FAMILIES OF PERIODIC SOLUTIONS OF RESONANT PDE's

Dario BAMBUSI*, Simone PALEARI*
Dipartimento di Matematica dell'Università,
Via Saldini 50, 20133 MILANO, Italy.

Abstract. We construct some families of small amplitude periodic solutions close to a completely resonant equilibrium point of a semilinear partial differential equation. To this end we construct, using averaging methods, a suitable functional in the unit ball of the configuration space. We prove that to each nondegenerate critical point of such a functional there corresponds a family of small amplitude periodic solutions of the system. The proof is based on Lyapunov–Schmidt decomposition. As an application we construct countable many families of periodic solutions of the nonlinear string equation $u_{tt}-u_{xx}\pm u^3=0$ with Dirichlet boundary conditions (and of its perturbations). We also prove that the fundamental periods of solutions belonging to the n^{th} family converge to $2\pi/n$ when the amplitude tends to zero.

1. Introduction

In this paper we study existence of families of small amplitude periodic solutions in some semilinear partial differential equations of the form

$$\frac{d^2u}{dt^2}+Au=f(u)\ ,\quad u\in\ell^2_s \tag{1.1}$$

where A is a positive self-adjoint operator with pure point spectrum that we assume to be completely resonant. ℓ^2_s is a suitable Hilbert space playing the role of the configuration space (for the precise definition see below). Concerning the map f we assume some smoothness to be specified below, that its Taylor expansion starts with terms of order higher or equal to two and that the first non-vanishing term of such a Taylor expansion is the gradient of a functional $F^{(0)}$. Consider the average $\langle F^{(0)}\rangle$ of $F^{(0)}$ with respect to the linear flow. Restrict $\langle F^{(0)}\rangle$ to a surface of constant harmonic energy; let x_0 be a critical point of the so obtained functional: we prove that if such a critical point is non-degenerate than correspondingly there exists a family of small amplitude periodic solutions of the system (1.1). Moreover the trajectories of such periodic solutions are close to those of the solutions of the linearized

---

* e-mail:`bambusi,simone@mat.unimi.it`

system with initial datum $u(0) = \epsilon x_0, \dot{u}(0) = 0$; ϵ is a small parameter that varies in a non countable set which accumulates at the origin. We also study the fundamental period T_ϵ of the so obtained solutions: we show that it cannot be essentially shorter than the fundamental period T_0 of the solution of the linearized system with initial datum $(x_0, 0)$. Conditions under which T_ϵ converges to T_0 are also obtained.

An application to the equation

$$u_{tt} - u_{xx} = \pm u^3 + \text{higher order terms} , \quad u(0,t) = u(\pi,t) = 0 \tag{1.2}$$

is also made. For such an equation we compute all the critical points of the considered functional and prove that they are non-degenerate. We obtain countable many families of periodic solutions which are described in a quite precise way and we find their fundamental periods. We show that for the solutions of the n^{th} family the fundamental period converges to $2\pi/n$.

Our proof is based on a Lyapunov-Schmidt decomposition, namely we study (1.1) as a functional equation in a Hilbert space of time periodic functions. As usual we decompose the problem into an equation on the range of the linear operator describing the linearized equation and an equation on its Kernel. In the case of PDE's the equation on the range (P–equation) usually involves small denominators (see [5,3,4]), however using a method already introduced in [2,7], we solve such an equation in a particularly simple way which avoids small denominators.

Then it remains to solve the equation on the Kernel (Q–equation). The technique we use to this end is the main contribution of the present paper (see sect. 3.2): it is based on the remark that the Q–equation is equivalent to an equation in the configuration space and that such an equation appears as the perturbation of the equation for the critical points of the functional introduced above. So, one can starts from a non-degenerate critical point of the above functional and apply the implicit function theorem to obtain a solution of the Q–equation.

We think that this establishes a relation between Lyapunov–Schmidt techniques and averaging theory that could be interesting in itself.

The abstract result of the present paper is an extension to partial differential equations of a result proved, for finite dimensional systems, in [8]. Our technique is completely different from that of [8] which seems not to extend to PDE's.

Finally we recall that families of periodic (and quasi-periodic) solutions of equations of the kind of (1.1) have been constructed by many authors (see e.g. [9,5,3,4,10,13]), but standard results are not (directly) applicable to the present case since they deal with nonresonant systems, while we deal here with resonant systems (for more details see the discussion of sect. 3). Moreover we need much less smoothness than that required in these papers. We also recall that a previous proof of existence of small oscillations in the particular case of the equation (1.2) is due to Lidskij and Shulmann[11] however the technique of [11] seems ad-hoc, while the present one is quite general and allows also to determine the fundamental periods of the periodic solutions.

The paper is organized as follows: in sect. 2 we give a precise statement of our abstract result on existence of periodic solutions. In sect. 3 we comment on its relations with previous results on families of small amplitude periodic solutions in PDE's. In sect. 4 we give the proof of the abstract theorem. Such a section is divided into two parts. The first deals with the P–equation, while the second is the most original one and deals with the Q–equation and with the relation between Lyapunov Schmidt methods and averaging methods. In sect. 5 we study the minimal periods of the periodic orbits we found. Finally in sect. 6 we apply the abstract results to the nonlinear string equation.

Acknowledgments We thank F. Verhulst for some interesting comments on a previous version of the paper and Susanna Terracini for some interesting discussion.

