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Recent Results on Lieb-Thirring Inequalities

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Abstract

We give a survey of results on the Lieb-Thirring inequalities for the eigenvalue moments of Schrödinger operators. In particular, we discuss the optimal values of the constants therein for higher dimensions. We elaborate on certain generalisations and some open problems as well.

1. Introduction

1. Let $H = H(V;\hbar)$ be the Schrödinger operator

$$H(V;\hbar) = -\hbar^2\Delta - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d).$$

For suitable real-valued potential wells V the negative spectrum $\{\lambda_n(V;\hbar)\}$ of H is semi-bounded from below and discrete. Put $\sigma \geq 0$ and $\hbar > 0$. The Lieb-Thirring inequalities

$$S_{\sigma,d}(V;\hbar) \leq R(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) \tag{1}$$

give, for appropriate pairs of σ and d, bounds on the moments of the negative eigenvalues[1]

$$S_{\sigma,d}(V;\hbar) = \operatorname{tr} H_-^\sigma(V;\hbar) = \sum_n (-\lambda_n(V;\hbar))^\sigma$$

in terms of averages of phase space volume

$$\begin{aligned} S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) &= \int\!\!\int h_-^\sigma(\xi,x)\frac{dx\,d\xi}{(2\pi\hbar)^d} & (2)\\ &= \frac{\Gamma(\sigma+1)}{2^d\pi^{d/2}\Gamma\left(\sigma+\frac{d}{2}+1\right)}\hbar^{-d}\int V_+^{\sigma+\frac{d}{2}}dx & (3)\end{aligned}$$

of the classical system with the correlated Hamilton function $h(\xi,x) = |\xi|^2 - V(x)$. The numerical factor in (3) is called the classical constant

$$L^{\mathrm{cl}}_{\sigma,d} = \frac{\Gamma(\sigma+1)}{2^d\pi^{d/2}\Gamma\left(\sigma+\frac{d}{2}+1\right)}$$

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[1] Here and in the sequel $x_\pm = (|x| \pm x)/2$ denote the positive and negative part of numbers, functions and hermitian matrices or operators.

and the so-called Lieb-Thirring constants $L_{\sigma,d}$ in the inequality

$$\sum_n (-\lambda_n(V;\hbar)) \leq L_{\sigma,d}\hbar^{-d}\int V_+^{\sigma+\frac{d}{2}}dx$$

evaluate as $L_{\sigma,d} = R(\sigma,d)L^{\mathrm{cl}}_{\sigma,d}$.

2. The intrinsic link between spectral quantities and their counterparts in (1) distinguishes these bounds from other variants of spectral estimates. In particlar, the r.h.s. of (1) captures the correct order of the semi-classical Weyl type asymptotics[2]

$$S_{\sigma,d}(V;\hbar) = (1+o(1))S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) \quad \text{as} \quad \hbar \to 0. \tag{4}$$

But in contrast to (4) the inequalities (1) are *uniform* in $\hbar > 0$. This allows one to extract information on the negative spectrum of Schrödinger operators from the classical systems in the *non-asymptotical* regime as well.

3. The parameter $\hbar$ can be scaled out from the bound (1). Up to a few asymptotical arguments we put in the sequel $\hbar = 1$ and drop it from the notation.

2. On the validity of the inequalities (1)

1. In the dimensions $d = 1, 2$ any arbitrary small attractive potential well will couple at least one bound state, see e.g. [4]. Hence, the quantity $S_{0,d}(V)$, being the counting function of the negative spectrum, is a positive integer for any non-trivial $V \geq 0$, while the phase space quantity $S^{\mathrm{cl}}_{0,d}(V)$ can be arbitrary small. This contradicts to (1) for $\sigma = 0$ and $d = 1, 2$.

Moreover, in the dimension $d = 1$ the unique weakly coupled negative bound state behaves as[3] [27]

$$(-\lambda_1(V;\hbar))^{1/2} = (2^{-1} + o(1))\hbar^{-1}\int V dx \quad \text{for} \quad \hbar \to \infty. \tag{5}$$

Hence $S_{\sigma,1}(V;\hbar) = O(\hbar^{-2\sigma})$ for large $\hbar$, while $S^{\mathrm{cl}}_{\sigma,1}(V;\hbar) = O(\hbar^{-1})$. This excludes (1) for $d = 1$ and $0 < \sigma < 1/2$. We conclude that

Fact 1 *The inequality (1) fails for* $0 \leq \sigma < 1/2$ *if* $d = 1$ *and for* $\sigma = 0$ *if* $d = 2$.

2. On the other hand it is known that

Fact 2 *The inequality (1) holds true for* $\sigma \geq 1/2$ *if* $d = 1$, *for* $\sigma > 0$ *if* $d = 2$ *and for* $\sigma \geq 0$ *if* $d \geq 3$.

