An inversion theorem in Fermi surface theory

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Abstract

We prove a perturbative inversion theorem for the map between the interacting and the noninteracting Fermi surface for a class of many fermion systems with strictly convex Fermi surfaces and short-range interactions between the fermions. This theorem gives a physical meaning to the counterterm function K that we use in the renormalization of these models: K can be identified as that part of the self–energy that causes the deformation of the Fermi surface when the interaction is turned on.

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1 Introduction

The Fermi surface is an important feature of the quantum field theory of solid state models. Besides being central to the theoretical analysis of such models it is also important from a conceptual point of view. In experiments, one observes and measures the Fermi surface of an interacting system (for brevity, we call this the *interacting Fermi surface*) – or more precisely, an approximation to it due to positive temperature effects, because the electrons interact with each other (say via a screened Coulomb interaction, phonons and so on). On the other hand, the theoretical analysis usually starts from a model of noninteracting electrons, moving in a crystal background, which exhibits the *noninteracting* Fermi surface. The effects of the electron–electron interaction are taken into account by 'turning on a coupling constant'. Thus, while the model of independent electrons exists only theoretically, important notions of solid state physics, for instance Fermi liquid theory, start from it and then incorporate the changes in the system caused by the interaction. One of these is a change in the dispersion relation, that gives the energy of a particle as a function of momentum. This results in the transformation of the Fermi surface from the noninteracting to the interacting one.

In this paper, we complete our perturbative analysis of the regularity properties of interacting nonspherical Fermi surfaces by proving an inversion theorem for the map between the interacting and the free dispersion relation that we used in the renormalization of these models. The main ingredients in the inversion theorem are an abstract iteration theorem that generalizes the usual contraction mapping theorem (which is not sufficient here) and a number of regularity estimates. The estimates are used to verify the hypotheses of this iteration theorem. The regularity estimates are an application of the methods and the results of [2], [3], and [4], referred to as `I`, `II`, and `III` in the following.

By 'perturbative analysis' we mean that the perturbation series is truncated at any finite order R (which may be arbitrarily large) in the coupling constant λ. There are situations where this expansion can be proven to converge, so that the limit $R \to \infty$ exists, but we do not give such bounds here.

In the remainder of this introduction, we define our class of models and state the inversion theorem. For a more detailed motivation, see the introductory sections of `I` and `II`.

1.1 The models

Let Γ be a nondegenerate lattice in $\mathbb{R}^d$ and

$$\Gamma^{\#} = \{\mathbf{b} \in \mathbb{R}^d \ : \ \mathbf{b} \cdot \gamma \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Gamma\} \tag{1}$$

its dual lattice. We denote the first Brillouin zone by $\mathcal{B}$ and choose it to be the d-dimensional torus $\mathcal{B} = \mathbb{R}^d/\Gamma^{\#}$. It is compact. For example, if $\Gamma = \mathbb{Z}^d$, then $\Gamma^{\#} = 2\pi\mathbb{Z}^d$ and $\mathcal{B} = \mathbb{R}^d/2\pi\mathbb{Z}^d$. We are interested in a class of models characterized by an action $\mathcal{A}(\psi, \bar\psi)$ that is a function of two variables $\psi = \big(\psi_{k,\sigma}\big)_{k\in\mathbb{R}\times\mathcal{B},\sigma\in\{\uparrow,\downarrow\}}$ and $\bar\psi = \big(\bar\psi_{k,\sigma}\big)_{k\in\mathbb{R}\times\mathcal{B},\sigma\in\{\uparrow,\downarrow\}}$. Note that $\bar\psi$ is not the complex conjugate of ψ. It is just another vector that is totally independent of ψ. The zero component k_0 of k is usually thought as an energy, the final d components $\mathbf{k}$ as (crystal) momenta and σ as a spin. There really should also be a sum over a band index n, but it will not play a role here and has been suppressed. In these models, the quantities one measures are represented by other functions $f(\psi,\bar\psi)$ of the same two vectors and the value of the observable $f(\psi,\bar\psi)$ in the model with action $\mathcal{A}(\psi,\bar\psi)$ is given formally by the ratio of integrals

$$\left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}} = \frac{\int f(\psi,\bar\psi)\, e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}} \tag{2}$$

The integrals are fermionic functional integrals. That is, linear functionals on a Grassmann algebra.

A typical action of interest is that corresponding to a gas of electrons, of strictly positive density, interacting through a two–body potential $u(\mathbf{x}-\mathbf{y})$. It is

$$\begin{aligned}\mathcal{A}_{\mu,\lambda} \;=\;& -\textstyle\sum_{\sigma\in\{\uparrow,\downarrow\}} \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \tfrac{d^{d+1}k}{(2\pi)^{d+1}}\, \big(ik_0 - \big(\tfrac{\mathbf{k}^2}{2m}-\mu\big)\big)\bar\psi_{k,\sigma}\psi_{k,\sigma} \\ & -\tfrac{\lambda}{2}\textstyle\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \textstyle\prod_{i=1}^4 \tfrac{d^{d+1}k_i}{(2\pi)^{d+1}} (2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4) \\ & \qquad \bar\psi_{k_1,\sigma}\psi_{k_3,\sigma}\hat u(\mathbf{k}_1-\mathbf{k}_3)\bar\psi_{k_2,\tau}\psi_{k_4,\tau}\end{aligned} \tag{3}$$

Here $\frac{\mathbf{k}^2}{2m}$ is the kinetic energy of an electron, μ is the chemical potential, which controls the density of the gas, and $\hat u$ is the Fourier transform of the two–body interaction. The coupling constant λ is assumed to be small, so that the interaction is weak.

More generally, when the electron gas is subject to a periodic potential due to the crystal lattice, Γ, and when the electrons are interacting with the motion of the crystal lattice through the mediation of harmonic phonons, the action is of the form

$$\begin{aligned}\mathcal{A}_{\lambda} \;=\;& -\textstyle\sum_{\sigma\in\{\uparrow,\downarrow\}} \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \tfrac{d^{d+1}k}{(2\pi)^{d+1}}\, \big(ik_0 - E(\mathbf{k})\big)\bar\psi_{k,\sigma}\psi_{k,\sigma} \\ & -\tfrac{\lambda}{2}\textstyle\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \textstyle\prod_{i=1}^4 \tfrac{d^{d+1}k_i}{(2\pi)^{d+1}} (2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4) \\ & \qquad \bar\psi_{k_1,\sigma}\psi_{k_3,\sigma}\hat v(k_{1,0}-k_{3,0},\mathbf{k}_1-\mathbf{k}_3)\bar\psi_{k_2,\tau}\psi_{k_4,\tau}\end{aligned} \tag{4}$$

where $E(\mathbf{k})$ is the dispersion relation minus the chemical potential μ.

1.2 The class of dispersion relations

Let $\mathcal{F}$ be a fundamental cell for the action of the translation group $\Gamma^\#$. In other words, $\mathcal{F}$ is an open set in $\mathbb{R}^d$ with the property that it together with its translates under $\Gamma^\#$ are dense in $\mathbb{R}^d$. For example, if $\Gamma=\mathbb{Z}^d$, then $\Gamma^\# = 2\pi\mathbb{Z}^d$

and we may choose $\mathcal{F} = (-\pi, \pi)^d$. Let $\mathcal{F}_2 = \{\mathbf{p} \in \mathcal{F} : 2\mathbf{p} \in \mathcal{F}\}$. For a continuous function E from $\mathcal{B}$ to $\mathbb{R}$ let

$$S_E = \{\mathbf{p} \in \mathcal{B} : E(\mathbf{p}) = 0\} \tag{5}$$

be the corresponding Fermi surface and $\mathcal{I}_E = \{\mathbf{p} \in \mathcal{B} : E(\mathbf{p}) < 0\}$ the corresponding Fermi sea. For $k \geq 2$ let

$$C_s^k(\mathcal{B}, \mathbb{R}) = \{E \in C^k(\mathcal{B}, \mathbb{R}) : E(-\mathbf{p}) = E(\mathbf{p}) \text{ for all } \mathbf{p} \in \mathcal{B}\} \tag{6}$$

With the norm $|f|_k = \sum_{|\alpha| \leq k} \|D^\alpha f\|_\infty$, it is a Banach space. For $E \in C_s^k(\mathcal{B}, \mathbb{R})$, let $B_\varepsilon^{(k)}(E) = \{e \in C_s^k(\mathcal{B}, \mathbb{R}) : |e - E|_k < \varepsilon\}$.

For positive constants $\delta_0, g_0, G_0, \omega_0$, let $\mathcal{E}_s = \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ be the set of all $E \in C_s^2(\mathcal{B}, \mathbb{R})$ that satisfy the following conditions

(i) $S_E \subset \mathcal{F}_2,\ \mathcal{I}_E \neq \emptyset,\ \mathcal{I}_E \neq \mathcal{B},\ d(S_E, \partial\mathcal{F}_2) > \delta_0,$

(ii) $|\nabla E(\mathbf{p})| > g_0$ for all $\mathbf{p} \in S_E$:

(iii) $|E|_2 < G_0$

(iv) $(\mathbf{t}(\mathbf{p}), E''(\mathbf{p})\mathbf{t}(\mathbf{p})) > \omega_0$ for all $\mathbf{p} \in S_E$ and all unit vectors $\mathbf{t}(\mathbf{p})$ tangent to S_E at $\mathbf{p}$

Since E is C^2, the condition that $\nabla E \neq 0$ on S_E implies that the Fermi surface S_E is a $(d-1)$–dimensional C^2-submanifold of $\mathcal{B}$, (in $d = 2$, the 'surface' is a curve). The condition $(\mathbf{t}(\mathbf{p}), E''(\mathbf{p})\mathbf{t}(\mathbf{p})) > \omega_0$ implies that S_E has strictly positive curvature everywhere.

The set $\mathcal{E}_s$ is open in $(C_s^2(\mathcal{B}, \mathbb{R}), |\cdot|_2)$. In this paper, we fix any $\delta_0 > 0,\ g_0 > 0,\ \omega_0 > 0$ and $G_0 > \max\{g_0, \omega_0\}$.

1.3 The class of interactions

We also define the class $\mathcal{V}$ of allowed interactions to be the set of all functions V, whose Fourier transforms $\hat{v}(p_0, \mathbf{p})$ obey

(i) $|\hat{v}|_2 \leq 1$

(ii) $\hat{v}(-p_0, \mathbf{p}) = \overline{\hat{v}(p_0, \mathbf{p})}$

(iii) $\hat{v}(p_0, -\mathbf{p}) = \hat{v}(p_0, \mathbf{p})$

(iv) There is a bounded function $\tilde{v} \in C^2(\mathcal{B}, \mathbb{R})$ and an $\alpha > 0$ such that

$$\limsup_{p_0 \to \infty} |p_0|^\alpha \sup_{\mathbf{p}} \left|\hat{v}(p_0, \mathbf{p}) - \tilde{v}(\mathbf{p})\right| < \infty$$

Condition (iv) is used only in the large k_0 regime. If an ultraviolet cutoff is placed on k_0, it may be omitted. Condition (i) implies that the interaction in momentum space, $\hat{v}$, is in $C^2(\mathbb{R}^{d+1})$. This is the case if the position space integral kernel $V(x-y)$ is bounded by $\frac{\text{const}}{1+|x-y|^{d+3+\varepsilon}}$ for some $\varepsilon > 0$. The 1 in the condition $|\hat{v}|_2 \leq 1$ is not a restriction, since V and λ appear only in the combination λV in the definition of the model, so a rescaling of V can be absorbed by a rescaling of λ.

1.4 The counterterm function

In `I`, we constructed a counterterm function K as a formal power series in λ,

$$K(e, \lambda V, \mathbf{p}) = \sum_{r=1}^{\infty} \lambda^r K_r(e, V, \mathbf{p}) \tag{7}$$

where $K_r : \mathcal{D} \times \mathcal{V} \times \mathcal{B} \to \mathbb{R}$ is defined for a set $\mathcal{D}$ of dispersion relations e with $\mathcal{E}_s \subset \mathcal{D}$. The conditions required for having a finite K_r for all r are much weaker than the conditions we impose here (see `I` and Section 4). K is constructed such that, for a model with action

$$\begin{aligned} \mathcal{A}_\lambda \;=\; & -\textstyle\sum_\sigma \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \; \big(ik_0 - e(\mathbf{k}) - K(e, \lambda V, \mathbf{k})\big) \bar{\psi}_{k,\sigma} \psi_{k,\sigma} \\ & -\tfrac{\lambda}{2} \textstyle\sum_{\sigma,\tau} \displaystyle\int_{\mathbb{R}\times\mathcal{B}} \textstyle\prod_{i=1}^{4} \tfrac{d^{d+1}k_i}{(2\pi)^{d+1}} (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \\ & \qquad\qquad \bar{\psi}_{k_1,\sigma} \psi_{k_3,\sigma} \hat{v}(k_{1,0} - k_{3,0}, \mathbf{k}_1 - \mathbf{k}_3) \bar{\psi}_{k_2,\tau} \psi_{k_4,\tau} \end{aligned} \tag{8}$$

the Fermi surface of the interacting model is fixed to S_e, independently of λ. The function $K(\mathbf{p})$ is real–valued, and under the symmetry hypotheses made here, $K(-\mathbf{p}) = K(\mathbf{p})$. By introducing the counterterm function, we removed the infrared divergences to all orders in the perturbation expansion in powers of λ. That is, when the expansion is truncated at any finite order R, all Green functions are finite almost everywhere. We showed in `I` that the counterterm function to any order R in λ,

$$K^{(R)}(e, \lambda V) = \sum_{r=1}^{R} \lambda^r K_r(e, V), \tag{9}$$

is differentiable in $\mathbf{p}$ and e (and, of course, C^∞ in λ since it is a polynomial for any finite R).

Thus a model that has an action whose quartic part (in the fields) is that corresponding to V and whose quadratic part is that corresponding to a dispersion relation E will have an interacting Fermi surface that is the zero set of a dispersion relation e if

$$e + K(e, \lambda V) = E. \tag{10}$$

In this paper we take a given E and V and solve

$$e + K^{(R)}(e, \lambda V) = E. \tag{11}$$

for $e = e^{(R)}(E, \lambda V)$. The dispersion relation e that appears in the propagator is only an auxiliary quantity, which is to be determined by (11). We shall solve (11) by iteration, starting from the given E. Clearly this requires having bounds with uniform constants on a set of dispersion relations that is mapped to itself by the function $\mathbb{1} + K^{(R)}$.