2. Statement of the main result

Fix $s \geq 0$, then we define the Hilbert space ℓ^2_s to be the space of the sequences u_j such that

$$|u|^2_s := \sum_{j\geq 1} j^{2s} u^2_j < \infty \ .$$

Consider a sequence $\Omega := (k_1, k_2, k_3, ...)$ of integers which will play the role of a frequency vector. Then we define the operator $A : \ell^2_s \supset D(A) \to \ell^2_s$ defining the linear part of the system by $(Au)_j := k^2_j u_j$, with $D(A)$ its maximal domain of definition.
Endowing $D(A)$ by the graph norm we obtain a Hilbert space that we will denote by $\mathcal{Q}$.
We assume that

1) The injection of $\mathcal{Q}$ into ℓ^2_s is compact. $\square$

In this section, for simplicity we also assume $k_1 = 1$; in section 5 we will show how to eliminate this assumption.
We will look for periodic mild solutions of the system of differential equations

$$\frac{d^2}{dt^2} u + Au = f^{(0)}(u) + f^{(1)}(u) \ , \quad u \in \ell^2_s \ . \tag{2.1}$$

where $f^{(0)}$, $f^{(1)}$ are nonlinear operators. We assume

2) $f^{(0)} : \ell^2_s \to \ell^2_s$ is a bounded homogeneous polynomial of degree $r \geq 2$, and there exists a Gateaux differentiable functional $F^{(0)} : \mathcal{Q} \to \mathbb{R}$ such that, denoting by $\nabla F^{(0)} := \left(\frac{\partial F^{(0)}}{\partial x_j}\right)$ its ℓ^2 gradient, we have $f^{(0)} = -\nabla F^{(0)}$. $\square$
3) $f^{(0)}$ leaves invariant the domain $D(A) \equiv \mathcal{Q}$ of A, and is bounded also as a polynomial from $\mathcal{Q}$ to $\mathcal{Q}$. $\square$
4) $f^{(1)}$ vanishes at zero, has a Lipschitz first derivative and satisfies the estimate

$$\sup_{\|u^i\|\leq 2\epsilon} \left| \mathrm{d}f^{(1)}(u^1) - \mathrm{d}f^{(1)}(u^2) \right|_s \leq C_1 \epsilon^{r-1} \left| u^1 - u^2 \right|_s \ ,$$

where $\mathrm{d}f^{(1)}$ denotes the differential of the map $f^{(1)}$. $\square$

Remark 2.1. *In particular* $f^{(1)}$ *has a zero of order* $r+1$ *at zero.*

To state our main theorem we have to define the functional whose critical points give rise to families of periodic solutions. To this end we need a few notations.
$x = \{x_j\}$ will denote a point of the configuration space ℓ^2_s or $\mathcal{Q} \equiv D(A)$;

$$E(x) := \sum_{j\geq 1} k_j^2 \frac{x_j^2}{2}$$

is a function whose level surfaces are the intersection of the energy level surfaces with the hyperplane $\dot{u} = 0$;

$$E^{-1}(1) := \{x \in \mathcal{Q} \ : \ E(x) = 1\}$$

is the level surface of E *in the space* $\mathcal{Q}$;

$$\Phi^t(x) := \sum_{j\geq 1} x_j \cos(k_j t) e_j$$

is the solution of the linearized equation with initial datum $u(0) = x, \dot{u}(0) = 0$; here $e_j \in \ell^2_s$ denotes the vector having the $j - th$ component equal to 1 and all other components equal to zero;

$$\langle F^{(0)}\rangle(x) := \frac{1}{2\pi}\int_0^{2\pi} F^{(0)}(\Phi^t(x))dt$$

is the average of $F^{(0)}$ with respect to the linear flow, and

$$\mathcal{F}^{(0)} := \langle F^{(0)}\rangle\big|_{E^{-1}(1)} \tag{2.2}$$

is its restriction to the surface $E(x) = 1$;
finally $W_\gamma \subset \mathbb{R}$ is the set of the real numbers (frequencies) ω fulfilling

$$|\omega l - k_j| \geq \frac{\gamma}{l} \ , \forall (j,l) \in \mathbf{N}\times\mathbf{N} \text{ such that } l \geq 1 \ , \quad k_j \neq l \tag{2.3}$$

Remark 2.2. *The periodic orbits we will find have frequencies* $\omega \in W_\gamma$.

Remark 2.3. *The set* W_γ *contains at least all the irrationals whose continued fraction expansion is of the form* $[1, a_1, a_2, ...]$ *fulfilling* $a_i < \gamma^{-1} - 2$ *for all* $i \geq 2$ *(see e.g. [14, p.23]). Such set is uncountable and in particular it accumulates at 1 from the right. The set* W_γ *contains also the irrationals whose continued fraction expansion is of the form* $[0, 1, a_2, a_3, ...]$, *fulfilling* $a_i < \gamma^{-1} - 2$ *for all* $i \geq 3$. *Such a second set accumulates to 1 from the left.*

Definition. *A critical point* x_0 *of* $\mathcal{F}^{(0)}$ *will be said to be non-degenerate if the 2–form* $\mathrm{d}^2\mathcal{F}^{(0)}(x_0)$ *is weakly non-degenerate, i.e. if*

$$\Big[\ Y \in T_{x_0}E^{-1}(1) \ , \quad \langle \mathrm{d}^2\mathcal{F}^{(0)}(x_0); X, Y\rangle = 0 \ , \quad \forall X \in T_{x_0}E^{-1}(1) \quad \Big] \Longrightarrow Y = 0$$

Here we denoted by $\mathrm{d}^2\mathcal{F}^{(0)}(x_0)$ *the second differential of* $\mathcal{F}^{(0)}$.

Remark 2.4. *Let x_0 be a critical point of $\mathcal{F}^{(0)}$, then there exists β_0, Lagrange multiplier, such that the following equation is satisfied*

$$\beta_0 \nabla E(x_0) - \nabla \langle F^{(0)} \rangle (x_0) = 0 \ .$$

Theorem 2.5. *Let $x_0 \in \mathcal{Q}$ be a non-degenerate critical point of $\mathcal{F}^{(0)}$ (cf. (2.2)), assume that the corresponding Lagrange multiplier β_0 is different from zero. Fix a positive $\gamma < 1/3$, then there exists a family $\{u_\epsilon(t)\}_{\epsilon \in \mathcal{E}_\gamma}$ of periodic solutions of (2.1), with the following properties*

1) The set $\mathcal{E}_\gamma \subset \mathbb{R}^+$ is uncountable and has zero as an accumulation point;

