Qualitative properties of the ground state in Nelson's massless scalar field model - A Gibbs measure-based approach

József Lőrinczi

Zentrum Mathematik, Technische Universität München
Gabelsbergerstr. 49, 80290 München, Germany
lorinczi@mathematik.tu-muenchen.de

Abstract

This is a summary of results based upon [2, 4, 5, 3] outlining how Gibbs measures can be associated with Nelson's massless scalar field model in its Euclidean representation. These Gibbs measures will be used for deriving qualitative properties of the Fock space ground state.

1 Introduction

Consider Nelson's model of a charged spinless point particle with position $q_t \in \mathbb{R}^d$ coupled to a massless scalar boson field given by the Hamiltonian

$$\begin{aligned} H_{\mathrm{N}} &= \left(-\frac{1}{2}\Delta + V(x)\right) \otimes 1 + 1 \otimes \int_{\mathbb{R}^d} |k| a^*(k) a(k) dk \\ &+ \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2|k|}} \left(\hat{\varrho}(k) e^{ikx} \otimes a(k) + \hat{\varrho}(-k) e^{-ikx} \otimes a^*(k)\right) dk. \end{aligned} \tag{1.1}$$

H_{N} acts on $L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$, $\mathcal{F}$ being the symmetric Fock space over $L^2(\mathbb{R}^d, dx)$. The first term is a Schrödinger operator on the Hilbert space $L^2(\mathbb{R}^d, dx)$ describing the non-interacting particle under a confining potential V. The external potential $V : \mathbb{R}^d \to \mathbb{R}$ is a multiplication operator assumed here of being bounded from below, continuous and having the asymptotics $V(x) = C|x|^{2s} + o(|x|^{2s})$ for large $|x|$ with some constant $C > 0$ and exponent $s > 1$. This confining potential makes sure that the Schrödinger operator has a unique strictly positive continuous eigenfunction ψ_0 lying at the bottom of its spectrum (ground state). The second term gives the free field energy with dispersion $|k|$ and is written in terms of the Bose creation and annihilation operators $a^*(k), a(k)$ on Fock space satisfying the canonical commutation relations $[a(k), a^*(k')] = \delta(k - k')$,

$[a(k), a(k')] = 0 = [a^*(k), a^*(k')]$, $\forall k, k'$. The last term in (1.1) accounts for the particle-field interaction. ϱ (its Fourier transform being denoted $\hat{\varrho}$) is the particle charge density that we take to be a radial, smooth, rapidly decreasing function whose integral is the full charge, i.e. $\int_{\mathbb{R}^d} \varrho(x)dx = e > 0$, playing here the role of the coupling constant. For reasons of mathematical well-definedness ϱ has to be chosen such that a cutoff is introduced at both ends of the energy spectrum. The condition $\int_{\mathbb{R}^d} dk|\hat{\varrho}(k)|^2/|k|^3 < \infty$ implements an infrared cutoff. Under mild assumptions on ϱ it is easily seen by the Kato-Rellich theorem that H_{N} is self-adjoint on the domain of the self-adjoint operator $(-1/2\Delta + V(x)) \otimes 1 + 1 \otimes \int |k| a^*(k)a(k)dk$, and it is also bounded from below.

An alternative to the Fock space representation of Nelson's model is a formulation in terms of Euclidean quantization. In this framework the particle and field are associated with specific stochastic processes. The Euclidean action for the particle is $S_{\mathrm{p}}(\{q_t\}) = \int_{\mathbb{R}}(\frac{1}{2}\dot{q}_t^2 + V(q_t))dt$, hence the particle is associated with the stationary $P(\phi)_1$-process q_t on $C(\mathbb{R}, \mathbb{R}^d)$ defined by the stochastic differential equation

$$dq_t = -\frac{\nabla\psi_0}{\psi_0}dt + dB_t \tag{1.2}$$

(dB_t denotes Brownian motion), and having the invariant measure $d\mathsf{N}^0 = \psi_0^2(x)dx$. The path measure of this process will be denoted by $\mathcal{N}^0$; for more details on this see [2] and references therein. The paths $Q = \{q_t\}$ are $\mathcal{N}^0$-almost surely continuous and q_t is a time-reversible Markov process. The field in its turn is described by a generalized Gaussian process $\xi(t,x)$ resulting from the Euclidean action $S_{\mathrm{f}}(\{\xi(t,x)\}) = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{1}{2}(\partial_t\xi(t,x)^2 + (\nabla\xi(t,x))^2)dxdt$. For any $f \in \mathcal{S}(\mathbb{R}^{d+1})$, i.e. the Schwartz space over $\mathbb{R}^d$, denote $\xi(f) = \int_{\mathbb{R}^{d+1}} f(t,x)\xi(t,x)dtdx$. The field $\{\xi(f), f \in \mathcal{S}(\mathbb{R}^{d+1})\}$ is described by the generalized Gaussian measure $\mathcal{G}$ on $\mathcal{S}(\mathbb{R}^{d+1})$ with zero mean and covariance