We proved in `I–III` that the following estimate holds (Theorem `III`.3.13). For all $r \geq 1$, there are constants $\kappa_r > 0$ such that, for all $e \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ and $V \in \mathcal{V}$, the contribution $K_r(e, V)$ is in $C_s^2(\mathcal{B}, \mathbb{R})$ and obeys

$$|K_r(e, V)|_2 \leq \kappa_r. \tag{12}$$

The constant κ_r depends only on $(\delta_0, g_0, G_0, \omega_0)$ and r. Consequently, $K^{(R)}$ satisfies

$$\left| K^{(R)}(e, \lambda V) \right|_2 \leq \sum_{r=1}^{R} |\lambda|^r \kappa_r \tag{13}$$

so $|K^{(R)}(e, \lambda V)|_2$ can be made arbitrarily small by decreasing λ. Because $\mathcal{E}_s$ is open in $|\cdot|_2$, $e + K^{(R)}(e, \lambda V) \in \mathcal{E}_s$ if $e \in \mathcal{E}_s$ and λ is sufficiently small.

1.5 The inversion theorem

To show that an iteration scheme for the solution converges, we need to have bounds for the distance between successive elements of the iteration sequence. For technical reasons that have nothing to do with the analysis of `I–III` and that will be explained later, we have to restrict to dispersion relations that have certain third order derivatives bounded, in order to control the distance between successive iterates. This is the reason why, in the following theorem, the starting E_0 is required to be in C^3.

Theorem 1 *Let* $\delta_0, g_0, \omega_0 > 0$, $G_0 > \max\{g_0, \omega_0\}$ *and* $R \in \mathbb{N}$. *Then there is a* $\lambda_R > 0$ *such that for each* $|\lambda| \leq \lambda_R$, *each* $E \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0) \cap C^3(\mathcal{B}, \mathbb{R})$ *and each* $V \in \mathcal{V}$, *there is a unique* $\mathrm{e}^{(R)} \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2)$ *solving* (11). *Moreover, there is a constant* $A_R > 0$ *such that*

$$\left| e^{(R)} - E \right|_2 \leq A_R\, |\lambda|. \tag{14}$$

Theorem 1 follows from the more detailed Theorem 2 below. We shall discuss the more detailed theorems about inversion in Section 5.

In this paper, we do not prove optimal bounds about the R–dependence of λ_R. For the models at hand, in particular because of the symmetry $E(\mathbf{p}) = E(-\mathbf{p})$, one expects that convergence does not hold at zero temperature. That is, one expects $\lambda_R \to 0$ as $R \to \infty$. The reason for this is that at temperatures below a critical temperature, the ground state of the system is superconducting, in which case the above perturbation expansion cannot converge. As noted in [6], at a positive temperature $T = \frac{1}{\beta} > 0$, one can expect convergence of the expansion for coupling constants λ in the region where $\lambda \log \beta$ is small enough, that is, for $T \geq T_0 \mathrm{e}^{-\lambda_0/|\lambda|}$ where λ_0 and T_0 are fixed constants (see [6] for a Fermi liquid criterion based on this convergence). For $d = 2$, a proof of this may be possible using the techniques of [7]. The bounds derived here do not change in an essential way at positive temperature. So a variant of our theorems can be expected to hold in this convergent positive temperature regime. Note, however that convergence of the expansion for K does not imply that the solution of the inversion equation can be expanded in λ. In fact, it can't. See [5, 8] for an informal explanation.

2 Preliminaries

2.1 Coordinates

Since e is going to change under the iteration, it is convenient to use momentum space coordinates that are independent of e. Under our assumptions, we can simply use polar coordinates in addition to the Fermi surface coordinates that we used in I–III. We shall review the latter shortly. It will be important that the angular variables θ are the same in both coordinate systems. Only the radial coordinate is different.

Polar coordinates: Consider a small ball B around an $E_0 \in \mathcal{E}_s$. Regard a small neighbourhood of the Fermi surface S_{E_0} as a subset of $\mathbb{R}^d$ instead of the torus $\mathcal{B}$ and introduce polar coordinates $(r,\theta) \in \mathbb{R}_0^+ \times S^{d-1}$, $\mathbf{p} = \mathbf{p}(r,\theta)$. For $d=2$, $\theta \in S^1$. In polar coordinates, the Fermi surface can be parametrized, for $e \in B$, as

$$S_e = \{\mathbf{p}(r_F(e,\theta),\theta) : \theta \in S^{d-1}\} \tag{15}$$

with $r_F : B \times S^{d-1} \to \mathbb{R}^+$. If $e \in C^k(\mathcal{B},\mathbb{R})$, then $r_F \in C^k(S^{d-1},\mathbb{R}^+)$.

Lemma 1 *Let $0 < k \le K$. Let S be a $(d-1)$–dimensional C^2 convex surface in $\mathbb{R}^d$ all of whose principal curvatures are between k and K. Let $\mathbf{c}_1, \mathbf{c}_2$ be any two maximally separated points of S. That is, $\mathbf{c}_1, \mathbf{c}_2 \in S$ with*

$$\|\mathbf{c}_1 - \mathbf{c}_2\| = \max\big\{\|\mathbf{p}_1 - \mathbf{p}_2\| \;:\; \mathbf{p}_1, \mathbf{p}_2 \in S\big\} \tag{16}$$

Set $\mathbf{c} = \frac{1}{2}\big(\mathbf{c}_1 + \mathbf{c}_2\big)$. Then, for every $\mathbf{p} \in S$,

$$\tfrac{1}{K} \le \|\mathbf{p} - \mathbf{c}\| \le \tfrac{1}{k} \tag{17}$$

and the angle $\theta(\mathbf{p})$ between $\mathbf{p} - \mathbf{c}$ and the outward pointing normal vector $\mathbf{n}(\mathbf{p})$ to S at $\mathbf{p}$ obeys

$$\cos(\theta(\mathbf{p})) \ge \tfrac{k}{K} \tag{18}$$

If, in addition, $-\mathbf{p} \in S$ for every $\mathbf{p} \in S$, then $\mathbf{c}$ is the origin.

Proof: See Appendix A ■

Lemma 2 *Let $\delta_0, g_0, \omega_0 > 0$ and $G_0 > \max\{g_0, \omega_0\}$. There are ε, r_0, $g_1 > 0$ such that, for every $E_0 \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ and every $e \in B_\varepsilon^{(2)}(E_0)$*

$$e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2) \tag{19}$$

and

$$\begin{aligned} \big|r_F(e,\theta) - r_F(E_0,\theta)\big| &\le r_0 && \text{for all} \quad \theta \in S^{d-1} && (20)\\ \tfrac{\partial}{\partial r} e(\mathbf{p}(r,\theta)) &> g_1 && \text{for all} \quad \big|r - r_F(E_0,\theta)\big| \le 2r_0,\ \theta \in S^{d-1} && (21) \end{aligned}$$

Note that the constants ε, r_0 and g_1 are independent of E_0.

Proof: See Appendix B ■

Let E_0, r_0 and ε be as in Lemma 2. Set

$$\begin{aligned} \underline{R}(\theta) &= r_F(E_0,\theta) - 2r_0 \\ \overline{R}(\theta) &= r_F(E_0,\theta) + 2r_0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} A &= \{(r,\theta) : \theta \in S^{d-1}, \underline{R}(\theta) < r < \overline{R}(\theta)\} \\ \tilde{A} &= \{\mathbf{p}(r,\theta) : (r,\theta) \in A\} \end{aligned} \tag{23}$$

Then, $S_e = \{\mathbf{p} : e(\mathbf{p}) = 0\} \subset \tilde{A}$ for all $e \in B_\varepsilon^{(2)}(E_0)$.

We use the notation $\tilde{F}(r,\theta) = F(\mathbf{p}(r,\theta))$ for functions in terms of the variables r and θ, for instance, $\tilde{e}(r,\theta) = e(\mathbf{p}(r,\theta))$. The above Lemma then states that for all $(r,\theta) \in A$, and all $e \in B_\varepsilon^{(2)}(E_0)$, $\partial_r \tilde{e}(r,\theta) > g_1 > 0$.

We could have introduced coordinates in the annulus $\tilde{A}$, based on any vector field that is transversal to S_{E_0}. This would only have changed the constant in the lower bound for $\partial_r e$.

Fermi surface coordinates: These are the coordinates used in `I–III`. They are the polar coordinate θ and $\rho = e(\mathbf{p})$, and thus obviously depend on e. We denote the corresponding inverse map, whose range is a neighbourhood of E_0's Fermi surface, by $\boldsymbol{\pi}$:

$$\boldsymbol{\pi} : (-\rho_0, \rho_0) \times S^{d-1} \to \mathcal{B}, \quad (\rho,\theta) \mapsto \boldsymbol{\pi}(\rho,\theta) \tag{24}$$

Clearly $e(\boldsymbol{\pi}(\rho,\theta)) = \rho$.

The projection to the Fermi surface is obtained by setting $\rho = 0$. In terms of the polar coordinates, it is constructed as follows. If $\tilde{F} : B_\varepsilon^{(2)}(E_0) \times A \to \mathbb{C}$ maps $(e,r,\theta) \mapsto \tilde{F}(e,r,\theta)$, then $\ell_e \tilde{F}(e,r,\theta) = \tilde{F}(e, r_F(e,\theta), \theta)$. Obviously, $\partial_r(\ell_e F) = 0$. Observe that $\boldsymbol{\pi}(0,\theta) = \mathbf{p}(r_F(e,\theta),\theta)$.

2.2 Norms

Let $\|\cdot\|_k$ be the seminorm $\|F\|_k = \sum\limits_{|\alpha|=k} \sup\limits_p |\partial^\alpha F(p)|$ and

$$|F|_k = \sum_{l=0}^{k} \|F\|_l. \tag{25}$$

It does not matter whether we use the norm in Cartesian or polar coordinates since the two are equivalent.

We define the *radial* norms for $p \geq 1$ as

$$|F|_{p,r} = |F|_{p-1} + \left\|\partial_r \tilde{F}\right\|_{p-1} \tag{26}$$

and denote the angular norms for $p \geq 0$ as $|F|_{p,\theta}$. In the latter norms, all derivatives are taken in the θ–directions.

Lemma 3 *1.* $|F|_{p,r} \le |F|_{p+1}$.

2. For all $e \in B_\varepsilon^{(2)}(E_0)$, $\partial_r \ell_e F = 0$, *and*

$$|\ell_e F|_{p,r} = |\ell_e F|_{p-1} = |\ell_e F|_{p-1,\theta}. \tag{27}$$

3.

$$\begin{aligned} |FG|_p &\le 2^p\, |F|_p\, |G|_p, \qquad (28)\\ |FG|_p &\le \|F\|_0 \|G\|_p + \|F\|_p \|G\|_0 + 2^{p+1} |F|_{p-1} |G|_{p-1}, \end{aligned}$$

4.

$$\begin{aligned} |FG|_{p+1,r} &\le 2^{p+2} |F|_p |G|_p + \|\partial_r F\|_p \|G\|_0 + \|F\|_0 \|\partial_r G\|_p \\ &\le 2^{p+2} \left(|F|_{p+1,r} |G|_p + |F|_p |G|_{p+1,r} \right). \qquad (29) \end{aligned}$$

Proof: The first statement is an immediate consequence of $\|\partial_r F\|_p \le \|F\|_{p+1}$. The second statement is an immediate consequence of the observation that the localization map ℓ_e does not depend on r. For the third and fourth statements, use the Leibniz rule and that $\prod \binom{\alpha_k}{\beta_k} \le \binom{p}{q}$ for all $\alpha_1 + \ldots + \alpha_n = p$ and $\beta_1 + \ldots + \beta_n = q$ (all nonnegative), to prove that

$$\|FG\|_p \le \sum_{q=0}^{p} \binom{p}{q} \|F\|_q \|G\|_{p-q} \tag{30}$$

and

$$\|\partial_r (FG)\|_p \le \sum_{q=0}^{p} \binom{p}{q} \left(\|\partial_r F\|_q \|G\|_{p-q} + \|F\|_{p-q} \|\partial_r G\|_q \right). \tag{31}$$

■

3 The iteration

Given e_0, e_1, and $t \in [0,1]$, denote $e_t = (1-t)e_0 + te_1$.

Theorem 2 *Let* $\delta_0, g_0, \omega_0 > 0$ *and* $G_0 > \max\{g_0, \omega_0\}$*. Let* $\varepsilon > 0$ *be as in Lemma 2.*

1. **Regularity.** *For each* $R \in \mathbb{N}$*, there is a constant* $D \geq 1$ *such that for all* $|\lambda| \leq 1$*, all* $V \in \mathcal{V}$ *and all* $e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2)$

$$\left|K^{(R)}(e)\right|_{3,r} = \left|K^{(R)}(e)\right|_2 < D|\lambda|. \tag{32}$$

2. **Norm bounds for the iteration.** *There is* $0 < \delta < 1$ *(independent of* $\delta_0, g_0, G_0, \omega_0$*) and, for each* $R \in \mathbb{N}$*, there are constants* $Q_0, Q_1 \geq 1$ *such that for all* $|\lambda| \leq 1$*, all* $V \in \mathcal{V}$*, all* $E \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ *and all* e_0 *and* $e_1 \in B^{(2)}_\varepsilon(E) \cap C^3$

$$\begin{aligned}
\left|K^{(R)}(e_1) - K^{(R)}(e_0)\right|_0 &\leq Q_0|\lambda|\,|e_1 - e_0|_0 && (33)\\
\left|K^{(R)}(e_1) - K^{(R)}(e_0)\right|_1 &\leq Q_0|\lambda|\Big[|e_1 - e_0|_0^\delta + |e_1 - e_0|_1\Big] && (34)\\
\left|K^{(R)}(e_1) - K^{(R)}(e_0)\right|_{3,r} &\leq Q_0|\lambda|\Big[|e_1 - e_0|_1^\delta + |e_1 - e_0|_2\Big] &&\\
&+ Q_1|\lambda| \sup_{0\leq t\leq 1} |e_t|_{3,r}\, |e_1 - e_0|_0. && (35)
\end{aligned}$$

In addition, if $V_1, V_2 \in \mathcal{V}$ *with* $|V_1 - V_2|_2 \leq 1$ *and* $e \in \mathcal{E}_s$ *then*

$$\left|K^{(R)}(e, \lambda V_1) - K^{(R)}(e, \lambda V_2)\right|_2 \leq Q_0|\lambda|\,|V_1 - V_2|_2. \tag{36}$$

3. **Existence of a unique solution to the inversion equation.** *Let* $E \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ *with* $|E|_{3,r} = G_3 < \infty$*. (This is the case if, for example,* $E \in \mathcal{E}_s \cap C^3$*). Set* $Q = \max\{Q_0 + Q_1(1 + G_3), D\}$*. Let*

$$B_{\mathrm{rad}} = \{e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2) : |e - E|_2 < \varepsilon, |e - E|_{3,r} < 1\} \tag{37}$$

and let $\lambda_R > 0$ *be such that* $Q\lambda_R < \min\{1, \varepsilon\}$*. Then for all* $|\lambda| \leq \lambda_R$ *and all* $V \in \mathcal{V}$*, there is a unique* $e^{(R)} \in B_{\mathrm{rad}}$ *such that* $E = e^{(R)} + K^{(R)}(e^{(R)}, \lambda V)$*. Moreover*

$$\left|e^{(R)} - E\right|_{3,r} \leq D\,|\lambda|. \tag{38}$$

4. **Continuity in E and V.** *Let* $E, E' \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0)$ *satisfy* $|E|_{3,r}$*,* $|E'|_{3,r} \leq G_3$ *and* $|E - E'|_{3,r} < \varepsilon/2$*. Then, for all* $|\lambda| \leq \lambda_R/2$ *and all* $V, V' \in \mathcal{V}$ *with* $|V - V'|_2 \leq 1$*,*

$$\begin{aligned}
&\left|e^{(R)}(E, \lambda V) - e^{(R)}(E', \lambda V')\right|_2\\
&\quad\leq 4\Big(|E - E'|_2 + |E - E'|_1^\delta + |E - E'|_0^{\delta^2} + |V - V'|_2^{\delta^2}\Big). \qquad (39)
\end{aligned}$$

Proof: Part 1 was proven in `I–III`: equation (32) follows directly from (13). The bound (33) follows from Theorem `I.3.5` by summation over $r \in \{1, \dots R\}$. We reexplain that argument briefly in the proof of Theorem 4 (Section 4.3). We shall shortly prove the remaining statements of part 2 from the more detailed estimates given in Theorem 3.

