Spectral Analysis of Quantum Field Models with a Particle Number Cutoff $^{(\star)}$

Vladimir Georgescu

Abstract. We determine the essential spectrum and prove the absence of singular continuous spectrum for a class of boson quantum field models with cutoffs. Our main purpose, however, is to show the power of an algebraic technique based on the study of the C^*-algebra generated by the operators which are natural candidates as hamiltonians of the system in a given physical situation.

1 Algebras of Creation-Anihilation Operators

Our aim is to study a system of at most $N < \infty$ bosons which interact between themselves and possibly also with some external system, creation and anihilation of particles being allowed. For this purpose I shall use a strategy developed in a joint work with A. Iftimovici, see its contribution to these proceedings and [GI]. Theorems 1.1 and 1.2 are the result of a collaboration with M. Mantoiu [GM]. I stress that my purpose here is not to get optimal results or to improve known results but rather to point out a new method of analysing the spectral properties of systems with a nontrivial algebraic structure. Although not considered here, the case $N = \infty$ can be treated by similar techniques.

Let $\mathcal{H}$ be the (complex, infinite dimensional) Hilbert space of one boson states. We denote $\mathcal{H}^{\vee n}$ the symmetric tensor power of order n of $\mathcal{H}$ and $\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}$ the Fock space associated to $\mathcal{H}$. Then $\Gamma_N(\mathcal{H}) = \bigoplus_{n=0}^{N} \mathcal{H}^{\vee n}$ is the state space of the system consisting of at most N bosons. The interaction between bosons is a (generalized) polynomial in $a(u)$, $a^*(u)$, the anihilation and creation operators of a boson in the state $u \in \mathcal{H}$ (the normalizations and other conventions are as in [DeG2]). Since we are interested only in situations when the number of bosons does not exceed N, we introduce the truncated anihilation and creation operators $a_N(u) = 1_N a(u) 1_N$, $a_N^*(u) = 1_N a^*(u) 1_N$, where 1_N the orthogonal projection $\Gamma(\mathcal{H}) \to \Gamma_N(\mathcal{H})$. The set $K(\Gamma_N(\mathcal{H}))$ of compact operators on $\Gamma_N(\mathcal{H})$ will be denoted $\mathcal{K}_N$. We state a preliminary result.

Theorem 1.1. *Let $\mathcal{A}_N$ be the unital C^*-algebra of operators on $\Gamma_N(\mathcal{H})$ generated by the operators $a_N(u)$ when u runs over $\mathcal{H}$. There is a unique unital morphism $\mathcal{A}_N \to \mathcal{A}_{N-1}$ such that the image of $a_N(u)$ be $a_{N-1}(u)$ for all $u \in \mathcal{H}$. This morphism is surjective and its kernel is equal to $\mathcal{K}_N$.*

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In other terms, $\mathcal{K}_N$ is a (closed, bilateral) ideal in $\mathcal{A}_N$ and we have a canonical identification of the quotient C^*-algebra: $\mathcal{A}_N/\mathcal{K}_N \cong \mathcal{A}_{N-1}$. We shall denote by $T \mapsto \widehat{T}$ the quotient map, thus $\widehat{\mathcal{A}_N} = \mathcal{A}_{N-1}$.

The computation of $\widehat{T}$ is important in applications to spectral analysis of the preceding result and we give now some examples. Let $\mathcal{A}$ be the unital $*$-algebra of (unbounded) operators on $\Gamma(\mathcal{H})$ (with domain $\bigcup_{k=0}^{\infty} \Gamma_k(\mathcal{H})$) generated by the $a(u)$. Clearly $1_N \mathcal{A} 1_N$ is a linear subspace of $\mathcal{A}_N$ (it is not dense and it is not a subalgebra) and in fact it coincides with the linear subspace of $\mathcal{A}_N$ generated by the Wick products

$$W_N = \prod_i a_N^*(u_i) \prod_j a_N(v_j).$$

One also has

$$W_N = 1_N \prod_i a^*(u_i) \prod_j a(v_j) 1_N \equiv 1_N W 1_N.$$

For Wick products it is very easy to compute the quotient:

$$[W_N]^\wedge = W_{N-1} \quad \text{or} \quad [1_N W 1_N]^\wedge = 1_{N-1} W 1_{N-1}.$$

