Asymptotic Electromagnetic Fields in Models of Quantum-Mechanical Matter Interacting with the Quantized Radiation Field

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Abstract

In models of (non-relativistic and pseudo-relativistic) electrons interacting with static nuclei and with the (ultraviolet-cutoff) quantized radiation field, the existence of asymptotic electromagnetic fields is established. Our results yield some mathematically rigorous understanding of Rayleigh scattering and of the phenomenon of relaxation of isolated atoms to their ground states. Our proofs are based on propagation estimates for electrons inspired by similar estimates known from N-body scattering theory.

I Introduction

In this paper we study the scattering of light at non-relativistic and pseudo-relativistic, quantum mechanical electrons moving under the influence of an external potential and minimally coupled to the soft modes of the quantized electromagnetic field. The external potential may be the Coulomb potential generated by a configuration of static nuclei. Our goal is to establish the *existence of asymptotic electromagnetic fields* on states of the system with the property that the velocities of all electrons present are smaller than the velocity of light, (in a sense to be made mathematically precise). This property is automatically satisfied if one chooses *relativistic kinematics* in the description of electrons, because the propagation velocity of a *massive* relativistic particle is smaller than the velocity of light. In contrast, if the kinematics of electrons is non-relativistic these particles can propagate arbitrarily fast, and the condition that the propagation velocities of electrons in a

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state of the system are smaller than the velocity of light is a very stringent one. It is satisfied if all electrons remain bound to nuclei. But, in realistic models, such "bound states" are *not dense* in the Hilbert space of the system.

The key physical idea underlying our analysis is very clear and simple: *Huygens' principle* for the electromagnetic field implies that, on states with the property that the propagation speeds of all charged particles are smaller than the velocity of light, the strength of interactions between the charged particles and the electromagnetic field tends to 0, as time t tends to $\pm\infty$, at an integrable rate. As a consequence, one can use a variant of *Cook's method* [Coo57] to prove existence of asymptotic electromagnetic fields; (a (strong) *LSZ asymptotic condition* holds for the electromagnetic field).

Huygens' principle was first applied in the context of scattering theory for the quantized electromagnetic field by Buchholz [Buc77]. More recently, it was used in [Spo97] to prove asymptotic completeness of Rayleigh scattering in some simple models of electrons permanently confined to nuclei.

Cook's method was used to prove existence of asymptotic fields in simple models of quantum field theory by Hoegh–Krohn in [HK69]. His arguments were inspired by work on scattering theory in the context of axiomatic field theory, in particular Haag–Ruelle scattering theory [Jos65] and Hepp's analysis of the LSZ asymptotic condition in massive quantum field theories [Hep65]. For models of the kind considered in this paper describing non–relativistic or pseudo–relativistic electrons interacting with *massive* photons, in a space–time of dimension four or more, *Cook's method* (in conjunction with field–operator domain estimates of the type proven in Lemma 5, Sect. II, below) is *all it takes* to construct asymptotic electromagnetic fields; (see e.g. [Frö73]). The reason is that the *amplitude* of a *spatially localized* excitation of the electromagnetic field propagates into the *interior* of a forward light cone if the mass of the photon is *strictly positive* and hence locally decays in time at an integrable rate, $\propto t^{-d/2}$, where d is the dimension of space. This is *not* so if the photon is *massless*. Then if d is *odd* Huygens' principle tells us that such excitations propagate along the *boundary* of a forward light cone where their amplitude decays in time only like $t^{-(d-1)/2}$, which is *not* integrable in dimension $d = 3$ [a)]!

If the propagation velocities of charged particles are smaller than the propagation velocity of light then every excitation of the electromagnetic field ends up propagating out of the region where the charged particles are localized and hence does not interact with them, anymore. This feature, along with a decay $\propto t^{-(d-1)/2}$ of its amplitude, suffices to rescue Cook's argument for proving the existence of asymptotic electromagnetic field operators.

Our discussion makes it clear what our main technical work has to consist in:

We must prove mathematically precise bounds on the velocity of propagation of electrons (*"propagation estimates"*, see Theorem 1, Sect. II) in physically realistic situations; and

we have to establish the *invariance* of certain domains in Hilbert space under the time evolution

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[a)] There has been some confusion on this point in the literature (see e.g. [Frö73]) which has been brought to our attention by Alessandro Pizzo on which propagation estimates hold and which are contained in the domain of definition of products of electromagnetic field operators (see Lemma 5, Sect. II).

The results proven in this paper are an essential ingredient in developing a mathematically rigorous theory of *Rayleigh scattering* in atomic and molecular physics and in proving that an isolated neutral atom or molecule prepared in an arbitrary, possibly highly excited *bound state* relaxes to its *ground state*[b)], as time t tends to infinity, by emitting photons. Relaxation to the ground state and the related phenomenon of "return to equilibrium" at positive temperature [JP96], [BFS00] play a key rôle in attempts to understand dissipative, irreversible behavior in the quantum theory of open systems. A rather fundamental ingredient in proving the relaxation of bound states of neutral atoms and molecules to their ground states is control over the nature of the spectrum of the basic Hamilton operator governing the dynamics of such systems, as achieved in [HS95], [BFS99], [BFSS99], [Ski98]. However, spectral results alone are not sufficient to exhibit relaxation of bound states to a ground state; they must be supplemented by results on Rayleigh scattering. Our results in this paper are a step in this direction; (see also [Spo97]). They are, moreover, one among several key ingredients in developing a scattering theory for unbound, freely moving charged particles ("infra–particles" [Sch63]) interacting with the quantized radiation field; (see [Frö73], [FMS79], [Buc82] for preliminary results in this direction).

Next, we introduce the models studied in this paper and summarize our main results in more precise terms.

We consider a system consisting of $N = 1, 2, 3, \ldots$ quantum–mechanical electrons with non–relativistic kinetimatics under the influence of an external potential, interacting among themselves via two–body Coulomb repulsion, and minimally coupled to the soft modes of the quantized electromagnetic field. Throughout this paper we describe the transverse degrees of freedom of the electromagnetic field in terms of the quantized electromagnetic vector potential, $A(x,t)$, in the *Coulomb gauge*, i.e., $(\nabla \cdot A)(x,t) = 0$.

Purely for reasons of notational simplicity we neglect electron spin.

The *Hamilton operator* generating the time evolution of this system is then given by

$$H \equiv H_N^{\mathrm{nr}} := \sum_{j=1}^{N} \frac{1}{2m}(p_j + eA(x_j))^2 + V_N + W_N + H_f$$

where V_N is a sum of one-body potentials, $v(x_j)$, describing, for example, the field of static nuclei, and W_N accounts for the interaction among the particles. Concerning W_N, we assume that the interaction of the j-th particle with all the other particles is repulsive and, of course, that W_N is symmetric with respect to permutations of the particle coordinates. The operator H_f denotes the field Hamiltonian (electromagnetic field energy), and $A(x)$ is the quantized UV-cutoff vector potential at the point x. The Hamiltonian H_N^{nr} acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$, where $\mathcal{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3; \mathbb{C}^2)$. We can also treat electrons with spin and work with $[\sigma\cdot(p+eA(x))]^2 = (p+eA(x))^2 + e\sigma\cdot B(x)$, instead of $(p+eA(x))^2$. This only burdens the formalism

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[b)] See [BFS99, GLL00] for results concerning the existence of ground states in atomic physics.

and is therefore omitted. The quantum statistics (Pauli principle) obeyed by the electrons turns out not to play any rôle in the following.

An important rôle is played by the threshold

$$\Sigma = \inf \sigma(H^{\mathrm{nr}}_{N-1}) + \liminf_{x\to\infty} v(x). \tag{1}$$

This is a lower bound for the total energy when one particle has been put to infinity. Note that $\Sigma = \infty$ occurs when $v(x)$ is confining. At energies below $\Sigma + mc^2/2$, where c is the speed of light, the kinetic energy of each particle is less than $mc^2/2$ and hence its speed is less than c. We prove a sharp propagation estimate which confirms this picture. Under this condition it follows that the interaction of the electrons with photons escaping to infinity decays integrably fast, as $t \to \infty$. As a result, states with asymptotic incoming and outgoing photons exist at energies below $\Sigma + mc^2/2$. More precisely, defining $a(h) = \sum_{\lambda=1,2} \int dk\, a_\lambda(k)\overline{h(k,\lambda)}$, and $a^*(h) = (a(h))^*$, the limits

$$a^{\#}_{+}(h)\varphi = \lim_{t\to\infty} e^{iHt} a^{\#}(h_t) e^{-iHt}\varphi \tag{2}$$

exist if $E < \Sigma + mc^2/2$, φ is in the range of the spectral projection $\chi(H \le E)$, and h belongs to $L^2(\mathbb{R}^3; (1+|k|^{-1})dk) \otimes \mathbb{C}^2$. The operator $a^{\#}_{+}(h)$ stands for either a creation operator, $a^*_+(h)$, or an annihilation operator, $a_+(h)$, and $h_t = e^{-i\omega t}h$, with $\omega(k) = c|k|$. Furthermore, the limit

$$a^{\#}_{+}(h_1)\dots a^{\#}_{+}(h_n)\varphi = \lim_{t\to\infty} e^{iHt} a^{\#}(h_{1,t})\dots a^{\#}(h_{n,t}) e^{-iHt}\varphi \tag{3}$$

exists if

$$E + \sum_i M_i < \Sigma + \frac{1}{2}mc^2$$

where the sum extends over those $i \ge 2$ only for which $a^{\#}(h_i)$ is a creation operator. Here $M_i = \sup\{|k| : h_i(k) \ne 0\}$. Hence, if $\Sigma = \infty$ asymptotic creation– and annihilation operators are densely defined. They can then be obtained by differentiating asymptotic Weyl operators. If $\Sigma = \infty$ the operator $\phi_+(h) = 1/\sqrt{2}(a_+(h) + a^*_+(h))$ can be shown to be essentially selfadjoint on the domain of the Hamilton operator, and

$$e^{i\phi_+(h)} = s - \lim_{t\to\infty} e^{iHt} e^{i\phi(h_t)} e^{-iHt}$$

exists. The asymptotic creation– and annihilation operators, $a^{\#}_{+}(h)$, determine a representation of the canonical commutation relations (CCR) on $\mathcal{H}$. If φ_0 is the unique ground state of the Hamilton operator H, i.e., φ_0 is the (up to a phase) unique normalized vector in $\mathcal{H}$ with $H\varphi_0 = E_0\varphi_0$, where $E_0 = \inf \sigma(H)$, (see [BFS99, GLL00, Hir00]), then $a_+(h)\varphi_0 = 0$, for an arbitrary function $h \in L^2(\mathbb{R}^3, (1+|k|^{-1})\, dk\,) \otimes \mathbb{C}^2$. Hence φ_0 is a *vacuum* for the asymptotic creation– and annihilation operators.

One expects that if $\Sigma = \infty$ *asymptotic completeness* (AC) holds, in the sense that the linear space , $\mathcal{H}_+$, spanned by vectors of the form

$$a^*_+(h_1)\dots a^*_+(h_n)\varphi_0, \quad n = 0, 1, 2, \dots, \tag{4}$$

is *dense* in $\mathcal{H}$. A result of this type, for a simple caricature of the models studied in this paper, has been proven in [Spo97].

If $\Sigma < \infty$ the situation is more subtle. We define $\mathcal{H}_{bs}$ to be the subspace of $\mathcal{H}$ consisting of "bound states": A vector $\varphi \in \mathcal{H}$ is a *bound state* iff it is in the domain of definition of the operator

$$\exp\left\{\varepsilon(\sum_{j=1}^{N}|x_j|)\right\},$$

for some $\varepsilon = \varepsilon(\varphi) > 0$. Thus, in a bound state, all electrons are exponentially well localized near the origin. The space $\mathcal{H}_+^{(R)}$ is defined to be the space spanned by all vectors of the form (4), which are strong limits of vectors of the form

$$e^{i(H-E_0)t}a^*(h_{1,t})\dots a^*(h_{n,t})\varphi_0,$$

as $t \to \infty$, where φ_0 is the ground state.

Asymptotic completeness of Rayleigh scattering is the statement that

$$\mathcal{H}_{bs} \subseteq \mathcal{H}_+^{(R)}. \tag{5}$$

We now show that every state of the form $\psi_t = e^{-iHt}\psi$, with $\psi \in \mathcal{H}_+^{(R)}$, relaxes to the ground state φ_0, as time t tends to ∞. If asymptotic completeness of Rayleigh scattering, Eq. (5), holds then the same is true for an *arbitrary bound state* $\psi \in \mathcal{H}_{bs}$.

We must first clarify what is meant by "*relaxation of ψ_t to the ground state*". Let $\mathcal{A}$ denote the C^* algebra of all bounded functions of the self-adjoint operators

$$\phi(h) = \frac{1}{\sqrt{2}}(a(h) + a^*(h)),$$

with $h \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^2)$, the Schwartz space of two–component test functions. By taking sums of tensor products of operators in $\mathcal{A}$ with arbitrary bounded operators acting on the Hilbert space of the N electrons one obtains a C^* algebra $\tilde{\mathcal{A}}$.

Let $\psi_t = e^{-iHt}\psi$. "Relaxation of ψ_t to the ground state" is the statement that

$$\lim_{t\to\infty}\langle\psi_t, A\psi_t\rangle = \langle\varphi_0, A\varphi_0\rangle\langle\psi,\psi\rangle, \tag{6}$$

for an arbitrary operator $A \in \tilde{\mathcal{A}}$.

Let us sketch the proof of (6); more details and a proof of different (generalized) versions of AC of Rayleigh scattering will appear elsewhere.

A vector $\psi \in \mathcal{H}_+^{(R)}$ can be approximated in norm by sums of vectors of the form

$$\psi_{n,+} := a_+^*(h_1)\dots a_+^*(h_n)\varphi_0, \tag{7}$$

$h_1, \dots, h_n$ in $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^2)$, which are strong limits of the vectors

$$e^{i(H-E_0)t}\psi_{n,t}, \tag{8}$$

where

$$\psi_{n,t} := a^*(h_{1,t}) \dots a^*(h_{n,t})\varphi_0, \tag{9}$$

as $t \to \infty$. Thus, it is enough to show that, for $\psi = \psi_{n,+}$ as in (7), Eq. (6) holds, with φ_0 as in (7).

It follows from (8) and (9) that

$$\lim_{t\to\infty} \|e^{-i(H-E_0)t}\psi_{n,+} - \psi_{n,t}\| = 0.$$

Thus, for an arbitrary operator $A \in \tilde{\mathcal{A}}$ (which is *bounded*, because $\tilde{\mathcal{A}}$ is a C^* algebra),

$$\begin{aligned} \lim_{t\to\infty} \langle e^{-iHt}\psi_{n,+}, Ae^{-iHt}\psi_{n,+}\rangle &= \lim_{t\to\infty} \langle e^{-i(H-E_0)t}\psi_{n,+}, Ae^{-i(H-E_0)t}\psi_{n,+}\rangle \\ &= \lim_{t\to\infty} \langle \psi_{n,t}, A\psi_{n,t}\rangle \\ &= \lim_{t\to\infty} \langle \prod_{j=1}^{n} a^*(h_{j,t})\varphi_0, A\prod_{i=1}^{n} a^*(h_{i,t})\varphi_0\rangle. \end{aligned} \tag{10}$$

Next, A is the norm–limit of operators $A_\alpha \in \tilde{\mathcal{A}}$, as $\alpha \to \infty$, with the properties that

i) $A_\alpha \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0$ is in the domain of definition of $\prod_{j=n}^{1} a(h_{j,t})$, for arbitrary t, and

ii) $s - \lim_{t\to\infty} [\prod_{j=n}^{1} a(h_{j,t}), A_\alpha] \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0 = 0$.

Property i) follows from easy domain estimates (of the kind of Lemma 5, Sect. II), and property ii) follows from the canonical commutation relations together with

$$\lim_{t\to\infty} \sum_{\lambda=1,2} \int dk\, \overline{h_j(k,\lambda)} g(k,\lambda) e^{i\omega(k)t} = 0,$$

for arbitrary $g \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^2)$ and $h_j \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^2)$, $j = 1,\dots,n$, and from the fact that $\prod_{j=n}^{1} a(h_{j,t})$ commutes with arbitrary bounded operators acting on the Hilbert space of the electrons.