$$\mathrm{cov}_{\mathcal{G}}(\xi(f_1), \xi(f_2)) = \left((-\partial_t^2 - \Delta)^{-1} f_1, f_2\right)_{L^2(\mathbb{R}^{d+1}, dx)} = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\hat{f}_1(k_0,k)\hat{f}_2^*(k_0,k)}{k_0^2 + k^2} dk_0 dk. \tag{1.3}$$

The probability space is $\mathcal{S}'(\mathbb{R}^{d+1})$ with its associated Borel σ-field. It will be convenient below to extend (1.3) to test functions of the form $f_t^{\varphi}(x',t') = \varphi(x')\delta(t'-t)$ with $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and view $\xi_t = \xi(t,\cdot)$ as a stochastic process taking values in a suitable Hilbert space $\mathcal{B}_D$. Thus we will use the stationary Gaussian process $\xi_t(\varphi) = \xi(f_t^{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $t \in \mathbb{R}$. Its stationary measure G defined on $\mathcal{S}'(\mathbb{R}^d)$ is again Gaussian with mean 0 and covariance $\mathrm{cov}_{\mathsf{G}}(\xi_0(\varphi_1), \xi_0(\varphi_2)) = 1/2\int_{\mathbb{R}^d} dk\hat{\varphi}_1(k)\hat{\varphi}_2^*(k)/k$. Moreover, $\xi_t(\varphi)$ is a time-reversible Markov process. The real Hilbert space $\mathcal{B}_D \subset \mathcal{S}'(\mathbb{R}^d)$ consists of functions with norm $||\xi||_{\mathcal{B}_D}^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d} D(k_1,k_2)\hat{\xi}(k_1)\hat{\xi}^*(k_2)dk_1dk_2$ (where $\hat{\xi}^*(k) = \hat{\xi}(-k)$). Here D is the operator with kernel $D(k_1,k_2) = |k_1|_1\bar{D}(k_1,k_2)|k_2|_1$, where $\bar{D} = (-\Delta_k + |k|^2)^{-(d+1)}$ and $|k|_1 = |k|$ if $|k| < 1$ and is 1 otherwise. The process ξ_t is continuous with respect to this norm with probability 1.

Finally, the Euclidean interaction becomes $S_{\mathrm{int}}(\{q_t, \xi_t\}) = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \varrho(x - q_t)\xi(t,x)dxdt \equiv \int(\xi_t * \varrho)(q_t)dt$. The joint particle-field process is realized on the space $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$ and it is described by the path measure $\mathcal{P}^0 = \mathcal{N}^0 \times \mathcal{G}$. The interacting system can be described by taking $\mathcal{P}^0$ as reference process and modifying it with a density given by

$$\frac{d\mathcal{P}_T}{d\mathcal{P}^0} = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt\right) \tag{1.4}$$

Here

$$Z_T = \int \exp\left(-\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt\right) d\mathcal{P}^0 = \int \exp\left(\int_{-T}^{T}\int_{-T}^{T} W(q_s - q_t, s-t) ds dt\right) d\mathcal{N}^0 \tag{1.5}$$

is the normalizing partition function with

$$W(q,t) = -\frac{1}{4}\int_{\mathbb{R}^d} \frac{|\hat{\varrho}(k)|^2}{|k|} e^{-|k||t|} \cos(k \cdot q)\, dk \tag{1.6}$$

