mathrm{d}t\langle\langle|X|^2\rangle\rangle_{\psi,H_\theta,t}\right)$ should behave like T^2. Theorem 1.1 then tells that this reasoning fails for the almost Mathieu model with Diophantine conditions.

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2 Proof of Strong Dynamical Localization

We shall denote by $H_{[x,y],\theta}$ the restriction of H_θ to the interval $[x,y]$ with zero boundary conditions at $x-1$ and $y+1$, and by $G^{\theta}_{[x,y]}(E,\cdot,\cdot)$ the corresponding Green function, that is the inverse of $H_{[x,y],\theta}-E$. We need the following definition.

Definition 2.1. *Let $\gamma > 0$, $k > 0$, and $E \notin \sigma(H_{[x,y],\theta})$. A point $z \in \mathbb{Z}$ is called (γ, E, θ, k)-regular at energy E if there exists an interval $[x, y]$, with $|x - y| \leq k$, containing z and such that*

$$\left| G^{\theta}_{[x,y]}(E, z, u) \right| < e^{-\gamma k}, \ u = x, y.$$

We also recall the following well known identity. Let $z \in [x, y]$, $E \notin \sigma(H_{[x,y],\theta})$, and $\varphi \in \ell^2(\mathbb{Z})$ such that $H_\theta \varphi = E\varphi$. One then has

$$\varphi(z) = G^{\theta}_{[x,y]}(E, z, x)\varphi(x - 1) + G^{\theta}_{[x,y]}(E, z, y)\varphi(y + 1). \tag{5}$$

The proof of Theorem 1.1 relies on the following lemma:

Lemma 2.1. *Assume that the hypotheses of Theorem 1.1 hold. Pick $\alpha \in (1, 2)$ and any $s > \alpha$. Define $\gamma_0 = \ln \frac{\lambda}{2} > 0$. Then there exist $L^*(\alpha, s)$, and for all $L > L^*(\alpha, s)$ and $y \in \mathbb{Z}$ a set $\Theta_L(y)$ of Lebesgue measure less than $9L^{-s+\alpha}$ such that, for any $\theta \notin \Theta_L(y)$, the following holds for all E:*

$$\left(\frac{L + 1}{2} \leq |x - y| < L^\alpha \right) \Longrightarrow (\text{either } x \text{ or } y \text{ is } (\gamma_0/2, E, \theta, L) - \text{regular}). \tag{6}$$

Lemma 2.1 above is a modified and improved version of Lemma 4 in [21]; it is also stated in a form that is closer to the core result of the multiscale analysis given in [10]. In section 3 we shall state another version of this Lemma 2.1: Lemma 3.1, which is closer to the one that can be found in [21]. We then show how to eploit the existing results to get Lemma 2.1.

The main difference between Lemma 2.1 and Lemma 4 in [21] (or equivalent lemmas in [19, 20]) is that Lemma 4 in [21] supplies a scale L^* that depends on θ *and* E. It is quite clear that one should be able to get rid of this double dependency in order to get the announced result, which was not possible in [21]. This is achieved in Section 3 below which is devoted to the proof of Lemma 2.1. The idea of Lemma 2.1 is to avoid the Borel-Cantelli Lemma systematically used in the previous proofs of Localization [19, 20, 21, 22, 13], and to work with finite scales L, supplying a set $\mathbb{T} \backslash \Theta_L(y)$ with Lebesgue measure close to one but not one, on which the conclusion (Line (6)) holds. The point is that this construction can be done for L large enough, large enough not depending on θ (nor E).

We finally point out that the Lebesgue measure of the set $\Theta_L(y)$ is independent of y. This is crucial for our proof of Strong Dynamical Localization (see e.g. Eq. (15)), and is due to the way we construct this set $\Theta_L(y)$ (see (18) below).

We turn to the proof of Strong Dynamical Localization: one has to handle sums of the form $\sum_{E^\theta} |\varphi^\theta_E(x)\varphi^\theta_E(y)|$. The basic idea is that for suitable x, y (6) of Lemma 2.1 asserts that either $\varphi^\theta_E(x)$ or $\varphi^\theta_E(y)$ is exponentially small. The other one can then be bounded by 1 (since $\varphi^\theta_E \in \ell^2(\mathbb{Z})$). However, one then ends up with an infinite sum over the energies E^θ, which diverges. To overcome the issue of an infinite summation, it has been proposed in [5], in the setting of random operators, to use the more detailed informations concerning the exponential decay of the eigenfunctions proved in [14] (namely the property SULE mentioned above).