To prove part 3, fix R, let $V \in \mathcal{V}$, and denote for brevity $K(E) = K^{(R)}(E, \lambda V)$ and $B = B_{\rm rad}$. Define $\Phi : B \to C^2_s(\mathcal{B}, \mathbb{R})$ by

$$\Phi(e) = E - K(e). \tag{40}$$

By (32) and the hypothesis on λ_R,

$$|\Phi(e) - E|_{3,r} = |K(e)|_{3,r} \le D|\lambda| \le Q\lambda_R < \min\{\varepsilon, 1\} < \varepsilon \tag{41}$$

so $\Phi(B) \subset B$. Thus the sequence $(e_n)_{n \ge 0}$ given by $e_0 = E$ and $e_{n+1} = \Phi(e_n)$ is well-defined. For $n \ge 1$, let $f_n = e_n - e_{n-1}$. Then $f_1 = -K(E)$, $e_n = E + \sum_{k=1}^n f_k$, and

$$f_{n+1} = \Phi(e_n) - \Phi(e_{n-1}) = K(e_{n-1}) - K(e_n). \tag{42}$$

Let $|\lambda| \le \lambda_R$. We show that, for all $n \ge 1$,

$$|f_n|_0 \le (Q|\lambda|)^n \tag{43}$$

$$|f_n|_1 \le B_R(\lambda)\ (Q|\lambda|)^{n\delta} \tag{44}$$

$$|f_n|_{3,r} \le C_R(\lambda)\ \max\{B_R(\lambda)^\delta, 1\}\ (Q|\lambda|)^{n\delta^2} \tag{45}$$

with

$$B_R(\lambda) = \frac{(Q|\lambda|)^{1-\delta}}{1 - (Q|\lambda|)^{1-\delta}}, \qquad C_R(\lambda) = \frac{(Q|\lambda|)^{1-\delta^2}}{1 - (Q|\lambda|)^{1-\delta^2}}. \tag{46}$$

Once this is done, (45) implies that $\sum f_n$ converges in $|\cdot|_{3,r}$. Thus $e^{(R)} = \lim_{n\to\infty} e_n$ exists. By (35), Φ is continuous in $|\cdot|_{3,r}$, so $\Phi(e^{(R)}) = e^{(R)}$ and hence, by (40), $E = e^{(R)} + K(e^{(R)})$. By (41), every e_n obeys $|e_n - E|_{3,r} \le D|\lambda|$, so $e^{(R)}$ satisfies (38). Since $Q_0|\lambda| < 1$, uniqueness follows from (33).

We prove (43)–(46) by induction on n. The statements are true for $n = 1$ because

$$|f_1|_0 \le |f_1|_1 \le |f_1|_{3,r} = |K(E)|_{3,r} \le D|\lambda| \le Q|\lambda| \tag{47}$$

and

$$\begin{aligned} (Q|\lambda|)^\delta B_R(\lambda) &= \frac{Q|\lambda|}{1 - (Q|\lambda|)^{1-\delta}} > Q|\lambda|, \\ (Q|\lambda|)^{\delta^2} C_R(\lambda) &= \frac{Q|\lambda|}{1 - (Q|\lambda|)^{1-\delta^2}} > Q|\lambda|. \end{aligned} \tag{48}$$

Assume (43)–(46) to hold for n. By (42), (33), and the inductive hypothesis (43),

$$|f_{n+1}|_0 = |K(e_n) - K(e_{n-1})|_0 \le Q_0|\lambda|\,|f_n|_0 \le Q_0|\lambda|\,(Q|\lambda|)^n \le (Q|\lambda|)^{n+1} \tag{49}$$

which proves (43) for $n+1$.

By (42), (34), (43) and the inductive hypothesis (44),

$$\begin{aligned}
|f_{n+1}|_1 &= |K(e_n)-K(e_{n-1})|_1 \\
&\le Q_0|\lambda|\left[|f_n|_0^\delta + |f_n|_1\right] \\
&\le Q|\lambda|\left[(Q|\lambda|)^{n\delta} + B_R(\lambda)\,(Q|\lambda|)^{n\delta}\right] \\
&= \left[(Q|\lambda|)^{1-\delta} + B_R(\lambda)\,(Q|\lambda|)^{1-\delta}\right](Q|\lambda|)^{(n+1)\delta}
\end{aligned} \tag{50}$$

Thus the induction goes through for (44) if

$$(Q|\lambda|)^{1-\delta} + B_R(\lambda)\,(Q|\lambda|)^{1-\delta} \le B_R(\lambda) \tag{51}$$

With the definition (46), equality holds in (51).

By (42), (33), (44) and the inductive hypothesis (45),

$$\begin{aligned}
|f_{n+1}|_{3,r} &= |K(e_n)-K(e_{n-1})|_{3,r} = |K(e_n)-K(e_{n-1})|_2 \\
&\le Q_0|\lambda||f_n|_1^\delta + Q_0|\lambda||f_n|_2 + Q_1(1+G_3)|\lambda||f_n|_0 \\
&\le Q|\lambda|\left[|f_n|_1^\delta + |f_n|_2\right] \qquad (52) \\
&\le Q|\lambda|\left[B_R(\lambda)^\delta\,(Q|\lambda|)^{n\delta^2} + C_R(\lambda)\max\{B_R(\lambda)^\delta,1\}\,(Q|\lambda|)^{n\delta^2}\right] \\
&\le \max\{B_R(\lambda)^\delta,1\}\left[(Q|\lambda|)^{1-\delta^2} + (Q|\lambda|)^{1-\delta^2}\,C_R(\lambda)\right](Q|\lambda|)^{(n+1)\delta^2}
\end{aligned}$$

Here we used that $e_n \in B$ implies $|e_n|_{3,r} \le 1 + |E|_{3,r} \le 1 + G_3$. Thus the induction goes through for (45) if

$$(Q|\lambda|)^{1-\delta^2} + (Q|\lambda|)^{1-\delta^2}\,C_R(\lambda) \le C_R(\lambda) \tag{53}$$

With the definition (46), equality holds in (53). This completes the proof of part 3.

We now prove part 4. Denote for brevity $e = e^{(R)}(E,\lambda V)$, $e' = e^{(R)}(E',\lambda V')$ and $K = K^{(R)}$. First, observe that both $e, e' \in B_{\text{rad}} \subset B_\varepsilon^{(2)}(E)$ because, by part 3,

$$\begin{aligned}
|e-E|_2 &\le D|\lambda| \le D\lambda_R/2 < \varepsilon/2 \\
|e'-E|_2 &\le |e'-E'|_2 + |E-E'|_2 \le D\lambda_R/2 + \varepsilon/2 < \varepsilon,
\end{aligned} \tag{54}$$

and because $|e-E|_{3,r} \le D|\lambda| < 1$ and $|e'-E|_{3,r} \le D|\lambda| + \varepsilon/2 < 1$ hold by (38). Thus $\max\{|e|_{3,r}, |e'|_{3,r}\} \le 1 + G_3$.

By definition, e and e' obey $E = e + K(e,\lambda V)$ and $E' = e' + K(e',\lambda V')$. Hence

$$E - E' = e - e' + K(e,\lambda V) - K(e',\lambda V') \tag{55}$$

$$= e - e' + K(e,\lambda V) - K(e',\lambda V) + K(e',\lambda V) - K(e',\lambda V')$$

so that, by (33) and (36),

$$|e-e'|_0 \le |E-E'|_0 + Q_0|\lambda|\,|e-e'|_0 + Q_0|\lambda|\,|V-V'|_2 \tag{56}$$

Recalling that $Q_0|\lambda| \le \frac{1}{2}Q\lambda_R < \frac{1}{2}$,

$$|e-e'|_0 \le 2\Big(|E-E'|_0 + \tfrac{1}{2}\,|V-V'|_2\Big) \le 2|E-E'|_0 + |V-V'|_2 \tag{57}$$

Similarly, by (34) and (36),

$$|e-e'|_1 \le |E-E'|_1 + Q_0|\lambda|\,\Big[|e-e'|_0^\delta + |e-e'|_1\Big] + Q_0|\lambda|\,|V-V'|_2 \tag{58}$$

and

$$\begin{aligned} |e-e'|_1 &\le 2\Big(|E-E'|_1 + \tfrac{1}{2}\,|e-e'|_0^\delta + \tfrac{1}{2}\,|V-V'|_2\Big) \\ &\le 2\Big(|E-E'|_1 + \tfrac{2^\delta}{2}\,|E-E'|_0^\delta + \tfrac{1}{2}|V-V'|_2^\delta + \tfrac{1}{2}\,|V-V'|_2\Big) \\ &\le 2\Big(|E-E'|_1 + |E-E'|_0^\delta + |V-V'|_2^\delta\Big) \end{aligned} \tag{59}$$

Similarly, by (33) and (36),

$$\begin{aligned} |e-e'|_2 &\le |E-E'|_2 + Q_0|\lambda|\,\Big[|e-e'|_1^\delta + |e-e'|_2\Big] \\ &+ Q_1|\lambda|\,(1+G_3)|e-e'|_0 + Q_0|\lambda|\,|V-V'|_2 \end{aligned} \tag{60}$$

and

$$\begin{aligned} |e-e'|_2 &\le 2\Big(|E-E'|_2 + \tfrac{1}{2}\,|e-e'|_1^\delta + \tfrac{1}{2}|e-e'|_0 + \tfrac{1}{2}\,|V-V'|_2\Big) \\ &\le 2\Big(|E-E'|_2 + \tfrac{2^\delta}{2}\,\Big[|E-E'|_1^\delta + |E-E'|_0^{\delta^2} + |V-V'|_2^{\delta^2}\Big] \\ &\qquad + \tfrac{1}{2}\Big[2|E-E'|_0 + |V-V'|_2\Big] + \tfrac{1}{2}\,|V-V'|_2\Big) \\ &\le 2\Big(|E-E'|_2 + |E-E'|_1^\delta + 2|E-E'|_0^{\delta^2} + 2|V-V'|_2^{\delta^2}\Big) \end{aligned} \tag{61}$$

$\blacksquare$

Part 2 of Theorem 2 is proven by a multiscale analysis in which the function $K^{(R)}(E,\lambda V)$ is represented as an infinite series

$$K^{(R)}(E,\lambda V) = \sum_{j<0} K_j^{(R)}(E,\lambda V), \tag{62}$$

where, very roughly speaking, $K_j^{(R)}$ is the contribution from integrating out those fermions that have an energy in the interval $[M^{j-1}, M^j]$. Here $M > 1$ and $j < 0$, so the limit $j \to \infty$ corresponds to momenta on the Fermi surface.

Theorem 3 *Let* $\delta_0, g_0, \omega_0 > 0$ *and* $G_0 > \max\{g_0, \omega_0\}$. *Let* $\varepsilon > 0$ *be as in Lemma 2. Let* $E_0 \in \mathcal{E}_s(\delta_0, g_0, G_0, \omega_0) \cap C^3$ *and let*

$$B_{\rm rad} = \{e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2) : |e - E_0|_2 < \varepsilon, |e - E_0|_{3,r} < 1\}. \tag{63}$$

There is a $0 < \gamma < 1$ *such that, for each* $R \in \mathbb{N}$, *there is* $Q_2 > 0$ *(*Q_2 *is uniform on* $\mathcal{E}_s$*!) such that for all* $e_0, e_1 \in B_{\rm rad}$ *and all* $j < 0$,

$$\left|K_j^{(R)}(e)\right|_{3,r} = \left|K_j^{(R)}(e)\right|_2 \le Q_2 |\lambda| M^{\gamma j} \tag{64}$$

$$\left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_1 \le Q_2|\lambda| \Big(M^{-1.1\,j} |e_1 - e_0|_0 + M^{\gamma j} |e_1 - e_0|_1 \Big) \tag{65}$$

and

$$\begin{aligned} \left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_{3,r} &= \left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_2 \qquad (66)\\ \le Q_2|\lambda| &\left(M^{-2.1\,j} |e_1 - e_0|_1 + M^{\gamma j} \sup_{t \in [0,1]} |e_t|_{3,r} |e_1 - e_0|_0 + M^{\gamma j} |e_1 - e_0|_2 \right) \end{aligned}$$

Moreover, for all $e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2)$ *and all* $V_1, V_2 \in \mathcal{V}$,

$$\left|K_j^{(R)}(e, \lambda V_1) - K_j^{(R)}(e, \lambda V_2)\right|_2 \le Q_2 |\lambda| M^{\gamma j} \; |V_1 - V_2|_2 \tag{67}$$

The proof of Theorem 3 is given in the next section. The factors $M^{-1.1j}$ and $M^{-2.1j}$ come from bounds of the type $M^{-j}|j|^\alpha \le$ const$(\alpha)\, M^{-1.1j}$.