This is obtained in the following way. For any fixed path $Q = \{q_t : t \in \mathbb{R}\} \in C(\mathbb{R}, \mathbb{R}^d)$ consider the conditional distribution $\mathcal{P}_T^Q = \mathcal{P}_T(\,\cdot\,|\{q_t\} = Q)$ as a probability measure on $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$. Since the interaction $\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt$ is linear in ξ_t, $\mathcal{P}_T^Q$ is itself a Gaussian measure. We have then for $\mathcal{P}_T$ on $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$ that $\mathcal{P}_T(E \times F) = \int_F \mathcal{P}_T^Q(E) d\mathcal{N}_T(Q)$, for all $T > 0$, where E and F are appropriate measurable sets. Here $\mathcal{N}_T$ is the probability measure on the particle path space $C(\mathbb{R}, \mathbb{R}^d)$ given by

$$\frac{d\mathcal{N}_T}{d\mathcal{N}^0} = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T}\int_{-T}^{T} W(q_s - q_t, s-t) ds dt\right) \tag{1.7}$$

with the pair interaction potential W appearing in (1.6). Thus for every $T > 0$ the finite-time Gibbs measure $\mathcal{N}_T$ is the marginal of the joint particle-field path measure $\mathcal{P}_T$ obtained by integrating over the Gaussian variables and it is the stationary measure for Brownian motion under external potential V and pair interaction potential W appearing as an effective potential for the particle on averaging over the Bose field. Since $\mathcal{P}_T^Q$ is a Gaussian measure, for understanding the properties of the joint particle-field path measure we only need to understand the properties of $\mathcal{N}_T$. We denote the single time distribution of $\mathcal{N}_T$ by N_T.

2 Gibbs measure

We will be interested whether the sequence of measures $\mathcal{P}_T$ has a limit as $T \to \infty$. We use the following notion of convergence. Let M be a metric space, and $C(\mathbb{R}, M)$ the space of continuous paths $\{X_t\}$ with values in M. For any interval $I_T = [-T, T] \subset \mathbb{R}$ let $\mathcal{M}_T \subset \mathcal{M}$ be a sub-σ field of the Borel σ-field $\mathcal{M}$ of M generated by the evaluations $\{X_t : t \in I_T\}$, i.e. $\mathcal{M}_T = \sigma(X_t, t \in I_T)$. We say that the sequence of probability measures $\{\mu_n\}$ on $C(\mathbb{R}, M)$ converges locally weakly to the probability measure μ if for any $0 \le T < \infty$ the restrictions $\mu_n|_{\mathcal{M}_T}$ converge weakly to the measure $\mu_{\mathcal{M}_T}$. We have then the following result.

Theorem 2.1 (Existence of particle-field path measure) *For $d \geq 3$ and $0 < e \leq e^*$, with sufficiently small $e^* > 0$, there exists the weak local limit $\mathcal{P}_T \to \mathcal{P}$ as $T \to \infty$, and it is a probability measure of a stationary reversible Markov process on the path space $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$. Moreover, the stationary distribution P of $\mathcal{P}$ can be obtained as the weak limit $\mathsf{P} = \lim_{T\to\infty} \mathsf{P}_T$, where P_T is the distribution of $\{q_0, \xi_0\}$ under $\mathcal{P}_T$.*

This theorem results from, as pointed out above, the properties of the Gibbs measure $\mathcal{N} = \lim_{T\to\infty} \mathcal{N}_T$ defined by (1.7).

Theorem 2.2 (Existence of Gibbs measure) *Take an arbitrary increasing sequence $\{T_n\}$ of positive numbers such that $\lim_{n\to\infty} T_n = \infty$, and suppose $0 < e \leq e^*$ with some sufficiently small $e^* > 0$. Then the local weak limit $\lim_{n\to\infty} \mathcal{N}_{T_n} = \mathcal{N}$ exists and is a Gibbs probability measure; the limit does not depend on the particular choice of the sequence $\{T_n\}$. Moreover, the measure $\mathcal{N}$ is invariant with respect to time shifts and time reflections.*

Theorem 2.3 (Uniqueness of the Gibbs measure) *On the domain of uniqueness of the reference measure the Gibbs measure $\mathcal{N}$ is unique in DLR-sense whenever it exists.*

Theorem 2.4 (Typical path behaviour) *For every $0 < e \leq e^*$ there exist a constant $C > 0$ and a function $F : C(\mathbb{R}, \mathbb{R}^d) \to \mathbb{R}$ of the path $Q = \{q_t : t \in \mathbb{R}\}$ such that for all $t \in \mathbb{R}$*

$$|q_t| \;\leq\; (C \log(|t| + 1|))^{1/(s+1)} + F(Q) \tag{2.1}$$

$\mathcal{N}$-almost surely.