Below we use a quite different and more straightforward method that relies on ideas developed in [13]. We show that a very natural way to get a bound for $\sum_{E^\theta} |\varphi^\theta_E(x)\varphi^\theta_E(y)|$ is to replace φ^θ_E by the generalized eigenfunctions $\widetilde{\varphi}^\theta_E$ that supplies the eigenfunctions expansion *à la* Berezanskii [2]. This has two crucial consequences. The first one is that the generalized eigenfunctions $\widetilde{\varphi}_E^\theta$ are polynomially bounded, and this, uniformly in energy ($|\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta$, see (8) below). The exponential decay of products of the form $|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|$ provided by Lemma 2.1 will not then be affected by that change. Second, and crucial, the sum over E^θ now converges. In other words, the loss of control on the eigenfunctions ($|\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta$ instead of $|\varphi_E^\theta(x)| \leq 1$) leads to a gain in the sum over E^θ. These three points (the definition of φ_E^θ and its two main consequences) are made more precise in respectively (7), (8) and (9) below.

We also note that although the spirit of this construction comes from the BKG expansion, there is no need to develop this theory in its full generality in the present setting. Indeed, since in our case the spectrum of H_θ is known to be pure point and simple, one can define directly those polynomially bounded eigenfunctions $\widetilde{\varphi}_E^\theta$ and work with them. The reader will find the general argument in [15] together with its application to classical waves and random Schrödinger operators. We finally point out that the general argument given in [15] allows one to get Strong Dynamical Localization directly, without proving Anderson Localization first.

We turn now to the proof and start with gathering the necessary ingredients. From [21] we know that under the hypotheses of the theorem the spectrum of $H_{\omega,\lambda,\theta}$ is pure point for all θ in a set $\widetilde{\Theta}$ of full Lebesgue measure [21]. In addition the spectrum is simple by a usual 1-D Wronskian argument. Let us denote by φ_E^θ the orthonormalized eigenfunctions of $H_{\omega,\lambda,\theta}$, with the energy E^θ, $\theta \in \widetilde{\Theta}$. Furthermore, as in [13], define

$$\widetilde{\varphi}_E^\theta := \varphi_E^\theta / \|B\varphi_E^\theta\|_{\ell^2}, \qquad \text{so that} \qquad \varphi_E^\theta = \|B\varphi_E^\theta\|_{\ell^2}\widetilde{\varphi}_E^\theta, \tag{7}$$

where B is the multiplication operator by $b(x) := (1+|x|)^{-\delta}$, $\delta > 1/2$. Note that $\widetilde{\varphi}_E^\theta$ is still in $\ell^2(\mathbb{Z})$. As briefly mentioned above, what makes those eigenfunctions of particular interest is the following: first, $\|B\widetilde{\varphi}_E^\theta\|_{\ell^2} = 1$ so that uniformly in θ and in energy E^θ, the following holds :

$$\forall\, x \in \mathbb{Z},\ |\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta. \tag{8}$$

Second, while considering a sum of the form $\sum_{n\geq 0} |\varphi_E^\theta(x)\varphi_E^\theta(y)|$ a factor $\|B\varphi_E^\theta\|_{\ell^2}^2$ will come out, and one notes that for all $\theta \in \widetilde{\Theta}$,

$$\sum_{E^\theta} \|B\varphi_E^\theta\|_{\ell^2}^2 = \sum_{x\in\mathbb{Z}} b(x)^2 \sum_{E^\theta} |\varphi_E^\theta(x)|^2 = \|b\|_{\ell^2}^2 < \infty. \tag{9}$$

The last quantity is actually nothing but the mass $\mu(\mathbb{R})$ of the spectral measure $\mu(\Delta) := \mathrm{tr}(BP(\Delta)B)$ that appears in the BKG expansion formula [2, 26, 23].