Proof of parts 1 and 2 of Theorem 2: Eq. (64) implies (32) when summed over j, with $D = Q_2 \frac{M^{-\gamma}}{1 - M^{-\gamma}}$. Eq. (33) was proven in `I` (Theorem `I`.3.5). Again by summation, (67) implies continuity in the interaction V.

Denote, for brevity, $K_j(e) = K_j^{(R)}(e, \lambda V)$. To prove (34), with $\delta = \frac{\gamma}{3}$, we split the sum over j in two parts. If j is such that $|e_1 - e_0|_0 \le M^{2j}$, then the inequality

$$|e_1 - e_0|_0 \le \left(M^{2j}\right)^{1 - \gamma/3} |e_1 - e_0|_0{}^{\gamma/3} \tag{68}$$

implies, by (65),

$$\begin{aligned} |K_j(e_1) - K_j(e_0)|_1 &\le Q_2|\lambda| \left(M^{(0.9 - 2\gamma/3)j} |e_1 - e_0|_0^{\gamma/3} + M^{\gamma j} |e_1 - e_0|_1 \right) \\ &\le Q_2|\lambda| \left(M^{0.2\,j} |e_1 - e_0|_0^{\gamma/3} + M^{\gamma j} |e_1 - e_0|_1 \right) \qquad (69) \end{aligned}$$

and hence

$$\sum_{\substack{j \le 0 \\ |e_1 - e_0|_0 \le M^{2j}}} |K_j(e_1) - K_j(e_0)|_1 \le Q_3 |\lambda| \left(|e_1 - e_0|_0^{\gamma/3} + |e_1 - e_0|_1 \right) \tag{70}$$

with $Q_3 = Q_2 \frac{1}{1-M^{-\gamma'}}$, where $\gamma' = \min\{0.2, \gamma/3\}$. If j is such that $|e_1 - e_0|_0 > M^{2j}$, then $|e_1 - e_0|_0^{-\gamma/3} \le M^{-2\gamma j/3}$ and therefore, by (64),

$$\frac{|K_j(e_1) - K_j(e_0)|_1}{|e_1 - e_0|_0^{\gamma/3}} \le 2M^{-2\gamma j/3} \max_{p=1,2}\{|K_j(e_p)|_1\} \le 2Q_2|\lambda| M^{\gamma j/3} \tag{71}$$

so

$$\sum_{\substack{j\le 0\\ |e_1-e_0|_0 > M^{2j}}} |K_j(e_1) - K_j(e_0)|_1 \le 2Q_3|\lambda|\,|e_1 - e_0|_0^{\gamma/3}. \tag{72}$$

To prove (35), with $\delta = \frac{\gamma}{4}$, we split the sum over j at $|e_1 - e_0|_1 = M^{3j}$. This time, writing $S_3 = \sup_{t\in[0,1]} |e_t|_{3,r}$, and using

$$|e_1 - e_0|_1 \le \left(M^{3j}\right)^{(1-\gamma/4)} |e_1 - e_0|_1^{\gamma/4} \tag{73}$$

when $|e_1 - e_0|_1 \le M^{3j}$ gives, by (66), for the j with $|e_1 - e_0|_1 \le M^{3j}$,

$$\begin{aligned} &|K_j(e_1) - K_j(e_0)|_{3,r} \\ &\quad\le\ Q_2|\lambda|\Big(M^{-2.1\,j}|e_1 - e_0|_1 + M^{\gamma j} S_3 |e_1 - e_0|_0 + M^{\gamma j}|e_1 - e_0|_2\Big) \\ &\quad\le\ Q_2|\lambda|\Big(M^{(0.9-3\gamma/4)j}|e_1 - e_0|_1^{\gamma/4} + M^{\gamma j} S_3 |e_1 - e_0|_0 + M^{\gamma j}|e_1 - e_0|_2\Big) \\ &\quad\le\ Q_2|\lambda|\Big(M^{0.15\,j}|e_1 - e_0|_1^{\gamma/4} + M^{\gamma j} S_3 |e_1 - e_0|_0 + M^{\gamma j}|e_1 - e_0|_2\Big) \end{aligned} \tag{74}$$

so

$$\begin{aligned} &\sum_{\substack{j\le 0\\ |e_1-e_0|_1 \le M^{3j}}} |K_j(e_1) - K_j(e_0)|_{3,r} \\ &\qquad\le\ Q_4|\lambda|\,\Big(|e_1 - e_0|_1^{\gamma/4} + S_3|e_1 - e_0|_0 + |e_1 - e_0|_2\Big) \end{aligned} \tag{75}$$

with $Q_4 = Q_2 \frac{1}{1-M^{-\gamma'}}$, where $\gamma' = \min\{0.15, \gamma/4\}$. If j is such that $|e_1 - e_0|_1 > M^{3j}$, then $|e_1 - e_0|_1^{-\gamma/4} \le M^{-3\gamma j/4}$ and therefore, by (64),

$$\frac{|K_j(e_1) - K_j(e_0)|_{3,r}}{|e_1 - e_0|_1^{\gamma/4}} \le 2M^{-3\gamma j/4} \max_{p=1,2}\{|K_j(e_p)|_{3,r}\} \le 2Q_2|\lambda| M^{\gamma j/4} \tag{76}$$

so

$$\sum_{\substack{j\le 0\\ |e_1-e_0|_1 > M^{3j}}} |K_j(e_1) - K_j(e_0)|_{3,r} \le 2Q_4|\lambda|\,|e_1 - e_0|_1^{\gamma/4}. \tag{77}$$

■

4 Bounds with scales – proof of Theorem 3

The counterterm is the localization of a selfenergy function,

$$K_j^{(R)}(e, \lambda V, \mathbf{p}) = \ell_e Y_j^{(R)}(e, \lambda V, p_0, \mathbf{p}). \tag{78}$$

The renormalized tree expansion gives $Y_j^{(R)}$ explicitly as

$$Y_j^{(R)}(e, \lambda V, p) = -\sum_{r=1}^{R} \lambda^r \sum_G \sum_{T \sim G} \prod_{f \in T} \frac{1}{n_f!} \sum_{J \in \mathcal{J}(T,j,G)} \mathrm{Val}\,(G^J)(p) \tag{79}$$

where G is summed over all one–particle irreducible (1PI) Feynman graphs with two external legs and r interaction vertices. We now briefly describe the genesis of this formula as well as the meaning of T, J, $\mathcal{J}(T,j,G)$ and $\mathrm{Val}\,(G^J)$. For the details, see, e.g., [2].

The formula is generated by successive applications of renormalization group maps, as follows (for details, see Section 2.3 of `I`). The covariance corresponding to the quadratic part of the action is expressed as an infinite sum $C = \sum_{j<0} C_j$, where the *single–scale covariance*, C_j, is supported in the subset of $\mathbb{R} \times \mathcal{B}$ where $M^{j-2} \le |\mathrm{i}p_0 - e(\mathbf{p})| \le M^j$ (see Section 2.1 of `I`). An infrared cutoff $I < 0$ is introduced by restricting the sum to $j \ge I$. Correspondingly, the Gaussian integral with the cutoff covariance is expressed as an $|I|$–fold integral

$$\int f(\varphi)\; d\mu_{\Sigma_{0<j\le I} C_j}(\varphi) = \int \cdots \int f\big(\textstyle\sum_{j=1}^{I} \varphi_j\big)\; \prod_{j=1}^{I} d\mu_{C_j}(\varphi_j) \tag{80}$$

with respect to the Gaussian measures of covariance $C_1, \; \cdots, \; C_I$. Fields with lower and lower energy scales are integrated out one scale after the other. The Gaussian integral with covariance C_j generates an *effective interaction* on scale j. The integral kernels of the effective action on scale j are given by a sum of values of Feynman graphs whose vertex functions are the integral kernels of the effective action on scale j and whose propagators are C_j.

The kernel of the part of the effective interaction on scale j that is quadratic in the fields is renormalized by subtracting from it the part of the counterterm whose value is ℓ_e applied to the kernel. The renormalized two–legged kernel is called an r–fork of scale j. The remaining part of the counterterm is the sum of all c–forks of scale j. See Section 2.3 of `I`.

The structure of the iteration is represented by GN (Gallavotti–Nicolò) trees in a natural way. Each graph G contributing to the effective interaction at scale j has associated to it a GN tree, T. Each fork, f, in the tree represents a connected subgraph G_f of G. The subgraph was introduced as a vertex contributing to the effective interaction of some scale j_f. Hence each fork of T carries a label, j_f, giving its scale and, if G_f is two–legged, a label specifying it as an r–fork or a c–fork. The fork of T corresponding to the entire graph G is called the root of T and its scale, j, the root scale of T. The lines of T give the partial ordering of the forks of T induced by the partial ordering of subgraphs of G by inclusion. If $\pi(f)$ is the fork immediately below f in the partial ordering of T, then

$$\begin{array}{ll} I \le j_f \le j_{\pi(f)} & \text{if } \pi(f) \text{ is a c–fork} \\ 1 \ge j_f > j_{\pi(f)} & \text{otherwise} \end{array} \tag{81}$$

The labelling J of G assigns a scale $0 < j_l \le I$ to every line l of G and a scale $0 < j_f \le I$ to every fork f of T. The set $\mathcal{J}(T, j, G)$ is the set of labellings determined by the requirements that (a) the root scale is j, (b) (81) is satisfied and (c) if G_f is the smallest of the subgraphs $G_{f'}$, $f' \in T$ that contain the line l, then $j_l = j_f$.

The value $\mathrm{Val}\,(G^J)(p)$ of a Feynman graph is the integral over momenta of the integrand which is a product of propagators associated to the lines and vertex functions associated to the vertices (see (I.2.54)). For now, the propagators are given by the covariances C_j. Later we shall combine strings of two–legged graphs into single lines, and thereby get more general propagators on the lines.

For each r, the coefficient of λ^r is a sum of only finitely many terms. Thus most perturbative questions can be reduced to bounding values of individual graphs. In some of our estimates in I, however, we also needed to avoid termwise bounds; this will also play a role in this paper.

It was shown in I that under general conditions, the limit

$$K^{(R)}(e, \lambda V, \mathbf{p}) = \lim_{I \to -\infty} \sum_{I \le j < 0} \ell_e Y_j^{(R)}(e, \lambda V, \mathbf{p}) \tag{82}$$

exists and is C^1 in $\mathbf{p}$ and Frèchet differentiable in e.

4.1 Proof of (64) and (67)

Eq. (64) is just a restatement of (III.3.110) in Theorem III.3.11. Because the function $\lambda_n(j, \varepsilon)$ in (III.3.110) is bounded by a constant times a power of $|j|$ by Lemma I.2.44 (v), any $\gamma < 1/3$ will do.

To see (67), we note that the value of any graph G contributing to $K_j^{(R)}$ in (79) contains a product of factors V associated to the vertices. The localization operator ℓ_e does not depend on V, and the expression (79) is linear in $\mathrm{Val}\,(G)$. Let G be a graph contributing to $K_j^{(R)}$. By the discrete product rule (II.3.126), the corresponding graph contributing to the difference on the left hand side of (67) has a difference $V_1 - V_2$ instead of V in one factor. Because all that happens to the vertex functions in the proofs is that they get differentiated (at most twice), and because the estimate is linear in each vertex function, (67) follows trivially from the proofs in I–III.

4.2 Weaker hypotheses for the proof of (33), (65), and (66)

The bounds (33), (65), and (66) hold under much weaker hypotheses than those stated in Theorem 3. In this section, we prove them under hypotheses that are only slightly stronger than those of `I`. In particular, we shall need neither convexity nor symmetry under $\mathbf{p} \to -\mathbf{p}$ nor the requirement that the Fermi surface be small in the sense that $S_E \subset \mathcal{F}_2$. In fact, it need not even be connected.

Let

> $\mathcal{N} \subset \mathcal{B}$ be an open set whose boundary has finitely many connected components, each of which is a C^∞ $(d-1)$–dimensional submanifold of $\mathcal{B}$
>
> u be a unit C^∞ vector field on a neighbourhood of the closure of $\mathcal{N}$ that is transverse to the boundary of $\mathcal{N}$
>
> $e_0, e_1 \in C^0(\mathcal{B}, \mathbb{R}) \cap C^2(\mathcal{N}, \mathbb{R})$

We assume that there are constants $\delta_0, u_0, Q_{\mathrm{vol}}, \gamma > 0$ such that, for all $s \in [0,1]$, $e_s = (1-s)e_0 + se_1$ has the following properties.

F1 The set $S_{e_s} = \{\mathbf{p} \in \mathcal{B} : e_s(\mathbf{p}) = 0\}$ satisfies $S_{e_s} \subset \mathcal{N}$ and the distance of S_{e_s} to $\mathcal{B} \setminus \mathcal{N}$ is bounded below by δ_0.

F2 For all $\mathbf{p} \in \mathcal{N}$,

$$\mathcal{D}_u e_s(\mathbf{p}) = u(\mathbf{p}) \cdot \nabla e_s(\mathbf{p}) > u_0. \tag{83}$$

F3 For $\varepsilon > 0$, let $\mathcal{U}(e, \varepsilon) = \{\mathbf{p} \in \mathcal{B} : |e(\mathbf{p})| \le \varepsilon\}$. For all $0 < \varepsilon_1 \le \varepsilon_2 \le \varepsilon_3$, and all $\mathbf{q} \in \mathcal{B}$,

$$\int\limits_{\mathcal{U}(e_s,\varepsilon_1)} d\mathbf{p}_1 \int\limits_{\mathcal{U}(e_s,\varepsilon_2)} d\mathbf{p}_2 \; \mathbb{1}\left(|e_s(\pm\mathbf{p}_1 \pm \mathbf{p}_2 + \mathbf{q})| \le \varepsilon_3\right) \; \le Q_{\mathrm{vol}} \varepsilon_1 \varepsilon_2 {\varepsilon_3}^{2\gamma}. \tag{84}$$

These hypotheses imply those imposed in `I` (the volume improvement exponent ϵ of `I` equals 2γ), so the results of `I` apply. Moreover, the stronger hypotheses stated in Section 1.2 imply **F1**–**F3** by the following Lemma.