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[2] This formula can be deduced for all $V \in C_0(\mathbb{R}^d)$. If for given σ and d the bound (1) holds, then (4) can be extended to all $V \in L^{\sigma+\frac{d}{2}}(\mathbb{R}^d)$. If on the other hand (1) fails, formula (4) does not apply to all $V \in L^{\sigma+\frac{d}{2}}(\mathbb{R}^d)$ in general [5].

[3] The potential V should have a positive mean value and $(1+|x|)V(x)$ should be integrable.

Estimates for Riesz means of eigenvalues have first been studied in [22]. There the cases $\sigma > 1/2$ for $d = 1$ and $\sigma > 0$ for $d \geq 2$ were settled. The methods of [22] do not cover the minimal admissible values of σ. In particular, for $\sigma = 0$ and $d \geq 3$ (1) turns into the celebrated Cwikel-Lieb-Rosenblum estimate on the number of bound states

$$S_{0,d}(V) \leq L_{0,d} \int V_+^{d/2} dx = R(0,d) S_{0,d}^{\mathrm{cl}}(V), \quad d \geq 3, \tag{6}$$

which has been shown in [8, 20, 24]. This bound on its turn implies (1) for $\sigma > 0$ and $d \geq 3$. Indeed,

$$\begin{aligned} S_{\sigma,d}(V) &= \int_0^\infty \#\{\lambda_n(V) < -t\} \frac{t^{\sigma-1} dt}{\sigma} = \int_0^\infty S_{0,d}(V-t) \frac{t^{\sigma-1} dt}{\sigma} \\ &\leq R(0,d) \int_0^\infty S_{0,d}^{\mathrm{cl}}(V-t) \frac{t^{\sigma-1} dt}{\sigma} \\ &= R(0,d) \int_0^\infty \left\{ \int \int_{h(\xi,x)<-t} \frac{dx d\xi}{(2\pi)^d} \right\} \frac{t^{\sigma-1} dt}{\sigma} \\ &= R(0,d) S_{\sigma,d}^{\mathrm{cl}}(V). \end{aligned}$$

This is a special case of an argument by Aizenman and Lieb [1], who show that

Fact 3 *If $R(\sigma,d)$ is finite for some d and some $\sigma \geq 0$ then*

$$R(\sigma',d) \leq R(\sigma,d) \quad \textit{for all} \quad \sigma' \geq \sigma. \tag{7}$$

The other remaining case $\sigma = 1/2$ and $d = 1$ has been settled in [28]. Here one finds in fact a *two-sided* estimate

$$S_{1,\frac{1}{2}}^{\mathrm{cl}}(V) \leq S_{1,\frac{1}{2}}(V) \leq 2 S_{1,\frac{1}{2}}^{\mathrm{cl}}(V), \quad V \geq 0,\ V \in L^1(\mathbb{R}). \tag{8}$$

The sharp lower bound in (8) was probably first observed in [12] and the sharp constant in the upper bound has been found in [15]. Comparing weak and strong coupling behaviours it is easy to see, that $\sigma = 1/2$ and $d = 1$ is the only case in the Lieb-Thirring scale, where such a two-sided estimate by the classical phase space average is possible.

3. Let us mention that the bound from below in (8) induces a lower estimate [12]

$$S_{0,2}(V) \geq S_{0,2}^{\mathrm{cl}}(V) \tag{9}$$

for non-negative *spherical symmetric* V in the dimension two. Moreover, for $d = 2$ the negative spectrum of $H(V)$ is infinite for *any* non-negative potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \backslash L^1(\mathbb{R}^2)$, that is whenever $S_{0,2}^{\mathrm{cl}}(V)$ is infinite, see [29]. It seems to be an interesting problem to clarify, in what way (9) can be extended to larger classes of potentials.

3. On the sharp values of the constants $R(\sigma, d)$.

1. While the validity of the bounds (1) is completely settled, the problem on the optimal values of the constants $R(\sigma, d)$ still posts some tantalising riddles. Namely, the inequality (7) shows that the functions $R(\sigma, d)$ are non-increasing in σ for fixed d. This corresponds to the understanding, that the eigenvalue moments $S_{\sigma,d}(V)$ should behave more regular and the constants $R(\sigma, d)$ should actually *improve* for higher values of σ.

On the other hand, all previously known methods of proofs of (1) rely on some initial estimates for $S_{0,d}(V-t)$, which are then modified and integrated to bounds for higher moments. These intermediate bounds for the counting functions do inevitably spoil the final estimates on $R(\sigma, d)$ for higher σ. Therefore, sharp results on $R(\sigma, d)$ require a *direct* approach to the eigenvalue moments. But the Birman-Schwinger principle, that is the technical key element for estimates on counting functions, does not extend to eigenvalue moments. The search for an appropriate detour is the core of the mathematical difficulties in the determination of the values of the constants $R(\sigma, d)$.