The following Lemma shows how Lemma 2.1 together with the construction (7)-(9) provide the needed exponential decay of the quantity $|\langle \delta_x, e^{-iH_\theta t}\delta_y\rangle|$, with x and y sufficiently far from each other (but not too far!). In other words, Lemma 2.2 below shows how exclusion of singular boxes together with (7)-(9) supplies the key dynamical step that implies (Strong) Dynamical Localization.

Lemma 2.2. *Let us pick $L > L^*(\alpha, s)$ and $y \in \mathbb{Z}$. Then there exists a constant $C_1(\delta, \alpha, \gamma_0) < \infty$, such that for any $\theta \notin \Theta_L(y)$ (given by Lemma 2.1), any $E = E^\theta$, and for all $x \in \mathbb{Z}$ such that $\frac{L+1}{2} \leq |x-y| < L^\alpha$, the following holds:*

$$|\varphi_E^\theta(x)\varphi_E^\theta(y)| \leq C_1(\delta, \alpha, \gamma_0)\|B\varphi_E^\theta\|_{\ell^2}^2 (1+|y|)^{2\delta} e^{-\gamma_0 L/4}. \tag{10}$$

As a consequence, for some constant $C_2(\delta,\alpha,\gamma_0)<\infty$*, and for all* $\theta\notin\Theta_L(y)$*,*

$$|\langle\delta_x,e^{-iH_\theta t}\delta_y\rangle|\le C_2(\delta,\alpha,\gamma_0)(1+|y|)^{2\delta}e^{-\gamma_0L/4}. \tag{11}$$

Proof of Lemma 2.2.

Let us pick x,y as in the Lemma. We first replace the normalized eigenfunctions φ_E^θ by the generalized eigenfunctions $\widetilde{\varphi}_E^\theta$:

$$|\varphi_E^\theta(x)\varphi_E^\theta(y)|=\|B\varphi_E^\theta\|_{\ell^2}^2\;|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|. \tag{12}$$

Now, from Lemma 2.1, if $\theta\notin\Theta_L(y)$, then at least one of x and y, with $\frac{L+1}{2}\le|x-y|<L^\alpha$, is $(\gamma_0/2,\theta,L)$-regular. Let us denote by $u=x$ or y, the regular point, and v the other one. Then, applying the identity (5) to $\widetilde{\varphi}_E^\theta$ at the point u, and taking into account the uniform polynomial bound (8), one has

$$|\widetilde{\varphi}_E^\theta(u)|\le 2(2+|u|+L)^\delta e^{-\gamma_0L/2}\le C(1+|u|)^\delta(1+L)^\delta e^{-\gamma_0L/2}. \tag{13}$$

Then we use (8) to bound the second term $\widetilde{\varphi}_E^\theta(v)$, and this leads to

$$|\widetilde{\varphi}_E^\theta(u)\widetilde{\varphi}_E^\theta(v)|\le C(1+|u|)^\delta(1+|v|)^\delta(1+L)^\delta e^{-\gamma_0L/2}. \tag{14}$$

So, in any case, whether x or y is $(\gamma_0/2,\theta,L)$-regular, one gets that for all $\theta\notin\Theta_L(y)$

$$|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|\le C(1+|y|)^{2\delta}(1+L)^{(\alpha+1)\delta}e^{-\gamma_0L/2}\le C_1(\delta,\alpha,\gamma_0)(1+|y|)^{2\delta}e^{-\gamma_0L/4}.$$

Together with (12), this proves (10). Combining (10) with (9), we obtain

$$\begin{aligned}|\langle e^{-iH_\theta t}\delta_x,\delta_y\rangle| &\le \sum_{E^\theta}|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|\le C_1(\delta,\alpha,\gamma_0)(1+|y|)^{2\delta}e^{-\gamma_0L/4}\sum_{E^\theta}\|B\varphi_E^\theta\|_{\ell^2}^2\\ &\le C_2(\delta,\alpha,\gamma_0)(1+|y|)^{2\delta}e^{-\gamma_0L/4}.\end{aligned}$$

□

Proof of Theorem 1.1.