As a second example, let p_k be the orthogonal projection onto $\mathcal{H}^{\vee k}$. So $1_N = \sum_{k=0}^{N} p_k$ and the particle number operator can be decomposed as $\mathbf{N}_N = \sum_{k=0}^{N} k p_k$. Then $p_k \in \mathcal{A}_N$ and we have $\widehat{p}_0 = 0$ and $\widehat{p}_k = p_{k-1}$ if $1 \leq k \leq N$. Hence $\mathbf{N}_N$ belongs to $\mathcal{A}_N$ and its quotient is $\widehat{\mathbf{N}_N} = \mathbf{N}_{N-1} + 1_{N-1}$. According to the teminology of [Gr] for example, $\mathbf{N}_N$ is the number of alive bosons and so $\mathbf{D}_N = N 1_N - \mathbf{N}_N$ is the number of dead bosons. Since $\widehat{1_N} = 1_{N-1}$ we see that $\widehat{\mathbf{D}_N} = \mathbf{D}_{N-1}$.

The class of hamiltonians which can be analysed by our methods with the help of Theorem 1.1 does not contain those which appear in the usual quantum field models. For this we introduce now a more general type of algebras. First a notation: if $S \in B(\mathcal{H})$ then $S^{\vee n} = S^{\otimes n} | \mathcal{H}^{\vee n}$ and

$$\Gamma_N(S) = \oplus_{n=0}^{N} S^{\vee n} \in B(\Gamma_N(\mathcal{H})).$$

Now if $\mathcal{C} \subset B(\mathcal{H})$ is a unital C^*-subalgebra, then $\mathcal{C}_N$ is the C^*-algebra of operators on $\Gamma_N(\mathcal{H})$ generated by the operators of the form $\Gamma_N(S)$ and $a_N(u)$ with $S \in \mathcal{C}$ and $u \in \mathcal{H}$. So $\mathcal{C}_N$ is the smallest C^*-algebra which contains $\Gamma_N(\mathcal{C}) \bigcup \mathcal{A}_N$. A rather natural notation for $\mathcal{C}_N$ is $\mathcal{A}_N \rtimes \mathcal{C}$ (to suggest crossed-product).

Theorem 1.2. *Assume that $\mathcal{C} \subset B(\mathcal{H})$ is a unital C^*-subalgebra with no nonzero finite rank projections. Then there is a unique morphism $\mathcal{P}_N : \mathcal{C}_N \to \mathcal{C} \otimes \mathcal{C}_{N-1}$ such that for all $u \in \mathcal{H}$ and $S \in \mathcal{C}$*

$$\begin{aligned} \mathcal{P}_N[a_N(u)] &= 1 \otimes a_{N-1}(u) \\ \mathcal{P}_N[\Gamma_N(S)] &= S \otimes \Gamma_{N-1}(S). \end{aligned}$$

One has $\ker \mathcal{P}_N = \mathcal{K}_N$.

Thus we have a canonical embedding

$$\mathcal{C}_N/\mathcal{K}_N \equiv \widehat{\mathcal{C}_N} \hookrightarrow \mathcal{C} \otimes \mathcal{C}_{N-1}.$$

As before we denote $T \mapsto \widehat{T}$ the canonical morphism of $\mathcal{C}_N$ onto $\widehat{\mathcal{C}_N}$ which will be identified with a C^*-subalgebra of $\mathcal{C} \otimes \mathcal{C}_{N-1}$. Note that this time the inclusion $\widehat{\mathcal{C}_N} \subset \mathcal{C} \otimes \mathcal{C}_{N-1}$ is strict.