It then follows from Eq. (10) and from i) and ii) that

$$\lim_{t\to\infty} \langle e^{-iHt}\psi_{n,+}, Ae^{-iHt}\psi_{n,+}\rangle = \lim_{t\to\infty} \langle A^*\varphi_0, \prod_{j=n}^{1} a(h_{j,t}) \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0\rangle. \tag{11}$$

Next, one shows that, for arbitrary functions $f_1,\dots,f_m$ in $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^2)$, $m = 1,2,\dots$,

$$s - \lim_{t\to\infty} \prod_{l=1}^{m} a(f_{l,t})\varphi_0 = 0.$$

This implies that

$$\begin{aligned} s - \lim_{t\to\infty} \prod_{j=n}^{1} a(h_{j,t}) \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0 &= \varphi_0 \lim_{t\to\infty} \langle \varphi_0, \prod_{j=n}^{1} a(h_{j,t}) \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0\rangle \\ &= \varphi_0 \lim_{t\to\infty} \langle \prod_{j=1}^{n} a^*(h_{j,t})\varphi_0, \prod_{i=1}^{n} a^*(h_{i,t})\varphi_0\rangle \\ &= \varphi_0 \langle \prod_{j=1}^{n} a_+^*(h_j)\varphi_0, \prod_{i=1}^{n} a_+^*(h_i)\varphi_0\rangle \\ &= \varphi_0 \langle \psi_{n,+}, \psi_{n,+}\rangle. \end{aligned} \tag{12}$$

Eqs. (11) and (12) show that

$$\lim_{t\to\infty}\langle e^{-iHt}\psi_{n,+}, Ae^{-iHt}\psi_{n,+}\rangle = \langle\varphi_0, A\varphi_0\rangle\langle\psi_{n,+},\psi_{n,+}\rangle,$$

which is what we have claimed we would prove! (More details will appear in a companion paper).

Next, we describe our model of electrons with relativistic kinematics interacting with the quantized radiation field (”pseudo–relativistic electrons”) and summarize our main results for this model. For simplicity (merely of notation!), we consider a one–electron system. The Hamilton operator is then given by

$$H^{\rm rel} = \sqrt{(p+eA(x))^2c^2+m^2c^4}+V+H_f.$$

An attractive feature of this model is the correct relativistic dispersion relation for the electron, as explained above. For all finite energies, the group velocity of the electron is strictly below the velocity of light, and hence the interaction of the electron with escaping photons decays, as $t\to\infty$. Again, we will prove a propagation estimate making this precise. As a consequence, states with incoming and outgoing photons exist for *arbitrary* energies. More precisely,

$$a_+^{\#}(h)\varphi = \lim_{t\to\infty} e^{iHt}a^{\#}(h_t)e^{-iHt}\varphi \tag{13}$$

exists, for arbitrary $\varphi\in D((H+i)^{1/2})$, and, for $\varphi\in D((H+i)^{n/2})$, Eq. (3) holds, with $H^{\rm nr}$ replaced by $H^{\rm rel}$.

To illustrate the main ideas behind our proofs for existence of (2) and (13) let us consider a massive, relativistic particle coupled to a quantized scalar field of bosons, linearly in annihilation and creation operators. Such a system is described by

$$H=\sqrt{p^2c^2+m^2c^4}+V+\phi(G_x)+H_f,$$

acting on $L^2(\mathbb{R}^3)\otimes\mathcal{F}$, where $p=-i\nabla_x$, x is the coordinate of the particle, $\phi(G_x)=1/\sqrt{2}(a(G_x)+a^*(G_x))$, $G_x(k)=e^{-ik\cdot x}\kappa(k)/\sqrt{|k|}$ and $\kappa\in C_0^\infty(\mathbb{R}^3)$. Let $h\in C_0^\infty(\mathbb{R}^3\backslash\{0\})$ be the wave function of a photon. To prove existence of $a_+^*(h)\varphi=\lim_{t\to\infty}e^{iHt}a^*(h_t)e^{-iHt}$, we need to show that the family of vectors $t\mapsto\varphi(t):=e^{iHt}a^*(h_t)e^{-iHt}\varphi$ satisfies the Cauchy criterion, as $t\to\infty$. A convenient sufficient condition for this, known as Cook's argument, is that $\int_1^\infty\|(d/dt)\,\varphi(t)\|\,dt<\infty$. From the equation $a^*(h_t)=\exp(-iH_ft)a^*(h)\exp(iH_ft)$ it is easy to derive that

$$\varphi'(t)=ie^{iHt}[\phi(G_x),a^*(h_t)]e^{-iHt}\varphi=\frac{i}{\sqrt{2}}e^{iHt}(G_x,h_t)e^{-iHt}\varphi, \tag{14}$$

where

$$(G_x,h_t)=\int dk\, e^{i(k\cdot x-\omega(k)t)}\frac{\kappa(k)}{\sqrt{|k|}}h(k),$$

with $\omega(k)=c|k|$. The problem is that $(G_x,h_t)\propto t^{-1}$, for $|x|\sim ct$, which is *not* integrable in t. The physical reason for this problem has been discussed at the beginning of this introduction. Away from $|x|=ct$, the inner product (G_x,h_t) decays faster than any negative power of t. In fact, by stationary phase arguments, one shows that

$$\sup_{||x|-t|\geq\varepsilon t}|(G_x,h_t)|\leq C_{n,\varepsilon}t^{-n},$$

while

$$\sup_x |(G_x, h_t)| \le K/t,$$

for some finite constants $C_{n,\varepsilon}$ and K. From these two estimates and (14) it follows that $\|\varphi'(t)\|$ *is integrable provided that*

$$\int_1^\infty dt\, \frac{1}{t} \|\chi(1-\varepsilon \le |x/ct| \le 1+\varepsilon) e^{iHt}\varphi\| < \infty. \tag{15}$$

It will turn out that it is enough to prove this for φ in a suitable dense subspace, such as the subspace of states of finite energy which are also in the domain of $|x|^{1/2}$.

Note that $|(G_x, h_t)| \le \text{const}\, t^{-3/2}$, which is integrable, if the bosons have a positive mass, i.e., if $\omega(k) = \sqrt{c^2k^2 + M^2c^4}$, with $M > 0$. Hence a sharp propagation estimate, such as the one in Eq. (15), must only be proven for massless bosons, as discussed at the beginning of this introduction.

Note also that (15) is trivial if φ is a *bound state*, i.e., if $\sup_x \||x|^\alpha \varphi_t(x)\| < \infty$, for some $\alpha > 0$. Then the integrand is of order $t^{-\alpha-1}$, which is integrable.

To prove (15) in the general case we use that the particle is massive and hence propagates with a velocity strictly less than c, as mentioned above. As a consequence, $\varphi_t(x)$, for $|x/ct|$ near 1, decays sufficiently fast for (15) to hold. In fact, we show that

$$\int_1^\infty \frac{dt}{t^\mu} \|\chi(|x/ct| \ge (1-\varepsilon)) e^{-iHt}\varphi\|^2 \le \text{const} \|\langle x\rangle^{1/2}\varphi\|^2 \tag{16}$$

for $\mu > 1/2$, φ with bounded energy distribution and ε small enough. From (16) and the Schwarz inequality the estimate (15) clearly follows if $\varphi \in D(\langle x\rangle^{1/2})$. Hence $a_+^*(h)\varphi$ exists, for such φ and $h \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$. For $\varphi \in D((H+i)^{1/2})$ and $h \in L^2((1+|k|^{-1})dk)$, existence then follows by simple approximation arguments.

The estimate (16) actually holds also for $\mu = 0$ and, at the expense of higher powers of $\langle x\rangle^{1/2}$ on the right side, one could accommodate even positive powers of t in the integral. To keep the proof short and simple we refrain from discussing these generalizations.

Sections II and III contain our main results on the non-relativistic and the relativistic model, respectively. The corresponding propagation estimates are also contained in these sections, but other technical prerequisites are deferred to various appendices. In Section IV we show that asymptotic Weyl operators $W_+(h)$ exist and that they are generated by asymptotic field operators $\phi_+(h)$. This is done for both models simultaneously. Our main results are Theorems 4, 6 in Section II, Theorems 13, 14 in Section III, and Theorem 15 in Section IV.

II Non–Relativistic QED

II.A Assumptions and Notations

To describe N non-relativistic particles interacting with the quantized radiation field we employ the Hamiltonian

$$H \equiv H_N := \sum_{j=1}^{N} \frac{1}{2}(p_j + A(x_j))^2 + (V + W_N) \otimes 1 + 1 \otimes H_f \tag{17}$$

acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$ where $\mathcal{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3; \mathbb{C}^2)$. The units are chosen in such a way that the mass of the particles, the speed of light, and Planck's constant are equal to one, and the charge has been absorbed in the definition of the quantized vector potential $A(x)$. The scalar potentials V and W_N are multiplications with real-valued, locally square integrable functions on $\mathbb{R}^{3N}$, the space of N-particle configurations $(x_1, \ldots, x_N)$. We assume that

$$V(x_1, \ldots, x_N) = \sum_{j=1}^{N} v(x_j)$$

and

$$W_N(x_1, \ldots, x_N) \geq W_{N-1}(x_1, \ldots, \hat{x}_j, \ldots, x_N)$$

for all $j \in \{1, \ldots, N\}$ and $N \geq 2$ $(W_1 = 0)$. Here $\hat{x}_j$ indicates that the variable x_j is omitted. Furthermore W_N is symmetric with respect to permutation of the particle coordinates and

$$V_- \leq \varepsilon(-\Delta) + C_\varepsilon \quad \text{for all} \quad \varepsilon > 0. \tag{18}$$

These assumptions are satisfied, for instance, if v is the Coulomb potential due to static nuclei and W is the Coulomb repulsion between charged identical particles. The operator H_f measures the field energy and is formally given by

$$H_f = \sum_{\lambda=1,2} \int dk\, |k| a_\lambda^*(k) a_\lambda(k). \tag{19}$$

The interaction between particles and field is described by the quantized UV-cutoff vector potential in Coulomb gauge

$$A(x) = \sum_{\lambda=1,2} \int dk\, \frac{\kappa(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left\{ e^{ik\cdot x} a_\lambda(k) + e^{-ik\cdot x} a_\lambda^*(k) \right\} = \phi(G_x) \tag{20}$$

where the polarization vectors $\varepsilon_\lambda(k)$ are perpendicular to k and the form factor κ is compactly supported. In addition we assume $\kappa \in C_0^\infty(\mathbb{R}^3)$. In Eq. (20), $\phi(G_x) = 1/\sqrt{2}(a(G_x) + a^*(G_x))$ where $G_x(k, \lambda) = \kappa(k)/\sqrt{|k|} \varepsilon_\lambda(k) e^{-ik\cdot x}$. The creation and annihilation operators $a^*(h)$ and $a(h)$ obey the usual CCR

$$[a(g), a^*(h)] = (g, h), \qquad [a^\sharp(g), a^\sharp(h)] = 0$$

where (g, h) denotes the inner product in $L^2(\mathbb{R}^3; \mathbb{C}^2)$. These operators are unbounded, they are however bounded w.r. to $(H_f + 1)^{1/2}$ when h belongs to the weighted L^2-space

$$L^2_\omega(\mathbb{R}^3; \mathbb{C}^2) := \left\{ h \in L^2 \,\middle|\, \|h\|^2_\omega := \sum_{\lambda=1,2} \int dk\, |h(k,\lambda)|^2 (1 + |k|^{-1}) < \infty \right\}$$

In fact, by definition of $a(h)$, H_f, by the Schwarz inequality, and by the CCR

$$\|a^\sharp(h)\varphi\| \leq \|h\|_\omega \|(H_f + 1)^{1/2}\varphi\|.$$

The Eq. (17) defines the Hamiltonian H as a symmetric operator on a suitable dense subspace of $\mathcal{H}$. In order to realize it self-adjointly we need that by Lemma 18 (see Appendix A)

$$V_- \leq \varepsilon H + D_\varepsilon \quad \text{for all} \quad \varepsilon > 0. \tag{21}$$

This implies that H is bounded from below and hence allows us to define a self-adjoint operator, also called H, by the Friedrichs' extension of H. Note that Lemma 18 also implies that all positive parts of H are form bounded with respect to H.

II.B A Sharp Propagation Estimate

As explained in the introduction at energies below $\Sigma + v^2/2 = \Sigma + mv^2/2$, Σ being the threshold defined in Eq. (1), the kinetic energy of each particle is less than $v^2/2$ and hence its speed is strictly smaller than v. The purpose of this section is to prove a sharp propagation estimate which implements this classical argument in our non-relativistic model of QED. For similar results in N-particle scattering see [SS88, Ski91, Gér92].

Theorem 1. *Suppose $f \in \mathcal{C}_0^\infty(\mathbb{R})$, $v > 0$, and $\sup\{\lambda | f(\lambda) \neq 0\} < \Sigma + v^2/2$. Let $\mu > 1/2$ be a fixed constant. Then there exists a constant C such that*

$$\int_1^\infty dt\, \frac{1}{t^\mu}\, \|\chi(|x_j| \geq vt) e^{-iHt} f(H)\varphi\|^2 \leq C\, \|(1 + |x_j|)^{1/2} f(H)\varphi\|^2$$

for all $j \in \{1, \ldots, N\}$ and all $\varphi \in \mathcal{H}$.

Remarks. 1) We are most interested in the case $v = 1 - \varepsilon$ where $\varepsilon > 0$ but may be chosen as small as we please. Then it suffices to assume $\operatorname{supp}(f) \subset (-\infty, \Sigma + 1/2)$.

2) The theorem actually holds for $\mu = 0$, but, for $\mu > 1/2$, the proof is easier, and this result is sufficient for our purposes.

Proof. We assume f is real-valued; the proof for complex-valued f is similar. Let $\varepsilon > 0$ with $\varepsilon \leq v$. Further restrictions will be imposed later. Pick $h \in \mathcal{C}^\infty(\mathbb{R})$, non-decreasing, with $0 \leq h \leq 1$, $h(s) = 1$ if $s \geq v$, and $h(s) = 0$ if $s \leq v - \varepsilon$. The function $\tilde{h}(s) = \int_0^s d\tau h^2(\tau)$ is a smooth version of $(s - v)\chi(s - v \geq 0)$. By construction of h,

$$\tilde{h}(s) \leq (s - (v - \varepsilon))h^2(s). \tag{22}$$

Henceforth h and $\tilde{h}$ are abbreviations for $h(\langle x_j\rangle/t)$ and $\tilde{h}(\langle x_j\rangle/t)$ respectively and $\langle x_j\rangle = (1+x_j^2)^{1/2}$. We will also use the notation $\varphi_t = e^{-iHt}\varphi$. The operator

$$\phi(t) = -f(H)\, t^{1-\mu}\tilde{h}\, f(H)$$

will play the role of the so called propagation observable. We will show that its Heisenberg derivative $D\phi(t) = [iH, \phi(t)] + \partial/(\partial t)\phi(t)$ satisfies

$$D\phi(t) \geq \delta\, t^{-\mu}\, f(H)h^2 f(H) + (\text{integrable w.r.t. } t) f(H)^2, \tag{23}$$

for $t \in [T_0, \infty)$, where $\delta > 0$ and $T_0 > 1$ is sufficiently large. It will follow for $T \geq T_0$ that

$$\begin{aligned} &\int_{T_0}^{T} dt\, t^{-\mu}\, \langle \varphi_t, f(H)h^2 f(H)\varphi_t\rangle \\ &\qquad \leq \frac{1}{\delta}\left\{\langle\varphi_T, \phi(T)\varphi_T\rangle - \langle\varphi_{T_0}, \phi(T_0)\varphi_{T_0}\rangle\right\} + C\|f(H)\varphi\|^2 \\ &\qquad \leq \frac{1}{\delta}|\langle\varphi_{T_0}, \phi(T_0)\varphi_{T_0}\rangle| + C\|f(H)\varphi\|^2, \end{aligned} \tag{24}$$

where, in the last step, we used that $\langle\varphi_T, \phi(T)\varphi_T\rangle \leq 0$, for all $T > 1$. This estimate proves the theorem because $h \geq \chi(|x_j| \geq vt)$ and because $|\langle\varphi_{T_0}, \phi(T_0)\varphi_{T_0}\rangle| \leq \text{const}\|(1+|x_j|)^{1/2} f(H)\varphi\|^2$ by Lemma 20 applied to $\exp(-iHT_0)g(H)$ with $g \in C_0^\infty(\mathbb{R})$ such that $g(H)f(H) = f(H)$. In order to prove (23) we note that $D\phi(t) = -f(H)\{[iH, t^{1-\mu}\tilde{h}] + (\partial/\partial t)(t^{1-\mu}\tilde{h})\}f(H)$, where by construction of $\tilde{h}$ and by (22)

$$-\frac{\partial}{\partial t}(t^{1-\mu}\tilde{h}) \geq (v-\varepsilon)\, t^{-\mu}\, h^2 \tag{25}$$

and, with the notation $\pi(x_j) = p_j + A(x_j)$,

$$\begin{aligned} [iH, t^{1-\mu}\tilde{h}] &= \frac{t^{1-\mu}}{2}\left\{\pi(x_j)\cdot\nabla\tilde{h} + \nabla\tilde{h}\cdot\pi(x_j)\right\} \\ &= \frac{t^{-\mu}}{2} h \left\{\pi(x_j)\cdot\frac{x_j}{\langle x_j\rangle} + \frac{x_j}{\langle x_j\rangle}\cdot\pi(x_j)\right\} h. \end{aligned} \tag{26}$$