Pick some $\alpha\in(1,2)$, and for any given $q>0$, take $s>\alpha(q+2)$. Lemma 2.1 then supplies a sequence of scales, $L_1=L^*(\alpha,s)$, $L_{k+1}=(2L_k-1)^\alpha$ (so that if $L_k=\frac{L+1}{2}$ then $L_{k+1}=L^\alpha$), and for any $y\in\mathbb{Z}$ some sets $\Theta_{L_k}(y)$ such that (6) and consequently (10) hold between L_k and L_{k+1}, $k\ge1$. Hence one has for all $t\in\mathbb{R}$, $k\ge1$ and x,y so that $L_k\le|x-y|<L_{k+1}$,

$$\int_{\mathbb{T}}\mathrm{d}\theta\left(|\langle e^{-iH_\theta t}\delta_x,\delta_y\rangle|\right)\le C_2(\delta,\alpha,\gamma_0)(1+|y|)^{2\delta}e^{-\gamma_0L_k/4}+9L_k^{-s+\alpha}. \tag{15}$$

Indeed, splitting the integral in two pieces : the first one over $\Theta_{L_k}^c(y)$ and the second one over $\Theta_{L_k}(y)$, one has, from Lemma 2.1 that, provided $L_k\le|x-y|<L_{k+1}$, $k\ge1$,

$$\int_{\mathbb{T}}\mathrm{d}\theta\left(|\langle e^{-iH_\theta t}\delta_x,\delta_y\rangle|\right)\le\int_{\Theta_{L_k}^c(x)}|\langle e^{-iH_\theta t}\delta_x,\delta_y\rangle|\mathrm{d}\theta+9L_k^{-s+\alpha}.$$

We used $|\langle e^{-iH_\theta t}x, \delta_y\rangle| \le 1$ for the second term. Lemma 2.2 then yields (15). One can therefore end the proof of the theorem as follows:

$$\begin{aligned}
&\int_{\mathbb{T}} \mathrm{d}\theta \left\| \, |X|^{q/2} e^{-iH_\theta t}\psi \right\|^2 \\
&\quad\le \sum_{y\in\mathbb{Z}} |\psi(y)| \left[2(|y| + L^*(\alpha,s))^q + |\langle e^{-iH_\theta t}\psi, \ |X|^q e^{-iH_\theta t}\delta_y\rangle|\right] \\
&\quad\le \sum_{y\in\mathbb{Z}} |\psi(y)| \left[2(|y| + L^*(\alpha,s))^q + \sum_{k\ge 1} \sum_{L_k \le |x-y| < L_{k+1}} |x|^q \int_{\mathbb{T}} \mathrm{d}\theta |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle|\right],
\end{aligned}$$

where we used the fact that $|\langle e^{-iH_\theta t}\psi, \delta_x\rangle| \le 1$. Then, by (15),

$$\begin{aligned}
&\int_{\mathbb{T}} \mathrm{d}\theta \left\| \, |X|^{q/2} e^{-iH_\theta t}\psi \right\|^2 \le C(q,\delta,\alpha,\gamma_0) \sum_{y\in\mathbb{Z}} |\psi(y)| \\
&\qquad \left[(|y| + L^*(\alpha,s))^q + \sum_{k\ge 1} (1+|y|)^q L_k^{\alpha(q+1)} \left[(1+|y|)^{2\delta} e^{-\gamma_0 L_k/4} + 9L_k^{-s+\alpha}\right]\right].
\end{aligned}$$

Since s is taken larger than $\alpha(q+2)$, the sum over k converges, and since ψ decays faster than any polynomial, so does the sum over y. □

3 Proof of Lemma 2.1

We first need some further definitions in order to state the two lemmas that will imply Lemma 2.1. We denote by $B(\theta, E)$ the one-step transfer matrix of the eigenvalue equation $H_{\theta,\lambda,\omega}u = Eu$:

$$B(\theta, E) = \begin{pmatrix} E - \lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the k-step transfer matrix is given by

$$M_k(\theta, E) = B(\theta + (k-1)\omega, E)\cdots B(\theta+\omega, E)B(\theta, E).$$

The Lyapunov exponent $\gamma(E)$ is defined as

$$\gamma(E) := \lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{T}} \ln \|M_k(\theta, E)\| \mathrm{d}\theta = \inf_k \frac{1}{k} \int_{\mathbb{T}} \ln \|M_k(\theta, E)\| \mathrm{d}\theta. \tag{16}$$