Theorem 2.5 (Single-time distributions) *For every $0 < e \leq e^*$ we have that*

1. *the distributions S_T under $\mathcal{N}_T$ of positions q_τ at arbitrary moments of time $\tau \in \mathbb{R}$ are absolutely continuous with respect to $d\mathsf{N}^0 = \psi_0^2 dx$, i.e. there exist $c_1, c_2 \in \mathbb{R}$ such that $c_1 \leq d\mathsf{S}_T/d\mathsf{N}^0(q_\tau) \leq c_2$, $\forall q \in \mathbb{R}^d, T > 0$. Moreover, the limit $d\mathsf{S}/d\mathsf{N}^0(x) = \lim_{T\to\infty} d\mathsf{S}_T/d\mathsf{N}^0(x)$ exists pointwise;*

2. *the conditional distributions $\mathcal{N}_T(\,\cdot\,|q_\tau = q)$ converge locally weakly to $\mathcal{N}(\,\cdot\,|q_\tau = q)$, $\forall q \in \mathbb{R}^d$.*

Theorem 2.6 (Degree of mixing) *For every $0 < e \leq e^*$ and any pair of bounded functions F, G on $\mathbb{R}^d$*

$$\mathbb{E}_{\mathcal{N}}[F(q_s)G(q_t)] - \mathbb{E}_{\mathcal{N}}[F(q_s)]\mathbb{E}_{\mathcal{N}}[G(q_t)] \;\leq\; C\,\frac{\sup|F|\sup|G|}{|s-t|^\gamma + 1} \tag{2.2}$$

with some $1 < \gamma \leq d - 1$, $d \leq 3$, and a suitable constant prefactor $C > 0$.

For proofs and details we refer to [4]. Except for the uniqueness result the proofs are based on cluster expansion. Note that the exponent γ appearing in Theorem 2.6 is expected to be equal to the rate of decay of the potential, i.e. $d - 1$. The statements listed above are actually true for a much larger class of pair interaction potentials than the W appearing in Nelson's model (see [4]) and therefore the results of this section are applicable also to other types of problems.

3 Ground state properties

Under the infrared (henceforth IR) cutoff condition $\int dk|\hat{\varrho}(k)|^2/|k|^3 < \infty$ and the conditions we have drawn on the external potential V, a unique strictly positive ground state Ψ of H_{N} exists in Fock space for all coupling constants [6]. However, Ψ actually exists for also more general V, possibly subject to restrictions on the coupling strength [1, 6].

The main point of this section is that particular ground state expectations (i.e., scalar products in $L^2(\mathbb{R}^d, dx) \times \mathcal{F}$) can be obtained as Gibbs averages of suitable functions related to the observables involved. Moreover, deriving the qualitative properties of the ground state, i.e. obtaining meaningful bounds on these expectations, becomes possible by estimating these Gibbs averages. Here are some of these properties obtained in [3, 5].

Theorem 3.1 (Boson distribution) *Suppose the IR condition holds and the Gibbs measure $\mathcal{N}$ exists. Denote by $\Psi_n = \Psi_n(k_1, ..., k_n, q)$, $n \geq 1$, the basis elements of the Fock space, and write $p_n = \int |\Psi_n(k_1, ..., k_n, q)|^2 dk_1...dk_n dq$ for the probability distribution of n bosons occurring in the ground state. Put $h(\beta) = \sum_{n=0}^{\infty} p_n e^{-\beta n} = (\Psi, e^{-\beta N}\Psi)$, where $\beta \geq 0$ and $N = 1 \otimes \int_{\mathbb{R}^d} a^*(k)a(k)dk$ is the boson number operator. Then*

1. *With W as under (1.6) we have*

$$h(\beta) = \mathbb{E}_{\mathcal{N}}\left[\exp\left((e^{-\beta} - 1)\int_{-\infty}^{0} dt \int_{0}^{\infty} ds\, W(q_t - q_s, t - s)\right)\right] \tag{3.1}$$

   Moreover, $h(\beta)$ has an analytic continuation into the whole complex plane and formula (3.1) extends everywhere. For large n we have asymptotically $p_n \sim b^n/n!$.