In the last equation we used that $\nabla\tilde{h} = h^2\, x_j/\langle x_j\rangle\, t^{-1}$ and then commuted one factor h to the left and the second one to the right. The commutators which arise cancel. By (26)

$$|\langle\varphi_t, f(H)[iH, t^{1-\mu}\tilde{h}]f(H)\varphi_t\rangle| \leq t^{-\mu}\, \|hf(H)\varphi_t\|\, \|\pi(x_j)hf(H)\varphi_t\|, \tag{27}$$

where we used Schwarz's inequality and $|x_j/\langle x_j\rangle| \leq 1$. Consider the factor $\|\pi(x_j)hf(H)\varphi_t\|$ on the r.h.s. of the last equation. From $|x_j| \geq (v-\varepsilon)t$ on the support of h, the definition of Σ, and $W_N - W_{N-1} \geq 0$ it is easy to see that

$$\begin{aligned} h\pi(x_j)^2 h &\leq 2h\left\{\frac{1}{2}\pi(x_j)^2 + v(x_j) + W_N - W_{N-1} + H_{N-1} - \Sigma + o(t^0)\right\} h \\ &= 2h\left\{H - \Sigma + o(t^0)\right\} h. \end{aligned} \tag{28}$$

Next pick $g = \bar{g} \in C_0^\infty(\mathbb{R})$ with $gf = f$ and $\text{supp}(g) \subset (-\infty, E+\varepsilon/2]$ where $E = \sup\{\lambda : f(\lambda) \neq 0\}$, so that $f(H) = f(H)g(H)$ and $g(H)Hg(H) \leq g(H)(E+\varepsilon/2)g(H)$. Then $[g(H), h] = O(t^{-1})$ implies that

$$f(H)hHhf(H) \leq f(H)h(E+\varepsilon/2)hf(H) + O(t^{-1})f(H)^2. \tag{29}$$

Equations (28) and (29) combined show that

$$\|\pi(x_j)hf(H)\varphi_t\| \leq \ (2(E-\Sigma+\varepsilon))^{1/2}\,\|hf(H)\varphi_t\| + C\,t^{-1/2}\,\|f(H)\varphi\|, \tag{30}$$

for t large enough. Inserting this into (27) we get

$$\begin{aligned}|\langle\varphi_t, f(H)[iH,\tilde{h}]f(H)\varphi_t\rangle| \leq t^{-\mu}\,(2(E-\Sigma+\varepsilon))^{1/2}\,\|hf(H)\varphi_t\|^2 \\ + C\,t^{-1/2-\mu}\,\|f(H)\varphi\|^2,\end{aligned} \tag{31}$$

which, together with (25), shows that

$$\begin{aligned}\langle\varphi_t, D\phi(t)\varphi_t\rangle \geq \ t^{-\mu}\,\{(v-\varepsilon) - \sqrt{2(E-\Sigma+\varepsilon)}\}\,\|hf(H)\varphi_t\|^2 \\ - C\,t^{-\mu-1/2}\,\|f(H)\varphi\|^2.\end{aligned} \tag{32}$$

We choose now $\varepsilon > 0$ so small that $E+\varepsilon < \Sigma + (v-\varepsilon)^2/2$. Then (23) follows from (32) with $\delta = \{(v-\varepsilon) - \sqrt{2(E-\Sigma+\varepsilon)}\} > 0$. □

II.C Existence of Asymptotic Field Operators

Next we use the propagation estimate from Theorem 1 to establish existence of the asymptotic field operators $a_+^*(h)$ and to show that scattering states with an arbitrary number of asymptotically free photons exist. The main results are summarized in the Theorems 4 and 6. An important and non-trivial technical ingredient for the scattering of n photons is the boundedness of $H_f^n(H+i)^{-n}$ which is proved in Appendix D.

In addition to Theorem 1 we need the following lemma (see [RS79, Theorem XI.18]).

Lemma 2. *Suppose $h \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$. Then there exists a constant C such that*

$$\sup_{x\in\mathbb{R}^3}\left|\int dk\,h(k)\,e^{i(k\cdot x-|k|t)}\right| \leq \frac{C}{|t|}, \tag{33}$$

for all $t \in \mathbb{R}$.

Proposition 3. *Suppose $f \in \mathcal{C}_0^\infty(\mathbb{R})$, with supp$(f) \subset (-\infty\,,\,\Sigma+1/2)$ and $h \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\};\mathbb{C}^2)$. Then, for all $\varphi \in \mathcal{H}$, the limit*

$$\lim_{t\to\infty} e^{iHt}a^\sharp(h_t)e^{-iHt}f(H)\varphi \tag{34}$$

exists.

Proof. Since $e^{iHt}a^\sharp(h_t)e^{-iHt}f(H)$ is bounded uniformly in t it suffices to prove existence of (34) for vectors φ from the dense subspace $D(\langle x\rangle^{1/2})$. We only consider creation operators, the proof for annihilation operators is similar. For given $\varphi \in \mathcal{H}$ let

$$\varphi(t) = e^{iHt}\,a^*(h_t)\,f\,e^{-iHt}\varphi,$$

where $f = f(H)$. By Cook's argument the existence of the limit (34) follows if

$$\int_1^\infty dt\,\|\frac{d}{dt}\varphi(t)\| < \infty.$$

In the following we will use the notation $\varphi_t = e^{-iHt}\varphi$. A straightforward computation shows that

$$\begin{aligned}\frac{d}{dt}\,\varphi(t) &= \frac{1}{2}\sum_{j=1}^{N} e^{iHt}\,[i(p_j + A(x_j))^2, a^*(h_t)]\, f\varphi_t\\ &= \frac{i}{\sqrt{2}}\sum_{j=1}^{N} e^{iHt}\,(G_{x_j}, h_t)\cdot(p_j + A(x_j))\, f\varphi_t.\end{aligned} \tag{35}$$

Next we fix $j \in \{1, \dots N\}$ and show that the corresponding term has norm which is integrable w.r.t. t. Choose $\varepsilon > 0$ and so small that $\sup\{\lambda : f(\lambda) \neq 0\} < \Sigma + (1-2\varepsilon)^2/2$. Pick then χ_1, $\chi_2 \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \chi_i \le 1$, $\chi_1^2 + \chi_2^2 = 1$, $\chi_1(s) = 0$ if $s \le 1 - 2\varepsilon$ and $\chi_1(s) = 1$ for $s \ge 1 - \varepsilon$. Let $\chi_1 = \chi_1(|x_j|/t)$ and $\chi_2 = \chi_2(|x_j|/t)$ henceforth. The term in the sum on the r.h.s. of (35) corresponding to the fixed j then has norm bounded by

$$\|e^{iHt}\,(G_{x_j}, h_t)\cdot(p_j + A(x_j))\, f\varphi_t\| \le \frac{1}{\sqrt{2}}\sum_{k=1}^{2} \|(G_{x_j}, h_t)\chi_k\|\,\|\chi_k(p_j + A(x_j))f\varphi_t\|. \tag{36}$$

Consider first the $k = 2$ term. Since $\|(p_j + A(x_j))f(H)\| < \infty$, and since, by a stationary phase argument, $\sup_{x_j} |(G_{x_j}, h_t)\chi_2| \le C_n/t^n$, for any $n \ge 1$, the term with $k = 2$ on the r.h.s. of (36) is integrable w.r.t. t. Consider now the term with $k = 1$. We first note, that

$$\|\chi_1\,(p_j + A(x_j))\, f\,\varphi_t\| \le \|(p_j + A(x_j))(H+i)^{-1}\|\,\|\chi_1\tilde{f}\varphi_t\| + C/t, \tag{37}$$

where $\tilde{f} = (H+i)f$. In order to prove the last equation, write $f = (H+i)^{-1}\tilde{f}$ and then commute the operator $(p_j + A(x_j))(H+i)^{-1}$ to the left of χ_1. The factor proportional to t^{-1} on the r.h.s. of (37) arises from the commutator of these two operators. Since, by Lemma 2, $\|\chi_1(G_{x_j}, h_t)\| \le C/t$, it follows that the term with $k = 1$ on the r.h.s. of (36) is bounded by $\mathrm{const}\, t^{-1}\,\|\chi_1\tilde{f}\varphi_t\| + \mathrm{const}\, t^{-2}$. The term proportional to t^{-2} is clearly integrable w.r.t. t and by the Schwarz inequality

$$\int_1^\infty dt\,\frac{1}{t}\,\|\chi_1\tilde{f}\varphi_t\| \le \left(\int_1^\infty dt\,\frac{1}{t^{1+2\delta}}\right)^{\frac{1}{2}}\left(\int_1^\infty dt\,\frac{1}{t^{1-2\delta}}\,\|\chi_1\tilde{f}\varphi_t\|^2\right)^{\frac{1}{2}}. \tag{38}$$

If we choose $\delta \in (0, 1/4)$ this is finite by Theorem 1 with $\mu = 1 - 2\delta > 1/2$ and $v = 1 - 2\varepsilon$, because $\tilde{f}(H)\varphi \in D(\langle x\rangle^{1/2})$ by Lemma 20. □

In the following theorem we list some important properties of the asymptotic field operators.

Theorem 4. *Suppose that $\varphi = \chi(H \le E)\varphi$, for some $E < \Sigma + 1/2$, and that $h, g \in L^2_\omega(\mathbb{R}^3; \mathbb{C}^2)$.*

i) The limit

$$a_+^\sharp(h)\varphi = \lim_{t\to\infty} e^{iHt} a^\sharp(h_t) e^{-iHt}\varphi$$

exists, and furthermore

$$\|\, a_+^\sharp(h)\chi(H \le E)\| \le C\,\|h\|_\omega, \tag{39}$$

for some finite constant $C > 0$.

ii) The canonical commutation relations

$$[a_+(g), a_+^*(h)] = (g,h) \qquad \text{and} \qquad [a_+^\sharp(h), a_+^\sharp(g)] = 0,$$

hold true, in form–sense, on $\chi(H \leq E)\mathcal{H}$.

iii) If in addition $\omega h \in L^2(\mathbb{R}^3; \mathbb{C}^2)$*, then*

$$[H, a_+^*(h)] = a_+^*(\omega h), \qquad [H, a_+(h)] = -a_+(\omega h),$$

in form–sense on $\chi(H \leq E)\mathcal{H}$*. Moreover if* $\Sigma = \infty$ *then the operator*

$$\phi_+(h) = \frac{1}{\sqrt{2}}\left(a_+(h) + a_+^*(h)\right)$$

is essentially self–adjoint on $\cup_{d>0}\chi(H \leq d)\mathcal{H}$*, and the domain of its closure contains* $D((H+i)^{1/2})$.

iv) Let $m = \inf\{|k| : h(k) \neq 0\}$ *and* $M = \sup\{|k| : h(k) \neq 0\}$*. Then*

$$\begin{aligned} a_+^*(h)\,\mathrm{Ran}\,\chi(H \leq E) &\subset \mathrm{Ran}\,\chi(H \leq E+M) \\ a_+(h)\,\mathrm{Ran}\,\chi(H \leq E) &\subset \mathrm{Ran}\,\chi(H \leq E-m). \end{aligned}$$

Proof. i) The existence of the limit for $h \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\}; \mathbb{C}^2)$ follows from Proposition 3. The operator bound (39) follows from $\|\, a^\sharp(h_t)(H_f+1)^{-1/2}\| \leq \|\, h\|_\omega$ and from the boundedness of $(H_f+1)^{1/2}(H+i)^{-1/2}$. Finally if $h \in L^2_\omega(\mathbb{R}^3; \mathbb{C}^2)$ the existence of the limit follows by an approximation argument, because $\mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$ is dense in $L^2_\omega(\mathbb{R}^3)$.

ii) Follows from i) and the CCR for $a(h)$ and $a^*(h)$.

iii) For φ and $\psi \in \chi(H \leq E)\mathcal{H}$

$$\begin{aligned} \langle \varphi, [H, a_+^*(h)]\psi\rangle &= \lim_{t\to\infty} \langle \varphi_t, [H, a^*(h_t)]\psi_t\rangle \\ &= \lim_{t\to\infty} \frac{1}{2}\sum_{j=1}^N \langle \varphi_t, [(p_j + A(x_j))^2, a^*(h_t)]\psi_t\rangle + \langle \varphi, a_+^*(\omega h)\psi\rangle \\ &= \langle \varphi, a_+^*(\omega h)\psi\rangle, \end{aligned}$$

because $|\langle \varphi_t, [(p_j + A(x_j))^2, a^*(h_t)]\psi_t\rangle| \leq C\, \sup_{x_j} |(G_{x_j}, h_t)| \to 0$ as $t \to \infty$. The proof of the second pull–through formula is similar. The proof of the essentially self–adjointness of $\phi_+(h)$, in the case $\Sigma = \infty$, is very similar to the proof of the corresponding statement for the pseudo–relativistic model (see Theorem 13, iii)).

iv) From the spectral theorem we know that $\psi \in \mathrm{Ran}\,\chi(H \leq \lambda)$ if and only if the function $\mathbb{R} \ni t \to e^{iHt}\psi$ has an analytic extension $z \to e^{iHz}\psi$, for $\mathrm{Im}\, z < 0$, which satisfies $\|e^{iHz}\psi\| \leq Ce^{|\,\mathrm{Im}\, z|\lambda}$. Now pick $\psi \in \mathrm{Ran}\,\chi(H \leq E)$. Then we have, from iii),

$$e^{iHt}a_+^*(h)\psi = a_+^*(e^{i\omega t}h)e^{iHt}\psi.$$

Since H and ω are bounded from below, it follows that the function $\mathbb{R} \ni t \to e^{iHt}a_+^*(h)\psi$ has an analytic extension, given by, $a_+^*(e^{i\omega z}h)e^{iHz}\psi$, which satisfies

$$\begin{aligned}\|a_+^*(e^{i\omega z}h)e^{iHz}\psi\| &\leq \|a_+^*(e^{i\omega z}h)\chi(H \leq E)\| \, \|e^{iHz}\psi\| \\ &\leq C\|e^{i\omega z}h\|_\omega e^{|\operatorname{Im} z|E} \\ &\leq Ce^{|\operatorname{Im} z|(E+M)},\end{aligned}$$

where we used the estimate (39). This proves that $a_+^*(h)\psi \in \operatorname{Ran}\chi(H \leq E+M)$. The statement for annihilation operators can be proved similarly.

□

Existence of scattering states with an arbitrary number of escaping photons does not obviously follow from Theorem 4 because creation and annihilation operators are unbounded. They are however bounded with respect to $(H_f+1)^{1/2}$ and by the following lemma powers of H_f are bounded w.r. to powers of H.

Lemma 5. *For any integer $n \geq 1$, the operators*

i) $[H_f^{n-1}, H](H+i)^{-n}$

ii) $H_f^n(H+i)^{-n}$

are bounded.

Remark. By an abstract interpolation argument it follows that also $H_f^{n/2}(H+i)^{-n/2}$ is bounded for all non-negative integers.

This lemma, whose proof is deferred to Appendix D, is the main technical ingredient, apart from Theorem 4, for the proof of the following theorem.