2) u_ϵ is periodic with period $2\pi/\omega_\epsilon$, and is of class $H^1([0, 2\pi/\omega_\epsilon], \ell^2_s)$;

3) there exists a strictly positive ω_* such that the frequency map ω_ϵ

$$\mathcal{E}_\gamma \ni \epsilon \mapsto \omega_\epsilon \in W \ , \quad \text{with} \quad W := \begin{cases} W_\gamma \cap [1, 1+\omega_*) & \text{if} \quad \beta_0 < 0 \\ W_\gamma \cap (1-\omega_*, 1] & \text{if} \quad \beta_0 > 0 \end{cases}$$

is monotone and one to one; moreover it fulfills the following estimate

$$|1 - \omega_\epsilon| \le C_2 \epsilon^{r-1} \ ; \tag{2.4}$$

4) u_ϵ is close to the solution of the linearized system with initial datum ϵx_0, precisely

$$\sup_{t \in \mathbb{R}} \left| u_\epsilon(t) - \epsilon \Phi^{t\omega^\epsilon}(x_0) \right|_s \le C_3 \epsilon^2 \ ; \tag{2.5}$$

5) u_ϵ has energy close to ϵ^2, precisely one has

$$\sup_{t \in \mathbb{R}} |E(u_\epsilon(0)) - \epsilon^2| \le C_4 \epsilon^{r+1} \ .$$

Remark 2.6. *We remark that the periodic solutions we have found have periods which are highly nonresonant with the period of the linear oscillations.*

The following remarks could be useful to verify the non-degeneracy condition of the theorem.

Remark 2.7. *Defining the linear operator $B_{x_0} : \mathcal{Q} \to \ell^2_s$ by*

$$B_{x_0} = \beta_0 A - \mathrm{d} \left[\nabla \langle F^{(0)} \rangle \right] (x_0) \ . \tag{2.6}$$

one has

$$\langle \mathrm{d}^2 \mathcal{F}^{(0)}(x_0); X, Y \rangle = \langle B_{x_0} X; Y \rangle_{\ell^2} \ , \quad \forall X, Y \in T_{x_0} E^{-1}(1) \ . \tag{2.7}$$

Remark 2.8. *The nondegeneracy of $\mathrm{d}^2 \mathcal{F}^{(0)}(x_0)$ is equivalent to*

$$\left\{ \beta A x_0 + B_{x_0} X = 0 \ , \quad X \in T_{x_0} E^{-1}(1) \ , \quad \beta \in \mathbb{R} \right\} \Longrightarrow X = \beta = 0 \ . \tag{2.8}$$

3. Discussion

In this section we add some comments on the relation of our result with the results by Kuksin[9], Craig & Wayne[5], Bourgain[3,4] (see also [13]). As anticipated above all these results do not apply directly to our resonant case, since their application requires the unperturbed system to be nonresonant. However, in principle it is possible to extend the above methods to deal with resonant systems. We comment now on this point.

A way to try to apply the results by Kuksin to the present case could be to follow the strategy of Dell'Antonio[8], i.e. (i) to use averaging theory to transform the original equation into a perturbation of the averaged system, (ii) to look for periodic solutions of the averaged system and then (iii) to use Kuksin's theorem to continue such approximate periodic solutions to exact periodic solutions of the system under consideration. However there is a difficulty in order to use this strategy: the application of Kuksin's theorem requires the existence of a system of coordinates where the Hamiltonian describing the linearization of the averaged system about its periodic orbit takes the form

$$\omega I + \sum_{k\geq 1} \omega_k \frac{p_k^2 + q_k^2}{2} \ .$$

It is possible to prove that in the finite dimensional case such a system of coordinates always exists, but in the infinite dimensional case it might not exist. We were not able to prove that in the situation we are studying such a coordinate system actually exists, so we abandoned this approach.

There is a very strict relation of the present work with the works by Craig & Wayne[5] and by Bourgain[3,4] and in particular with the work [6]. Indeed the starting point of these works is a Lyapunov–Schmidt decomposition which, in the present resonant case, takes exactly the same form of that we perform. The main contribution of the works [5,3,4] consists of the technique they use to solve the P–equation (the equation on the range); probably the techniques of those papers are applicable also to the present resonant case but we do not have a clear idea on that. Moreover the techniques of [5,3,4] are much more complicate than ours. Then, after solving the P–equation there remains to solve the Q-equation, which in the present case is infinite dimensional, while in the cases of [5,3,4,6] is finite dimensional and very simple; the main contribution of the present paper consists in the technique we introduce to solve the Q–equation. In particular we reduce the problem of solving the Q-equation to the problem of finding nondegenerate critical points of the functional $\mathcal{F}^{(0)}$.

On the contrary (as anticipated in the introduction), the technique we use to solve the P–equation is not new: it was already introduced in the paper [2], and is inspired by that used by de la LLave to prove variationally the existence of nonresonant periodic orbits in nonlinear wave equations (see [7]).

4. Proof of theorem 2.5

3.1 General setting and solution of the P-equation.

We look for periodic solutions of (2.1) of the form

$$u(t) = q(\omega t)$$

with $q(t)$ of the form

$$q(t) := \sum_{j\geq 1} \left\{ q_{0j} + \sum_{l\geq 1} q_{lj}\cos(lt) \right\} e_j \ , \tag{4.1}$$

so that its frequency is ω. Define $\mathcal{H}_s \subset H^1([-\pi,\pi],\ell^2_s)$ to be the Hilbert space of the functions q of the form (4.1) fulfilling

$$\|q\|^2_{\mathcal{H}_s} := \sum_{j\geq 1} \left\{ 2j^{2s}q^2_{0j} + \sum_{l\geq 1} q^2_{lj} j^{2s}(1+l^2) \right\} = \frac{1}{\pi}\int_0^{2\pi} \left(|q(t)|^2_s + |\dot{q}(t)|^2_s \right) dt < \infty$$

endowed by the natural scalar product which will be denoted by $\langle .;. \rangle_{\mathcal{H}_s}$.