2 Some General Facts

We summarize several definitions and results from [ABG] and [DaG]. A self-adjoint operator H on a Hilbert space $\mathscr{H}$ is called *affiliated* to a C^*-algebra $\mathscr{C}$ of operators on $\mathscr{H}$ if $(H-z)^{-1} \in \mathscr{C}$ for some $z \in \mathbb{C}\setminus\sigma(H)$. This implies $\varphi(H) \in \mathscr{C}$ for all $\varphi \in C_0(\mathbb{R})$, so one can associate to H a morphism $\varphi \mapsto \varphi(H)$ of the C^*-algebra $C_0(\mathbb{R})$ into $\mathscr{C}$ (by "morphism" between two C^*-algebras we understand $*$-morphism and $C_0(\mathbb{R})$ is the space of continuous convergent to zero at infinity complex functions on $\mathbb{R}$). Such a morphism is called *observable affiliated to* $\mathscr{C}$. This notion makes sense even if $\mathscr{C}$ is an abstract C^*-algebra (i.e. is not realized on a Hilbert space) and has the advantage that one can speak of the image of an observable through a morphism $\mathcal{P} : \mathscr{C} \to \mathscr{D}$ (this is just the composition of the two morphisms). Note that if $\mathscr{C} \subset B(\mathscr{H})$ then there are observables affiliated to $\mathscr{C}$ which do not correspond to self-adjoint operators on $\mathscr{H}$ (if one requires, as usual, that such an operator be densely defined). However, if the observable is *strictly affiliated* to $\mathscr{C}$, then it is represented by a (densely defined) self-adjoint operator in each non-degenerate representation of $\mathscr{C}$ (see [DaG]). We also refer to [DaG] for a quite powerful and easy to use criterion which assures that an operator of the form $H = H_0 + V$ is (strictly) affiliated to an algebra if one knows that H_0 has the corresponding property (all the situations considered below and in [GI], [GM] are covered by this result).

Assume now that a C^*-algebra $\mathscr{C}$ of operators on a Hilbert space $\mathscr{H}$ is given and that $\mathscr{C}$ contains the set of compact operators $K(\mathscr{H})$. From the physical point of view, this is the C^*-algebra generated by the operators which are natural candidates as hamiltonians for some system in a given physical situation; we call it *C^*-algebra of energy observables* of the system. Then $K(\mathscr{H})$ is an ideal (closed, bilateral) in $\mathscr{C}$ and one can construct the quotient C^*-algebra $\widehat{\mathscr{C}} = \mathscr{C}/K(\mathscr{H})$. If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ then one can consider its image $\widehat{H}$ through the canonical morphism $\mathscr{C} \to \widehat{\mathscr{C}}$. Then $\widehat{H}$ is an observable affiliated to $\widehat{\mathscr{C}}$ and it is easy to show that the essential spectrum of H is equal to the spectrum of $\widehat{H}$ (the spectrum of an observable being defined in a natural way). Now assume that H is strictly affiliated to $\mathscr{C}$ and that a faithful non-degenerate realization of $\widehat{\mathscr{C}}$ on some Hilbert space $\widehat{\mathscr{H}}$ is given. Then the observable $\widehat{H}$ is realized as a self-adjoint operator (that we denote also by $\widehat{H}$) on $\widehat{\mathscr{H}}$ and clearly we have $\sigma_{ess}(H) = \sigma(\widehat{H})$. This is the method that we use for the computation of the essential spectrum of the hamiltonians considered in [GI], [GM].

The absence of the singular continuous spectrum is proved with the help of the Mourre method in the version developed in [ABG]. The main difficulty is to prove the Mourre estimate, so I shall indicate here the main steps. I think that one gets a particularly elegant presentation by working at a purely algebraic level (as in [ABG]), but I shall avoid this in order not to introduce too many abstract objects.