Lemma 4 *Let $B = B^{(2)}_\varepsilon(E_0)$ be the ball of Lemma 2, $\mathcal{N}$ be the annulus $\tilde{A}$ defined in* (23) *and $u = \hat{r}$, the radial vector field of polar coordinates. Then there are constants $\delta_0, u_0, Q_{\mathrm{vol}}, \gamma > 0$ such that* **F1**–**F3** *hold for all $e_0, e_1 \in B$.*

Proof: B is convex, so for all $s \in [0,1]$, $e_s = (1-s)e_0 + se_1 \in B \subset \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2)$. **F1** is obvious by the definition of $\mathcal{E}_s$. **F2** follows directly from Lemma 2, with $u_0 = g_1$. **F3** follows from Theorem `II.1.1` by the usual Taylor expansion which is described in (`I.A.2`)–(`I.A.6`). ■

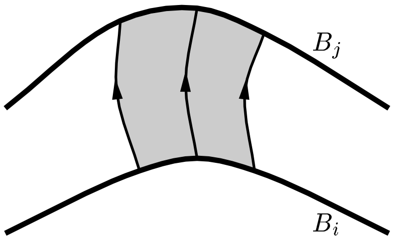
F2 implies that there is $g_0 > 0$ such that for all $s \in [0,1]$ and all $\mathbf{p} \in \mathcal{N}$, $|\nabla e_s(\mathbf{p})| > g_0$. For a fixed e, the converse is proven in Lemma `I.2.1`.

Lemma 5 *Let $\mathcal{N}_c$ be a connected component of $\mathcal{N}$ which has a nonempty intersection with S_{e_s} for some $0 \le s \le 1$.*

1. *The boundary of $\mathcal{N}_c$ has precisely two connected components. These two components are diffeomorphic.*

2. *Denote by S one of the two components of the boundary of $\mathcal{N}_c$. There is, for each $0 \le s \le 1$, a C^2 bijection $\boldsymbol{\pi}_s$ from a neighbourhood of $\{0\} \times S$ in $\mathbb{R} \times S$ to $\mathcal{N}_c$ such that $e_s(\boldsymbol{\pi}_s(\rho,\theta)) = \rho$, $\frac{\partial \boldsymbol{\pi}_s}{\partial \rho}(\rho,\theta)$ is parallel to $u(\boldsymbol{\pi}_s(\rho,\theta))$ and*
$$\frac{1}{\sup_{s,\mathbf{p}} |\nabla e_s|} \le \left| \frac{\partial \boldsymbol{\pi}_s}{\partial \rho} \right| \le \frac{1}{u_0}. \tag{85}$$

Proof: Denote by $B_1, \ldots, B_n$, the connected components of the boundary of $\mathcal{N}$. Since $e_0 \in C^0(\mathcal{B}, \mathbb{R})$ and $\mathcal{B}$ is compact, e_0 is bounded above and below on $\mathcal{N}$. By **F2**, the value of e_0 changes at a rate of at least u_0 per unit time along each trajectory of the vector field u. Hence each trajectory must start on some B_i and end on some B_j. Because u is transverse to the boundary of $\mathcal{N}$ and B_i and B_j do not themselves have boundaries, each trajectory starting on B_i and ending on B_j has an open neighbourhood in $\mathcal{N}$ that is a union of trajectories starting on B_i and ending on B_j. Let, for each $1 \le i,j \le n$, $\mathcal{N}_{i,j}$ be the set of all points of $\mathcal{N}$ that lie on a trajectory which starts on B_i and ends on B_j. Then the $\mathcal{N}_{i,j}$'s are all open and mutually disjoint and their union is $\mathcal{N}$. Hence each $\mathcal{N}_{i,j}$ is either empty or a connected component of $\mathcal{N}$.



We claim that if $\mathcal{N}_{i,j}$ has a nonempty intersection with S_{e_s}, then $i \ne j$. By **F1**, e_s may not vanish in a neighbourhood of the boundary of $\mathcal{N}$ and hence must be of uniform sign near each B_k. If e_s has the same sign, say positive, near both B_i and B_j (as will certainly be the case if $i = j$) then, as it vanishes somewhere in $\mathcal{N}_{i,j}$, e_s must have a local minimum somewhere in $\mathcal{N}_{i,j}$. This violates **F2**.

Suppose that $\mathcal{N}_c = \mathcal{N}_{i,j}$. Then $i \ne j$ and the components of the boundary of $\mathcal{N}_c$ are B_i and B_j. The map which associates to each $\mathbf{p} \in B_i$ the unique point of B_j that is on the same trajectory as $\mathbf{p}$ is a diffeomorphism, so we have completed the proof of part 1. For each $\mathbf{p} \in \mathcal{N}_c$, denote by $\Theta(\mathbf{p})$ the unique point of B_i that is on the same trajectory of u as $\mathbf{p}$. As B_i is a C^∞ manifold, u is transverse to B_i and the trajectories are C^∞ in their dependence on time and initial conditions, $\Theta(\mathbf{p})$ is C^∞. The map $\mathbf{p} \mapsto \big(e_s(\mathbf{p}), \Theta(\mathbf{p})\big)$ is defined and C^2 on $\mathcal{N}_c$, injective (as e_s is strictly monotone on each trajectory and each trajectory hits a different point of B_i) onto a neighbourhood of $\{0\} \times S$ (since e_s is of opposite sign near B_i and B_j it must vanish once on each trajectory). Furthermore the Jacobian of this map is nonsingular at each $\mathbf{p} \in \mathcal{N}_c$ by **F2** and the transversality of u at B_i. We may thus take $\boldsymbol{\pi}_s$ to be the inverse of this map.

■

Let $\mathbf{P}(e_s, \mathbf{p}) = \boldsymbol{\pi}_s\big(0, \Theta(\mathbf{p})\big)$ be the projection on S_{e_s}, and let ℓ_{e_s} denote the localization operator for e_s, as given by Definition I.2.6. Then $\mathcal{D}_u \ell_{e_s} = 0$ for all $s \in [0,1]$. Under the hypotheses of Section 1, and if u is chosen to be the radial field $u = \hat{r}$, $\mathbf{P}$ agrees with the projection $\mathbf{p}(r,\theta) \mapsto \mathbf{p}(r_F(e_s,\theta),\theta)$ in a neighbourhood of the Fermi surface.

We now take a fixed $V \in \mathcal{V}$ and prove bounds that are uniform on $\mathcal{V}$. Thus we again drop the λV from the notation.

Theorem 4 *Under the hypotheses F1–F3, there are constants $\tilde{Q}_0$ and $\tilde{Q}_1$, depending on $G = \sup_s |e_s|_2$, $Q_{\rm vol}$, γ, R, r_0, and u_0, such that*

$$\left|K^{(R)}(e_1) - K^{(R)}(e_0)\right|_0 \;\le\; \tilde{Q}_0 |\lambda|\, |e_1 - e_0|_0 \tag{86}$$

$$\left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_1 \;\le\; \tilde{Q}_1 |\lambda| \Big(M^{-1.1\,j} |e_1 - e_0|_0 + M^{\gamma j} |e_1 - e_0|_1 \Big). \tag{87}$$

If for all $s \in [0,1]$ the norm $|e_s|_{3,r} = |e_s|_2 + \|\mathcal{D}_u e_s\|_2$ is finite, then

$$\left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_{3,r} = \left|K_j^{(R)}(e_1) - K_j^{(R)}(e_0)\right|_2 \tag{88}$$

$$\le\; \tilde{Q}_1 |\lambda| \left(M^{-2.1j} |e_1 - e_0|_1 + M^{\gamma j} \sup_{t \in [0,1]} |e_t|_{3,r} |e_1 - e_0|_0 + M^{\gamma j} |e_1 - e_0|_2 \right).$$

By Lemma 4, Theorem 4 implies (33), (65), and (66), with $Q_0 = \tilde{Q}_0$ and $Q_2 = \tilde{Q}_1$.

4.3 Proof of Theorem 4

Dropping uniform constants in the notation: We introduce the notation $A \lesssim B$ meaning that $A \le$ const B where the constant depends only on G, $Q_{\rm vol}$, γ, R, r_0, and u_0 (thus in particular the constant is uniform on $\mathcal{E}_s$). For instance, we have, for $p \le 3$, $|FG|_p \lesssim |F|_p |G|_p$, and $|e|_2 \lesssim 1$ if $e \in \mathcal{E}_s$.

For a function F that depends on e, let $D_h F$ denote the directional derivative of F with respect to e, $D_h F = \frac{\partial}{\partial \alpha} F(e + \alpha h)\,|_{\alpha=0}$. We proved in I that K is Fréchet differentiable in e, so these derivatives exist. Moreover, Fréchet differentiability holds for all quantities in which there is an infrared cutoff.

Proof of (86)

By (79), for any $s \in [0,1]$,

$$\left| D_h \left(\ell_{e_s} \sum_{I \le j < 0} Y_j^{(R)}(e_s) \right) \right|_0$$

$$\le \sum_{r=1}^{R} |\lambda|^r \sum_G \left| \sum_{I \le j < 0} \sum_{T \sim G} \prod_{f \in T} \frac{1}{n_f!} \sum_{J \in \mathcal{J}(T,j,G)} D_h \left(\ell_{e_s} \, \mathrm{Val}\,(G^J)(e_s) \right) \right|_0 \qquad (89)$$

By (I.3.35), there is a constant, depending only on G and on the constants given in the Lemma, such that

$$\left| D_h \left(\ell_{e_s} \sum_{I \le j < 0} Y_j^{(R)}(e_s) \right) \right|_0 \le \sum_{r=1}^{R} |\lambda|^r \sum_G \text{ const } (G) |h|_0 . \qquad (90)$$

For fixed R, the sum over graphs G contains finitely many terms, so

$$\left| D_{e_1 - e_0} \left(\ell_{e_s} \sum_{I \le j < 0} Y_j^{(R)}(e_s) \right) \right|_0 \lesssim |\lambda| \, |e_1 - e_0|_0 \qquad (91)$$

uniformly in I and s. Thus (86) follows by

$$\begin{aligned} \sum_{j<0} (K_j^{(R)}(e_1) - K_j^{(R)}(e_0)) &= \int_0^1 \mathrm{d}s \, \frac{\partial}{\partial s} \ell_{e_s} \sum_{j<0} Y_j^{(R)}(e_s) \\ &= \int_0^1 \mathrm{d}s \, D_{e_1 - e_0} \ell_{e_s} \sum_{j<0} Y_j^{(R)}(e_s). \qquad (92) \end{aligned}$$

Preliminaries for the proof of (87) **and** (88)

To prove the single–scale bounds (87) and (88), we show that for $k \le 2$, the seimnorms $\| K_j^{(R)}(e_1) - K_j^{(R)}(e_0) \|_k$ obey bounds with the same right hand side as in (87) and (88). Note that even the bound for $k = 0$ does not follow from (91) because we are now considering a fixed scale j, not a sum over scales, and the summation over scales provided a cancellation that was important in the proof of Theorem I.3.5. However, the proof does not require very detailed estimates because the coefficient of $|e_1 - e_0|_{k-1}$ is (up to factors $|j|$, which we bound by $M^{-0.1j}$) a factor M^{-kj} larger than the undifferentiated power counting behaviour M^j of a single–scale selfenergy contribution like $Y_j^{(R)}$. This is naive power counting behaviour. The estimates will again follow by applying bounds already proven in I.

We now interpolate the difference of the two K functions. The derivative of ℓ_e with respect to e was calculated in Lemma I.3.1. The interpolation gives

$$K_j^{(R)}(e_1) - K_j^{(R)}(e_0) = \int_0^1 \mathrm{d}s \; (\mathcal{Y}_1(s) - \mathcal{Y}_2(s)) \qquad (93)$$

with

$$\begin{aligned} \mathcal{Y}_1(s) &= \ell_{e_s}\left(D_{e_1-e_0}Y_j^{(R)}(e_s)\right) && (94)\\ \mathcal{Y}_2(s) &= \ell_{e_s}\left[(e_1-e_0)\frac{1}{\mathcal{D}_u e_s}\mathcal{D}_u Y_j^{(R)}(e_s)\right], && (95) \end{aligned}$$

with $\mathcal{D}_u$ defined in (83). Because K_j is the localization of Y_j, $\mathcal{D}_u K_j = 0$, so the first equality in (88) holds. Thus we have to bound $\|\mathcal{Y}_i\|_k$ for $k \in \{0,1,2\}$. In the following, we drop the superscript R from $Y_j^{(R)}$.

Estimates for $\|\mathcal{Y}_2\|_k$

Let $k = 0$. The bound $|\mathcal{Y}_2|_0 \le |e_1-e_0|_0\frac{1}{u_0}|\mathcal{D}_u Y_j|_0$ and Theorem I.2.46 (i) imply that

$$|\mathcal{Y}_2|_0 \lesssim |j|^R M^{2\gamma j}|e_1-e_0|_0 \lesssim M^{\gamma j}|e_1-e_0|_0. \tag{96}$$

Let $k = 1$. Because

$$\frac{\partial}{\partial p_\alpha}(\ell_{e_s}F)(p) = \frac{\partial}{\partial p_\alpha}F(0,\mathbf{P}(e_s,\mathbf{p})) = \sum_\beta \frac{\partial \mathbf{P}_\beta}{\partial p_\alpha}(e_s,\mathbf{p})\left[\frac{\partial}{\partial q_\beta}F(0,\mathbf{q})\right]_{\mathbf{q}=\mathbf{P}(e_s,\mathbf{p})}, \tag{97}$$

we have

$$\frac{\partial}{\partial p_\alpha}\mathcal{Y}_2(s)(p) = \sum_\beta \frac{\partial \mathbf{P}_\beta}{\partial p_\alpha}(e_s,\mathbf{p})\ \mathcal{X}_\beta(\mathbf{P}(e_s,\mathbf{p})) \tag{98}$$

with

$$\mathcal{X}_\beta(\mathbf{q}) = \frac{\partial}{\partial q_\beta}\left[(e_1-e_0)(\mathbf{q})\ \frac{1}{\mathcal{D}_u e_s(\mathbf{q})}\mathcal{D}_u Y_j(0,\mathbf{q})\right]. \tag{99}$$

Thus

$$\begin{aligned} \|\mathcal{Y}_2(s)\|_1 &\le d\,\|\mathbf{P}(e_s)\|_1\Big(\|e_1-e_0\|_1\tfrac{1}{u_0}|\mathcal{D}_u Y_j|_0 \\ &\quad + |e_1-e_0|_0\tfrac{1}{u_0^2}\|\mathcal{D}_u e_s\|_1|\mathcal{D}_u Y_j|_0 \\ &\quad + |e_1-e_0|_0\tfrac{1}{u_0}\|\mathcal{D}_u Y_j\|_1\Big) \\ &\lesssim |e_1-e_0|_1|\mathcal{D}_u Y_j|_0 + |e_1-e_0|_0\|\mathcal{D}_u Y_j\|_1 && (100) \end{aligned}$$

because $\|\mathbf{P}(e_s)\|_1 \lesssim 1$ and $\|\mathcal{D}_u e_s\|_1 \lesssim |e_s|_2 \lesssim 1$.