Remark 4.1. *One has*

$$q_{0j} = \frac{1}{2\pi}\int_0^{2\pi} \langle q(t); e_j\rangle_{\ell^2} dt \ , \quad q_{lj} = \frac{1}{\pi}\int_0^{2\pi} \langle q(t); e_j\rangle_{\ell^2} \cos lt dt \ .$$

Remark 4.2. *The Nemitski operator associated to $f^{(0)}$, i.e. the map $q \mapsto f^{(0)} \circ q$ is analytic as a map from $\mathcal{H}_s$ to itself.*

Remark 4.3. *The Nemitski operator associated to $f^{(1)}$ is a Lipschitz map from $\mathcal{H}_s$ to itself; moreover one has*

$$\sup_{\substack{\|q\|_{\mathcal{H}_s}\leq\epsilon \\ \|\tilde{q}\|_{\mathcal{H}_s}\leq\epsilon}} \left\| f^{(1)}(q) - f^{(1)}(\tilde{q}) \right\|_{\mathcal{H}_s} \leq C_5 \epsilon^r \|q-\tilde{q}\|_{\mathcal{H}_s}$$

Define the operator L_ω by

$$L_\omega : \mathcal{H}_s \supset D(L_\omega) \to \mathcal{H}_s \ ; \qquad L_\omega q := \omega^2 \frac{d^2}{dt^2} q + Aq \ .$$

Then any solution of the equation

$$L_\omega q = f^{(0)}(q) + f^{(1)}(q) \ , \qquad q \in \mathcal{H}_s \tag{4.2}$$

is a periodic solution with frequency ω of (2.1).

We look for periodic solution with frequency close to 1.

We will denote $K := Ker(L_1)$, and $R := K^\perp$; correspondingly we will use the projectors $Q : \mathcal{H}_s \to K$ and $P = \mathbb{1} - Q$.

Remark 4.4. *K is generated by the functions $\cos(k_j t)e_j \equiv \Phi^t(e_j)$.*

Remark 4.5. *K is isomorphic in a natural way to the "configuration space" $\mathcal{Q}$. Indeed, given $x \in \mathcal{Q}$ we define the isomorphism I by $q \equiv I(x)$ with $q(t) := \Phi^t(x)$. To make this isomorphism isometric we change the norm of $\mathcal{Q}$ from the graph norm to the following equivalent norm*

$$\|x\|^2_{\mathcal{Q}} := \sum_{j:k_j\neq 0} x_j^2 j^{2s}(1+k_j^2) + \sum_{j:k_j=0} 2j^{2s}x_j^2 \ .$$

We look for a solution q of (4.2) of the form

$$q = \epsilon v + \epsilon^r q_\perp \ , \text{ with } \quad q_\perp \in R \ , \quad v \in K \tag{4.3}$$

and ϵ a (small) parameter. We put

$$\omega^2 = 1 + 2\beta\epsilon^{r-1} \tag{ω}$$

with a still undetermined β.

Projecting (4.2) on K and R, and taking into account that $f^{(0)}$ is homogeneous of degree r, we obtain the system

$$L_\omega q_\perp = Pf^{(0)}(v) + P\left[f^{(0)}(v+\epsilon^{r-1}q_\perp) - f^{(0)}(v) + \frac{1}{\epsilon^r}f^{(1)}(\epsilon v + \epsilon^r q_\perp)\right] \tag{P}$$

$$-2\beta A v = Qf^{(0)}(v) + Q\left[f^{(0)}(v+\epsilon^{r-1}q_\perp) - f^{(0)}(v) + \frac{1}{\epsilon^r}f^{(1)}(\epsilon v + \epsilon^r q_\perp)\right] \tag{Q}$$

It is useful to consider $\beta, \omega, \epsilon, q_\perp$ and v as independent quantities related only by the system (P, Q, ω).

We start by studying the (P)-equation.

Lemma 4.6. *Fix $\omega \in W_\gamma$, then $L_\omega\big|_R$ admits a bounded inverse L_ω^{-1} satisfying*

$$\left\|L_\omega^{-1}\right\| \leq C_6\gamma^{-1} \tag{4.4}$$

with a C_6 independent of ω. Moreover, for any positive ρ there exists $\epsilon_1 > 0$ dependent on ω only through γ and a Lipschitz function

$$\begin{aligned} q_\perp^\omega : B_\rho \times (-\epsilon_1, \epsilon_1) &\to R \\ (v,\epsilon) &\mapsto q_\perp^\omega(v,\epsilon) \end{aligned}$$

which solves (P); here we denoted by $B_\rho \subset K$ the ball of radius ρ in K. Moreover one has

$$\left\|q_\perp^\omega(v,\epsilon) - L_\omega^{-1}Pf^{(0)}(v)\right\|_{\mathcal{H}_s} \leq C_7\epsilon \ . \tag{4.5}$$

Proof. To invert the operator L_ω remark that its eigenvalues λ_{lj} are given by

$$\lambda_{lj} := -l^2\omega^2 + k_j^2 = (-l\omega + k_j)(l\omega + k_j) \ ,$$

so that (by the very definition of W_γ),

$$|\lambda_{lj}| \geq \frac{\gamma}{2} \ , \quad \forall (l,j) \in N^2 \ , \quad l \neq k_j$$

and therefore L_ω has a bounded inverse on R fulfilling (4.4). Apply L_ω^{-1} to (P), then the implicit function theorem gives the result. □

3.2 Solution of the Q–equation via averaging methods

To study the (Q)-equation we will use the isomorphism I of K with the configuration space $\mathcal{Q}$. Remark that the Q–equation is a perturbation of the equation

$$-2\beta A v - Q f^{(0)}(v) = 0 \ . \tag{4.6}$$

To pull back this equation to the configuration space it is useful to split $\mathcal{Q}$ into the kernel of A and its orthogonal. So we denote

$$\mathcal{N} := \{x \in \mathcal{Q} \ : \ Ax = 0\} \ ,$$

and $\mathcal{R} := \mathcal{N}^\perp \equiv span\{e_j \ : \ k_j \neq 0\}$ its orthogonal. We also denote

$$\tilde{\mathcal{N}} := I(\mathcal{N}) \ , \quad \tilde{\mathcal{R}} := I(\mathcal{R}) \ ,$$

and by $\mathcal{P}_{\tilde{\mathcal{N}}}, \mathcal{P}_{\tilde{\mathcal{R}}}$ the orthogonal projectors (in K) on $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{R}}$ respectively.