Let $\mathscr{C}$ be the same C^*-algebra as above and A a self-adjoint operator on $\mathscr{H}$ such that $\mathrm{e}^{-itA}\mathscr{C}\mathrm{e}^{itA} = \mathscr{C}$ for each real t. Assume also that the map $t \mapsto \mathrm{e}^{-itA}S\mathrm{e}^{itA}$ is norm continuous for each $A \in \mathscr{C}$. Since $\mathrm{e}^{-itA}K(\mathscr{H})\mathrm{e}^{itA} = K(\mathscr{H})$, there is a norm continuous one-parameter group of automorphisms α_t of $\widehat{\mathscr{C}}$ such that $\left[\mathrm{e}^{-itA}S\mathrm{e}^{itA}\right]^\wedge = \alpha_t(\widehat{S})$ for all t and $S \in \mathscr{C}$. Finally, assume that the group α_t is unitarily implemented in the representation on $\widehat{\mathscr{H}}$, more explicitly: there is a self-adjoint operator $\widehat{A}$ on $\widehat{\mathscr{H}}$ such that, for all t and $S \in \mathscr{C}$,

$$\left[\mathrm{e}^{-itA}S\mathrm{e}^{itA}\right]^\wedge = \mathrm{e}^{-it\widehat{A}}\,\widehat{S}\,\mathrm{e}^{it\widehat{A}}. \tag{2.1}$$

We have to recall now some general facts concerning the Mourre estimate in the version of [ABG]. A self-adjoint operator H is of class $C^1_{\mathrm{u}}(A)$ if the map $t \mapsto \mathrm{e}^{-itA}(H+i)^{-1}\mathrm{e}^{itA}$ is norm C^1. We define two lower semi-continuous functions $\rho^A_H, \widehat{\rho}^A_H : \mathbb{R} \to (-\infty, \infty]$ by the following rules: $\widehat{\rho}^A_H(\lambda)$ is the upper bound of the numbers a for which there are a real function $\varphi \in C_c(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and a compact operator K such that

$$\varphi(H)[H, iA]\varphi(H) \geq a\varphi(H)^2 + K$$

(one can rigorously give a meaning to the sesquilinear form from the l.h.s. above as a bounded symmetric operator). In other terms, $\widehat{\rho}^A_H(\lambda)$ is the best constant in the Mourre estimate. Similarly, $\rho^A_H(\lambda)$ is the upper bound of the numbers a such that the preceding inequality holds for some φ and $K = 0$. The set $\tau_A(H)$ of real λ where $\widehat{\rho}^A_H(\lambda) \leq 0$ is closed and will be called the set of *A-thresholds* of H. The (closed) set $\kappa_A(H)$ of *A-critical points* of H is given by the condition $\rho^A_H(\lambda) \leq 0$.

An important property of the functions ρ^A_H and $\widehat{\rho}^A_H$ is the following. Say that $\lambda \in \mathbb{R}$ is a *M-eigenvalue* of H if it is an eigenvalue and $\widehat{\rho}^A_H(\lambda) > 0$. By the virial theorem, these eigenvalues are of finite multiplicity and are not accumulation points of eigenvalues of H. Theorem 7.2.13 from [ABG] says that $\rho^A_H(\lambda) = 0$ if λ is a M-eigenvalue of H and otherwise $\rho^A_H(\lambda) = \widehat{\rho}^A_H(\lambda)$. Thus the functions $\rho^A_H, \widehat{\rho}^A_H$ differ only on a small set: the (discrete) set of M-eigenvalues. Also, $\rho^A_H(\lambda) > 0$ if and only if $\widehat{\rho}_H(\lambda) > 0$ and $\lambda \notin \sigma_{\mathrm{p}}(H)$, or $\kappa_A(H) = \tau_A(H) \cup \sigma_{\mathrm{p}}(H)$.