2. *The average number of bosons in the ground state is*

$$(\Psi, N\Psi) = -\frac{dh}{d\beta}|_{\beta=0} = \mathbb{E}_{\mathcal{N}}\left[\int_{-\infty}^{0} dt \int_{0}^{\infty} ds\, W(q_t - q_s, t - s)\right] \tag{3.2}$$

   Moreover,

$$(\Psi, a^*(k)a(k)\Psi) \;\leq\; \frac{|\hat{\varrho}(k)|^2}{|k|^3} \tag{3.3}$$

Theorem 3.2 (Average field and field fluctuations) *Suppose the IR condition holds. Then the average field in the ground state is*

$$(\Psi, (a(\varphi) + a^*(\varphi))\Psi) = \mathbb{E}_{\mathcal{N}}\left[\int_{-\infty}^{\infty} dt \int_{\mathbb{R}^d \times \mathbb{R}} \frac{\hat{\varrho}(k)}{\sqrt{|k|}}\hat{\varphi}(k')e^{-|k||t|}\, \cos(k - k')q_t\, dkdk'\right] \tag{3.4}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. If, moreover, φ is concentrated on a point $x \in \mathbb{R}^d$, then for large $|x|$ we have asymptotically $(\Psi, (a(\varphi) + a^*(\varphi))\Psi) \sim 1/|x|$. Furthermore, field fluctuations increase by turning on the particle-field interaction, i.e. $\mathrm{var}_{\mathcal{P}^0}(\cdot, \xi) > \mathrm{var}_{\mathcal{P}}(\cdot, \xi)$.

Theorem 3.3 (Particle localization) *Suppose the ground state* Ψ *exists and denote* $\Lambda(q) = ||\Psi(q,\cdot)||^2_{\mathcal{B}_D} = \int \Psi(q,\xi)^2 d\mathsf{G}(\xi)$. *There exist some constants* $C, \alpha > 0$ *such that*

$$\Lambda(q) \;\leq\; Ce^{-\alpha|q|^{s+1}}, \quad \forall q \in \mathbb{R}^d. \tag{3.5}$$

If $0 < e \leq e^*$, *then there exist some constants* $c_2 \geq c_1 > 0$ *such that* $c_1\psi_0^2 \leq \Lambda \leq c_2\psi_0^2$.

By using the above Gibbs-averages representation of ground state expectations we can show that these bounds stay valid even under removing the IR condition [3].

To conclude we draw a comparison between Fock space quantization and Euclidean quantization. The first is made by taking the Hilbert space $\mathcal{H}^0 = L^2(\mathbb{R}^d \times \mathcal{B}_D, d\mathsf{N}^0 \times d\mathsf{G})$ with Hamiltonian H which is unitarily equivalent with H_{N} and which gives rise to the semigroup e^{-tH} by $(F, e^{-tH}G)_{\mathcal{H}^0} = \mathbb{E}_{\mathcal{P}^0}[F(\xi_0,q_0)G(\xi_t,q_t)e^{-\int_0^t(\xi_s*\varrho)(q_s)ds}]$. In the second we take the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d \times \mathcal{B}_D, d\mathsf{P})$. Then the semigroup T_t associated with the time reversible Markov stochastic process $\{\xi_t, q_t\}$ distributed by $\mathcal{P}$ is defined through $(F, T_t G)_{\mathcal{H}} = \mathbb{E}_{\mathcal{P}}[F(\xi_0,q_0)G(\xi_t,q_t)]$, $\forall t > 0$, and is a symmetric contractive semigroup. Hence there is a self-adjoint semibounded operator H_{euc} generating it by $T_t = e^{-tH_{\mathrm{euc}}}$, which we will view by definition as the Hamiltonian of the system obtained in Euclidean quantization. The function $1 \in \mathcal{H}$ is the ground state of H_{euc}.

Theorem 3.4 (3D infrared divergence) *Take* $d = 3$ *and suppose* $0 < e \leq e^*$. *If* $\int_{\mathbb{R}^d} |\hat{\varrho}(k)|^2/|k|^3 dk = \infty$ *then* H *has no ground state in* $\mathcal{H}^0$. *In particular,* H *and* H_{euc} *are not unitarily equivalent and* H_N *has no ground state in Fock space.*

This means that in this case, unless an infrared cutoff is introduced, there are infinitely many bosons in the ground state which however cannot be accomodated in Fock space. We remark that this does not occur for $d \geq 4$: under sufficiently weak couplings we can find a unitary map $\Gamma : \mathcal{H} \to \mathcal{H}^0$ such that $\Gamma^{-1}(H - E_0)\Gamma = H_{\mathrm{euc}}$, where $E_0 = \inf \operatorname{Spec} H > -\infty$. For further details we refer to [5].

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