Lemma 4.7. *Let $F : \mathcal{Q} \to$ IR be a Gateaux differentiable functional such that $\nabla F : \mathcal{Q} \to \mathcal{Q}$, then one has*

$$\begin{aligned} \mathcal{P}_{\tilde{\mathcal{R}}} I \left(\nabla\langle F\rangle(x)\right) &= \frac{1}{2}\mathcal{P}_{\tilde{\mathcal{R}}} Q \left[\nabla F(I(x))\right] \ , \\ \mathcal{P}_{\tilde{\mathcal{N}}} I \left(\nabla\langle F\rangle(x)\right) &= \mathcal{P}_{\tilde{\mathcal{N}}} Q \left[\nabla F(I(x))\right] \ . \end{aligned} \tag{4.7}$$

Proof. Denote $q \equiv I(x)$, and by $[\nabla F(q)]_{k_j j}$, the component on $\cos(k_j t) e_j$ of $\nabla F(q) \equiv (\nabla F) \circ q$. For j such that $k_j \neq 0$ one has (see remark 4.1)

$$[\nabla F(q)]_{k_j j} = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial F}{\partial x_j}(q(t)) \cos(k_j t) dt \ , \tag{4.8}$$

and remark that

$$\mathcal{P}_{\tilde{\mathcal{R}}} Q[\nabla F(q)] = \sum_{j:k_j\neq 0} [\nabla F(q)]_{k_j j} \cos(k_j t) e_j \ . \tag{4.9}$$

On the other hand one has

$$\begin{aligned}\frac{\partial\langle F\rangle}{\partial x_j}(x) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_i \frac{\partial F}{\partial x_i}(\Phi^t(x)) \frac{\partial[\Phi^t]^i}{\partial x_j}(x)dt = \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial F}{\partial x_j}(q(t))\cos(k_j t)dt \ ,\end{aligned}$$

which coincides with one half of (4.8); applying I this gives the first of (4.7). The second one is proved in the same way. $\square$

Then remark that $-\beta A I(x) = I(-\beta A x)$, so that projecting (4.6) on $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{R}}$, one has

$$\mathcal{P}_{\tilde{\mathcal{N}}}\left(I\left(-2\beta A x\right)\right) = -\mathcal{P}_{\tilde{\mathcal{N}}}\left(I\nabla\langle F^{(0)}\rangle(x)\right) \tag{4.10}$$

$$\mathcal{P}_{\tilde{\mathcal{R}}}\left(I\left(-2\beta A x\right)\right) = -\mathcal{P}_{\tilde{\mathcal{R}}}\left(2I\nabla\langle F^{(0)}\rangle(x)\right) \tag{4.11}$$

and since $Ax \in \mathcal{R}$ the l.h.s. of (4.10) vanishes. It follows that (4.10) is also equivalent to

$$\mathcal{P}_{\tilde{\mathcal{N}}}\left(I\left(-2\beta A x\right)\right) = -2\mathcal{P}_{\tilde{\mathcal{N}}}\left(I\nabla\langle F^{(0)}\rangle(x)\right)$$

so that (4.6) is equivalent to the following equation in the configuration space $\mathcal{Q}$:

$$-\beta A x = -\nabla\langle F^{(0)}\rangle(x) \ , \tag{4.12}$$

Moreover one has $Ax = \nabla E(x)$, so (4.12) is the equation for the critical points of $\langle F^{(0)}\rangle$ restricted to a surface $E(x)$ =constant, β appearing as a Lagrange multiplier. Actually, since $f^{(0)}$ (and therefore also $\langle F^{(0)}\rangle$) is a homogeneous polynomial, its critical points on a surface $E(x)$ =constant are obtained just by rescaling the critical points on the surface $E(x) = 1$. So, we will look at (4.12) as the equation for the critical points of $\langle F^{(0)}\rangle$ restricted to the surface $E(x) = 1$.

Inserting $q_\perp^\omega(v,\epsilon)$ in the Q–equation and transporting this equation to $\mathcal{Q}$ we get the equation

$$\beta\nabla E(x) - \nabla\langle F^{(0)}\rangle(x) + G^{(1)}(x,\epsilon) = 0 \ , \tag{Q'}$$

where

$$\begin{aligned}&G^{(1)}(x,\epsilon) \\ &= \frac{1}{2} I^{-1} Q\left(f^{(0)}(I(x) + \epsilon^{r-1} q_\perp^\omega(I(x),\epsilon)) - f^{(0)}(I(x)) + \frac{1}{\epsilon^r} f^{(1)}(\epsilon I(x) + \epsilon^r q_\perp^\omega(I(x),\epsilon))\right) \ .\end{aligned}$$

In particular remark that

$$\left\|G^{(1)}(x,\epsilon)\right\|_{\mathcal{Q}} \leq C_8\epsilon$$

We will use the implicit function theorem to solve the Q'-equation. To this end we will assume that a critical point of $\langle F^{(0)}\rangle|_{E^{-1}(1)}$ actually exists and is nondegenerate.