By the subadditive ergodic Theorem (*e.g.* [4]) the limits exist and the second equality holds. Pick any $s > \alpha$ and let us define, for $L \ge 1$

$$\widetilde{\Theta}_L^{(1)} := \left\{\theta \in \mathbb{T} \mid \exists |j| \le 2L^\alpha, \ \left|\sin 2\pi\left(\theta + \frac{j}{2}\omega\right)\right| \le \frac{1}{L^s}\right\}. \tag{17}$$

Since for any fixed $j \in \mathbb{Z}$, $\left|\{\theta \in \mathbb{T} \mid \left|\sin 2\pi\left(\theta + \frac{j}{2}\omega\right)\right| \le L^{-s}\}\right| \le 2L^{-s}$, one immediately gets that $\left|\widetilde{\Theta}_L^{(1)}\right| \le 8L^{-s+\alpha}$. Then define the set of "bad" angles θ associated to $L \ge 1$ and $y \in \mathbb{Z}$ as follows:

$$\Theta_L^{(1)}(y) = \widetilde{\Theta}_L^{(1)} - y\omega. \tag{18}$$

Obviously, one still has

$$\left|\Theta_L^{(1)}(y)\right| \le \frac{8}{L^{s-\alpha}}. \tag{19}$$

Lemma 2.1 will be a consequence of the following two lemmas.

Lemma 3.1. [**21**] *Let $\varepsilon \in (0, \gamma_0/2)$, $\alpha \in (1,2)$, $s > \alpha$ and ω be a Diophantine irrational. Let L be a positive integer and assume that there exists a set $\Theta_L^{(2)}$ such that for all $k \ge L$ and $\theta \notin \Theta_L^{(2)}$, $\|M_k(\theta, E)\| \le e^{k(\gamma(E)+\varepsilon)}$.*
Then there exists $L_1(\varepsilon, \alpha, \omega)$ such that for any $L \ge L_1(\varepsilon, \alpha, \omega)$ and any given y and $\theta \notin \left(\Theta_L^{(1)}(y) \cup \Theta_L^{(2)}\right)$, where $\Theta_L^{(1)}(y)$ is given by (18), the following holds for all E:

$$\left(x \text{ and } y \text{ are } (\gamma(E)-\varepsilon, E, \theta, L) - \text{singular and } |x-y| > \frac{L+1}{2} \right) \Longrightarrow |x-y| > L^\alpha. \tag{20}$$

Lemma 3.2. [**3, 16**] *Assume that ω is Diophantine. Then for $\sigma < 1/2$, there exists $L(\sigma, \omega)$ such that, for all E, $k \ge L(\sigma, \omega)$ implies*

$$\left| \left\{ \theta \in \Theta \ \left| \frac{1}{k} \ln \|M_k(\theta, E)\| - \gamma(E) \right| \ge \frac{1}{k^\sigma} \right\} \right| \le e^{-k^{1-2\sigma}}. \tag{21}$$

The reader will not find these two lemmas exactly stated in this form in respectively [21] and [3, 16] but they are very close to what has been proved there. In particular they do not require any new arguments, only a precise re-reading together with some slight modifications. For this reason we state Lemma 3.1 and Lemma 3.2 without a proof. However, we shall make below some comments in order to make precise what these required "slight modifications" are. We first show how to derive Lemma 2.1 and then turn to those comments.

Proof of Lemma 2.1.
Let α, ω, s and ε be as in Lemma 3.1. Lemma 3.2 will provide the control on $\|M_k(\theta, E)\|$ that Lemma 3.1 requires. Define $\Theta_k := \{\theta \in \mathbb{T} \ \left| \frac{1}{k} \ln \|M_k(\theta, E)\| - \gamma(E) \right| \ge \frac{1}{k^\sigma}\}$ and $\Theta_L^{(2)} := \cup_{k \ge L} \Theta_k$. Lemma 3.2 then provides a scale $L_2(\sigma, \omega, \varepsilon)$, such that for any $L \ge L_2(\sigma, \omega, \varepsilon)$ one has
a) $|\Theta_L^{(2)}| \le C_\sigma e^{-L^{1-2\sigma}}$, and
b)

$$\forall\, \theta \notin \Theta_L^{(2)}, \forall\, E,\ k \ge L \Longrightarrow \|M_k(\theta, E)\| \le e^{k(\gamma(E)+k^{-\sigma})} \le e^{k(\gamma(E)+\varepsilon)}. \tag{22}$$