Theorem 6. *Suppose $\varphi = \chi(H \leq E)\varphi$, $h_i \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$ and let $M_i = \sup\{|k| : h_i(k) \neq 0\}$, for $i \in \{1,\dots,n\}$. Then*

$$a_+^\sharp(h_1)\dots a_+^\sharp(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})e^{-iHt}\varphi, \tag{40}$$

provided that $E + \sum_{i\geq 2} M_i < \Sigma + 1/2$, where the sum $\sum_{i\geq 2} M_i$ runs over those i's bigger than 1 for which $a^\sharp(h_i)$ is a creation operator. Furthermore there exists a constant C_n, independent of E such that

$$\|a_+^\sharp(h_1)\dots a_+^\sharp(h_n)(H+i)^{-n/2}\chi(H \leq E)\| \leq C_n\|h_1\|_\omega \dots \|h_n\|_\omega. \tag{41}$$

Proof. By Theorem 4, iv) and because of the assumption $E + \sum_i M_i < \Sigma + 1/2$, it follows that, for each $l \in \{2,\dots,n\}$ the vector $a_+^\sharp(h_l)\dots a_+^\sharp(h_n)\varphi$ is in the range of a spectral projector $\chi(H \leq E')$, for some $E' < \Sigma + 1/2$. Thus $a_+^\sharp(h_{l-1})\dots a_+^\sharp(h_n)\varphi$ is well defined and

$$a_+^\sharp(h_l)\dots a_+^\sharp(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^\sharp(h_{l,t})e^{-iHt}a_+^\sharp(h_{l+1})\dots a_+^\sharp(h_n)\varphi. \tag{42}$$

Now we want to prove the equality (40). We consider only the case with n creation operators, the other cases being similar. Assume first that $h_i \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\};\mathbb{C}^2)$, for all $i \in \{1,\dots,n\}$. We proceed then by induction over n. The statement for $n=1$ follows from Theorem 4. Now we assume that the statement holds for any integer less than a given n and we prove that it holds also for the product of n creation operators. To this end we write

$$\begin{aligned} e^{iHt}a^*(h_{1,t})\dots a^*(h_{n,t})e^{-iHt}\varphi - a_+^*(h_1)\dots a_+^*(h_n)\varphi \\ = (e^{iHt}a^*(h_{1,t})e^{-iHt} - a_+^*(h_1))a_+^*(h_2)\dots a_+^*(h_n)\varphi \\ + e^{iHt}a^*(h_{1,t})e^{-iHt}\times \\ \left\{e^{iHt}a^*(h_{2,t})\dots a^*(h_{n,t})e^{-iHt} - a_+^*(h_2)\dots a_+^*(h_n)\right\}\varphi. \end{aligned} \tag{43}$$

The norm of the first term on the r.h.s of the last equation converges, by (42), to 0 as $t\to\infty$. To handle the second term on the r.h.s. of (43) we insert the operator $id = (H+i)^{-1}(H+i)$ just in front of the braces, and we commute the factor $(H+i)$ through the operators within the braces. The second term on the r.h.s. of (43) becomes then

$$\begin{aligned} & e^{iHt}a^*(h_{1,t})e^{-iHt}(H+i)^{-1}\times \\ & \quad \{\,(e^{iHt}a^*(h_{2,t})\dots a^*(h_{n,t})e^{-iHt} - a_+^*(h_2)\dots a_+^*(h_n))(H+i)\varphi \\ & \quad + \sum_{l=2}^{n}(e^{iHt}a^*(h_{2,t})\dots a^*(\omega h_{l,t})\dots a^*(h_{n,t})e^{-iHt} - a_+^*(h_2)\dots a_+^*(\omega h_l)\dots a_+^*(h_n))\varphi \\ & \quad + \sum_{l=2}^{n}\sum_{j=1}^{N} e^{iHt}a^*(h_{2,t})\dots \frac{1}{2}[(p_j + A(x_j))^2, a^*(h_{l,t})]\dots a^*(h_{n,t})e^{-iHt}\varphi\}\,. \end{aligned} \tag{44}$$

Now the term in front of the braces is bounded uniformly in t. The first term inside the braces, and each factor in the first sum converge to 0, as $t\to\infty$, by induction hypothesis. Finally the terms in the last sum inside the braces have norm which is bounded by

$$\frac{1}{\sqrt{2}}\|(G_{x_j},h_{l,t})\|\,\|a^*(h_{2,t})\dots a^*(h_{l-1,t})(p_j+A(x_j))a^*(h_{l+1,t})\dots a^*(h_{n,t})(H+i)^{-n}\|\,\|(H+i)^n\varphi\|. \tag{45}$$

The first factor in (45) converges to 0, as $t\to\infty$. By Lemma 5 it follows that the second factor in (45) is bounded uniformly in t. We have thus shown that both terms on the r.h.s. of (43) converge to 0, as $t\to\infty$. This proves equation (40) if $h_i \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\};\mathbb{C}^2)$, for all $i\in\{1,\dots,n\}$. For $h_i \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$ equation (40) follows now by an approximation argument, because $\mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$ is dense in $L^2_\omega(\mathbb{R}^3)$, by Lemma 17 and because

$$\|a_+^*(h_1)\dots a_+^*(h_n)\chi(H\le E)\| \le C\|h_1\|_\omega\dots\|h_n\|_\omega,$$

provided $E+\sum_{i\ge 2}M_i < \Sigma + 1/2$. The last estimate follows from the bound (39) and from Theorem 4 iv), which implies

$$\begin{aligned} a_+^*(h_1)\dots a_+^*(h_n)\chi(H\le E) = a_+^*(h_1)\chi(H\le E+\Sigma_{l\ge 2}M_l)\dots \\ \dots a_+^*(h_{n-1})\chi(H\le E+M_n)a_+^*(h_n)\chi(H\le E). \end{aligned}$$

Finally (41) follows from (40), Lemma 17, and Lemma 5. This completes the proof of the theorem.

□

In the case $\Sigma = \infty$ the results of Theorem 6 can be refined to get following theorem.

Theorem 7. *Suppose* $\Sigma = \infty$, $h_i \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$ *for* $i = 1, \dots, n$ *and* $\varphi \in D((H+i)^{n/2})$. *Let* $a^\sharp_+(h_i)$ *denote the closure of the asymptotic operators* $a^\sharp_+(h_i)$ *obtained in Theorem 4 and defined on* $\cup_{d>0}\chi(H \leq d)\mathcal{H}$. *Then* $\varphi \in D(a^\sharp_+(h_1)\dots a^\sharp_+(h_n))$,

$$\|a^\sharp_+(h_1)\dots a^\sharp_+(h_n)(H+i)^{-n/2}\| \leq C_n\|h_1\|_\omega \dots \|h_n\|_\omega$$

and

$$a^\sharp_+(h_1)\dots a^\sharp_+(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})e^{-iHt}\varphi.$$

Proof. We proceed by induction over n and assume the statement holds for n replaced by $n-1$. Let $\varphi_d = \chi(H \leq d)\varphi$ where $d \in \mathbb{R}$.Then $(H+i)^{n/2}(\varphi_d - \varphi) \to 0$ as $d \to \infty$. Hence by Theorem 6 the limit

$$\lim_{d\to\infty} a^\sharp_+(h_1)\dots a^\sharp_+(h_n)\varphi_d$$

exists and by the induction hypothesis

$$\lim_{d\to\infty} a^\sharp_+(h_2)\dots a^\sharp_+(h_n)\varphi_d = a^\sharp_+(h_2)\dots a^\sharp_+(h_n)\varphi.$$

Since $a^\sharp_+(h_1)$ is closed this proves $a^\sharp_+(h_2)\dots a^\sharp_+(h_n) \in D(a^\sharp_+(h_1))$ and the first equation from

$$\begin{aligned} a^\sharp_+(h_1)\dots a^\sharp_+(h_n)\varphi &= \lim_{d\to\infty} a^\sharp_+(h_1)\dots a^\sharp_+(h_n)\varphi_d \\ &= \lim_{d\to\infty}\lim_{t\to\infty} e^{iHt}a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})e^{-iHt}\varphi_d \\ &= \lim_{t\to\infty} e^{iHt}a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})e^{-iHt}\varphi. \end{aligned}$$

The second equation follows from Theorem 6 and the last one from the fact that, by Theorem 6, the convergence as $d \to \infty$ is uniform in t. □

III Pseudo–Relativistic QED

III.A Definition and Properties of the Model

This section is devoted to scattering of photons at a single electron in a pseudo-relativistic model of UV-cutoff QED. The merits of this model have been discussed in Section I. In units where the mass of the particle, the speed of light, and Planck's constant are equal to one, the Hamiltonian is given by

$$H = \sqrt{(p + A(x))^2 + 1} + V(x) \otimes 1 + 1 \otimes H_f, \tag{46}$$

and acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$. Here $p = -i\nabla_x$ and the quantized vector potential $A(x)$, the field energy H_f, and the Fock space $\mathcal{F}$ are as in Section II. To define $\sqrt{(p + A(x))^2 + 1}$ take the Friedrichs' extension of the symmetric and positive operator $(p+A(x))^2+1$ and then use the spectral theorem. The scalar potential V is defined by multiplication with the real-valued, locally square integrable function $V(x)$ on $\mathbb{R}^3$. We assume V is operator-bounded w.r. to $\Omega(p) = \sqrt{p^2+1}$ with bound less than 1. That is, there exist constants $a < 1$ and $b \in \mathbb{R}$ such that

$$\|V\psi\| \le a\|\Omega(p)\psi\| + b\|\psi\| \qquad \text{for all} \qquad \psi \in C_0^\infty(\mathbb{R}^3). \tag{47}$$

Using first the diagmanetic inequality and then the Schrödinger representation of Fock space (see [AHS78] and [FFG96]) one shows that $V \otimes 1$ is also operator-bounded w.r. to $\Omega(p+A) = \sqrt{(p+A(x))^2+1}$ with bound less than 1. Hence the operator given by Eq. (46) is bounded from below and may be self-adjointly realized by the Friedrichs' extension. We denote this extension again by H. Moreover it follows that H_f and $\Omega(p+A)$ are form-bounded w.r. to H. For reasons given in Section II this is not enough for the purpose of constructing scattering states with more than one asymptotically free photon. We need to bound higher powers of H_f by powers of H.

Lemma 8. *For any integer $m \ge 1$, the operators*

i) $[H_f^{m-1}, H](H+i)^{-m}$

ii) $H_f^m(H+i)^{-m}$

are bounded.

The proof of this lemma is long and difficult because of the complicated form of the interaction between photons and electron. It is sketched in Appendix E. By Lemma 8, in particular $H_f(H+i)^{-1}$ is bounded, which leads to the following corollary.

Corollary 9. *Assuming (47) the operator $\Omega(p+A)(H+i)^{-1}$ is bounded.*

Proof. We use Ω as an abbreviation for the operator $\Omega(p+A(x))$. For each $d > 0$ we have

$$\begin{aligned}\Omega(H+i)^{-1} &= H(H+i)^{-1} - H_f(H+i)^{-1}\\ &\quad - V(x)(\Omega+d)^{-1}(\Omega+d)(H+i)^{-1},\end{aligned}$$

which shows that

$$(1+V(x)(\Omega+d)^{-1})\Omega(H+i)^{-1} \text{ is bounded.}$$

The corollary now follows, because, by (47), $\|V(x)(\Omega+d)^{-1}\| < 1$ for $d > 0$ sufficiently large. □

III.B A Sharp Propagation Estimate

We begin to prove a propagation estimate similar to Theorem 1. Unlike in the non–relativistic case now the group velocity of the particle is always less than the speed of light for finite energies. As a consequence $a_+^\sharp(h)$ will exist on a dense subspace.

Theorem 10. *Let $f \in C_0^\infty(\mathbb{R})$. Then, for each $\varepsilon > 0$ sufficiently small and for each $\mu > 1/2$, there exists a constant C such that, for all $\varphi \in \mathcal{H}$,*

$$\int_1^\infty dt\, \frac{1}{t^\mu}\, \|\chi(|x| \ge (1-\varepsilon)t)e^{-iHt}f(H)\varphi\|^2 \le C\,\|(1+|x|)^{1/2}f(H)\varphi\|^2.$$

Remark. The theorem actually holds for $\mu = 0$ but for $\mu > 1/2$ the proof is easier and this result is sufficient for our purposes.

Proof. This proof follows a similar pattern as the proof of Theorem 1. Some of the explanations given in that proof are not repeated here. We may assume f is real-valued. Pick $h \in \mathcal{C}_0^\infty(\mathbb{R})$ as in the proof of Theorem 1 with the choice $v = 1 - \varepsilon$. That is, h is non–decreasing, $0 \le h \le 1$, $h(s) = 0$ if $s \le (1 - 2\varepsilon)$ and $h(s) = 1$ for $s \ge (1 - \varepsilon)$. Here $\varepsilon > 0$ and small. Again $\tilde{h}$ is defined by $\tilde{h}(s) = \int_0^s d\tau h^2(s)$ and obeys

$$\tilde{h}(s) \le (s - (1 - 2\varepsilon))h^2(s). \tag{48}$$

The propagation observable is given by

$$\phi(t) = -f(H)\, t^{1-\mu}\tilde{h}(\langle x\rangle/t)\, f(H),$$

and the theorem follows if we show that

$$\begin{aligned} D\phi(t) &= -f(H)\{[iH, t^{1-\mu}\tilde{h}] + (\partial/\partial t)(t^{1-\mu}\tilde{h})\}f(H) \\ &\ge \delta\, t^{-\mu}\, f(H)h^2 f(H) + (\text{integrable w.r.t. } t)f(H)^2, \end{aligned} \tag{49}$$

for $t \in [T_0, \infty)$, where $\delta > 0$ and $T_0 > 1$ is sufficiently large. By construction of $\tilde{h}$ and by (48)

$$-\frac{\partial}{\partial t}(t^{1-\mu}\tilde{h}) \ge (1 - 2\varepsilon)\, t^{-\mu}\, h^2. \tag{50}$$

To compute $[iH, \tilde{h}]$ note that $\tilde{h}(s) = s - c$ for $s \ge 1 - \varepsilon$ where c is constant. Hence there exists a function $g \in C_0^\infty(\mathbb{R})$ such that $\tilde{h}(s) = g(s) + s - c$ for $s > 0$. This makes Lemma 19 applicable and leads to

$$\begin{aligned} [iH, t^{1-\mu}\tilde{h}] &= t^{1-\mu}\,[i\Omega, \tilde{h}] \\ &= \frac{t^{1-\mu}}{2}\left(\nabla\Omega \cdot \nabla\tilde{h} + \nabla\tilde{h} \cdot \nabla\Omega + O(t^{-2})\right). \end{aligned}$$

Using next that $\nabla\tilde{h} = x/\langle x\rangle\, t^{-1}h^2$ and $[\nabla\Omega, h] = O(t^{-1})$ one finds

$$[iH, t^{1-\mu}\tilde{h}] = \frac{t^{-\mu}}{2}\, h\left(\nabla\Omega \cdot \frac{x}{\langle x\rangle} + \frac{x}{\langle x\rangle} \cdot \nabla\Omega\right) h + O(t^{-1-\mu})$$

by commuting one factor of h to the left of $\nabla\Omega$, respectively to the right of $\nabla\Omega$. Since $|x/\langle x\rangle| \le 1$ and by the Schwarz inequality we conclude

$$|\langle\varphi_t, f(H)[iH, t^{1-\mu}\tilde{h}]f(H)\varphi_t\rangle| \le t^{-\mu}\, \|hf(H)\varphi_t\|\, \|\, |\nabla\Omega|\, hf(H)\varphi_t\| + O(t^{-1-\mu})f(H)^2. \tag{51}$$

To estimate the factor $\|\, |\nabla\Omega|\, hf(H)\varphi_t\|$ we pick $g \in C_0^\infty(\mathbb{R})$ with $gf = f$ and then commute h with $g(H)$ using $[g(H), h] = O(t^{-1})$. This shows that

$$f(H)\, h\, |\nabla\Omega|^2\, h\, f(H) \le f(H)\, h\, g(H)\, |\nabla\Omega|^2\, g(H)\, h\, f(H) + O(t^{-1})f(H)^2. \tag{52}$$

Next write $|\nabla\Omega|^2 = 1 - \Omega^{-2}$ and let $P = \chi(H \in \operatorname{supp} g)$. From $1 \le \Omega \le \alpha H + \beta$ for some constants α, β, it follows that $\Omega^{-1} \ge (\alpha H + \beta)^{-1}$, and hence that

$$P\frac{1}{\Omega^2}P \ge (P\frac{1}{\Omega}P)^2 \ge (8\varepsilon - 16\varepsilon^2)P,$$

if $\varepsilon > 0$ is small enough. These remarks in conjunction with (52) show that

$$f(H)\,h\,|\nabla\Omega|^2\,h\,f(H) \leq (1-4\varepsilon)^2\,f(H)\,h^2\,f(H) + O(t^{-1})f(H)^2, \tag{53}$$

where we commuted h and $g(H)$ once again. We now insert this result in (51) and we get

$$\begin{aligned}|\langle\varphi_t, f(H)[iH, t^{1-\mu}\tilde{h}]f(H)\varphi_t\rangle| \leq{}& t^{-\mu}(1-4\varepsilon)\,\|hf(H)\varphi_t\|^2\\ &+ O(t^{-1/2-\mu})\,\|f(H)\varphi\|^2.\end{aligned}$$

From the last equation, and from (50), it follows that

$$\begin{aligned}\langle\varphi_t, D\phi(t)\varphi_t\rangle \geq{}& \{(1-2\varepsilon)-(1-4\varepsilon)\}\,t^{-\mu}\,\|hf(H)\varphi_t\|^2\\ &+ C\,t^{-\mu-1/2}\,\|f(H)\varphi\|^2.\end{aligned}$$

This implies (49), with $\delta = 2\varepsilon > 0$, and completes the proof of the theorem. □

III.C Existence of Asymptotic Field Operators

Following the general strategy outlined in the introduction we next use the propagation estimate to prove existence of $a_+^*(h)$ by the Cook method. Existence of scattering states with more than one photon then follows with the help of Lemma 8. The proofs are similar as in the non–relativistic case and therefore kept short.