Proposition 4.8. *Let x_0 be a nondegenerate critical point of $\mathcal{F}^{(0)}$ with Lagrange multiplier $\beta_0 \neq 0$; then there exist a positive $\epsilon_\sharp$ and a Lipschitz map*

$$\begin{aligned} (-\epsilon_\sharp, \epsilon_\sharp) &\to \mathbb{R} \times E^{-1}(1) \\ \epsilon &\mapsto (\beta(\epsilon), \bar{x}(\epsilon)) \ , \end{aligned}$$

which solves equation (Q'). Moreover one has

$$\|\bar{x} - x_0\|_{\mathcal{Q}} \leq C_9\epsilon \ , \quad |\beta - \beta_0| \leq C_{10}\epsilon \ .$$

Proof. Consider the map

$$\mathbb{R} \times \mathbb{R} \times E^{-1}(1) \ni (\epsilon, \beta, x) \mapsto \beta\nabla E(x) - \nabla\langle F^{(0)}\rangle(x) + G^{(1)}(x,\epsilon) \in \ell^2_s \ ;$$

it is Lipschitz, differentiable at $\epsilon = 0$, and equal to zero at $(0, \beta_0, x_0)$. In order to apply the implicit function theorem we have to show that its differential with respect to (β, x) at $(0, \beta_0, x_0)$ is an isomorphism. The differential with respect to (β, x) at $(0, \beta_0, x_0)$ is given by the map

$$\mathbb{R} \times T_{x_0}[E^{-1}(1)] \ni (\beta, X) \mapsto \beta A x_0 + B_{x_0} X \in \ell^2_s \ , \tag{4.13}$$

where B_{x_0} is the operator defined by (2.6). So, we have to show that $\forall Y \in \ell^2_s$ the equation

$$\beta A x_0 + B_{x_0} X = Y \tag{4.14}$$

is solvable. We prove that the left hand side is a Fredholm operator. First remark that A is Fredholm, and that, by the assumption that $f^{(0)} : \mathcal{Q} \to \mathcal{Q}$ is smooth, the operator $\mathrm{d}[\nabla\langle F^{(0)}\rangle](x_0)$ is compact as an operator from $\mathcal{Q}$ to ℓ^2_s. Therefore, by $\beta_0 \neq 0$ the operator B_{x_0} is the sum of a Fredholm operator and a compact operator, so it is Fredholm. The operator $\beta A x_0$ has finite rank, and therefore also the l.h.s. of (4.14) is a Fredholm operator. By nondegeneracy it is also surjective. □

Taking $v = I(\bar{x})$ we satisfy the Q equation. It remains to satisfy the (ω)–equation. This can be done exactly as in the nonresonant case treated in [2], we repeat here the argument for the sake of completeness. Recall that ω is fixed, so eq.(ω) is an equation for ϵ. Denoting $\beta_1(\epsilon) := \beta(\epsilon) - \beta_0$ we have

$$\omega^2 = 1 - 2\epsilon^{r-1}(\beta_0 + \beta_1(\epsilon)) \ . \tag{4.15}$$

Rewrite (4.15) as

$$\mu = \frac{1-\omega^2}{2\beta_0} - \mu\frac{\beta_1(\mu^{1/(r-1)})}{\beta_0} \ ,$$

with $\mu = \epsilon^{r-1}$.

Fix a positive $\delta < 1$, then provided ω is close enough to 1, the r.h.s. is a contraction of a ball centered at $(1-\omega^2)/2\beta_0$ having radius $((1-\omega^2)/2\beta_0)\delta$, and therefore, for any fixed ω there exists a *unique* $\mu = \epsilon^{r-1}$ fulfilling such an equation and the estimate

$$\left| \epsilon^{r-1} - \frac{1-\omega^2}{2\beta_0} \right| \leq C_{11} \left(\frac{1-\omega^2}{2\beta_0} \right) \delta$$

with a suitable C_{11}. This concludes the proof of theorem 2.5.

5. Minimal periods

We remark that the theorem 2.5 gives no information on the *minimal period* T_ϵ of the periodic solutions u_ϵ. We first prove that T_ϵ cannot be (essentially) shorter than the minimal period of the solution of the linearized system with initial datum x_0.

Proposition 5.1. *Let x_0 be a critical point of $\mathcal{F}^{(0)}$ fulfilling the assumptions of theorem 2.5, denote by u_ϵ the periodic solutions whose existence is ensured by that theorem, and denote by T_ϵ their fundamental periods; fix an integer n such that $\forall m > n$ one has*

$$\Phi^{2\pi/m}(x_0) \neq x_0 \ ,$$

then, there exists a constant $C_{12} > 0$ and an integer $n_0 \leq n$ such that

$$\left| T_\epsilon - \frac{2\pi}{n_0} \right| \leq C_{12}\epsilon^{r-1} \ , \quad \forall \epsilon \in \mathcal{E}_\gamma \ . \tag{5.1}$$

Proof. The existence of $n_0 \geq 1$ such that (5.1) holds follows from (2.4). Assume by contradiction that $n_0 > n$, take $\eta > 0$ and, for $t \in (2\pi/n_0 - \eta, 2\pi/n_0 + \eta)$, consider

$$\begin{aligned} \frac{|u_\epsilon(t) - u_\epsilon(0)|_s}{\epsilon} &\geq \left| \Phi^{2\pi/n_0}(x_0) - x_0 \right|_s - \left[\left| \frac{1}{\epsilon} u_\epsilon(t) - \Phi^{\omega^\epsilon t}(x_0) \right|_s \right. \\ &\left. + \left| \Phi^{\omega^\epsilon t}(x_0) - \Phi^{2\pi/n_0}(x_0) \right|_s + \left| x_0 - \frac{1}{\epsilon} u_\epsilon(0) \right|_s \right] \end{aligned}$$

Using inequality (2.5), the fact that ω^ϵ converges to 1 and defining $\kappa := |\Phi^{2\pi/n_0}(x_0) - x_0|_s$ one has that the above quantity is larger than

$$\kappa - [C_3\epsilon + o(\eta) + o(\epsilon) + C_3\epsilon] \ ,$$

where we also used the continuity of the linear flow with respect to time. So we have

$$T_\epsilon \notin \left(\frac{2\pi}{n_0} - \eta, \frac{2\pi}{n_0} + \eta\right) .$$

It follows $n_0 \le n$, which is the thesis. □

Under some additional assumptions we can also ensure that the fundamental period of u_ϵ actually converges to that of the solution of the linearized system with initial datum $(x_0, 0)$.