Let $k = 2$. Because

$$\begin{aligned} \frac{\partial^2}{\partial p_\gamma \partial p_\alpha}\mathcal{Y}_2(p) &= \sum_\beta\Big[\mathcal{X}_\beta(\mathbf{P}(e_s,\mathbf{p}))\frac{\partial^2 \mathbf{P}_\beta}{\partial \mathbf{p}_\gamma \partial \mathbf{p}_\alpha}(e_s,\mathbf{p}) \\ &\quad + \sum_\rho(\partial_\rho\mathcal{X}_\beta)(\mathbf{P}(e_s,\mathbf{p}))\partial_\gamma\mathbf{P}_\rho(e_s,\mathbf{p})\partial_\alpha\mathbf{P}_\beta(e_s,\mathbf{p})\Big], && (101) \end{aligned}$$

we have

$$\|\mathcal{Y}_2(s)\|_2 \le d\,\|\mathbf{P}(e_s)\|_2\max_\beta|\mathcal{X}_\beta|_0 + \|\mathbf{P}(e_s)\|_1{}^2\sum_{\beta,\rho}|\partial_\rho\mathcal{X}_\beta|_0. \tag{102}$$

Because $\|\mathbf{P}(e_s)\|_2 \lesssim 1$ and

$$\begin{aligned}\left|\frac{\partial}{\partial q_\rho}\mathcal{X}_\beta(\mathbf{q})\right| &\lesssim |e_1-e_0|_2 |\mathcal{D}_u Y_j|_0 + |e_1-e_0|_0 |\mathcal{D}_u Y_j|_0 \|\mathcal{D}_u e_s\|_2 \\ &+ |e_1-e_0|_1 \|\mathcal{D}_u Y_j\|_1 + |e_1-e_0|_0 \|\mathcal{D}_u Y_j\|_2, \qquad (103)\end{aligned}$$

we have

$$\begin{aligned}\|\mathcal{Y}_2(s)\|_2 &\lesssim |e_1-e_0|_2 \, |\mathcal{D}_u Y_j|_0 + |e_1-e_0|_1 \, \|\mathcal{D}_u Y_j\|_1 \\ &+ |e_1-e_0|_0 \left(|\mathcal{D}_u Y_j|_0 \, \|\mathcal{D}_u e_s\|_2 + \|\mathcal{D}_u Y_j\|_2\right). \qquad (104)\end{aligned}$$

The term $\|\mathcal{D}_u e_s\|_2$ is the reason why we have to deal with functions that have bounded radial derivatives. Because it arises only from the derivative of the localization operator, it has got nothing to do with the scale dependence of Y_j.

By (79), it suffices to bound the contribution from every 1PI two–legged graph G separately. That is, we may replace Y_j by $W = \sum_{J\in\mathcal{J}(T,j,g)} \mathrm{Val}\,(G^J)$ in (96), (100), and (104) if we take a maximum over G and T and multiply by the number of graphs and the number of possible T's. By Theorem I.2.46 (i), and using $\lambda_n(j,\gamma)M^{\gamma j} \lesssim 1$, we have

$$\begin{aligned}\|\mathcal{Y}_2(s)\|_1 &\lesssim |e_1-e_0|_1 \, M^{\gamma j} + |e_1-e_0|_0 \, M^{(\gamma-1)j} \qquad (105)\\ \|\mathcal{Y}_2(s)\|_2 &\lesssim |e_1-e_0|_2 \, M^{\gamma j} + |e_1-e_0|_1 \, M^{(\gamma-1)j} \\ &+ |e_1-e_0|_0 \left(M^{\gamma j} \|\mathcal{D}_u e_s\|_2 + \|\mathcal{D}_u Y_j\|_2\right). \qquad (106)\end{aligned}$$

Thus $\sup_s \|\mathcal{Y}_2(s)\|_k$ obey bounds that imply (87) and (88) if we can prove that

$$\|\mathcal{D}_u Y_j\|_2 \lesssim M^{-2.1\,j} \qquad (107)$$

and that

$$\|\mathcal{Y}_1(s)\|_1 \lesssim M^{-1.1j}|e_1-e_0|_0, \quad \|\mathcal{Y}_1(s)\|_2 \lesssim M^{-2.1j}|e_1-e_0|_0. \qquad (108)$$

To do this, we need to exhibit the structure of the graphs G that contribute to Y_j in a little bit more detail.

Graphical tools

Let G be a graph contributing to (79), T a rooted tree compatible to G, with an r and c labelling assigned to the forks, and $\mathcal{J}(T,j,G)$ the set of labellings of G compatible with T and root scale j. Let ϕ be the root of T. To every fork $f \in T$ there corresponds a connected subgraph G_f of G, which is a proper subgraph of G for $f > \phi$. We call f an m–legged fork if G_f has m external legs. In the following we construct a graph Γ, a tree T' compatible with Γ, and a set of labellings $\mathcal{J}'$ with the following properties.

- Γ is two–legged and 1PI, and Γ has only four–legged vertices with vertex functions $\hat{v}$.

- The associated tree T' has no 2–legged forks.
- The scale assignments in $\mathcal{J}'$ are $j_f > j_{\pi(f)}$ for all $f \in T'$. With propagators associated to Γ in the way given below,

$$\sum_{J \in \mathcal{J}(T,j,G)} \mathrm{Val}\,(G^J) = \sum_{J' \in \mathcal{J}'(T',j,\Gamma)} \mathrm{Val}\,(\Gamma^{J'}) \tag{109}$$

Summation over the trees gives

$$\sum_{T \sim G} \prod_{f \in T} \frac{1}{n_f!} \sum_{J \in \mathcal{J}(T,j,G)} \mathrm{Val}\,(G^J) = \sum_{T' \sim \Gamma} \prod_{f' \in T'} \frac{1}{n_{f'}!} \sum_{J' \in \mathcal{J}'(T',j,\Gamma)} \mathrm{Val}\,(\Gamma^{J'}). \tag{110}$$

This construction is similar to that of Remark I.2.45, only simpler, because here we do not aim at tight bounds for the powers of $|j|$ generated by scale sums of four–legged subdiagrams.

If no $f > \phi$ is two–legged, then $\Gamma = G$, $T' = T$, $\mathcal{J}' = \mathcal{J}$. Otherwise, let $f_1, \ldots f_n > \phi$ be all minimal two–legged forks of T. That is, there is no two–legged fork f' with $\phi < f' < f_i$. Let $\tilde{T}$ be the tree where the subtrees T_i rooted at the forks f_i are replaced by leaves λ_i. To obtain the corresponding graph $\tilde{G}$, replace G_{f_i} by a two–legged vertex v_i with ($j_{\pi(f_i)}$–dependent) vertex function

$$A_i = \mathcal{P}_i \sum_{j_i} \sum_{J_i \in \mathcal{J}(T_i, j_{\pi(f_i)}, G_{f_i})} \mathrm{Val}\,(G_{f_i}^{J_i}). \tag{111}$$

The projection $\mathcal{P}_i$ is ℓ_{e_s} if f_i is a c–fork and $1 - \ell_{e_s}$ if F_i is an r–fork of T. The summation range is $j_i > j_{\pi(f_i)}$ if f_i is an r–fork and $j_i \le j_{\pi(f_i)}$ if f_i is a c–fork.

Because all c–forks have now been replaced by vertices (or hidden inside two–legged vertices), $\tilde{\mathcal{J}} = \{J|_{\tilde{T}} : J \in \mathcal{J}(T,j,G)\}$ consists only of labellings with $j_f > j_{\pi(f)}$ for all $f \in \tilde{T}$. With the standard definition of the value of a labelled graph (see, e.g., (I.2.54)),

$$\sum_{J \in \mathcal{J}(T,j,G)} \mathrm{Val}\,(G^J) = \sum_{\tilde{J} \in \tilde{\mathcal{J}}(\tilde{T},j,\tilde{G})} \mathrm{Val}\,\left(\tilde{G}^{\tilde{J}}\right). \tag{112}$$

The graph $\tilde{G}$ is not yet what we want because the graph G_{f_i} whose value appears in (111) is not necessarily 1PI and because $\tilde{G}$ may contain two–legged vertices. In order to apply Theorem I.2.46, we want to reduce all vertex functions of two–legged vertices to sums over values of 1PI graphs.

If f_i is a c–fork, G_{f_i} is 1PI because otherwise ℓ_{e_s} of its value would vanish. If f_i is an r–fork, G_{f_i} may be 1PR; then $\mathcal{P}_i\ \mathrm{Val}\,(G_{f_i}^J) = \mathrm{Val}\,(G_{f_i}^J)$ and it is a string of two–legged subgraphs, some of which may be single–scale insertions (SSI's) defined in Remark I.2.45. Momentum conservation, the scale structure on T, and the support properties of the cutoff function fix the scale of the lines connecting the 1PI pieces to $j_{\pi(f_i)} + 1$. When every r–fork corresponding to an 1PR graph is replaced by its string as above, the only changes to $\tilde{G}$

are that additional two–legged vertices may appear and that, besides the cases $\mathcal{P}_i = \ell_{e_s}, 1-\ell_{e_s}$, there is the third case $\mathcal{P}_i = 1$ for SSI's, with the scale sum for a SSI consisting only of the one term where all scales are $j_{\pi(f_i)}+1$ (see Remark I.2.45 for details).

Let Γ be the graph where all strings of two–legged subgraphs are replaced by single lines, and T' be the tree in which all leaves of $\tilde{T}$ that correspond to two–legged vertices of $\tilde{G}$ are removed. For a line ℓ of Γ, let j_ℓ be the minimum over all $j_{\tilde{\ell}}$, where $\tilde{\ell}$ runs over the lines of $\tilde{G}$ on the string σ_ℓ in $\tilde{G}$ replaced by ℓ. The propagator associated to ℓ is

$$S_{\ell,j_\ell}(p) = \sum_{(j_{\tilde{\ell}})_{\tilde{\ell} \text{ on } \sigma_\ell}} \prod_{\tilde{\ell} \text{ on } \sigma_\ell} C_{j_{\tilde{\ell}}}(p) \prod_{v \text{ on } \sigma_\ell} A_v \tag{113}$$

where the summation is over all scale assignments $j_{\tilde{\ell}} \in \{j_\ell, j_\ell+1\}$ that are compatible with $\tilde{T}$, and, if n propagators appear in the product, $n-1$ factors A_v appear. By construction, (109) and (110) hold.

Lemma 6 *Let α be a multiindex with $w = |\alpha| \le 1$. Then the propagators S_{ℓ,j_ℓ} given by* (113) *satisfy*

$$|D^\alpha S_{\ell,j_\ell}(p)| \lesssim M^{-j_\ell(1+w)+j_\ell\gamma g} \, 1\!\!1\left(|\mathrm{i}p_0 - e_s(\mathbf{p})| \le M^{j_\ell}\right) \tag{114}$$

where g is the number of c–forks plus the number of SSI on the string σ_ℓ corresponding to S_{ℓ,j_ℓ}.

Proof: The support condition follows directly from that of C_{j_ℓ}. We now bound the functions A_v and their first derivatives. This is a direct application of Theorem I.2.46 (i), which states (with $\varepsilon = 2\gamma$) that if G is two–legged and 1PI, then for all $r \in \{0,1,2\}$,

$$\sum_{J \in \mathcal{J}(T,j,G)} \left| \operatorname{Val} G^J \right|_r \lesssim |j|^{n_G} M^{j(1+2\gamma-r)} \lesssim M^{j(1+\gamma-r)}. \tag{115}$$

Let $w \in \{0,1\}$ and α be a multiindex with $|\alpha| = w$. For v corresponding to an r–fork and for p such that $|\mathrm{i}p_0 - e_s(\mathbf{p})| \le M^{j_\ell}$,

$$|D^\alpha A_v(p)| \lesssim \sum_{j > j_\ell} \left| \sum_{J \in \mathcal{J}(T_i,j,G_{f_i})} D^\alpha (1-\ell_{e_s}) \operatorname{Val}(G^J_{f_i})(p) \right| \tag{116}$$

For $w = 0$, Taylor expansion gives the renormalization gain M^{j_ℓ} and one derivative acting on $\operatorname{Val}(G^J_{f_i})$. By (115), with $r = 1 + w = 1$,

$$|D^\alpha A_v(p)| \lesssim M^{j_\ell} \sum_{j > j_\ell} M^{j(1+\gamma-1)} \lesssim M^{j_\ell}. \tag{117}$$

For $w = 1$, we estimate the 1 and ℓ_{e_s} terms separately. By (115),

$$|D^\alpha A_v(p)| \lesssim 2 \sum_{j > j_\ell} M^{j(1+\gamma-1)} \lesssim 1. \tag{118}$$

For v corresponding to a c–fork,

$$|A_v|_w \lesssim \sum_{j \le j_\ell} \left| \sum_{J \in \mathcal{J}(T_i, j, G_{f_i})} \ell_{e_s} \ \mathrm{Val}\,(G^J_{f_i}) \right|_w , \tag{119}$$

so (115) implies

$$|A_v|_w \lesssim \sum_{j \le j_\ell} M^{j(1+\gamma-w)} \lesssim M^{j_\ell(1+\gamma-w)}. \tag{120}$$

The estimate for v corresponding to an SSI is similar to that of a c–fork, except that there is not even a scale sum to do because the scales are all fixed in an SSI. Using the product rule for derivatives acting on (113) and using that

$$|D^\alpha C_j(p)| \lesssim M^{-j(1+|\alpha|)} \ 1\!\!1 \left(|ip_0 - e_s(\mathbf{p})| \le M^j \right) \tag{121}$$

we get the statement of the Lemma. ∎

Lemma 6 gives us control over first order derivatives of the propagators S_{ℓ, j_ℓ} with respect to momentum. The next lemma will imply that we can always arrange the integral for the value of a graph contributing to Y_j such that every line of the graph gets differentiated at most once, even if we take three derivatives with respect to the external momentum.