We go back now to the setting described before and consider a self-adjoint operator H on $\mathscr{H}$ strictly affiliated to $\mathscr{C}$ and of class $C^1_{\mathrm{u}}(A)$. It is clear then that $\widehat{H}$ is of class $C^1_{\mathrm{u}}(\widehat{A})$ and a straightforward computation gives $\widehat{\rho}^A_H = \rho^{\widehat{A}}_{\widehat{H}}$.

For a complete determination of the threshold set in the cases of interest for us here we need a last formula from [ABG]. Let $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ and H_i, A_i self-adjoint operators in $\mathscr{H}_i$ such that H_i is bounded from below and of class $C^1_{\mathrm{u}}(A_i)$. Consider the self-adjoint operators $H = H_1 \otimes 1 + 1 \otimes H_2$ and $A = A_1 \otimes 1 + 1 \otimes A_2$ on $\mathscr{H}$. Then H is of class $C^1_{\mathrm{u}}(A)$ and

$$\rho^A_H(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2} \left[\rho^{A_1}_{H_1}(\lambda_1) + \rho^{A_2}_{H_2}(\lambda_2)\right]. \tag{2.2}$$

3 Spectral Analysis of Hamiltonians Affiliated to $\mathcal{C}_N$

We apply now the results described in Section 2 to the self-adjoint operators afiliated to the C^*-algebra $\mathcal{C}_N$ constructed in Section 1. In order to make clear the connection with the quantum field models studied in the literature, we consider in detail the case of a spin zero boson with $\mathbb{R}^s$ as configuration space (however, it should be clear that the next arguments are of an abstract nature and so apply to much more general situations). Then the one boson Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^s)$ and we choose $\mathcal{C} = \{\psi(P) \mid \psi \in C_0(\mathbb{R}^s)\}$ where $P = -i\nabla$ is the momentum observable. Let $\omega : \mathbb{R}^s \to \mathbb{R}$, the one boson kinetic energy, be a continuous divergent function. The field operators on $\varGamma(\mathcal{H})$ are denoted $\varphi(x)$ and $\varphi_\chi(x)$ is a regularized version of the field given by an ultraviolet cutoff χ. Finally, let $g : \mathbb{R}^s \to \mathbb{R}$ be a spatial cutoff. Then the total hamiltonian of the boson field has the form

$$H = d\varGamma(\omega(P)) + \int_{\mathbb{R}^s} g(x) :\!\mathrm{Pol}[\varphi_\chi(x)]\!: \mathrm{d}x \equiv d\varGamma(\omega(P) + W$$

where Pol is a real polynomial and : : denotes Wick product. We also recall that if S is a self-adjoint bounded from below operator on $\mathcal{H}$ then the self-adjoint operator $d\varGamma(S)$ on $\varGamma(\mathcal{H})$ is defined by $\exp(-d\varGamma(S)) = \varGamma(\exp(-S))$ (for details on this formalism, see the papers of J. Derezinski and C. Gérard quoted in the references). From now on, in order to simplify the notations, we set $\omega = \omega(P)$.

We shall now add a particle number cutoff, so we take $\mathscr{H} = \varGamma_N(\mathcal{H})$ and $\mathscr{C} = \mathcal{C}_N$ in Section 2 and consider the operator $H_N = 1_N H 1_N$. Set $d\varGamma_N(\omega) = 1_N d\varGamma(\omega) 1_N$ and $W_N = 1_N W 1_N$, so $H_N = d\varGamma_N(\omega) + W_N$. Then, under quite general conditions on g, χ, the operator H_N is self-adjoint on $\varGamma_N(\mathcal{H})$ and strictly affiliated to $\mathcal{C}_N$. Indeed, $d\varGamma_N(\omega)$ is always strictly affiliated to $\mathcal{C}_N$ and W_N is in fact a bounded operator in $\mathcal{A}_N$; one gets $(H_N + \lambda)^{-1} \in \mathcal{C}_N$ for large positive λ by making a Neumann expansion. An operator obtained as norm resolvent limit of operators of the form H_N is still affiliated to $\mathcal{C}_N$; this allows one to remove the ultraviolet cutoff if $s = 1$.