To give the precise statement we define the subspace $\mathcal{Q}_n \subset \mathcal{Q}$ of the initial data giving rise to solutions of the linearized system with period equal to $2\pi/n$:

$$\mathcal{Q}_n := \left\{ x \in \mathcal{Q} \ : \ \Phi^{2\pi/n}(x) = x \right\} .$$

Remark 5.2. *The space $\mathcal{Q}_n$ is generated by the vectors e_j such that k_j is an integer multiple of n.*

Correspondingly we define the functional

$$\mathcal{F}_n^{(0)} := \langle F^{(0)} \rangle \big|_{E^{-1}(1) \cap \mathcal{Q}_n} . \tag{5.2}$$

Finally we define the new set of frequencies $W_\gamma^n \subset \mathbb{R}$, by

$$W_\gamma^n := \left\{ \omega \in \mathbb{R} \ : \ |\omega l - k_j| \ge \frac{\gamma}{l} \quad \forall (l, j) : nl \ne k_j \right\} .$$

Theorem 5.3. *Let $x_0 \in \mathcal{Q}$ be a critical point of $\mathcal{F}^{(0)}$ fulfilling the assumptions of theorem 2.5. Assume that the minimal period of $\Phi^t(x_0)$ is $2\pi/n$. Assume furthermore that x_0 is nondegenerate as a critical point of $\mathcal{F}_n^{(0)}$; then there exists a family of periodic solutions $\{u_\epsilon\}_{\epsilon \in \mathcal{E}_\gamma}$, which fulfills the statements 1–5 of theorem 2.5 with the set W substituted by*

$$W^n := \begin{cases} W_\gamma^n \cap [n, n + \omega_*) & \text{if} \quad \beta_0 < 0 \\ W_\gamma^n \cap (n - \omega_*, n] & \text{if} \quad \beta_0 > 0 \end{cases}$$

and with equation (2.4) substituted by $|n - \omega_\epsilon| \le C_{13} \epsilon^{r-1}$. Moreover the fundamental period of u_ϵ is $T_\epsilon := 2\pi/\omega_\epsilon$.

Remark 5.4. *The proof consists in looking for periodic solutions of system (2.1) with frequency close to n. So, the solutions we find might not coincide with those constructed by theorem 2.5. We expect also the solutions found in theorem 2.5 to have fundamental period close to $2\pi/n$.*

Remark 5.5. *By the same technique we use for the proof of this theorem one can generalize theorem 2.5 to the case $k_1 \ne 0$.*

Proof. We repeat the proof of theorem 2.5 just looking for periodic solutions with frequency close to n. So we define $K_n := Ker(L_n)$, and $R_n := K_n^{\perp}$, and denote by Q_n and P_n respectively the corresponding projectors. The space K_n is isomorphic to the space $\mathcal{Q}_n$, the isomorphism I_n being defined by

$$q \equiv I_n(x) \qquad q(t) \equiv \Phi^t_n(e_j) := \Phi^{t/n}(e_j) \ .$$

with Φ^t as above. Then one proceeds essentially as in the proof of theorem 2.5. The main difference being that in equations (4.7) of lemma 4.7 the average of the functional F with respect to the linear flow is substituted by

$$\langle F \rangle_n (x) := \frac{1}{2\pi} \int_0^{2\pi} F(\Phi^t_n(x)) dt \ , x \in \mathcal{Q}_n . \tag{5.3}$$

Concerning this point remark that, since $\Phi^{t/n}$ restricted to $\mathcal{Q}_n$ coincides with Φ^t_n, $\langle F \rangle|_{\mathcal{Q}_n}$ is just the average over n periods of Φ^t_n of the restriction of F to $\mathcal{Q}_n$. So one has that $\langle F \rangle|_{\mathcal{Q}_n} = \langle F \rangle_n$. Therefore a critical point of $\mathcal{F}^{(0)}$ is also a critical point of $\langle F^{(0)} \rangle|_{E^{-1}(1)}$. Now, if the critical point x_0 is nondegenerate also as a critical point of $\mathcal{F}^{(0)}_n$, as assumed in the statement, then following the proof of theorem 2.5, one obtains a family of periodic orbits fulfilling the statement of the theorem and having frequency close to n. By the argument of proposition 5.1 this is also the frequency corresponding to the fundamental period. $\square$

6. Application to nonlinear wave equations

Consider the nonlinear wave equation

$$u_{tt} - u_{xx} = \psi(u) \qquad u(0,t) = u(\pi, t) = 0 \ , \quad u \in H^1([0,\pi], \mathbb{R}) \tag{6.1}$$

with a nonlinearity of class C^4 fulfilling

$$\psi(0) = \psi'(0) = \psi''(0) = 0 \ , \quad \alpha := \psi'''(0) \neq 0$$

so that (6.1) is of the kind (1.2) (as one sees rescaling the variable u). Moreover, passing to sin–Fourier series it is immediate to realize that (6.1) is of the kind of (2.1) with the index s of the space equal to 1.

We also recall that (6.1) is Hamiltonian with Hamiltonian function given by

$$\int_0^{\pi} \frac{1}{2} \left(\dot{u}^2(x) + u_x^2(x) \right) dx + \int_0^{\pi} \Psi(u(x)) dx \ ,$$

with Ψ such that $\Psi' = -\psi$. In particular the main part of the perturbation admits a potential $F^{(0)}$ given by

$$F^{(0)}(u) = -\frac{\alpha}{4}\int_0^\pi u^4(x)dx \ .$$

To give our result on eq. (6.1) we will use elliptic functions (for their properties see e.g. [1]). We denote by $\mathtt{K}, \mathtt{E}$ the complete elliptic integrals of the first and of the second kind respectively, moreover we fix a constant $\mathtt{m}$ to be the only solution of the equation

$$\frac{\mathtt{E}(\mathtt{m})}{\mathtt{K}(\mathtt{m})} = \frac{7+\mathtt{m}}{6} \ ,$$

and define a further constant $\mathtt{w} := 2\mathtt{K}(\mathtt{m})/\pi$.