In I, Definition 2.19, we introduced the notion of overlapping graphs. A graph is overlapping if there is a line ℓ of G which is part of two independent (non self–intersecting) loops. We say that the line ℓ is part of the two overlapping loops.

Lemma 7 *Let G be a two–legged 1PI graph with two external vertices v_1 and v_2. Let all vertices of G have an even incidence number. Let T be any spanning tree of G, and let θ be the linear subtree of T corresponding to the unique path from v_1 to v_2 over lines of T. Then every line $\ell \in \theta$ is part of two overlapping loops generated by lines $\ell_1 \notin T$ and $\ell_2 \notin T$. For $i \in \{1,2\}$, the graph T_i, obtained from T by removing ℓ and adding ℓ_i, is a spanning tree for G.*

Proof: Let ℓ be a line of θ. Cut ℓ to get a four–legged graph $F = G - \ell$. Because G is 1PI, F is connected, so there is a (nonselfintersecting) path π in F that joins the endpoints of ℓ. Because T is a tree, $T - \ell$ has two connected components, T_1 and T_2. As $T_1 \cup T_2 \cup \pi$ is connected, one of the lines on π, say ℓ_1, joins T_1 and T_2, but is not in T. Thus ℓ is on the loop generated by ℓ_1. Go back to G and cut ℓ_1. The result is a four–legged graph $F' = G - \ell_1$. Because $\ell_1 \notin T$, T is still a spanning tree for F'. Cutting ℓ does not disconnect F' because if it did, each of the connected components would have to have three external lines – one of G's original external lines, one end of ℓ_1 and one end of ℓ (as all vertices of G have even incidence number, all connected graphs must have an even number of external lines). Let ℓ_2 be a line on the shortest path in $F' - \ell$ connecting the endpoints of ℓ with ℓ_2 joining T_1 and T_2 but not in T.

Then ℓ is in the loop generated by ℓ_2. Thus the loops generated by ℓ_1 and ℓ_2 overlap on ℓ. ■

It would not have been a loss of generality to assume that G has no proper two–legged subgraphs. In that case, Remark `I.2.23` implies that F' is also 1PI. If T is chosen such that θ is a shortest path from v_1 to v_2 in G, the statement of the Lemma is an obvious consequence of Lemma `III.2.5` (see Figures `III.2.3`–`III.2.6`; note that the lines from v_r to v_{r+1} and from v_s to v_{s+1} can be any pair of lines on θ).

The bound for $\|\mathcal{D}_u Y_j\|_2$

Because $\|\mathcal{D}_u Y_j\|_2 \le |Y_j|_3$, it suffices to prove that

$$|Y_j|_3 \lesssim M^{-2.1j}. \tag{122}$$

By (79), it suffices to prove the same bound for

$$\mathcal{W} = \sum_{J \in \mathcal{J}(T,j,G)} \mathrm{Val}\,(G^J). \tag{123}$$

All graphs that contribute are two–legged and 1PI, so by (115),

$$|\mathcal{W}|_2 \lesssim M^{j(1+2\gamma-2)}(1+|j|^R) \lesssim M^{-j}M^{\gamma j} \lesssim M^{-2j}, \tag{124}$$

so it suffices to bound $\|\mathcal{W}\|_3$. Let Γ be the graph associated to G with the properties (109) and (110), then

$$\|\mathcal{W}\|_3 \le \sum_{J \in \mathcal{J}(T',j,\Gamma)} \|\ \mathrm{Val}\,(\Gamma^J)\|_3. \tag{125}$$

Let T be a spanning tree for Γ. The only factors in the integrand for $\mathrm{Val}\,\Gamma^J$ that can depend on the external momentum q are

- vertex functions $\hat{v}$; the dependence is of the form $\hat{v}(q-p)$ where p is a loop momentum or a sum of loop momenta because G is 1PI and two–legged (it can happen that $\hat{v}$ does not depend on any loop momentum; this is, however, only the case for tadpoles, in which case only $\hat{v}(0)$ appears).
- propagators S_{ℓ,j_ℓ} for those ℓ that are in the path on T connecting the external vertices (if there is only one external vertex, no propagator depends on q).

We now take three derivatives of $\mathrm{Val}\,(\Gamma^J)$ and use the above lemmas to avoid having two derivatives acting on any propagator and three on any vertex function, as follows.

If Γ has only one external vertex and is not a tadpole, we first route q through the $\hat{v}$ of the external vertex. We let two derivatives act and then change variables from p to $q-p$ in the loop integral in which $\hat{v}(q-p)$ appears. The third derivative can then not act on this vertex function any more. It can act on another vertex function or on a propagator.

If Γ has two external vertices, there are two cases, depending on where the first derivative acted.

1. The first derivative acts on a vertex function. Take another derivative. If it acts on the same vertex function, change variables from p to $q-p$ in the loop integral in which $\hat{v}(q-p)$ appears. The third derivative can then not act on this vertex function any more. If the second derivative acts on the propagator S_{ℓ,j_ℓ}, we change the spanning tree using Lemma 7. The third derivative can then not act on S_{ℓ,j_ℓ} any more.

2. The first derivative acts on the propagator S_{ℓ,j_ℓ}. We change the spanning tree to T_1 by replacing ℓ with another line ℓ_1 (this is possible by Lemma 7) and take another derivative. It can act on a propagator on a line ℓ' on the path in T_1 that connects the external vertices ($\ell' = \ell_1$ is possible). By Lemma 7, there are two lines, ℓ_1' and ℓ_2', such that for $i \in \{1,2\}$, T_i', obtained by replacing ℓ' by ℓ_i' in T_1, is still a spanning tree for Γ. At most one of ℓ_1' and ℓ_2' may be ℓ, so we may change to a spanning tree that contains neither ℓ nor ℓ'. Once this is done, the third derivative cannot act on the propagators associated to the lines ℓ and ℓ'.

In summary, the net effect of taking three derivatives in the way just described is, by Lemma 6, at most a factor M^{-3j}, as compared to standard power counting (a factor M^{-3j} arises only if all three derivatives act on propagators; when vertex functions get differentiated, no factor M^{-j} is produced). Because the GN tree T' associated to Γ has no 2–legged forks, the scale sum converges by standard arguments (see Lemma I.2.4 and Remark I.2.5), and is bounded by $|j|^R M^j$. Thus

$$\|\mathcal{W}\|_3 \lesssim |j|^R M^j M^{-3j} \lesssim M^{-2.1j}. \tag{126}$$

The bound for $\|\mathcal{Y}_1(s)\|_2$

In the following bounds we keep the tree sums inside of the norms. By (110), we thus need to estimate

$$\left\| \sum_{T'\sim\Gamma} \prod_{f\in T'} \frac{1}{n_f!} \sum_{J\in\mathcal{J}(T',j,\Gamma')} D_h \operatorname{Val}(\Gamma^J) \right\|_k \tag{127}$$

for $k \in \{0,1,2\}$, with $h = e_1 - e_0$. By construction of Γ, D_h acts only on the propagators S_{ℓ,j_ℓ}.

Lemma 8 *For all $s \in [0,1]$ and all lines ℓ of Γ*

$$|D_h S_{\ell,j_\ell}(p)| \lesssim |h|_0 \, M^{-2j_\ell} \, \mathbb{1}\left(|ip_0 - e_s(\mathbf{p})| \le M^{j_\ell}\right) . \tag{128}$$

Proof: By definition (113), D_h can act on factors (a) C_j, (b) A_v coming from an r–fork, (c) A_v coming from a c–fork, (d) A_v coming from an SSI. In the last three cases, by (111), we have to estimate the norms of

$$\tilde{\mathcal{W}}_i = \mathcal{P}_i \sum_{j_i} \sum_{T_i \sim G_{f_i}} \prod_{f \in T_i} \frac{1}{n_f!} \sum_{J_i \in \mathcal{J}(T_i, j_{\pi(f_i)}, G_{f_i})} \mathrm{Val}\,(G_{f_i}^{J_i}). \tag{129}$$

(a) by (I.3.44),

$$\left|D_h C_{j_{\tilde\ell}}(p)\right| \lesssim |h|_0 \; M^{-2j_\ell} \; \mathbb{1}\left(|ip_0 - e_s(\mathbf{p})| \le M^{j_{\tilde\ell}}\right) \; . \tag{130}$$

(b) By Lemma I.3.1,

$$D_h(\ell_{e_s}\mathcal{W}_i)(e_s) = \ell_{e_s}(D_h\mathcal{W}_i) - \ell_{e_s}\left(\tfrac{h}{\mathcal{D}_u e_s}\mathcal{D}_u\mathcal{W}_i\right)(e_s) \tag{131}$$

so

$$D_h(1-\ell_{e_s})\mathcal{W}_i = (1-\ell_{e_s})D_h\mathcal{W}_i + \ell_{e_s}\left(\tfrac{h}{\mathcal{D}_u e_s}\mathcal{D}_u\mathcal{W}_i\right). \tag{132}$$

If p is such that $|ip_0 - e_s(\mathbf{p})| \le M^{j_\ell}$, then by Taylor expansion

$$|(1-\ell_{e_s})D_h\mathcal{W}_i(p)| \lesssim M^{j_\ell}|D_h\mathcal{W}_i|_1. \tag{133}$$

By (I.3.42), this is

$$\lesssim |h|_0 \; M^{j_\ell} \sum_{j > j_\ell} M^{j(2\gamma-1)}|j_\ell|^{R_i} \lesssim |h|_0 |j_\ell|^{R_i} M^{2\gamma j_\ell} \lesssim |h|_0 \tag{134}$$

with R_i the number of vertices of G_{f_i}. The second term in (132) is bounded by

$$\left|\tfrac{h}{\mathcal{D}_u e_s}\mathcal{D}_u\mathcal{W}_i\right|_0 \lesssim |h|_0|\mathcal{D}_u\mathcal{W}_i|_0 \lesssim |h|_0|\mathcal{W}_i|_1 \lesssim |h|_0 \tag{135}$$

(in the last step, we used (115)).

(c) Eq. (I.3.41) (with depth $P \le R$) implies that

$$|D_h A_v|_0 \lesssim |h|_0. \tag{136}$$

(d) Eq. (I.3.42) again implies (136).

Thus in all cases, the derivative produces at most an additional factor $\lesssim M^{-j_\ell}$ in the bounds. Applying (117), (120) with $w = 0$, and (121) with $|\alpha| = 0$, counting up factors M^{j_ℓ}, now implies the bound. ■

Thus the effect of a derivative with respect to the dispersion relation acting on the propagator S_{ℓ,j_ℓ} can be bounded in exactly the same way as a derivative with respect to momentum (see Lemma 6), except that γ (which was never actually used) has been replaced by zero. By Lemma 7 we can again prevent the at most two derivatives that appear in the norms from acting on $D_h S_{\ell,j_\ell}$.

Thus, repeating the argument from (125) to (126), again using Lemma `I`.2.4 and Remark `I`.2.5, and using $|j|^R \lesssim M^{-0.1j}$, we have

$$\begin{aligned} \|\mathcal{Y}_1(s)\|_1 &\lesssim M^{-1.1j}|e_1 - e_0|_0, \qquad (137)\\ \|\mathcal{Y}_1(s)\|_2 &\lesssim M^{-2.1j}|e_1 - e_0|_0. \qquad (138)\end{aligned}$$

Summing the seminorms $\|\cdot\|_k$, we get (87) and (88).

5 Discussion

In this section, we briefly discuss the role of the various hypotheses we used in our proofs, to summarize which parts of our argument extend easily to general Fermi surface geometries and where more work is needed. We also discuss the role of the symmetry condition $e(-\mathbf{p}) = e(\mathbf{p})$ because cases where this symmetry does not hold are interesting from a physical point of view.

The two main ingredients for the iteration by which we construct the solution to (11) are

1. the existence of an invariant set for the map $e \mapsto e + K^{(R)}$,
2. the contraction–like bounds (33), (34), and (35).

To prove item 2, we needed only rather weak hypotheses on the Fermi surface geometry. In particular, we neither used a symmetry $e(-\mathbf{p}) = e(\mathbf{p})$ in that part of the proof, nor any assumption about strict convexity, nor that $S_e \subset \mathcal{F}_2$. With a different localization operator, defined as in [10], one can even drop **F3** in the proof of (87) and (88) (recall that these bounds imply (34) and (35) by Theorem 3 and Lemma 4). However, **F3** is also necessary for the Lipschitz continuity, eq. (33), in $|\cdot|_0$, proven in `I`, which is essential for our iteration estimates. One should also keep in mind that if **F3** does not hold, the selfenergy Σ and the function K will in general not even be C^1 (in one dimension, where there are no curvature effects, Σ is not C^1; this is the source of anomalous decay exponents of the two–point function).

The result that requires the most restrictive hypotheses is that, for $e \in \mathcal{E}_s$, the bound (32) for $\left|K^{(R)}\right|_2$ holds. This provides an invariant set for the iteration. The proof of (32), contained in `II` and `III`, uses very detailed geometric estimates which require convexity and positive curvature, as well as the condition $S_e \subset \mathcal{F}_2$.

The conditions stated in Section 1.2 (including, in particular, the symmetry (Sy): $e(-\mathbf{p}) = e(\mathbf{p})$ for all $\mathbf{p}$) imply hypotheses $(\mathrm{H2})_{2,0}$, (H3), (H4), and (H5) of `II` and thus imply (32). In the *asymmetric case*, where the condition (Sy) is dropped, the regularity proof of `II` and `III` requires an additional hypothesis, stated as (H4') in `II`, which imposes a minimal rate of change of the curvature of the Fermi surface at those points where the curvature coincides with that at the antipode. This condition (H4') is not stable under an iteration in $|\cdot|_{3,r}$. It is, however, only needed to estimate the contributions to K of a very special class of graphs (the so–called wicked ladders; see Section `II`.4). We shall analyze these contributions in a further paper, to extend our regularity proof, and thus the inversion theorem, to the asymmetric case. The asymmetry plays a critical role in the proof of the existence of a two–dimensional Fermi liquid at zero temperature that was announced in [9].