2. Below we summarise the available information on this topic and begin with the case $d = 1$. Sharp constants in the dimension one appear already in [22] and [1]. There it has been shown that

$$R(\sigma, 1) = 1 \quad \text{for all} \quad \sigma \geq 3/2. \tag{10}$$

Since the asymptotical behaviour (4) implies that

$$R(\sigma, d) \geq 1$$

for all admissible σ and d, the constants (10) are clearly best possible. The original deduction of (10) uses a *trace identity* for $\sigma = 3/2$ and the monotonicity argument (7). We discussed this more in detail in section 5.

Another case in the dimension $d = 1$ has been settled in [15] with

$$R\,(1/2, 1) = 2. \tag{11}$$

This constant reflects the weak coupling limit behaviour (5). Moreover, if $V(x) = \delta(x)$ then $H(\delta)$ has the unique negative eigenvalue $\lambda_1(\delta) = -1/4$. Up to translation and scaling this is the only potential for which the constant (11) is achieved [15]. The result (11) is based on a monotonicity principle for partial eigenvalue moments of a modified Birman-Schwinger operator.

The optimal values of $R(\sigma, 1)$ for $1/2 < \sigma < 3/2$ are unknown. An analysis of the lowest bound state shows that here

$$R(\sigma, 1) = \sup_{V \in L^{\sigma+\frac{1}{2}}} \frac{S_{\sigma,1}(V)}{S^{\mathrm{cl}}_{\sigma,1}(V)} \geq \sup_{V \in L^{\sigma+\frac{1}{2}}} \frac{(-\lambda_1(V))^{\sigma}}{S^{\mathrm{cl}}_{\sigma,1}(V)} = 2\left(\frac{\sigma - \frac{1}{2}}{\sigma + \frac{1}{2}}\right)^{\sigma - \frac{1}{2}}. \tag{12}$$

The maximising potential is

$$V(x) = (\sigma^2 - 1/4)\cosh^{-2} x.$$

Lieb and Thirring conjectured in [22] that $R(\sigma,1)$ is actually equal to the term in the r.h.s. of (12). The result (11) in conjunction with (7) implies at least $R(\sigma,1)\leq 2$.

3. Until recently only sparse knowledge was available on sharp constants $R(\sigma,d)$ in higher dimensions. The first related result concerns the special case of the eigenvalues $\{\mu_k\}$ of the Dirichlet Laplacian $H^D_\Omega=-\Delta$ in an open domain $\Omega\subset\mathbb{R}^d$. In 1972 Berezin [3] showed that

Fact 4 *For all $\sigma\geq 1$, $\Lambda\geq 0$, $d\in\mathbb{N}$ and any open domain Ω it holds*

$$\sum_k(\mu_k-\Lambda)^\sigma_-\leq\frac{1}{(2\pi)^d}\int_\Omega dx\int_{\mathbb{R}^d}d\xi(|\xi|^2-\Lambda)^\sigma_-=L^{cl}_{\sigma,d}\mathrm{vol}(\Omega)\Lambda^{\sigma+\frac{d}{2}}. \tag{13}$$

Remark 1 Choosing a potential $V_\Omega(x)=\Lambda$ for $x\in\Omega$ and $V_\Omega(x)=-\infty$ for $x\notin\Omega$ the bound (13) can be rewritten as

$$S_{\sigma,d}(V_\Omega)\leq S^{\mathrm{cl}}_{\sigma,d}(V_\Omega),\quad \sigma\geq 1,\, d\in\mathbb{N}, \tag{14}$$

and is a special case of (1). Note that for this class of potentials the constant in (14) is the semi-classical one.

We outline a short proof of (13) from [16]. Let $\{\phi_k\}$ be an ortho-normed eigenfunctions of H^D_Ω, which we continue by 0 on $\mathbb{R}^d\backslash\Omega$. Let $\{\tilde{\phi}_k\}$ be the Fourier transformed of $\{\phi_k\}$, which form an ortho-normed system in $L^2(\mathbb{R}^d)$. Applying Jensen's inequality with respect to the measures $|\tilde{\phi}_k(\xi)|^2d\xi$ on R^{d} we find

$$\begin{aligned}\sum_k(\mu_k-\Lambda)^\sigma_- &= \sum_k\left(\int_{\mathbb{R}^d}(|\xi|^2-\lambda)|\tilde{\phi}_k(\xi)|^2d\xi\right)^\sigma_- &(15)\\ &\leq \int_{\mathbb{R}^d}(|\xi|^2-\Lambda)^\sigma_-\sum_k|\tilde{\phi}_k(\xi)|^2d\xi. &(16)\end{aligned}$$