The following lemma will serve us to compute commutators with $\Omega = \sqrt{(p+A)^2+1}$.

Lemma 11. *Let B be an operator on $\mathcal{H}$. Then*

$$[\Omega, B] = \frac{1}{\pi}\int_1^\infty dy\,\frac{\sqrt{y-1}}{y+(p+A)^2}\,[(p+A)^2, B]\,\frac{1}{y+(p+A)^2}. \tag{54}$$

Proof. Using the representation

$$\Omega^{-1} = (1+(p+A)^2)^{-1/2} = \frac{1}{\pi}\int_1^\infty dy\,\frac{1}{\sqrt{y-1}}\,\frac{1}{y+(p+A)^2},$$

we get

$$\begin{aligned}[\Omega, B] &= \Omega^2\,[\Omega^{-1}, B] + [(p+A)^2, B]\Omega^{-1}\\ &= \frac{1}{\pi}\int_1^\infty dy\,\frac{1}{\sqrt{y-1}}\,\left\{-((p+A)^2+1)(y+(p+A)^2)^{-1}+1\right\}\\ &\qquad\times[(p+A)^2, B](y+(p+A)^2)^{-1}.\end{aligned}$$

The statement of the lemma follows from the last equation since the factor in the braces, on the r.h.s. of the last equation, is equal to $(y-1)\,(y+(p+A)^2)^{-1}$. □

The following proposition establishes existence of $a_+^*(h)$ on the subspace $\cup_{d>0}\chi(H \leq d)\mathcal{H}$ and for $h \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\};\mathbb{C}^2)$.

Proposition 12. *Suppose $f \in \mathcal{C}_0^\infty(\mathbb{R})$ and $h \in \mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$. Then, for all $\varphi \in \mathcal{H}$, the limit*

$$\lim_{t\to\infty} e^{iHt}a^\sharp(h_t)e^{-iHt}f(H)\varphi \tag{55}$$

exists.

Proof. Since $e^{iHt}a^\sharp(h_t)e^{-iHt}f(H)$ is bounded uniformly in t it suffices to prove existence of (55) for $\varphi \in D(\langle x\rangle^{1/2})$. We only consider creation operators, the proof for annihilation operators is similar. For given $\varphi \in \mathcal{H}$ let

$$\varphi(t) = e^{iHt}\, a^*(h_t)\, e^{-iHt}\, f\, \varphi,$$

where $f = f(H)$. By Cook's argument the existence of the limit (55) follows if

$$\int_1^\infty dt\, \|\frac{d}{dt}\varphi(t)\| < \infty.$$

In the following we will use the notation $\varphi_t = e^{-iHt}\varphi$. A straightforward computation shows that

$$\begin{aligned} \frac{d}{dt}\varphi(t) &= e^{iHt}\,[i\Omega\,, a^*(h_t)]\, f\varphi_t \\ &= \frac{1}{\pi}\int_1^\infty dy\, \sqrt{y-1}\, e^{iHt}R(y)[i(p+A)^2, a^*(h_t)]R(y)f\varphi_t, \end{aligned} \tag{56}$$

where we expanded the commutator with Ω using Lemma 11, and where $R(y) = (y + (p+A)^2)^{-1}$. Since

$$[(p+A)^2, a^*(h_t)] = \sqrt{2}(G_x, h_t)\cdot(p+A),$$

we see that

$$\|\frac{d}{dt}\varphi(t)\| \le \frac{\sqrt{2}}{\pi}\int_1^\infty dy\, \sqrt{y-1}\, \|R(y)\|\, \|(G_x, h_t)\cdot(p+A)\, R(y)f\varphi_t\|. \tag{57}$$

Next choose $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R})$, with $0 \le \chi_1 \le 1$, $\chi_1(s) = 0$ if $s \le 1-2\varepsilon$ and $\chi_1(s) = 1$ for $s \ge 1-\varepsilon$. Here $\varepsilon > 0$ and so small that the propagation estimate (Theorem 10) holds for the cut–off function $(s+i)f(s)$ and ε. Let $\chi_2^2 = 1 - \chi_1^2$ and let $\chi_1 = \chi_1(|x|/t)$, $\chi_2 = \chi_2(|x|/t)$ henceforth. Inserting

$$(G_x, h_t) = \chi_1^2\,(G_x, h_t) + \chi_2^2\,(G_x, h_t)$$

in the r.h.s. of (57), we find

$$\|\frac{d}{dt}\varphi(t)\| \le \frac{\sqrt{2}}{\pi}\sum_{k=1,2}\int_1^\infty dy\, \frac{\sqrt{y-1}}{y}\, \|(G_x, h_t)\chi_k\|\, \|\chi_k(p+A)R(y)f\varphi_t\|. \tag{58}$$

Consider first the term with $k = 2$. Since

$$\|(p+A)R(y)f\| \le \|(p+A)\Omega^{-1}\|\, \|R(y)\Omega f\| \le C/y,$$

and since, by a stationary phase argument,

$$\|\chi_2\,(G_x, h_t)\| \le \frac{C_n}{t^n},$$

for any $n \ge 1$, the term with $k = 2$ on the r.h.s. of (58) is bounded by

$$\text{const} \int_1^\infty dy\, \frac{\sqrt{y-1}}{y^2}\, t^{-2} \le \frac{\text{const}}{t^2},$$

and is therefore integrable w.r.t. t. Consider now the term with $k = 1$. We first note that

$$\begin{aligned} \|\chi_1(p+A)R(y)f\varphi_t\| &\le \|(p+A)R(y)(H+i)^{-1}\|\,\|\chi_1\tilde f\varphi_t\| + \frac{C}{y\,t} \\ &\le \frac{C}{y}\,\|\chi_1\tilde f\varphi_t\| + \frac{C}{y\,t}, \end{aligned} \tag{59}$$

where $\tilde f = (H+i)f$. To see this write $f = (H+i)^{-1}\tilde f$ and then commute the operator $(p+A)R(y)(H+i)^{-1}$ to the left of χ_1. The second term on the r.h.s. of (59) arises from the commutator of these two operators. Since, by Lemma 2, $\|\chi_1(G_{x_j}, h_t)\| \le C/t$, it follows that the term with $k = 1$ on the r.h.s. of (58) is bounded by $\mathrm{const}\, t^{-1}\,\|\chi_1\tilde f\varphi_t\| + \mathrm{const}\, t^{-2}$. The term proportional to t^{-2} is clearly integrable w.r.t. t and by the Schwarz inequality

$$\int_1^\infty dt\,\frac{1}{t}\,\|\chi_1\tilde f\varphi_t\| \le \left(\int_1^\infty dt\,\frac{1}{t^{1+2\delta}}\right)^{\frac12}\left(\int_1^\infty dt\,\frac{1}{t^{1-2\delta}}\,\|\chi_1\tilde f\varphi_t\|^2\right)^{\frac12}. \tag{60}$$

If we choose $\delta \in (0, 1/4)$ this is finite by Theorem 10 with $\mu = 1-2\delta > 1/2$, because $\tilde f\varphi \in D(\langle x\rangle^{1/2})$ by Lemma 20. □

Using Proposition 12 and an approximation argument we now extend existence of $a_+^*(h)\varphi$ to $h \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$ and to $\varphi \in D((H+i)^{1/2})$.

Theorem 13. *Suppose $g, h \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$. Then*

i)

$$a_+^\sharp(h)\varphi = \lim_{t\to\infty} e^{iHt}a^\sharp(h_t)e^{-iHt}\varphi$$

exists for all $\varphi \in D((H+i)^{1/2})$.

ii)

$$[a_+(g), a_+^*(h)] = (g,h) \qquad \text{and} \qquad [a_+^\sharp(h), a_+^\sharp(g)] = 0,$$

in form-sense on $D((H+i)^{1/2})$.

iii) If $\omega h \in L^2(\mathbb{R}^3;\mathbb{C}^2)$, in addition to $h \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$, then

$$[H, a_+^*(h)] = a_+^*(\omega h), \qquad [H, a_+(h)] = -a_+(\omega h),$$

in form-sense on $D(H)$, and

$$\phi_+(h) = \frac{1}{\sqrt{2}}(a_+^*(h) + a_+(h))$$

is essentially self adjoint on $\cup_{d>0}\chi(H \le d)\mathcal{H}$.

iv) Set $M = \sup\{|k| : h(k) \ne 0\}$ and $m = \inf\{|k| : h(k) \ne 0\}$. Then

$$\begin{aligned} a_+^*(h)\,\mathrm{Ran}\,\chi(H \le E) &\subset \mathrm{Ran}\,\chi(H \le E+M) \\ a_+(h)\,\mathrm{Ran}\,\chi(H \le E) &\subset \mathrm{Ran}\,\chi(H \le E-m). \end{aligned}$$

Proof. i) Follows from $\mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\}) \subset L^2_\omega(\mathbb{R}^3)$ dense, from $\|a^\sharp(h)(H+i)^{-1/2}\| \leq C\|h\|_\omega$ and because $\cup\chi(H\leq d)\mathcal{H}$ is a form-core for H.

ii) Follows from i) and the CCR for $a(h)$ and $a^*(h)$.

iii) For $\varphi, \psi \in D(H)$, by ii),

$$\begin{aligned}\langle\varphi,[H,a_+^*(h)]\psi\rangle &= \lim_{t\to\infty}\langle\varphi_t,[H,a^*(h_t)]\psi_t\rangle\\ &= \lim_{t\to\infty}\langle\varphi_t,[\Omega,a^*(h_t)]\varphi_t\rangle + \langle\varphi, a_+^*(\omega h)\psi\rangle.\end{aligned}$$

Now, by Lemma 11,

$$[\Omega, a^*(h_t)] = \frac{\sqrt{2}}{\pi}\int_1^\infty dy\,\frac{\sqrt{y-1}}{y+(p+A)^2}(G_x,h_t)\cdot(p+A)\frac{1}{y+(p+A)^2}.$$

Since $\sup_x|(G_x,h_t)| \to 0$ for $t\to\infty$, by Lemma 2, it follows that $[\Omega,a^*(h_t)]\,\Omega^{-1} \to 0$ as $t\to\infty$. This proves the first pull through formula, the proof of the second is similar. The essential self-adjointness of $\phi_+(h)$ follows from $[H,\phi_+(h)] = -i\phi_+(i\omega h)$ and from Nelson's commutator theorem ([RS75], Theorem X.37), since $\phi_+(h)$ is symmetric and bounded w.r.t. $(H+i)^{1/2}$.

iv) Follows in the same way as in the non–relativistic case (see Theorem 4, iv)).

□

The following theorem, on the existence of scattering states with an arbitrary number of asymptotically free photons, together with Theorem 13 is the main result of this section.

Theorem 14. *Suppose* $h_i \in L^2_\omega(\mathbb{R}^3)$, *for all* $i\in\{1,\dots,n\}$ *and* $\varphi \in D((H+i)^{n/2})$. *Then* $\varphi \in D(a_+^\sharp(h_1)\cdots a_+^\sharp(h_n))$,

$$a_+^\sharp(h_1)\dots a_+^\sharp(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})e^{-iHt}\varphi, \tag{61}$$

and

$$\|a_+^\sharp(h_1)\dots a_+^\sharp(h_n)(H+i)^{-n/2}\| \leq C\|h_1\|_\omega\cdots\|h_n\|_\omega. \tag{62}$$

The proof follows the same lines as in the non–relativistic case and is given in the Appendix E. Technically the most difficult part is to show Lemma 8 which, together with Theorem 13, is the main ingredient.

IV Weyl Operators

The purpose of this section is to establish existence of the asymptotic Weyl operators $W_+(h) = s-\lim_{t\to\infty} e^{iHt}W(h_t)e^{-iHt}$ and to show that $W_+(h) = e^{i\phi_+(h)}$. We will treat the relativistic and the non-relativistic model simultaneously, as most of the arguments are independent of the model. However, in the non-relativistic case we assume that $\Sigma=\infty$. The Weyl operators are defined by

$$W(h) = e^{i\phi(h)}, \qquad h\in L^2(\mathbb{R}^3;\mathbb{C}^2)$$

where $\phi(h)$ denotes the closure of $1/\sqrt{2}(a(h)+a^*(h))$, which is self-adjoint. They obey the Weyl relations

$$W(g+h) = W(g)W(h)e^{i/2\,\mathrm{Im}(g,h)}. \tag{63}$$

Theorem 15. *Suppose $H = H^{rel}$ or $H = H^{nr}$ on the appropriate Hilbert space and assume $\Sigma = \infty$ in the second case. If $g, h \in L^2_\omega$ then*

- *(i) The strong limits $W_+(h) = s - \lim_{t\to\infty} e^{iHt}W(h_t)e^{-iHt}$ exist.*
- *(ii) The asymptotic Weyl operators obey the Weyl relations*

$$W_+(g+h) = W_+(g)W_+(h)e^{i/2\,\mathrm{Im}(g,h)}.$$

- *(iii) The mapping $\mathbb{R} \ni s \mapsto W_+(sh)$ defines a strongly continuous, one parametric group of unitary operators generated by the closure of $\phi_+(h)$.*

Proof. (i) Suppose first $h \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$ and pick $f \in C_0^\infty(\mathbb{R})$. Then existence of

$$\lim_{t\to\infty} e^{iHt}W(h_t)e^{-iHt}f(H)\varphi \tag{64}$$

is proved as in (3) and (12) with only small modifications: rather than $[(p+A(x))^2, a^*(h_t)] = \sqrt{2}\,(p+A(x))\cdot(G_x, h_t)$ as in equations (35) and (56) one now has $[(p+A(x))^2, W(h_t)]$ for which we use the formula

$$[(p+A(x))^2, W(h_t)] = -W(h_t)\left\{2\,\mathrm{Im}(G_x,h_t)\cdot(p+A(x)) - [\mathrm{Im}(G_x,h_t)]^2\right\}. \tag{65}$$

New is the factor $W(h_t)$ in front, which does not affect the subsequent estimates since $\|W(h_t)\| = 1$, and the additional term $[\mathrm{Im}(G_x,h_t)]^2$ which is dealt with similarly as the first term in braces.

Next if $\varphi = \chi(D \le d)\varphi$ and $h \in L^2_\omega$ pick $f \in C_0^\infty(\mathbb{R})$ with $f(H)\varphi = \varphi$. Existence of (64) then follows by an approximation argument using a sequence $\{h_n\} \subset C_0^\infty(\mathbb{R}^3\backslash\{0\})$ with $\|h_n - h\|_\omega \to 0$ and Lemma 16. This establishes existence of $W_+(h)$ as a strong limit on the dense subspace $\bigcup_{d>0}\chi(H \le d)\mathcal{H}$. Statement (i) now follows by an other approximation argument.