We have now to compute $\widehat{H_N}$. Note first that Theorem 1.2 gives an embedding $\widehat{\mathcal{C}_N} \subset \mathcal{C} \otimes \mathcal{C}_{N-1}$ hence a faithful and nondegenerate realization of $\widehat{\mathcal{C}_N}$ on the Hilbert space $\widehat{\mathscr{H}} \equiv \mathit{\Gamma}_N(\mathcal{H})^\wedge = \mathcal{H} \otimes \mathit{\Gamma}_{N-1}(\mathcal{H})$. This is the Hilbert space on which we shall represent $\widehat{H_N}$. From Theorem 1.2 and the examples given after Theorem 1.1 we obtain $\mathcal{P}_N(W_N) = W_{N-1}$. Then we compute the quotient of $d\mathit{\Gamma}_N(\omega)$ by noticing that it is uniquely defined by the property

$$\exp(-\mathcal{P}_N(d\mathit{\Gamma}_N(\omega))) = \mathcal{P}_N(\exp(-d\mathit{\Gamma}_N(\omega)) = \mathcal{P}_N(\mathit{\Gamma}_N(\exp(-\omega))$$

and by using Theorem 1.2 once again. This gives

$$[d\mathit{\Gamma}_N(\omega)]^\wedge = \omega \otimes 1_{N-1} + 1 \otimes d\mathit{\Gamma}_{N-1}(\omega).$$

Finally, by making a Neumann expansion of $(H_N+\lambda)^{-1}$ and observing that $\widehat{H_N}$ is characterized by the relation $\mathcal{P}_N[(H_N+\lambda)^{-1}] = (\widehat{H_N}+\lambda)^{-1}$, we obtain

$$\widehat{H_N} = \omega \otimes 1_{N-1} + 1 \otimes H_{N-1}. \tag{3.1}$$

As we explained in Section 2, this relation has important consequences. First, it allows us to compute the essential spectrum of H_N. We recall that $\sigma(S\otimes 1 + 1\otimes T) = \sigma(S) + \sigma(T)$ if S, T are operators bounded from below. Let $m = \min \omega$ be the least value of the function ω; then $\sigma(\omega(P)) = [m, \infty)$ hence

$$\sigma_{ess}(H_N) = [m + \min \sigma(H_{N-1}), \infty). \tag{3.2}$$

Now we look for a conjugate operator in the sense of Mourre for the hamiltonian H_N. We start, as in [Gr], with a self-adjoint operator $\mathfrak{a}$ on $\mathcal{H}$ such that $\omega(P)$ is of class $C^1_{\mathrm{u}}(\mathfrak{a})$ and take $A \equiv A_N = d\mathit{\Gamma}_N(\mathfrak{a})$. Since $\mathrm{e}^{itA_N} = \mathit{\Gamma}_N(\mathrm{e}^{it\mathfrak{a}})$ we have $\mathrm{e}^{-itA_N} a_N(u) \mathrm{e}^{itA_N} = a_N(\mathrm{e}^{-it\mathfrak{a}}u)$, so the algebra $\mathcal{A}_N$ is stable under the group of automorphisms associated to A_N. But $\mathrm{e}^{-itA_N}\mathit{\Gamma}_N(S)\mathrm{e}^{itA_N} = \mathit{\Gamma}_N(\mathrm{e}^{-it\mathfrak{a}} S \mathrm{e}^{it\mathfrak{a}})$, so we have to assume that $\mathcal{C}$ is stable under the automorphisms associated to $\mathfrak{a}$ in order that the condition $\mathrm{e}^{-itA_N}\mathcal{C}_N\mathrm{e}^{itA_N} = \mathcal{C}_N$ be satified. For example, this holds if the function ω is of class C^1 and we take $\mathfrak{a} = F(P)Q + QF(P)$, where F is a vector field on $\mathbb{R}^s$ proportional to $\nabla\omega$ (cf. (7.6.15) and Lemma 7.6.4 from [ABG]).