This means that Lemma 3.1 applies for $L \ge L_2(\sigma, \omega, \varepsilon)$ and the set of phases $\Theta_L^{(2)}$. Define $\Theta_L(y) := \Theta_L^{(1)}(y) \cup \Theta_L^{(2)}$. Thus one has for L larger than some $L_3(\sigma, \alpha, \omega, \varepsilon, s)$

$$|\Theta_L(y)| \le \frac{9}{L^{s-\alpha}}.$$

Let us take ε small enough so that $\gamma_0/2 \le \gamma_0 - \varepsilon$. Then, since for all E one has $\gamma(E) \ge \ln \frac{\lambda}{2} = \gamma_0$ [4], (20) of Lemma 3.1 holds with the set of phases $\Theta_L(y)$, $L \ge L^*(\alpha, \omega, \varepsilon, s) = \max(L_1(\alpha, \omega, \varepsilon), L_3(\sigma, \alpha, \omega, \varepsilon, s))$, but with the rate $\gamma_0/2$ instead of $\gamma(E) - \varepsilon$. This in turn implies Lemma 2.1. □

We now come back to Lemma 3.1 and Lemma 3.2.

Lemma 3.1 is a sort of intermediate statement between Lemma 2.1 and Lemma 4 of [21]. The difference between our Lemma 3.1 and Lemma 4 of [21] is slim but crucial. To get a control on the dependency on θ, the idea is to avoid the Borel-Cantelli Lemma that leads to a set Θ of full measure in [19, 20, 21, 22, 13], and to find for a given scale L a suitable non-resonant set of θ with a measure close to one. This will precisely be the set $\Theta_L^{(1)}(y)$ (see (18) and (19)). More technically we set $\Theta_L^{(1)}(y)$ is defined so that the conclusion of Lemma 13 in [21] holds for any scale large enough, uniformly both in x such that $\frac{L+1}{2} \leq |x-y| < L^\alpha$ and in $\theta \notin \Theta_L^{(1)}(y)$. As one can see from its definition (18), the set of resonances $\Theta_L^{(1)}(y)$ where Lemma 13 of [21] fails does depend on y. The crucial point for our proof of Strong Dynamical Localization (see e.g. Eq. (15)) is that its *measure* does not depend on y.

Lemma 3.2 takes care of the (non) dependency on the energy E of the constant L^* of Lemma 2.1. Actually the proof of Dynamical Localization in [22, 13] could only handle values of $\lambda \geq 15$ (and not $\lambda > 2$), in part, because of the lack of information on the rate of convergence in (16). The missing information is actually provided by the combination of the two recent works [3] and [16], which establish a large deviation theorem in the context of quasi-periodic potentials [3], and the continuity of the Lyapunov exponent $\gamma(E)$ [16]. Indeed, the reader will easily convince himself that Lemma 3.2 above follows directly form Lemma 1.1 in [3] and Lemma 4.2 in [16]. However, the Diophantine condition that is used in [16] is stronger than the one we allowed ourselves in the present paper (see (2)). However, unlike in [16], we do not need sharp estimates on the speed of convergence of $\frac{1}{k}\int_{\mathbb{T}} \ln \|M_k(\theta,E)\| \mathrm{d}\theta$ toward $\gamma(E)$ (but just the uniformity in E). That means one can adapt the proof of Lemma 4.2 of [16], and get, under our assumption (2), that, for any fixed $\varepsilon > 0$ and E,

$$\left| \frac{1}{k} \int_{\mathbb{T}} \ln \|M_k(\theta,E)\| \mathrm{d}\theta - \gamma(E) \right| \leq \frac{C_\varepsilon}{k^{1-\varepsilon}}, \tag{23}$$

where C_ε is a constant uniform in E. Again getting (23) does not require any arguments but the one already contained in [16]. The estimate (23), together with large deviation Theorem (Lemma 1.1 in [3]) is then clearly sufficient to get Lemma 3.2.

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