Statement (ii) follows easily from the Weyl relations (63) and from (i).

(iii). From Lemma 16 and (i) it follows that the mapping $L^2_\omega \ni h \mapsto W_+(h)(H+i)^{-1/2}$ is continuous. This implies that $L^2_\omega \ni h \mapsto W_+(h)$ is strongly continuous and hence so is $\mathbb{R} \ni s \to W_+(sh)$. Unitarity and the group structure follow from (i) and (ii).

Finally, because of the essential self-adjointness of $\phi_+(h)$ on $D(H)$, it only remains to show that

$$\lim_{s\to 0}\frac{1}{s}(W_+(sh) - 1)\varphi = \phi_+(h)\varphi$$

for all $\varphi \in D(H)$. By (i), Theorem 4 in the non-relativistic case, and by Theorem 13 in the relativistic case, this is equivalent to

$$\lim_{s\to\infty}\lim_{t\to\infty} e^{iHt}\frac{1}{s}[W(sh_t) - 1 - i\phi(sh_t)]e^{-iHt}\varphi = 0.$$

But this follows from

$$\|[W(sh_t) - 1 - i\phi(sh_t)](H+i)^{-1}\| \le \frac{1}{2}\|\phi(sh)^2(H+i)^{-1}\| \le Cs^2$$

which holds uniformly in t. Here we used Lemma 16 again and Lemma 17. □

Lemma 16.

$$(i) \quad \|[W(h) - 1 - i\phi(h)]u\| \le \frac{1}{2}\|\phi(h)^2 u\|$$

$$(ii) \quad \|[W(h_1) - W(h_2)]u\| \le \|h_1 - h_2\|\|h_1 + h_2\|\|u\| + 2\|h_1 - h_2\|_\omega \|(H_f + 1)^{1/2}u\|$$

Proof. (i) follows from $|e^{it} - 1 - it| \le t^2/2$ and the spectral theorem. (ii) follows from

$$W(h_1) - W(h_2) = W(h_1)\left(1 - e^{i/2\,\mathrm{Im}(h_1,h_2)}\right) + W(h_1)e^{i/2\,\mathrm{Im}(h_1,h_2)}(1 - W(h_2 - h_1)). \tag{66}$$

□

A Estimating Field Operators

Recall from Section II that $L^2_\omega(\mathbb{R}^3;\mathbb{C}^2) = L^2(\mathbb{R}^3, (1 + 1/|k|)\,dk) \otimes \mathbb{C}^2$ and that $\|\,.\,\|_\omega$ denotes the norm of this space.

Lemma 17. *Suppose $h_i \in L^2_\omega(\mathbb{R}^3;\mathbb{C}^2)$ for $i = 1, \dots, n$. Then*

$$\|a^\sharp(h_1)\dots a^\sharp(h_n)(H_f+1)^{-n/2}\| \le C_n \prod_{i=1}^n \|h_i\|_\omega,$$

with a finite constant C_n.

Proof. Let $\bar\omega(k) = \min\{|k|, 1\}$, $d\Gamma(\omega) = H_f$ and

$$d\Gamma(\bar\omega) = \sum_{\lambda=1,2}\int dk\,\bar\omega(k)a^*_\lambda(k)a_\lambda(k).$$

Since $\bar\omega \le \omega$ we have $\|(d\Gamma(\bar\omega)+1)^{n/2}(d\Gamma(\omega)+1)^{-n/2}\| \le 1$ and hence it suffices to show that the lemma holds with H_f replaced by $d\Gamma(\bar\omega)$. For the case $n = 1$ see e.g. [BFS98]. If $n > 1$ let $\Lambda = (d\Gamma(\bar\omega) + 1)$ and note that

$$a^\sharp(h_1)\dots a^\sharp(h_n)\Lambda^{-n/2} = \prod_{k=1}^n \Lambda^{(k-1)/2}a^\sharp(h_k)\Lambda^{-k/2}. \tag{67}$$

Next, in each factor $\Lambda^{(k-1)/2}a^\sharp(h_k)\Lambda^{-k/2}$ commute as many factors of Λ to the right as possible. For k even one gets

$$\Lambda^{(k-1)/2}a^\sharp(h_k)\Lambda^{-k/2} = \Lambda^{-1/2}a^\sharp(h_k) + \Lambda^{-1/2}[\Lambda^{k/2}, a^\sharp(h_k)]\Lambda^{-k/2} \tag{68}$$

and a similar formula holds for odd k. From (68), the commutator expansion (71) for $[\Lambda^{k/2}, a^\sharp(h)]$, and $\mathrm{ad}^l_\Lambda(a^\sharp(h)) = (\pm 1)^l a^\sharp(\bar\omega^l h)$ it follows that

$$\|\Lambda^{(k-1)/2}a^\sharp(h_k)\Lambda^{-k/2}\| \le C_k\|h_k\|_\omega$$

which, together with (67) proves the lemma. □

In particular $A(x)^2$ is form-bounded with respect to $H_f + 1$. As a consequence we have the following lemma.

Lemma 18. *Suppose* $H = H^{nr}$*, and that for each* $\varepsilon > 0$ *there exists a* $C_\varepsilon > 0$ *such that* $V_- \leq \varepsilon(-\Delta) + C_\varepsilon$ *in form-sense. Then*

$$V_- \leq \varepsilon H + D_\varepsilon \qquad \textit{for all} \quad \varepsilon > 0 \tag{69}$$

with another constant D_ε*.*

Proof. It is enough to show that $-\Delta = \sum_{i=i}^N p_j^2$ is form-bounded with respect to H. Since $A(x)^2 \leq C(H_f + 1)$ by lemma 17, and from $p^2 \leq 2(p + A(x))^2 + 2A(x)^2$ it follows that

$$\sum_{j=1}^N p_j^2 \leq a(H + V_-) + b \tag{70}$$

for some $a, b > 0$. Combined with the assumption on V_- this shows that $\sum_{i=i}^N p_j^2$ is form-bounded with respect to H and hence proves the lemma. $\square$

B Commutator Estimates

In this section we collect some techniques which are useful to compute and to estimate commutators and which are frequently used in this paper.

B.A Commutator Expansions

First we recall that, given two operators A, B acting on the same space,

$$[A^n, B] = \sum_{l=1}^n \begin{pmatrix} n \\ l \end{pmatrix} \mathrm{ad}_A^l(B) A^{n-l} \tag{71}$$

$$\mathrm{ad}_A^n(BC) = \sum_{l=0}^n \begin{pmatrix} n \\ l \end{pmatrix} \mathrm{ad}_A^l(B) \mathrm{ad}_A^{n-l}(C) \tag{72}$$

where $\mathrm{ad}_A^n(B)$ is defined by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^n(B) = [A, \mathrm{ad}_A^{n-1}(B)]$ and where C is a third operator.

In the case where the two operators to be commuted are defined by multiplication with functions in position and momentum space respectively, the following lemma is very useful.

Lemma 19. *Suppose* $f \in \mathcal{S}(\mathbb{R}^d)$*,* $g \in \mathcal{C}^2(\mathbb{R}^d)$ *and* $\sup_{|\alpha|=2} \|\partial^\alpha g\|_\infty < \infty$*. Let* $p = -i\nabla$*. Then*

$$\begin{aligned} i[g(p), f(x)] &= \nabla f(x) \cdot \nabla g(p) + R_1 \\ &= \nabla g(p) \cdot \nabla f(x) + R_2 \end{aligned}$$

where, for $j = 1, 2$*,*

$$\|R_j\| \leq C \sup_{|\alpha|=2} \|\partial^\alpha g\|_\infty \int dk\, |k|^2 |\hat{f}(k)|.$$

Proof. Let $f(x) = \int dk\, e^{ikx}\hat{f}(k)$. The first equation follows from

$$\begin{aligned} g(p)e^{ikx} - e^{ikx}g(p) &= e^{ikx}\left[e^{-ikx}g(p)e^{ikx} - g(p)\right] \\ &= e^{ikx}[g(p+k) - g(p)] \end{aligned} \tag{73}$$

and Taylor's formula

$$\begin{aligned} g(p+k) - g(p) = \nabla g(p)\cdot k& \\ &+ \int_0^1 dt\,(1-t)\sum_{|\alpha|=2}(\partial^\alpha g)(p+tk)k^\alpha. \end{aligned}$$

To obtain the second equation write

$$g(p)e^{ikx} - e^{ikx}g(p) = -[g(p-k) - g(p)]e^{ikx}$$

instead of (73). □

Note that the Lemma basicly follows from $[p_i, x_j] = -i\delta_{ij}$. In particular the statement also holds for p replaced by $p + A(x)$.

B.B Helffer–Sjöstrand Functional Calculus

Suppose $f \in C_0^\infty(\mathbb{R};\mathbb{C})$ and A is a self-adjoint operator. A convenient representation of $f(A)$ is then given by

$$f(A) = -\frac{1}{\pi}\int dxdy\,\frac{\partial\tilde{f}}{\partial\bar{z}}(z)\,(z-A)^{-1}, \qquad z = x+iy,$$

which holds for any extension $\tilde{f} \in C_0^\infty(\mathbb{R}^2;\mathbb{C})$ of f with $|\partial_{\bar{z}}\tilde{f}| \le C|y|$,

$$\tilde{f}(z) = f(z) \qquad \text{and} \qquad \frac{\partial\tilde{f}}{\partial\bar{z}}(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(z) = 0 \qquad \text{for all} \qquad z \in \mathbb{R}. \tag{74}$$

Such a function $\tilde{f}$ is called an *almost analytic extension* of f. A simple example is given by $\tilde{f}(z) = (f(x) + iyf'(x))\,\chi(z)$ where $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\chi = 1$ on some complex neighborhood of supp$f$. Sometimes we need faster decay of $|\partial_{\bar{z}}\tilde{f}|$ as $|y| \to 0$ in the form $|\partial_{\bar{z}}\tilde{f}| \le C|y|^n$. In that case we work with the almost analytic extension

$$\tilde{f}(z) = \left(\sum_{k=0}^{n} f^{(k)}(x)\frac{(iy)^k}{k!}\right)\chi(z)$$

where χ is as above. For more details and extensions of this functional calculus the reader is referred to [HS00] or [Dav95].

C Invariance of Domains

The following lemma provides us with a dense subspace on which $(1+x^2)^{1/4}f(H)$ is defined. This is important in the proofs of the Propositions 3 and 12.

Lemma 20. *Let $H = H^{rel}$ or $H = H^{nr}$ and let $\langle x\rangle = (1+x^2)^{1/2}$, where $x^2 = \sum_{j=1}^N x_j^2$ in the non-relativistic case. If $f \in C_0^\infty(\mathbb{R})$ then $[\langle x\rangle^{1/2}, f(H)]$ is a bounded operator and hence*

$$f(H)D(\langle x\rangle^{1/2}) \subset D(\langle x\rangle^{1/2}).$$

Proof. We only prove the lemma for the case $H = H^{rel}$. The case $H = H^{nr}$ is similar and easier.

The second statement follows from the first one by an argument using the self-adjointness of $\langle x\rangle^{1/2}$. To show that $[\langle x\rangle^{1/2}, f(H)]$ is a bounded operator let $\tilde{f}$ be an almost analytic extension of f with $|\partial_{\bar{z}}\tilde{f}| \le C|y|^2$ (see Appendix B.B). Then

$$\begin{aligned}[\langle x\rangle^{1/2}, f(H)] &= -\frac{1}{\pi}\int dudv\, \partial_{\bar{w}}\tilde{f}(w)\, [\langle x\rangle^{1/2}, (w-H)^{-1}] \\ &= -\frac{1}{\pi}\int dudv\, \partial_{\bar{w}}\tilde{f}(w)\, (w-H)^{-1}[\langle x\rangle^{1/2}, \Omega]\, (w-H)^{-1},\end{aligned} \tag{75}$$

where $w = u+iv$. By Lemma 11

$$\begin{aligned}[\langle x\rangle^{1/2}, \Omega] = \frac{i}{2\pi}\int_1^\infty dy\, \sqrt{y-1}\, (y+(p+A)^2)^{-1} \\ \times \left\{(p+A)\cdot\frac{x}{\langle x\rangle^{3/2}} + \frac{x}{\langle x\rangle^{3/2}}\cdot(p+A)\right\}(y+(p+A)^2)^{-1}.\end{aligned} \tag{76}$$

Insert this into (75). From

$$\begin{aligned}&\|(w-H)^{-1}(y+(p+A)^2)^{-1}(p+A)\| \\ &\qquad \le \|(w-H)^{-1}\Omega\|\|(y+(p+A)^2)^{-1}\|\|\Omega^{-1}(p+A)\| \\ &\qquad \le \frac{C}{y}(\frac{1+|w|}{|v|}),\end{aligned}$$

and the boundedness of $x/\langle x\rangle^{3/2}$ it then follows that

$$\|\,[\langle x\rangle^{1/2}, f(H)]\,\| \le \text{const}\int dudv\, |\partial_{\bar{w}}\tilde{f}|\, (\frac{1+|w|}{|v|^2})\int_1^\infty dy\, \frac{\sqrt{y-1}}{y^2}.$$

This is finite by Appendix B.B and hence $[\langle x\rangle^{1/2}, f(H)]$ is a bounded operator. □

D Non–Relativistic QED: Higher Order Estimates

In this section we want to prove Lemma 5, which ensures the boundedness of H_f^n w.r.t. H^n, where H_f and H are the field energy and the total energy of the system respectively. Such higher order estimates are needed to keep control of products of creation or annihilation operators. In particular they are essential in the proof of Theorem 6. We begin by showing the boundedness of $H_f(H+i)^{-1}$. This will give us the first step in the inductive proof of Lemma 5.

Lemma 21. *The operator $H_f(H+i)^{-1}$ is bounded.*

Proof. We write

$$\begin{aligned} H_f(H+i)^{-1} = (H+i)^{-1/2} H_f (H+i)^{-1/2} \\ + \, [H_f, (H+i)^{-1/2}]\,(H+i)^{-1/2}. \end{aligned} \tag{77}$$

The first term on the r.h.s. of (77) is bounded, because $H_f^{1/2}(H+i)^{-1/2}$ is bounded. To show that also the second term on the r.h.s. of (77) is bounded, we use the expansion

$$(H+i)^{-1/2} = \frac{1}{\pi}\int_0^\infty dy\, \frac{1}{\sqrt{y}}\,\frac{1}{y+H+i},$$

which yields

$$\begin{aligned} &[H_f,(H+i)^{-1/2}]\,(H+i)^{-1/2} \\ &\quad = \frac{1}{\pi}\int_0^\infty dy\,\frac{1}{\sqrt{y}}\,\frac{1}{y+H+i}[H,H_f]\frac{1}{y+H+i}(H+i)^{-1/2} \\ &\quad = \frac{i}{2\pi}\sum_{j=1}^{N}\int_0^\infty dy\,\frac{1}{\sqrt{y}}\,\frac{1}{y+H+i}\{(p_j+A(x_j))\cdot\phi(i\omega G_{x_j})+h.c.\} \\ &\quad\quad \times \frac{1}{y+H+i}(H+i)^{-1/2}. \end{aligned}$$

Here and henceforth we use the notation $\phi(h) = 1/\sqrt{2}(a(h)+a^*(h))$. The r.h.s. of the last equation is clearly a bounded operator, because $\|(y+H+i)^{-1}(p_j+A(x_j))\|$ and $\|\phi(\omega G_{x_j})(H+i)^{-1/2}\|$ are bounded by some constants, while the second factor $(y+H+i)^{-1}$, which has norm less than $((y-c)^2+1)^{-1/2}$, for some constant $c>0$, ensures absolute convergence of the integral. □

Now we are ready to prove Lemma 5.

Proof of Lemma 5. We proceed by induction over n. For $n=1$ the boundedness of i) is trivial and that of ii) follows from Lemma 21. Next we assume that the statement of the Lemma holds for positive integers less or equal to a given n and prove it for $n+1$.