It is easy to deduce from Theorem 1.2 that in the present situation the relation (2.1) is satisfied with $\widehat{A_N} = \mathfrak{a} \otimes 1_{N-1} + 1 \otimes A_{N-1}$. We stress that the hat on A_N should not be interpreted in the sense of Section 1 (because this operator is not affiliated to the algebra $\mathcal{C}_N$). Now one has to put some conditions on the cutoffs g and χ which assure that W_N is of class $C^1_{\mathrm{u}}(A_N)$; this is quite easy, see again the quoted papers of Derezinski and Gérard (our purpose here is not to get optimal results but rather to point out a strategy of proof). Assuming this, we see that H_N is of the same class, so we may associate to it the functions $\rho_N = \rho^{A_N}_{H_N}$ and $\widehat{\rho}_N = \widehat{\rho}^{A_N}_{H_N}$. As was explained in Section 2, $\widehat{\rho}_N$ coincides with the ρ function associated to the couple $\widehat{A_N}, \widehat{H_N}$. Let $\rho = \rho^{\mathfrak{a}}_{\omega}$; then (2.2) gives the recursion relation

$$\widehat{\rho}_N(\lambda) = \inf_{\lambda = \lambda_1 + \lambda_2} \left[\rho(\lambda_1) + \rho_{N-1}(\lambda_2)\right]. \tag{3.3}$$

This has to be used in conjunction with the relation: $\rho_{N-1}(\lambda) = \widehat{\rho}_{N-1}(\lambda)$ if λ is not a M-eigenvalue of H_{N-1}; $\rho_{N-1}(\lambda) = 0 < \widehat{\rho}_{N-1}(\lambda)$ otherwise.

We shall compute now the threshold sets by induction on N under the asumption $\rho \geq 0$. Note that (3.3) and the last remark imply $0 \leq \rho_N \leq \widehat{\rho}_N$ for all N. The functions ρ and ρ_{N-1} are equal to ∞ below the spectra of the operators ω and H_{N-1} respectively and the function $\mu \mapsto \rho(\lambda - \mu) + \rho_{N-1}(\mu)$ is lower semi-continuous, hence it attains its lower bound on each compact set. Thus the infimum in (3.3) is attained. In particular, $\widehat{\rho}_N(\lambda) = 0$ if and only if one can write $\lambda = \lambda_1 + \lambda_2$ with $\rho(\lambda_1) = 0$ and $\rho_{N-1}(\lambda_2) = 0$. Equivalently $\tau_{A_N}(H_N) = \kappa_{\mathfrak{a}}(\omega) + \kappa_{A_{N-1}}(H_{N-1})$, or

$$\tau_{A_N}(H_N) = \kappa_{\mathfrak{a}}(\omega) + \tau_{A_{N-1}}(H_{N-1}) \cup \sigma_{\mathrm{p}}(H_{N-1}). \tag{3.4}$$

Since $\tau_{A_0}(H_0) = \emptyset$ and $\sigma_{\mathrm{p}}(H_0) = \{0\}$ we get successively $\tau_{A_1}(H_1) = \kappa_{\mathfrak{a}}(\omega)$, $\tau_{A_2}(H_2) = \kappa_{\mathfrak{a}}(\omega) + \kappa_{\mathfrak{a}}(\omega) \cup \sigma_{\mathrm{p}}(H_1)$ and so on.

The simplest situation is that when one can choose $\mathfrak{a}$ such that $\rho(\lambda) > 0$ if $\lambda \neq 0$ and $\rho(0) = 0$ (this happens in the usual models, when the function ω has only one critical point hence only one critical value). Then $\tau_{A_N}(H_N) = \bigcup_{n=0}^{N-1} \sigma_{\mathrm{p}}(H_n)$. In particular, $\tau(H_N) \equiv \tau_{A_N}(H_N)$ is a closed countable set and the eigenvalues of H_N which do not belong to $\tau(H_N)$ are M-eigenvalues, so of finite multiplicity and with no accumulation points outside $\tau(H_N)$. If H_N is of a slightly stronger regularity class with respect to A_N, then H_N will not have singularly continuous spectrum.