On the other hand, $\tilde{\phi}_k(\xi)$are the complex conjugates of the Fourier coefficients of $(2\pi)^{-d/2}e^{ix\xi}$ on Ω with respect to $\{\phi_k\}$ in $L^2(\Omega)$. Hence,

$$\sum_k|\tilde{\phi}_k(\xi)|^2=(2\pi)^{-d}\int_\Omega|e^{ix\xi}|^2dx=(2\pi)^{-d}\int_\Omega dx. \tag{17}$$

If we insert (17) into (16) we obtain (13) . □

Let us consider the Legendre transformation [4] of the inequality (13) for $\sigma=1$. It is easy to see that

$$(\sum_k(\mu_k-x)_-)^\wedge(p)=(p-[p])\mu_{[p]+1}+\sum_{k=1}^{[p]}\mu_k$$

while

$$\left(L^{\mathrm{cl}}_{1,d}\mathrm{vol}(\Omega)x^{1+\frac{d}{2}}\right)^\wedge(p)=p^{1+\frac{2}{d}}\left(L^{\mathrm{cl}}_{0,d}\mathrm{vol}(\Omega)\right)^{-\frac{2}{d}}\frac{d}{2+d}.$$

---

[4] We recall that the Legendre transformation $f^\wedge(p)$ of a convex, non-negative function $f(x)$ on $\mathbb{R}_+$ is given by $f^\wedge(p)=\sup_{x\geq 0}(px-f(x))$, $p\geq 0$.

Since $f(x) \leq g(x)$ for all $x \geq 0$ implies $g^\wedge(p) \leq f^\wedge(p)$ for all $p \geq 0$, from (13) with $x = \Lambda$ for $p = n \in \mathbb{N}$ one recovers a result by Li and Yau [19]

$$\sum_{k=1}^{n} \mu_k \geq n^{1+\frac{2}{d}} \left(L_{0,d}^{\mathrm{cl}} \mathrm{vol}(\Omega) \right)^{-\frac{2}{d}} \frac{d}{2+d}.$$

4. The harmonic oscillator is another example which has been studied in connection with Lieb-Thirring inequalities. Put $m = (m_1, \ldots, m_d)$ and

$$V_m(x) = \Lambda - \sum_{k=1}^{d} m_k^2 x_k^2, \quad \Lambda > 0, \quad m_k > 0. \tag{18}$$

Then the operator $H(V_m,)$ has the eigenvalues

$$\lambda_\tau(V_m) = -\Lambda + \sum_{k=1}^{d} m_k(1 + 2\tau_k), \quad \tau = (\tau_1, \ldots, \tau_d),$$

with $\tau_k = 0, 1, 2, \ldots$ In the dimension $d = 1$ for $\sigma = 1$ the classical phase space average equals to

$$S_{1,1}^{\mathrm{cl}}(V_m) = \Lambda^2/(4m_1). \tag{19}$$

On the other hand,

$$\begin{aligned} S_{1,1}(V_m) &= \sum_{k \geq 0} (m_1(1+2k) - \Lambda)_- \\ &= m_1 \left(\Lambda^2 (2m_1)^{-2} - t^2 \right) \leq S_{1,1}^{\mathrm{cl}}(V_m), \end{aligned}$$

where $t = 1 + \left[\frac{\Lambda}{2\hbar m_1} - \frac{1}{2} \right] - \frac{\Lambda}{2\hbar m_1}$. With the Lieb-Aizenman argument we conclude that

$$S_{\sigma,1}(V_m) \leq S_{\sigma,1}^{\mathrm{cl}}(V_m) \quad \text{for all} \quad \sigma \geq 1. \tag{20}$$

A similar evaluation in the d-dimensional case is more involved and has been carried out by De la Bretèche [9]. We present an alternative generalisation to higher dimensions, see [17]. Put $x' = (x_1, \ldots, x_{d-1})$ and $V(x) = W(x') - m_d^2 x_d^2$. Integration in the dth coordinates of x and ξ gives

$$\begin{aligned} S_{\sigma,d}^{\mathrm{cl}}(V) &= \int\!\!\int \left(|\xi|^2 + m_d^2 x_d^2 - W(x') \right)_-^\sigma \frac{dx d\xi}{(2\pi)^d} \\ &= \frac{1}{2(\sigma+1) m_d} S_{\sigma+1,d-1}^{\mathrm{cl}}(W). \end{aligned}$$