First note that the boundedness of ii) follows from that of i), since

$$\begin{aligned} H_f^{n+1}(H+i)^{-n-1} = H_f(H+i)^{-1}H_f^n(H+i)^{-n} \\ - H_f(H+i)^{-1}[H_f^n,H](H+i)^{-n-1}. \end{aligned}$$

So it is enough to show the boundedness of the operator i) for n replaced by $n+1$. To this end we write, using the commutator expansion (71),

$$\begin{aligned} [H_f^n,H](H+i)^{-n-1} &= \frac{1}{2}\sum_{j=1}^{N}[H_f^n,(p_j+A(x_j))^2]\,(H+i)^{-n-1} \\ &= \frac{1}{2}\sum_{j=1}^{N}\sum_{l=1}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\mathrm{ad}_{H_f}^l((p_j+A(x_j))^2)H_f^{n-l}(H+i)^{-n-1}. \end{aligned} \tag{78}$$

Now, using $[H_f, \phi(h)] = -i\phi(i\omega h)$, we find, using the commutator expansion (72)

$$\begin{aligned}\mathrm{ad}^l_{H_f}((p_j + A(x_j))^2) = (-i)^l \sum_{m=1}^{l-1} \begin{pmatrix} l \\ m \end{pmatrix} \phi(i^m\omega^m G_{x_j}) \cdot \phi(i^{l-m}\omega^{l-m} G_{x_j}) \\ + 2\,(-i)^l\,\phi(i^l\omega^l G_{x_j}) \cdot (p_j + A(x_j)) \\ + (-i)^l \sum_{k=1}^{3} [(p_{j,k} + A_{j,k}), \phi(i^l\omega^l G_{x_j,k})].\end{aligned} \tag{79}$$

If we insert the terms in the sum over m on the r.h.s. of (79) into the r.h.s. of (78) we find contributions to $[H_f^n, H](H+i)^{-n-1}$ which are proportional to

$$\begin{aligned}\phi(i^m\omega^m G_{x_j}) \cdot \phi(i^{l-m}\omega^{l-m} G_{x_j}) H_f^{n-l}(H+i)^{-n-1} \\ = \{\phi\,(i^m\omega^m G_{x_j}) \cdot \phi(i^{l-m}\omega^{l-m} G_{x_j})(H_f+1)^{-1}\,\} \\ \times \left\{(H_f+1)H_f^{n-l}(H+i)^{-n-1}\right\}.\end{aligned}$$

These contributions are all bounded. To see this note that the operator in the first braces on the r.h.s. of the last equation is bounded, by Lemma 17, and that the operator in the second braces is bounded, too, by induction assumption. Similarly we can see that if we insert the terms in the sums over k on the r.h.s. of (79) into the r.h.s. of (78) we find contributions to $[H_f^n, H](H+i)^{-n-1}$ which are bounded. Finally, if we insert the second term on the r.h.s. of (79) into the r.h.s. of (78), we find a contribution proportional to

$$\begin{aligned}\phi(i^l\omega^l G_{x_j}) \cdot (p_j + A(x_j)) H_f^{n-l}(H+i)^{-n-1} \\ = \phi(i^l\omega^l G_{x_j})(H_f+d)^{-1}(p_j + A(x_j))(H_f+d)H_f^{n-l}(H+i)^{-n-1} \\ + \text{bounded},\end{aligned} \tag{80}$$

where we inserted the operator $(H_f+d)^{-1}(H_f+d)$ on the left side, between the operators $\phi(i^l\omega^l G_{x_j})$ and $(p_j + A(x_j))$, and then commuted the factor (H_f+d) to the right of $(p_j + A(x_j))$. The contribution due to the commutator of these two terms is bounded. Now we commute one of the $n+1$ resolvents $(H+i)^{-1}$ to the left of the field Hamiltonians, in order to bound the factor $(p_j + A(x_j))$. The term on the r.h.s. of (80) becomes then

$$\begin{aligned}\phi(i^l\omega^l G_{x_j})(H_f+d)^{-1}(p_j + A(x_j))(H_f+d)H_f^{n-l}(H+i)^{-n-1} \\ = \phi(i^l\omega^l G_{x_j})(H_f+d)^{-1}(p_j + A(x_j))(H+i)^{-1} \\ \times\{(H_f+d)H_f^{n-l}(H+i)^{-n} + d\,[H_f^{n-l}, H](H+i)^{-n-1} \\ + [H_f^{n-l+1}, H](H+i)^{-n-1}\}.\end{aligned} \tag{81}$$

For $l > 1$ the term on the r.h.s. of (81) is bounded, by induction hypothesis. For $l = 1$ the operator corresponding to the sum of the first two terms in the braces on the r.h.s. of (81) is also bounded by induction hypothesis. It follows, from (78) and (79), that

$$\begin{aligned}[H_f^n, H](H+i)^{-n-1} = -\,i\sum_{j=1}^{N} \phi(i\omega G_{x_j})(H_f+d)^{-1}(p_j + A(x_j))(H+c)^{-1} \\ \times [H_f^n, H](H+i)^{-n-1} + \text{bounded}.\end{aligned}$$

If we define A_d to be the operator to the left of $[H_f^n, H](H+i)^{-n-1}$ in the first term on the r.h.s. of last equation, we have

$$(1 - A_d)[H_f^n, H](H+i)^{-n-1} = \text{bounded}.$$

Since the norm of $\phi(i\omega G_{x_j})(H_f + d)^{-1}$ can be made arbitrarily small, by choosing d sufficiently large, it follows that $\|A_d\| < 1$ for suitable d, and thus also the operator $[H_f^n, H](H+i)^{-n-1}$ has to be bounded.

□

E Pseudo–relativistic QED: Higher Order Estimates.

In this section we want to prove Lemma 8, which is the analogous of Lemma 5 for the pseudo–relativistic model. This Lemma ensures the boundedness of higher powers of the field Hamiltonian H_f w.r.t. higher powers of the total Hamiltonian H. Then we will apply this lemma to prove Theorem 14, which establishes the existence of asymptotic states with an arbitrary number of free photons. Also here, as in the non–relativistic case, we begin by showing the boundedness of $H_f(H+i)^{-1}$. This result will be used as first step in the inductive proof of Lemma 8.

Lemma 22. *The operator $H_f(H+i)^{-1}$ is bounded.*

Proof. We use

$$\begin{aligned} H_f^2 &= \sum_\lambda \int dk\, |k| a_\lambda^*(k) a_\lambda(k) H_f \\ &= \sum_\lambda \int dk\, |k| a_\lambda^*(k)(H_f + |k|) a_\lambda(k) \end{aligned}$$

to write

$$\|H_f(H+c)^{-1}\psi\|^2 = \sum_\lambda \int dk\, |k|\, \|(H_f+|k|)^{1/2} a_\lambda(k)(H+c)^{-1}\psi\|^2. \tag{82}$$

Now we apply the pull–through–formula

$$\begin{aligned} a_\lambda(k)(H+c)^{-1} = (H+|k|+c)^{-1} a_\lambda(k) \\ + (H+|k|+c)^{-1}\,[\Omega, a_\lambda(k)]\,(H+c)^{-1}, \end{aligned}$$

which can be easily proved by commuting the Hamiltonian H with $a_\lambda(k)$, and we find, from (82)

$$\begin{aligned} \|H_f(H+c)^{-1}\psi\|^2 \le\ & 2\sum_\lambda \int dk\, |k| \|(H_f+|k|)^{1/2}(H+|k|+c)^{-1} a_\lambda(k)\psi\|^2 \\ &+2\sum_\lambda \int dk\, |k| \|(H_f+|k|)^{1/2}(H+|k|+c)^{-1}\,[\Omega, a_\lambda(k)](H+c)^{-1}\psi\|^2. \end{aligned} \tag{83}$$

The first term on the r.h.s. of (83) is bounded by $2\|\psi\|^2$, for $c > 0$ sufficiently large. To see this, note that

$$\begin{aligned} &\|(H_f + |k|)^{1/2}(H + |k| + c)^{-1} a_\lambda(k)\psi\|^2 \\ &\qquad = \langle \psi, a_\lambda^*(k)(H + |k| + c)^{-1}(H_f + |k|)(H + |k| + c)^{-1} a_\lambda(k)\psi\rangle \\ &\qquad \leq \langle \psi, a_\lambda^*(k)(H_f + |k| + 1)^{-1} a_\lambda(k)\psi\rangle \\ &\qquad \leq \langle \psi, a_\lambda^*(k) a_\lambda(k)(H_f + 1)^{-1}\psi\rangle, \end{aligned} \tag{84}$$

where we used that $(H_f + |k|) \leq (H + |k| + c)$ and that $(H + |k| + c)^{-1} \leq (H_f + |k| + 1)^{-1}$ for $c > 0$ sufficiently large. From (84), after integration over k and sum over λ, it follows that the first term on the r.h.s. of (83) is bounded by $2\,\|\psi\|^2$.

We consider now the second term on the r.h.s. of (83). We have

$$\begin{aligned} &\|(H_f + |k|)^{1/2}(H + |k| + c)^{-1}\,[\Omega, a_\lambda(k)](H + c)^{-1}\psi\| \\ &\qquad \leq \|(H_f + |k|)^{1/2}(H + |k| + c)^{-1/2}\|\,\|(H + |k| + c)^{-1/2}\Omega^{1/2}\| \\ &\qquad\quad \times \|\Omega^{-1/2}[\Omega, a_\lambda(k)]\Omega^{-1/2}\|\,\|\Omega^{1/2}(H + c)^{-1}\|\,\|\psi\| \\ &\qquad \leq \text{const}\,\|\Omega^{-1/2}[\Omega, a_\lambda(k)]\Omega^{-1/2}\|\,\|\psi\| \end{aligned} \tag{85}$$

for all k and λ. Now we expand the commutator in the term on the r.h.s. of the last equation, using Lemma 11, and we get

$$\begin{aligned} \|\Omega^{-1/2}[\Omega, a_\lambda(k)]\Omega^{-1/2}\| &\leq \frac{\sqrt{2}}{\pi}\int_1^\infty dy\sqrt{y-1}\|(y + (p + A)^2)^{-1}\|^2 \\ &\qquad \times \|\Omega^{-1/2}(p + A)\cdot G_{\lambda,x}(k)\Omega^{-1/2}\| \\ &\leq \text{const}\,\|\Omega^{-1/2}(p + A)\cdot G_{\lambda,x}(k)\Omega^{-1/2}\| \\ &\leq \text{const}\,\frac{\kappa(k)}{\sqrt{|k|}}\,\|\Omega^{-1/2}(p + A)e^{-ik\cdot x}\Omega^{-1/2}\|. \end{aligned} \tag{86}$$

After commutation of the exponential $e^{-ik\cdot x}$ in the term on the r.h.s. of (86) to the right of the factor $\Omega^{-1/2}$ we find

$$\|\Omega^{-1/2}(p + A)e^{-ik\cdot x}\Omega^{-1/2}\| \leq \text{const}(1 + |k| + |k|^2).$$

Inserting this into the r.h.s. of (86), the resulting bound into the r.h.s. of (85) and then in (83) we find, after integration over k (which gives no problem because of the UV–cutoff), and after sum over λ, that also the second term on the r.h.s. of (83) is bounded by $C\,\|\psi\|^2$, for some finite constant $C > 0$. This proves the lemma. □

We are now ready to prove Lemma 8. The ideas of the proof are the same as in the non-relativistic case but the estimates are more involved because of the complicated form of the interaction between photons and electron.

Proof of Lemma 8. We proceed by induction over n. For $n = 1$ the boundedness of i) is trivial and the boundedness of ii) follows from Lemma 22. Now assume that the statement of the lemma holds true for any integer m less or equal to a given n. We prove the statement for $m = n+1$. As in the non-relativistic case it suffices to prove the boundedness of the operator i). To this end we use the expansion given in Lemma 11 and the commutator expansion (71) to write

$$\begin{aligned}
[H_f^n, H](H+i)^{-n-1} &= [H_f^n, \Omega]\,(H+i)^{-n-1} \\
&= \frac{-1}{\pi}\int_1^\infty dy\,\sqrt{y-1}\,[H_f^n, (y+(p+A)^2)^{-1}]\,(H+i)^{-n-1} \\
&= \frac{-1}{\pi}\sum_{l=1}^{n}\binom{n}{l}\int_1^\infty dy\,\sqrt{y-1}\,\mathrm{ad}_{H_f}^l((y+(p+A)^2)^{-1})H_f^{n-l}(H+i)^{-n-1}.
\end{aligned} \tag{87}$$

We set $\Gamma = (y+(p+A)^2)^{-1}$ and $\Lambda_m = \mathrm{ad}_{H_f}^m((p+A)^2)$. Now we note that each factor $\mathrm{ad}_{H_f}^l((y+(p+A)^2)^{-1})$ on the r.h.s. of (87) can be expanded in a sum of terms like

$$\mathrm{const}\,\Gamma\Lambda_{m_1}\Gamma\ldots\Gamma\Lambda_{m_r}\Gamma, \quad \text{with} \quad m_i \in \{1,\ldots,l\}, \quad m_1+\cdots+m_r = l.$$

Using

$$\begin{aligned}
\mathrm{ad}_{H_f}^l((p+A(x))^2) = (-i)^l &\sum_{m=1}^{l-1}\binom{l}{m}\phi(i^m\omega^m G_x)\cdot\phi(i^{l-m}\omega^{l-m}G_x) \\
&+ 2\,(-i)^l\,\phi(i^l\omega^l G_x)\cdot(p+A(x)) \\
&+ (-i)^l\sum_{k=1}^{3}[(p_k+A_k),\phi(i^l\omega^l G_{x,k})],
\end{aligned}$$

we see that the r.h.s. of (87) can be written as a sum of terms like

$$\mathrm{const}\int_1^\infty dy\,\sqrt{y-1}\,\Gamma\Lambda_1'\Gamma\ldots\Gamma\Lambda_r'\Gamma H_f^{n-l}(H+i)^{-n-1}, \tag{88}$$

with $1 \le r \le l$, $l \in \{1,\ldots,n\}$, and where each Λ_i' is either the product of two field–operators, as $a^\sharp(i^j\omega^j G_x)\cdot a^\sharp(i^s\omega^s G_x)$, or the product of a field–operator with a factor $(p+A)$, as $a^\sharp(i^m\omega^m G_x)\cdot(p+A)$ or $(p+A)\cdot a^\sharp(i^m\omega^m G_x)$. We consider first the terms like (88) with $r \ge 2$. To this end expand each field operator $a^\sharp(i^j\omega^j G_x)$ in the Λ_i', writing

$$a^*(i^j\omega^j G_x) = i^j\sum_\lambda\int dk|k|^j\frac{\kappa(k)}{\sqrt{|k|}}\epsilon_\lambda(k)e^{-ik\cdot x}a_\lambda^*(k),$$

and similarly for the annihilation operators. Next commute each operator–valued distribution $a_\lambda^\sharp(k)$ to the right of all the resolvents Γ, using the commutation relations

$$\begin{aligned}
[a_\lambda^\sharp(k),(p+A)] &= \pm\frac{1}{\sqrt{2}}G_{\lambda,x}^\sharp(k) \\
[a_\lambda^\sharp(k),\Gamma] &= \pm\frac{1}{\sqrt{2}}\Gamma\left\{(p+A)\cdot G_{\lambda,x}^\sharp(k) + G_{\lambda,x}^\sharp(k)\cdot(p+A)\right\}\Gamma.
\end{aligned}$$

At the end we can write each operator like (88), with $r \ge 2$, as a sum of terms like

$$\begin{aligned}
\mathrm{const}\int_1^\infty dy\sqrt{y-1}&\sum_{\lambda_1,\ldots,\lambda_m}\int dk_1\ldots dk_m\,\frac{\kappa(k_1)\ldots\kappa(k_m)}{\sqrt{|k_1|\ldots|k_m|}} \\
&\times\Gamma\delta_1\Gamma\ldots\Gamma\delta_{r'}\Gamma a_{\lambda_1}^\sharp(k_1)\ldots a_{\lambda_m}^\sharp(k_m)H_f^{n-l}(H+i)^{-n-1},
\end{aligned} \tag{89}$$

where $r' \geq 2$ (since we started with $r \geq 2$ and by the commutations of the $a^*_\lambda(k)$ the number of resolvents Γ could only get larger), $m \leq 2l$ (since in (88) we had at most $2r \leq 2l$ operators $a^\sharp(i\omega G)$ and since by the commutations of the $a^*_\lambda(k)$ the number of such fields could only get smaller), and where each operator δ_i is either a bounded operator or the product of a factor $(p+A)$ with a bounded operator. In both cases we have $\|\Gamma^{1/2}\delta_i\Gamma^{1/2}\| \leq \mathrm{const}\, 1/\sqrt{y}$, and thus

$$\|\Gamma\delta_1\Gamma\dots\Gamma\delta_{r'}\Gamma\| \leq \mathrm{const}\,\frac{1}{y^{\frac{r'}{2}+1}}.$$