We indicate now the procedure which has to be used when the function ω has many critical values. Take $\mathfrak{a} = F(P)Q + QF(P)$ with $F(p) = \frac{1}{2}\theta(\omega(p))|\nabla\omega(p)|^2$, where $\theta \geq 0$ is a C^∞ function, $\theta(\lambda) = 1$ if $\lambda < s$ and $\theta(\lambda) = 0$ if $\lambda > s+1$ for some large positive s. Then $\rho(\lambda) = \theta(\lambda)\min_{\omega(p)=\lambda}|\nabla\omega(p)|^2$. We see that $\rho \geq 0$ and on $(-\infty, s]$ one has $\rho(\lambda) > 0$ except when λ is a critical value of ω. We define the *threshold set* of H_N recursively by

$$\tau(H_N) = \kappa(\omega) + \tau(H_{N-1}) \cup \sigma_{\mathrm{p}}(H_{N-1}), \tag{3.5}$$

where $\kappa(\omega)$ is the set of critical values of the function ω. Note that this is a closed set depending only on the hamiltonian. Outside it H_N admits locally conjugate operators, so $\sigma_{\mathrm{p}}(H_N) \setminus \tau(H_N)$ consists of M-eigenvalues. If $\kappa(\omega)$ is countable and H_N satifies a certain regularity condition, then H_N has no singularly continuous spectrum. Concerning ω, it suffices to assume that it is of polynomial growth with derivatives bounded by $C(1 + |\omega|)$.

4 Coupling of Two Systems

A system of bosons is often considered in interaction with an external system, like in the spin-boson case or the more general Pauli-Fierz models (see [Gr] and [DeG1]. We state briefly an algebraic result which allows one to study such a situation.

Assume that two physical systems are given with state spaces $\mathscr{H}_1, \mathscr{H}_2$ and "algebras of energy observables" $\mathscr{C}_1 \subset B(\mathscr{H}_1), \mathscr{C}_2 \subset B(\mathscr{H}_2)$ such that $K(\mathscr{H}_i) \subset \mathscr{C}_i$, and that at least one of the algebras $\mathscr{C}_i$ is nuclear. We shall define the algebra of energy observables of the coupled system as being $\mathscr{C}_1 \otimes \mathscr{C}_2$. Since $K(\mathscr{H}_1) \otimes K(\mathscr{H}_2) = K(\mathscr{H}_1 \otimes \mathscr{H}_2)$ we are in a situation similar to the preceding ones: $K(\mathscr{H}_1 \otimes \mathscr{H}_2) \subset \mathscr{C}_1 \otimes \mathscr{C}_2 \subset B(\mathscr{H}_1 \otimes \mathscr{H}_2)$. We denote, as in Section 2, by a hat the quotient with respect to the ideal of compact operators. The main fact is that there is a canonical embedding

$$[\mathscr{C}_1 \otimes \mathscr{C}_2]^\wedge \subset \left[\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2\right] \oplus \left[\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}\right].$$

For example, in order to treat a boson field coupled with a confined system (like in the spin-boson or Pauli-Fierz case) we take $\mathscr{C}_1 = \mathcal{C}_N$ and $\mathscr{C}_2 = K(\mathscr{H}_2)$. Then $\widehat{\mathscr{C}_2} = 0$ which makes things quite simple. One can also couple the field with a non-relativistic many-body system for which the algebra of energy observables is known (see [DaG] for example) and the quotient is not trivial.

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Address

VLADIMIR GEORGESCU, CNRS and Département de Mathématiques de l'Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, F 95302 Cergy-Pontoise Cedex, France
E-MAIL: Vladimir.Georgescu@math.u-cergy.fr

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