Separation of variables implies that $\lambda_{\tau',\tau_d}(V) = \lambda_{\tau'}(W) + m_d(1 + 2m_d)$. Carrying out the sum over $\tau_d \geq 0$ first, from (20) for $\sigma \geq 1$ it follows that

$$\begin{aligned} S_{\sigma,d}(V) &= \sum_{\tau',\tau_d} (\lambda_{\tau'}(W) + m_d(1 + 2\tau_d))_-^\sigma \\ &= \sum_{\tau'} S_{\sigma,1}(\lambda_{\tau'}(W) - m_d^2 x_d^2) \leq \sum_{\tau'} S_{\sigma,1}^{\mathrm{cl}}(\lambda_{\tau'}(W) - m_d^2 x_d^2) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\tau'} \int\int (|\xi|^2 + m_d^2 x_d^2 + \lambda_{\tau'}(W))_-^\sigma \frac{dx_d d\xi_d}{(2\pi)} \\
&\leq \frac{1}{2(\sigma+1)m_d} \sum_{\tau'} (-\lambda_{\tau'}(W))^{\sigma+1} = \frac{1}{2(\sigma+1)m_d} S_{\sigma+1,d-1}.
\end{aligned}
$$

Hence, we have

$$
\frac{S_{\sigma,d}(V)}{S^{\mathrm{cl}}_{\sigma,d}(V)} \leq \frac{S_{\sigma+1,d-1}(W)}{S^{\mathrm{cl}}_{\sigma+1,d-1}(W)}. \tag{21}
$$

This induction procedure gives

Fact 5 *For the harmonic oscillator* $V = \Lambda - \sum_{k=1}^d m_k^2 x_k^2$ *it holds*

$$
S_{\sigma,d}(V) \leq S^{cl}_{\sigma,d}(V) \quad \textit{for all} \quad \sigma \geq 1 \quad \textit{and all} \quad d \in \mathbb{N}. \tag{22}
$$

Remark 2 From Theorem 1 in section 4 it will follow that in fact (22) holds true for all $\sigma \geq 1$, all $d \in \mathbb{N}$ and all potentials of the type $V(x) = W(x_1, \ldots, x_{d-1}) - m_d^2 x_d^2$ for which $S^{\mathrm{cl}}_{\sigma,d}(V)$ is finite.

5. The harmonic oscillator provides also certain counterexamples. In particular, one can show that if $\sigma < 1$ then for certain parameter Λ and m the strict inequality $S_{\sigma,d}(V_m) > S^{\mathrm{cl}}_{\sigma,d}(V_m)$ holds [13], and thus

$$
R(\sigma, d) > 1 \quad \text{for all} \quad \sigma \leq 1, \quad d \in \mathbb{N}. \tag{23}
$$

An analysis of the lowest eigenvalue via the respective Sobolev embedding constant shows, that for $d = 2$ the bound (23) extends to all $\sigma < \sigma_0$ with $\sigma_0 \approx 1.16$ [22].

The methods of [22, 20] give certain explicite upper bounds on the constants $R(\sigma, d)$. In particular, the best known estimates on $R(0, d)$ can be found in [20]. There have been minor improvements for certain cases of higher moments, see e.g. [6].

In [22] Lieb and Thirring posed the

Conjecture 1 *In any dimension* d *there exists a finite critical value* $\sigma_{cr}(d)$*, such that* $R(\sigma, d) = 1$ *for all* $\sigma \geq \sigma_{cr}(d)$*. It is expected that* $\sigma_{cr}(d) = 1$ *for* $d \geq 3$*.*

In the sequel we state our results towards the solution of these conjectures.

4. Lieb-Thirring Inequalities for Operator Valued Potentials

1. Our results are based on the following generalisation of the Lieb-Thirring inequalities (1). Namely, let G be a separable Hilbert space, let 1_G be the identity operator on G and let V be a function on $\mathbb{R}^d$ which takes a.e. compact self-adjoint operators $V(x)$ on G as its values. We shall study the negative eigenvalues $\{\lambda_n(V; \hbar)\}$ of the operator

$$
H(V; \hbar) = -\hbar^2 \Delta \otimes 1_G - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d) \otimes G.
$$

We shall find bounds

$$S_{\sigma,d}(V;\hbar) \le r(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) \tag{24}$$

of the eigenvalue moments

$$S_{\sigma,d}(V;\hbar) = \mathrm{tr}_{L^2(\mathbb{R}^d)\otimes G} H^\sigma_-(V;\hbar) = \sum_n (-\lambda_n(V;\hbar))^\sigma$$

in terms of the classical counterparts

$$S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) = \int\int \mathrm{tr}_G h^\sigma_-(\xi,x)\frac{dx d\xi}{(2\pi\hbar)^d} = L^{\mathrm{cl}}_{\sigma,d}\hbar^{-d}\int \mathrm{tr}_G V_-^{\sigma+\frac{d}{2}}(x)dx,$$

where $h(\xi,x) = |\xi|^2 \otimes 1_G - V(x)$. The constants $r(\sigma,d)$ should not depend on G and (24) should hold whenever the r.h.s. is finite. It is obvious that (1) is a special case of (24) and

$$1 \le R(\sigma,d) \le r(\sigma,d).$$

If not needed we put $\hbar = 1$ and omit it from the notation.