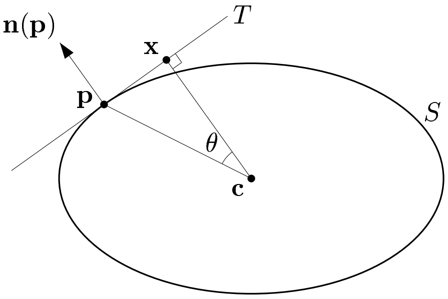
The set $\mathcal{E}_s(\delta_0, g_0, G_0, \omega_0) \cap C^3(\mathcal{B}, \mathbb{R})$ of starting E allowed in Theorem 1 is not an open subset of $C_s^2(\mathcal{B}, \mathbb{R})$. However, a look at the more detailed Theorem 2 shows that the inversion map really maps the ball $B_{\rm rad}$, defined in (37), which is open in $|\cdot|_{3,r}$, to itself (see (38)). Thus in the space of functions with bounded radial derivatives, there is an open set for which the inversion equation has a solution. Observe that, for our inversion theorem, in contrast to the KAM theorem, there is no diophantine condition for irrationality of frequencies.

As mentioned above, we needed the norm $|\cdot|_{3,r}$ instead of $|\cdot|_2$ merely for apparently rather technical reasons. A superficial look at part 4 of Theorem 2 even seems to suggest that one can extend the inversion map to balls in $\mathcal{E}_s$ that are open in $|\cdot|_2$ However, this is not the case because λ_R depends on G_3, so (39) does not imply that the inverse map is defined on a dense subset of $B^{(2)}_{\varepsilon/2}$.

A Proof of Lemma 1

We first show that (18) follows from (17). Fix any $\mathbf{p} \in S$. Let T be the tangent plane to S at $\mathbf{p}$ and let $\mathbf{x}$ be the point of T nearest $\mathbf{c}$. Since S is convex it lies on one side of T. So the sphere of radius $\frac{1}{K}$ centered on $\mathbf{c}$, which by (17) is inside S, also lies on one side of T. Hence $\|\mathbf{x} - \mathbf{c}\| \ge \frac{1}{K}$. The vector $\mathbf{x} - \mathbf{c}$ is normal to T and hence parallel to $\mathbf{n}(\mathbf{p})$. So $\theta(\mathbf{p})$ is the angle between $\mathbf{x} - \mathbf{c}$ and $\mathbf{p} - \mathbf{c}$ and

$$\cos\theta(\mathbf{p}) = \tfrac{\|\mathbf{x}-\mathbf{c}\|}{\|\mathbf{p}-\mathbf{c}\|} \ge \tfrac{1/K}{1/k} = \tfrac{k}{K} \tag{139}$$



We now prove (17), starting with $\|\mathbf{p}-\mathbf{c}\| \ge \frac{1}{K}$. This is a variant of a classical result. See, for example, §24 of [1]. Let $L > K$ and define, for each $\mathbf{p} \in S$,

$$\tilde{\mathbf{p}}(\mathbf{p}) = \mathbf{p} - \tfrac{1}{L}\mathbf{n}(\mathbf{p}) \tag{140}$$

Set

$$\tilde{S} = \left\{ \tilde{\mathbf{p}}(\mathbf{p}) \ : \ \mathbf{p} \in S \right\} \tag{141}$$

Then $\tilde{S}$ is a C^1 surface.

We claim further that $\mathbf{n}(\mathbf{p})$ is normal to $\tilde{S}$ at $\tilde{\mathbf{p}}(\mathbf{p})$. To see this, let $\mathbf{t}$ be a unit vector that is a principal direction for S at $\mathbf{p}$. Call the corresponding principal curvature κ. Let $\mathbf{q}(s)$ be a curve on S that is parametrized by arc length, passes through $\mathbf{p}$ at $s = 0$ and has tangent vector $\mathbf{t}$ there. Then $s \mapsto \tilde{\mathbf{p}}\big(\mathbf{q}(s)\big) = \mathbf{q}(s) - \frac{1}{L}\mathbf{n}\big(\mathbf{q}(s)\big)$ is a curve on $\tilde{S}$ that passes through $\tilde{\mathbf{p}}(\mathbf{p})$ at $s = 0$ and has tangent vector

$$\tfrac{d}{ds}\tilde{\mathbf{p}}\big(\mathbf{q}(s)\big)\Big|_{s=0} = \mathbf{t} - \tfrac{1}{L}\tfrac{d}{ds}\mathbf{n}\big(\mathbf{q}(s)\big)\Big|_{s=0} = \mathbf{t} - \tfrac{\kappa}{L}\mathbf{t} \tag{142}$$

there. Since $\kappa < L$, $\mathbf{t}$ is also a tangent vector to $\tilde{S}$ at $\tilde{\mathbf{p}}(\mathbf{p})$. As this is the case for all principal directions $\mathbf{t}$, the tangent plane to $\tilde{S}$ at $\tilde{\mathbf{p}}(\mathbf{p})$ is parallel to the tangent plane to S at $\mathbf{p}$.

Since S is strictly convex, with principal curvatures bounded away from zero, the Gauss map $\mathbf{p} \in S \mapsto \mathbf{n}(\mathbf{p})$ is bijective and has a C^1 inverse $\mathbf{n} \in S^{d-1} \mapsto \mathbf{p}(\mathbf{n}) \in S$. The map $\mathbf{n} \in S^{d-1} \mapsto \tilde{\mathbf{p}}\big(\mathbf{p}(\mathbf{n})\big)$ is then C^1 and surjective. Furthermore, the normal to $\tilde{S}$ at $\tilde{\mathbf{p}}\big(\mathbf{p}(\mathbf{n})\big)$ is the same as the normal to S at $\mathbf{p}(\mathbf{n})$, which is $\mathbf{n}$. Consequently, $\tilde{S}$ is convex.

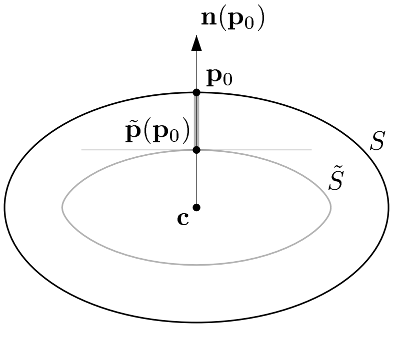
As the chord $\mathbf{c}_1 - \mathbf{c}_2$ is of maximal length, it must be parallel to both $\mathbf{n}(\mathbf{c}_1)$ and $\mathbf{n}(\mathbf{c}_2)$. Thus

$$\mathbf{n}(\mathbf{c}_1) = \frac{\mathbf{c}_1 - \mathbf{c}_2}{\|\mathbf{c}_1 - \mathbf{c}_2\|} = -\mathbf{n}(\mathbf{c}_2) \tag{143}$$

so that

$$\mathbf{c} = \tfrac{1}{2}\big(\mathbf{c}_1 + \mathbf{c}_2\big) = \tfrac{1}{2}\big(\mathbf{c}_1 - \tfrac{1}{L}\mathbf{n}(\mathbf{c}_1)\big) + \tfrac{1}{2}\big(\mathbf{c}_2 - \tfrac{1}{L}\mathbf{n}(\mathbf{c}_2)\big) \tag{144}$$

is also the midpoint of a line joining two points of $\tilde{S}$. By convexity, $\mathbf{c}$ is inside $\tilde{S}$. The convexity of $\tilde{S}$ also implies that $\tilde{S}$ lies on one side of the tangent plane at $\tilde{\mathbf{p}}(\mathbf{p})$, the side opposite $\mathbf{n}(\mathbf{p})$. Hence $\mathbf{c}$, which is inside $\tilde{S}$ and $\mathbf{p} \in S$ are on opposite sides of the tangent plane to $\tilde{\mathbf{p}}(\mathbf{p})$. In particular, the straight line from $\mathbf{c}$ to the nearest point, say $\mathbf{p}_0$, of S is parallel to $\mathbf{n}(\mathbf{p}_0)$ and coincides, in part, with the line from $\tilde{\mathbf{p}}(\mathbf{p}_0)$ to $\mathbf{p}_0$, which is of length $\frac{1}{L}$. We conclude that $\|\mathbf{p} - \mathbf{c}\| \geq \frac{1}{L}$ for every $L > K$ and every $\mathbf{p} \in S$.



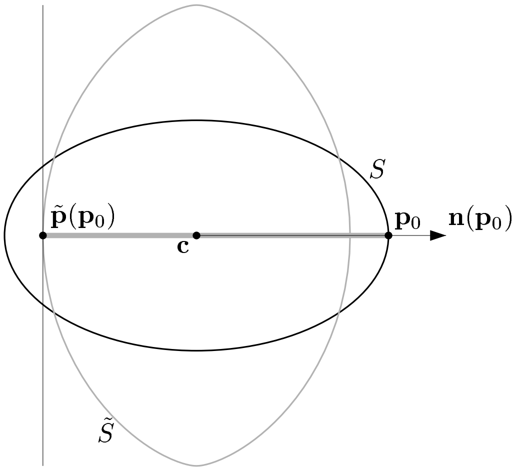
The proof that $\|\mathbf{p}\| \leq \frac{1}{k}$ is similar. This time, one lets $\ell < k$ and defines

$$\tilde{\mathbf{p}}(\mathbf{p}) = \mathbf{p} - \tfrac{1}{\ell}\mathbf{n}(\mathbf{p}) \tag{145}$$

and sets

$$\tilde{S} = \big\{\tilde{\mathbf{p}}(\mathbf{p}) \ : \ \mathbf{p} \in S\big\} \tag{146}$$

This time, $\tilde{S}$, and hence $\mathbf{c}$, lies on the same side of the tangent plane at $\tilde{\mathbf{p}}(\mathbf{p})$ as $\mathbf{n}(\mathbf{p})$. So the straight line from $\mathbf{c}$ to the farthest point, say $\mathbf{p}_0$, of S is contained in the line from $\tilde{\mathbf{p}}(\mathbf{p}_0)$ to $\mathbf{p}_0$, which is of length $\frac{1}{\ell}$.



When S is invariant under inversion in the origin, $\mathbf{n}(\mathbf{c}_1) = -\mathbf{n}(\mathbf{c}_2)$ implies that $\mathbf{c}_1 = -\mathbf{c}_2$ so that $\mathbf{c} = \frac{1}{2}\big(\mathbf{c}_1 + \mathbf{c}_2\big) = \mathbf{0}$.

B Proof of Lemma 2

Let $\mathbf{p}$ be any point of S_{E_0} and let $\mathbf{t}$ be any principal direction for S_{E_0} at $\mathbf{p}$. Let $\mathbf{q}(s)$ be a curve on S_{E_0} that is parametrized by arc length, passes through $\mathbf{p}$ at $s = 0$ and has tangent vector $\mathbf{t}$ there. The principal curvature κ corresponding to $\mathbf{t}$ obeys

$$\kappa\mathbf{t} = \frac{d}{ds}\frac{\nabla E_0\big(\mathbf{q}(s)\big)}{\|\nabla E_0\big(\mathbf{q}(s)\big)\|}\Bigg|_{s=0} = \frac{E_0{}''(\mathbf{p})\mathbf{t}}{\|\nabla E_0(\mathbf{p})\|} + \nabla E_0\big(\mathbf{p}\big)\ \frac{d}{ds}\frac{1}{\|\nabla E_0\big(\mathbf{q}(s)\big)\|}\Bigg|_{s=0} \tag{147}$$

and hence

$$\kappa = \frac{\big(\mathbf{t}, E_0{}''(\mathbf{p})\mathbf{t}\big)}{\|\nabla E_0(\mathbf{p})\|} \tag{148}$$

Consequently, S_{E_0} is a convex surface that is invariant under inversion in the origin and has all principal curvatures between $\frac{\omega_0}{G_0}$ and $\frac{G_0}{g_0}$. By Lemma 1,

$$\frac{\partial}{\partial r}E_0(\mathbf{p}(r,\theta)) = \nabla E_0(\mathbf{p}(r,\theta)) \cdot \frac{\partial \mathbf{p}}{\partial r}(r,\theta) \geq \|\nabla E_0(\mathbf{p}(r,\theta))\| \frac{\omega_0/G_0}{G_0/g_0} \geq \frac{\omega_0 {g_0}^2}{{G_0}^2} \tag{149}$$

for all $r = r_F(E_0,\theta)$. Choose $g_1 = \frac{\omega_0 {g_0}^2}{4{G_0}^2}$ and $r_0 = \min\{\frac{g_1}{G_0}, \delta_0\}$. Then

$$\begin{aligned} \frac{\partial}{\partial r}E_0(\mathbf{p}(r,\theta)) \quad &= \quad \nabla E_0\big(\mathbf{p}(r_F(E_0,\theta),\theta)\big) \cdot \tfrac{\partial \mathbf{p}}{\partial r}(r,\theta) \qquad (150) \\ &\qquad + \Big[\nabla E_0\big(\mathbf{p}(r,\theta)\big) - \nabla E_0\big(\mathbf{p}(r_F(E_0,\theta),\theta)\big)\Big] \cdot \tfrac{\partial \mathbf{p}}{\partial r}(r,\theta) \end{aligned}$$

and

$$\Big|\Big[\nabla E_0\big(\mathbf{p}(r,\theta)\big) - \nabla E_0\big(\mathbf{p}(r_F(E_0,\theta),\theta)\big)\Big] \cdot \tfrac{\partial \mathbf{p}}{\partial r}(r,\theta)\Big| \leq G_0\big|r - r_F(E_0,\theta)\big| \tag{151}$$

so

$$\tfrac{\partial}{\partial r}E_0(\mathbf{p}(r,\theta)) \geq 2g_1 \qquad \text{for all} \quad \big|r - r_F(E_0,\theta)\big| \leq 2r_0,\ \theta \in S^{d-1} \tag{152}$$

Similarly, if $|e - E_0|_1 \leq g_1$,

$$\tfrac{\partial}{\partial r}e(\mathbf{p}(r,\theta)) \geq g_1 \qquad \text{for all} \quad \big|r - r_F(E_0,\theta)\big| \leq 2r_0,\ \theta \in S^{d-1} \tag{153}$$

This verifies (21). We merely need to choose $\varepsilon < g_1$.

To verify (20), observe that if $|e - E_0|_0 \leq r_0 g_1$, then $\big|e\big(r_F(E_0,\theta),\theta\big)\big| \leq r_0 g_1$ and hence

$$\big|r_F(e,\theta) - r_F(E_0,\theta)\big| \leq r_0 \tag{154}$$

by (21).

The same argument that shows that $\mathcal{E}_s$ is open in $(C^2_s(\mathcal{B},\mathbb{R}), |\cdot|_2)$ also yields $e \in \mathcal{E}_s(\delta_0/2, g_0/2, 2G_0, \omega_0/2)$, if we choose ε small enough, depending only on δ_0, g_0, G_0 and ω_0.

Acknowledgements

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