It follows that each term like (89) has norm bounded by

$$\begin{aligned}\mathrm{const}\int_1^\infty dy\,\frac{\sqrt{y-1}}{y^{\frac{r'}{2}+1}}\,\{\sum_{\lambda_1,\dots,\lambda_m}\int dk_1\dots dk_m\frac{\kappa(k_1)\dots\kappa(k_m)}{\sqrt{|k_1|\dots|k_m|}}\\ \times\|a^\sharp_{\lambda_1}(k_1)\dots a^\sharp_{\lambda_m}(k_m)(H_f+1)^{-l}\|\}\,\|(H_f+1)^lH_f^{n-l}(H+i)^{-n-1}\|.\end{aligned} \tag{90}$$

By Lemma 17, and because we have at most $2l$ operators $a^\sharp_\lambda(k)$, we find, that the integrals and the sums inside the braces are bounded by some finite constant. Moreover, since $r' \geq 2$, the integration over y yields another finite constant. Finally the factor $\|(H_f+1)^lH_f^{n-l}(H+i)^{-n-1}\|$ is bounded by induction hypothesis. It follows that (90) is bounded by a finite constant. We want now to consider the terms like (88) with $r=1$. These terms are of the form

$$\mathrm{const}\int_1^\infty dy\sqrt{y-1}\Gamma\Lambda\Gamma H_f^{n-l}(H+i)^{-n-1}, \tag{91}$$

where again Λ is either the product of two fields–operators or the product of a field operator with a factor $(p+A)$. If Λ is the product of two fields $a^\sharp(i^j\omega^jG_x)$ then we can proceed as before, expanding the fields in integrals of operator–valued distributions $a^\sharp_\lambda(k)$, commuting these distributions to the right, and using the two resolvents Γ to ensure the convergence of the integral over y. It remains to consider the case where Λ is the product of a field $a^\sharp(i^j\omega^jG_x)$ with a factor $(p+A)$. We consider the case $\Lambda = a^*(i^j\omega^jG_x)\cdot(p+A)$, the other cases, with a^* and $(p+A)$ interchanged, and with an annihilation operator a instead of a^*, being similar. To this end we expand the field $a^*(i^j\omega^jG_x)$ in an integral over k and we commute the factor $a^*_\lambda(k)(p+A)$ to the right of the resolvent. This yields

$$\begin{aligned}&\mathrm{const}\int_1^\infty dy\,\sqrt{y-1}\,\Gamma a^*(i^j\omega^jG_x)\cdot(p+A)\Gamma H_f^{n-l}(H+i)^{-n-1}\\ &=\mathrm{const}\int_1^\infty dy\sqrt{y-1}\sum_\lambda\int dk\,\frac{\kappa(k)}{\sqrt{|k|}}\epsilon_{\lambda,j}(k)\,\Gamma\,e^{-ik\cdot x}\,\Gamma\\ &\quad\times a^*_\lambda(k)(p+A)H_f^{n-l}(H+i)^{-n-1}\\ &\quad+\text{bounded}\ ,\end{aligned} \tag{92}$$

because the commutator of Γ with $a^*_\lambda(k)(p+A)$ gives factors which can be handled as we did with the terms with $r>1$ (and whose norm is hence bounded). Now we insert, in the operator on the r.h.s. of (92), between the factors $a^*_\lambda(k)$ and $(p+A)$, the operator $(H_f+d)^{-1}(H_f+d)$, for some

$d > 0$, and then we commute the term $(H_f + d)$ to the right of $(p + A)$. The operator on the r.h.s. of (92) can then be written as

$$\begin{aligned} &\text{const} \int_1^\infty dy \sqrt{y-1} \sum_\lambda \int dk \, \frac{\kappa(k)}{\sqrt{|k|}} \epsilon_{\lambda,j}(k) \, \Gamma \, e^{-ik\cdot x} \, \Gamma \\ &\quad \times a_\lambda^*(k)(H_f + d)^{-1}(p_j + A_j)(H_f + d)H_f^{n-l}(H + i)^{-n-1} \\ &\quad + \text{bounded}, \end{aligned} \tag{93}$$

because the commutator of $(H_f + d)$ with the factor $(p + A)$ gives only another field $\phi(i\omega G_x)$ which can be bound by another resolvent $(H_f + 1)^{-1}$. Finally we commute one of the $n + 1$ resolvents $(H + i)^{-1}$ to the left of the field Hamiltonians in order to bound the term $(p + A)$. The operator in (93) is then equal to

$$\begin{aligned} &\text{const} \int_1^\infty dy \sqrt{y-1} \sum_\lambda \int dk \, \frac{\kappa(k)}{\sqrt{|k|}} \epsilon_{\lambda,j}(k) \, \Gamma \, e^{-ik\cdot x} \, \Gamma \, a_\lambda^*(k)(H_f + d)^{-1} \\ &(p_j + A_j)(H + i)^{-1}\{(H_f + d)H_f^{n-l}(H + i)^{-n} \\ &+ d\,[H_f^{n-l}, H](H + i)^{-n-1} + [H_f^{n-l+1}, H](H + i)^{-n-1}\}. \end{aligned} \tag{94}$$

If $l > 1$ the operator in (94) is bounded by induction hypothesis, since the presence of two factors Γ ensures the absolute convergence of the integral over y. If $l = 1$ the contributions arising from the first two terms inside the braces are bounded, too, by induction hypothesis. The only contributions which could be unbounded are those arising from the third term in the braces in (94) if $l = 1$. Adding all these potentially unbounded contributions together (such contributions arise from terms like (91), with $l = 1$ in both cases, wheater Λ contains a creation or an annihilation operator), we find, from (87),

$$\begin{aligned} [H_f^n, H](H + i)^{-n-1} = &\text{const} \int_1^\infty dy \sqrt{y-1} \sum_\lambda \int dk \, \frac{\kappa(k)}{\sqrt{|k|}} \epsilon_{\lambda,j}(k) \, \Gamma \, e^{-ik\cdot x} \, \Gamma \\ &\times (a_\lambda^*(k) + a_\lambda(k))(H_f + d)^{-1}(p_j + A_j)(H + i)^{-1}[H_f^n, H](H + i)^{-n-1} \\ &+ \text{bounded}. \end{aligned}$$

Now we define A_d to be the operator to the left of $[H_f^n, H](H+i)^{-n-1}$ in the first term on the r.h.s. of last equation. Then it follows

$$(1 - A_d)\,[H_f^n, H](H + i)^{-n-1} = \text{bounded}.$$

Since the norm of $\phi(i\omega G_{x_j})(H_f + d)^{-1}$ can be made arbitrarily small, by choosing d sufficiently large, it follows that $\|A_d\| < 1$ for suitable d, and thus, by the last equation, also the operator $[H_f^n, H](H + i)^{-n-1}$ has to be bounded.

□

With the help of Lemma 8, we can now give the proof of Theorem 14.

Proof of Theorem 14. We first assume that $h_i \in \mathcal{C}_0^\infty(\mathbb{R}^3 \backslash \{0\})$, for all $i \in \{1, \dots n\}$, and that $\varphi = \chi(H \le E)\varphi$ for some $E < \infty$. We prove then the limit (61) by induction over n. For $n = 1$, (61) follows by Theorem 13. Assume now that (61) holds true for any integer less than a given n. We want to prove it for n fields $a_+^\sharp(h_i)$. We consider the case where we have n creation operators: the other cases are then similar. Since $\varphi = \chi(H \le E)\varphi$, since the functions h_i have compact support, and because of Theorem 13, iv), the vector $a_+^*(h_l) \dots a_+^*(h_n)\varphi$ is well defined, and

$$a_+^*(h_l) \dots a_+^*(h_n)\varphi = \lim_{t \to \infty} e^{iHt} a^*(h_{l,t}) e^{-iHt} a_+^*(h_{l+1}) \dots a_+^*(h_n)\varphi, \tag{95}$$

for each $l \in \{1, \dots, n\}$. To show that the limit (61) holds true we consider now the difference

$$\begin{aligned} e^{iHt} a^*(h_{1,t}) \dots a^*(h_{n,t}) e^{-iHt}\varphi - a_+^*(h_1) \dots a_+^*(h_n)\varphi \\ = (e^{iHt} a^*(h_{1,t}) e^{-iHt} - a_+^*(h_1)) a_+^*(h_2) \dots a_+^*(h_n)\varphi \\ + e^{iHt} a^*(h_{1,t}) e^{-iHt} \times \\ \left\{ e^{iHt} a^*(h_{2,t}) \dots a^*(h_{n,t}) e^{-iHt} - a_+^*(h_2) \dots a_+^*(h_n) \right\} \varphi. \end{aligned} \tag{96}$$

The norm of the first term on the r.h.s of the last equation converges, by (95), to 0 as $t \to \infty$. To handle the second term on the r.h.s. of (96) we insert the operator $id = (H+i)^{-1}(H+i)$ just in front of the braces, and we commute the factor $(H+i)$ through the whole braces. The second term on the r.h.s. of (96) becomes then

$$\begin{aligned} & e^{iHt} a^*(h_{1,t}) e^{-iHt} (H+i)^{-1} \times \\ & \big\{ (e^{iHt} a^*(h_{2,t}) \dots a^*(h_{n,t}) e^{-iHt} - a_+^*(h_2) \dots a_+^*(h_n))(H+i)\varphi \\ & + \sum_{l=2}^n (e^{iHt} a^*(h_{2,t}) \dots a^*(\omega h_{l,t}) \dots a^*(h_{n,t}) e^{-iHt} \\ & \qquad - a_+^*(h_2) \dots a_+^*(\omega h_l) \dots a_+^*(h_n))\varphi \\ & + e^{iHt} [\Omega, a^*(h_{2,t}) \dots a^*(h_{n,t})] \, e^{-iHt} \varphi \big\} . \end{aligned} \tag{97}$$

Now the term in front of the braces is bounded, uniformly in t. The first term inside the braces, and each factor in the sum over l converges to 0, as $t \to \infty$, by induction hypothesis. Hence (61) follows if we show that the norm of the last term inside the braces converges to zero as $t \to \infty$. To this end we expand the commutator with Ω in an integral, as in Lemma 11. Using the commutation relation $[a^*(h_t), (p+A)^2] = -\sqrt{2}(G_x, h_t) \cdot (p+A)$ we find

$$\begin{aligned} & \| [\Omega, a^*(h_{2,t}) \dots a^*(h_{n,t})] e^{-iHt} \varphi \| \\ & \le \frac{\sqrt{2}}{\pi} \sum_{j=2}^n \int_1^\infty dy \, \sqrt{y-1} \, \| (y + (p+A)^2)^{-1} \| \, \| (G_x, h_{j,t}) \| \\ & \times \| a^*(h_{2,t}) \dots a^*(h_{j-1,t}) (p+A) a^*(h_{j+1,t}) \dots a^*(h_{n,t}) (y + (p+A)^2)^{-1} \varphi_t \|, \end{aligned} \tag{98}$$

where $\varphi_t = e^{-iHt}\varphi$. Since $\|(G_x, h_{j,t})\|$ converges to zero as $t \to \infty$ it only remains to show that the vector

$$a^*(h_{2,t}) \dots a^*(h_{j-1,t}) (p+A) a^*(h_{j+1,t}) \dots a^*(h_{n,t}) (y + (p+A)^2)^{-1} \varphi_t \tag{99}$$

has norm bounded by C/y, for some $C < \infty$, and for all $y \geq 1$ (note that the factor $1/y$ is necessary to make the y–integral absolut convergent). To this end we consider the operator which act on φ_t in (99) and we commute, first of all, the factor $(p + A)$ to the very left, using the commutation rule $[a^*(h_{i,t}), (p + A)] = -1/\sqrt{2}\,(G_x, h_{i,t})$. We find that the operator which act on φ_t in (99) can be written as

$$\begin{aligned}(p + A)a^*(h_{2,t}) \dots a^*(h_{n,t})(y + (p + A)^2)^{-1}\\ - \frac{1}{\sqrt{2}} \sum_{l=2}^{j-1} (G_x, h_{l,t}) a^*(h_{2,t}) \dots a^*(h_{n,t})(y + (p + A)^2)^{-1}.\end{aligned} \tag{100}$$

Next we commute, in the first term as well as in each term in the sum in (100), all the fields $a^*(h_{m,t})$ to the right of the resolvent $(y + (p + A)^2)^{-1}$. Here we use the commutation relation

$$\begin{aligned}{}[a^*(h_{m,t}), (p + A)] &= -\frac{1}{\sqrt{2}}\,(G_x, h_{m,t})\\ [a^*(h_{m,t}), (y + (p + A)^2)^{-1}] &= \sqrt{2}\,(y + (p + A)^2)^{-1}\\ &\qquad \times (p + A) \cdot (G_x, h_{m,t})\,(y + (p + A)^2)^{-1}.\end{aligned}$$

At the end each term in the sum over l in (100) will be written as a sum of terms like

$$(y + (p + A)^2)^{-1} \times B \times a^*(h_{i_1,t}) \dots a^*(h_{i_m,t}), \tag{101}$$

where B is some bounded operator, and the number of field is at most $n - 3$. The first factor in (100), on the other hand will be written as a sum of factors like

$$(p + A)(y + (p + A)^2)^{-1/2} \times B \times (y + (p + A)^2)^{-1} a^*(h_{i_1,t}) \dots a^*(h_{i_m,t}), \tag{102}$$

where, again B is a bounded operator and $m \leq n - 3$, plus the factor

$$(p + A)(y + (p + A)^2)^{-1} a^*(h_{2,t}) \dots a^*(h_{n,t}). \tag{103}$$

Each term like (101) or like (102) gives a contribution to the vector (99) whose norm is bounded, uniformly in t, by C/y, for a finite constant C. This follows from the bound $\|a^\sharp(h_{1,t}) \dots a^\sharp(h_{n,t})\varphi_t\| \leq C$, for each $t \in \mathbb{R}$. This estimate follows from Lemma 17, in Appendix A, and from the boundedness of $H_f^n(H + 1)^{-n}$, which follows by Lemma 8.

It only remains to consider the contribution from the operator (103). To this end we commute the factor $(p + A)$ through the resolvent $(y + (p + A)^2)^{-1}$. The contribution from the term which contains the commutator between $(p + A)$ and $(y + (p + A)^2))^{-1}$ can be handled as we did with operators like (101), and has therefore a norm bounded by C/y. It only remains to consider the contribution to the norm of the vector (99) arising from the term (103) with $(p + A)$ and $(y + (p + A)^2)^{-1}$ interchanged. This is bounded by

$$\begin{aligned}&\|(y + (p + A)^2)^{-1}(p + A)a^*(h_{2,t}) \dots a^*(h_{n,t})\varphi_t\|\\ &\qquad \leq 1/y\,\|(p + A)a^*(h_{2,t}) \dots a^*(h_{n,t})\varphi_t\|\\ &\qquad \leq C/y,\end{aligned} \tag{104}$$

where, in the last step we used that $\|(p+A)a^\sharp(h_{1,t})\dots a^\sharp(h_{n,t})\varphi_t\| \leq C$, $\forall t > 0$, which follows, after some manipulations, from the boundedness of the operator $[H_f^n, H](H+i)^{-n-1}$ (see Lemma 8). This completes the proof of the limit (61), for $h_i \in \mathcal{C}_0^\infty(\mathbb{R}^3 \backslash \{0\})$, and for $\varphi \in \operatorname{Ran}\chi(H \leq E)$. The bound (62) follows now from the corresponding bound for products of the usual fields $a^\sharp(h)$. This bound permits also to show that the limit (61) also holds for wave–functions $h_i \in L^2_\omega(\mathbb{R}^3; \mathbb{C}^2)$, because $\mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$ is dense in L^2_ω. Moreover the limit (61) also holds for $\varphi \in D((H+i)^{n/2})$, because $\cup_{E>0}\chi(H \leq E)\mathcal{H}$ is a core for $(H+i)^{n/2}$.

□

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