2. In [18] we prove the following result, which confirms the first part of the conjecture by Lieb and Thirring with $\sigma_{\mathrm{cr}} \le 3/2$.

Theorem 1 [A. Laptev, T. Weidl] *The identity*

$$R(\sigma,d) = r(\sigma,d) = 1$$

holds true for all $\sigma \ge 3/2$ and all $d \in \mathbb{N}$.

One of the most interesting case for applications is $\sigma = 1$ and $d = 3$. Here the best know estimate was $R(1,3) \le 5.24$ [6]. In [14] we show

Theorem 2 [D. Hundertmark, A. Laptev, T. Weidl] *The bounds*

$$R(\sigma,d) \le r(\sigma,d) \le \begin{cases} 4 & \text{for } \frac{1}{2} \le \sigma < 1 \\ 2 & \text{for } 1 \le \sigma < \frac{3}{2} \end{cases}$$

hold true in all dimensions $d \in \mathbb{N}$. In particular, if $d = 1$ then

$$R(1/2,1) = r(1/2,1) = 2 \quad \text{for} \quad \sigma = 1/2, \tag{25}$$

$$R(\sigma,1) \le r(\sigma,1) \le 2 \quad \text{for} \quad 1/2 < \sigma < 3/2. \tag{26}$$

Remark 3 The method of [22] extends to systems and shows (24) for $\sigma > 0$ if $d \ge 2$ and for $\sigma > 1/2$ if $d = 1$ with the same upper bounds on the constants $r(\sigma,d)$ as are given there for $R(\sigma,d)$. The validity of (24) in the case $d \ge 3$, $\sigma = 0$ has not been settled yet.

It turns out that $R(\sigma,d) = r(\sigma,d)$ in all cases, where the sharp values of these constants are known. We formulate

Conjecture 2 *The bound (24) holds for all pairs σ, d for which (1) holds, and the optimal values of the constants $R(\sigma,d)$ and $r(\sigma,d)$ coincide.*

3. We sketch now the proof of Theorem 1. First we establish the bound (24) with the identity $r(\sigma,1)=1$ for $d=1$ and $\sigma\geq 3/2$. By [1] it suffices to study the case $\sigma=3/2$. Moreover, by a density argument one can reduce the problem to finite-dimensional Hilbert spaces G and smooth, compactly supported matrix functions V. We apply then a generalisation of trace formulae [7, 11] to matrix valued potentials [18]. Some more details will be given in section 5.

Recently Benguria and Loss found an independent proof of this special case based on the Darboux transformation [2].

4. In the second stage of the proof we apply an iteration with respect to the spatial dimension d. Namely, a standard variational argument implies that

$$\begin{aligned} S_{\sigma,d}(V) &= \operatorname{tr}_{L^2(\mathbb{R}^d)\otimes G}\left(-\frac{\partial^2}{\partial x_d^2}\otimes 1_G-(\Delta'+V(x',x_d))\right)_-^\sigma \\ &\leq \operatorname{tr}_{L^2(\mathbb{R})\otimes\tilde G}\left(-\frac{d^2}{dx_d^2}\otimes 1_{\tilde G}-W_-(x_d)\right), \end{aligned}$$

where $x'=(x_1,\ldots,x_{d-1})$, Δ' is the Laplacian in the coordinates x' and $W(x_d)$ is the operator $W(x_d)=-\Delta'\otimes 1_G-V(x';x_d)$ on $\tilde G=L^2(\mathbb{R}^{d-1})\otimes G$ with the frozen coordinate parameter x_d. Put $\sigma\geq 1$. We apply (24) for $d=1$ with $r(\sigma,1)=1$ and the internal Hilbert space $\tilde G$ and find

$$\begin{aligned} S_{\sigma,d}(V) &\leq L^{\mathrm{cl}}_{\sigma,1}\int \operatorname{tr}_{\tilde G} W_-^{\sigma+\frac12}(x_d)\,dx_d \\ &\leq L^{\mathrm{cl}}_{\sigma,1}\int S_{\sigma+\frac12,d-1}(V(\cdot;x_d))\,dx_d. \end{aligned}$$

We continue this induction and find in the final step with $\tilde x=(x_2,\ldots,x_d)$

$$\begin{aligned} S_{\sigma,d}(V) &\leq \prod_{k=0}^{d-2} L^{\mathrm{cl}}_{\sigma+\frac{k}{2},1}\int S_{\sigma+\frac{d-1}{2},1}(V(\cdot;\tilde x))\,d\tilde x \\ &\leq \prod_{k=0}^{d-1} L^{\mathrm{cl}}_{\sigma+\frac{k}{2},1}\int \operatorname{tr}_G V_+^{\sigma+\frac{d}{2}}(x)\,dx. \end{aligned}$$

Since $L^{\mathrm{cl}}_{\sigma,d}=\prod_{k=0}^{d-1}L^{\mathrm{cl}}_{\sigma+\frac{k}{2},1}$ and $S^{\mathrm{cl}}_{\sigma,d}(V)=L^{\mathrm{cl}}_{\sigma,d}\int\operatorname{tr}_G V_+^{\sigma+\frac{d}{2}}dx$, we find that $S_{\sigma,d}(V)\leq S^{\mathrm{cl}}_{\sigma,d}(V)$. □

5. Trace formulae and further estimates

1. Put $G=\mathbb{C}^n$ and consider the system of ordinary differential equations

$$-\left(d^2/dx^2\otimes 1_G\right)y(x)-V(x)y(x)=k^2y(x),\quad x\in\mathbb{R}. \tag{27}$$

Here V is a compactly supported, smooth (not necessary sign definite) Hermitian matrix-valued function. Define $x_{\min}=\min\operatorname{supp}V$ and $x_{\max}=\max\operatorname{supp}V$. Then for any $k\in\mathbb{C}\backslash\{0\}$ there exist unique $n\times n$ matrix-solutions $y(x)=F(x,k)$ and $y(x)=G(x,k)$ of (27) satisfying

$$F(x,k)=e^{ikx}1_G \quad\text{as}\quad x\geq x_{\max},\qquad G(x,k)=e^{-ikx}1_G\quad\text{as}\quad x\leq x_{\min}.$$

The pairs of matrices $F(x,k)$, $F(x,-k)$ and $G(x,k)$, $G(x,-k)$ form complete systems of independent solutions of (27). Hence the matrix $F(x,k)$ can be expressed as a linear combination of $G(x,k)$ and $G(x,-k)$

$$F(x,k) = G(x,k)B(k) + G(x,-k)A(k). \tag{28}$$

The matrix functions $A(k)$ and $B(k)$ are uniquely defined by (28).

2. The Buslaev-Faddeev-Zakharov [7] trace formulae can be generalised to matrix-valued potentials [18]. The first three identities read as follows

$$\begin{aligned} L^{\mathrm{cl}}_{\frac{1}{2},1} \int \mathrm{tr}_G V dx &= S_{\frac{1}{2},1}(V) - I_0 && (29)\\ L^{\mathrm{cl}}_{\frac{3}{2},1} \int \mathrm{tr}_G V^2 dx &= S_{\frac{3}{2},1}(V) + 3I_2 && (30)\\ L^{\mathrm{cl}}_{\frac{5}{2},1} \int \mathrm{tr}_G V^3 dx - J &= S_{\frac{5}{2},1}(V) - 5I_4 && (31) \end{aligned}$$

where $J = \frac{1}{2} L^{\mathrm{cl}}_{\frac{5}{2},1} \int \mathrm{tr}_G \left(dV/dx\right)^2 dx$ and

$$I_j = (2\pi)^{-1} \int k^j \ln |\det A(k)| dk, \quad j = 0, 2, 4.$$

For $k \in \mathbb{R}$ it holds $A(k)A^*(k) = 1_G + B(-k)B^*(-k)$ and $|\det A(k)| \geq 1$. Hence $I_j \geq 0$. If we drop the term $3I_2$ from (30) we immediately find (24) with $r(3/2,1) = 1$.

Similarly (29) and (25) lead to the lower bound in

$$L^{\mathrm{cl}}_{\frac{1}{2},1} \int \mathrm{tr}_G V(x) dx \leq S_{\frac{1}{2},1}(V) \leq 2L^{\mathrm{cl}}_{\frac{1}{2},1} \int \mathrm{tr}_G V_+(x) dx. \tag{32}$$

3. We put now $V \geq 0$. The upper bound in (32) and (29) imply

$$I_0 = S_{\frac{1}{2},1}(V) - L^{\mathrm{cl}}_{\frac{1}{2},1} \int \mathrm{tr}_G V dx \leq L^{\mathrm{cl}}_{\frac{1}{2},1} \int \mathrm{tr}_G V dx.$$

Moreover, from (24) for $d = 1$, $\gamma = 5/2$, $r(5/2,1) = 1$ and (31) it follows that

$$5I_4 = S_{\frac{5}{2},1}(V) - L^{\mathrm{cl}}_{\frac{5}{2},1} \int \mathrm{tr}_G V^3 dx + J \leq J.$$

For the scalar case the last inequalities was found in [22].

We apply now Hölder's inequality $I_2^2 \leq I_0 I_4$ and insert the resulting estimate on I_2 back into (30). In view of (24) for $d = 1$, $\sigma = 3/2$ and $r(3/2,1) = 1$, after rescaling $\hbar$ back into the estimate we find that

$$0 \leq S^{\mathrm{cl}}_{\frac{3}{2},1}(V;\hbar) - S_{\frac{3}{2},1}(V;\hbar) \leq R(V) \tag{33}$$

with

$$R(V) = \frac{3}{16} \sqrt{\int \mathrm{tr}_G V dx} \cdot \sqrt{\int \mathrm{tr}_G (\frac{dV}{dx})^2 dx}.$$

The term on the r.h.s. of (33) does not depend on $\hbar$, while the quantities $S^{\mathrm{cl}}_{\frac{3}{2},1}(V;\hbar)$ and $S_{\frac{3}{2},1}(V;\hbar)$ show the asymptotical order $O(\hbar^{-1})$ as $\hbar \to 0$. Hence, the inequality (33) provides an uniform estimate on the remainder term to the Weyl asymptotics for $S_{\frac{3}{2},1}(V;\hbar)$ for sign-defined perturbations. By continuity it extends to sign-defined operator-valued potentials on infinite-dimensional Hilbert spaces G, for which the terms $R(V)$ and $S_{\frac{3}{2},1}(V)$ are finite.

6. Polyharmonic operators

1. Another natural generalisation of (1) are Lieb-Thirring inequalities for the operators

$$H_l(V) = (-\Delta)^l - V(x), \quad l \in \mathbb{N}$$

on $L^2(\mathbb{R}^d)$. Let $\{\lambda_n(V)\}_n$ be the negative eigenvalues of $H_l(V)$. We study the inequalities

$$S_{\sigma,d,l}(V) \le R(\sigma,d,l) S^{\mathrm{cl}}_{\sigma,d,l}(V), \tag{34}$$

where $S_{\sigma,d,l}(V) = \sum_n (-\lambda_n(V))^\sigma$ and

$$S^{\mathrm{cl}}_{\sigma,d,l}(V) = \int\int (h_l(\xi,x))^\sigma_- \frac{dx d\xi}{(2\pi)^d} = L^{\mathrm{cl}}_{\sigma,d,l} \int V_+^{1+\kappa} dx$$

with

$$L^{\mathrm{cl}}_{\sigma,d,l} = \frac{\Gamma(\sigma+1)\Gamma(\kappa+1)}{2^d \pi^{d/2} \Gamma\left(l\kappa+1\right)\Gamma(\kappa+\sigma+1)}$$

and $\kappa = d/2l$.

2. The validity of (34) is settled by

Fact 6 *The inequality (34) holds true if and only if*

$$\begin{array}{lll} \sigma \ge 0 & \textit{for} & \kappa > 1, \\ \sigma > 0 & \textit{for} & \kappa = 1, \\ \sigma \ge 1-\kappa & \textit{for} & \kappa < 1. \end{array}$$

The case $\sigma = 0$ for $\kappa > 1$ has been settled in [8, 24]. In particular, the techniques of [8] apply to non-integer l as well. Using the Lieb-Aizenman trick we can then raise (34) to all $\sigma \ge 0$. For $\sigma > 1-\kappa$ with $\kappa \le 1$ one can easily adapt the approach of Lieb and Thirring [22], see also [10]. These methods do also extend to non-integer values of l.

The critical case for *integer* values of l has been solved in [23]. By analogy with $\sigma = 1/2$ for $l = d = 1$ we have a two-sided estimate

$$\tilde{L}_{1-\kappa,d,l} \hbar^{-d} \int V dx \le S_{1-\kappa,d,l}(V;\hbar) \le L_{1-\kappa,d,l} \hbar^{-d} \int V_+ dx \tag{35}$$

with appropriate positive, finite constants $\tilde{L}_{1-\kappa,d,l}$ and $L_{1-\kappa,d,l}$. For non-integer values of l the validity of (35) has not been settled yet. Comparing the asymptotical behaviour for $S_{\sigma,d,l}(V;\hbar)$ as $\hbar \to 0, \infty$, we see that $\sigma = 1-\kappa > 0$ is the only power, for which a two-sided bound by the phase space average might exist.

Counterexamples to (34) in the corresponding cases can be found in the limit $\hbar \to \infty$. Now one might have a family of weak coupling states, but the contribution of the lowest one is leading. This analysis leads to the

Conjecture 3 *We have*

$$\tilde{L}_{1-\kappa,d,l} = L^{cl}_{1-\kappa,d,l} \quad \text{and} \quad L_{1-\kappa,d,l} = \frac{\pi\kappa}{\sin \pi\kappa} L^{cl}_{1-\kappa,d,l} \quad \text{for} \quad \kappa < 1.$$

3. Constants in Lieb-Thirring inequalities for higher order operators are much less studied than their counterparts for $l = 1$. No sharp values of the constants are known, not even in the dimension $d = 1$. It is also not clear, whether the bound (35) extends to operator-valued potentials. A more detailed investigation of Lieb-Thirring bounds for general l might pay off with new insights for the special but most interesting case $l = 1$.

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