For any $n \geq 1$ consider the following solution of the *linearized system*

$$v_n(x,t) := \frac{V}{n}\left[\mathrm{sn}(n\mathtt{w}(x+t)|\mathtt{m}) - \mathrm{sn}(n\mathtt{w}(-x+t)|\mathtt{m})\right] \tag{6.2}$$

(which is easily seen to satisfy the boundary conditions), with V a normalization constant such that $E(v_n(.,0)) = 1$.

Theorem 6.1. *For any positive $n \geq 1$ and any positive $\gamma < 1/3$ there exists a family $\{u^n_\epsilon(t)\}_{\epsilon\in\mathcal{E}^n_\gamma}$ of periodic solutions of (6.1), with the following properties*

1) *The set $\mathcal{E}^n_\gamma \subset \mathbb{R}^+$ is uncountable and has zero as an accumulation point;*
2) *u^n_ϵ is periodic with minimal period $2\pi/\omega^n_\epsilon$, and is of class $H^1([0, 2\pi/\omega_\epsilon], H^1)$;*
3) *there exists a strictly positive ω^n_* such that the frequency map ω^n_ϵ*

$$\mathcal{E}^n_\gamma \ni \epsilon \mapsto \omega^n_\epsilon \in W^n \ , \quad \text{with} \quad W^n := \begin{cases} W^n_\gamma \cap [n, n+\omega^n_*) & \text{if} \quad \alpha < 0 \\ W_\gamma \cap (n-\omega^n_*, n] & \text{if} \quad \alpha > 0 \end{cases}$$

is monotone and one to one; moreover it fulfills the following estimate

$$|n - \omega^n_\epsilon| \leq C_{14}\epsilon^2 \ ;$$

4) *u^n_ϵ is close to the solution $\epsilon v_n(x,t)$ of the linearized system, precisely*

$$\sup_{t\in\mathbb{R}} \|u^n_\epsilon(t) - \epsilon v_n(.,\omega^n_\epsilon t)\|_{H^1} \leq C_{15}\epsilon^2 \ ; \tag{6.3}$$

5) *u^n_ϵ has energy close to ϵ^2, precisely one has*

$$\sup_{t\in\mathbb{R}} |E(u^n_\epsilon(.,0)) - \epsilon^2| \leq C_{16}\epsilon^4 \ .$$

Remark 6.2. *In principle theorem 2.5 applies also to the case of periodic boundary condition, but we performed explicit computations only in the case of Dirichlet boundary conditions.*

Remark 6.3. *The function $\psi(u)$ in (6.1) does not need to be odd. However, if it is odd and has $k+1$ Lipschitz derivatives, then one can easily prove that $u^n_\epsilon \in H^k([0, 2\pi/\omega^n_\epsilon], H^1)$.*

The proof of this theorem requires the easy verification of the smoothness assumptions of theorem 5.3, the computation of the function $\langle F^{(0)}\rangle$, the computation of the critical points of $\mathcal{F}^{(0)}$, the proof that they are nondegenerate, and non-degenerate also for $\mathcal{F}^{(0)}_n$.

The critical points of $\mathcal{F}^{(0)}$ have been studied in [12], where practical stability of v_1 was also proved. The results obtained in that paper are summarized in the forthcoming proposition 6.4. For its very technical proof we refer to [12]. Denoting by [.] the average over $[0, \pi]$ of the function in brackets, i.e.

$$[g] := \frac{1}{\pi}\int_0^\pi g(x)dx \ ,$$

we have

Proposition 6.4. *Consider the nonlinear wave equation (1.2), then the functional $\langle F^{(0)}\rangle$ is given by*

$$\langle F^{(0)}\rangle(u) = \frac{\pi}{32}\left([u^4] + 3[u^2]^2\right) \ ; \tag{6.4}$$

the critical points V_n of $\mathcal{F}^{(0)}$ are given by

$$V_n(x) = \frac{V}{n}\mathrm{sn}(\mathtt{w}nx|\mathtt{m}) \ ,$$

where V is such that $E(V_n) = 1$. The second differential of $\mathcal{F}^{(0)}$ at V_n is given by

$$\left\langle \mathrm{d}^2\mathcal{F}^{(0)}(V_n); h, k\right\rangle = \langle B_{V_n}h, k\rangle_{L^2}$$

with

$$B_{V_n}h := m_n h_{xx} + \frac{3}{8}\left(V_n^2 + [V_n^2]\right)h + \frac{3}{4}[V_n h]V_n \ ,$$

where m_n is a constant depending on n and fixed by the value of $\mathtt{K}$ and $\mathtt{m}$ (its precise value is not important in the following). The critical points V_n are nondegenerate for $\mathcal{F}^{(0)}$.

To obtain theorem 6.1 we will apply theorem 5.3. Remarking that the fundamental period of the solution (6.2) of the linearized system is $2\pi/n$, it remains to verify that the critical points V_n are nondegenerate also for $\mathcal{F}^{(0)}_n$.

Observe first that in the present case

$$\mathcal{Q}_n = \mathrm{span}\{\sin(nkx)\ , \quad k \geq 1\} \ .$$

Then the proof of nondegeneracy is based on the following lemma

Lemma 6.5. *Define $\mathcal{Q}_n^\perp$ to be the L^2 orthogonal to $\mathcal{Q}_n$; then one has*

$$B_{V_n}\mathcal{Q}_n \subset \mathcal{Q}_n \ , \qquad B_{V_n}\mathcal{Q}_n^\perp \subset \mathcal{Q}_n^\perp$$

Proof. Due to the form of B_{V_n} and to the remark that $[V_n h] = 0$ if $h \in \mathcal{Q}_n^{\perp}$, it is enough to verify that

$$V_n^2 h \in \mathcal{Q}_n \quad \text{if } h \in \mathcal{Q}_n \ , \qquad V_n^2 h \in \mathcal{Q}_n^{\perp} \quad \text{if } h \in \mathcal{Q}_n^{\perp}$$

We prove the first equation; the second is obtained in the same way. So, fix $h \in \mathcal{Q}_n$ and remark that a function h is in $\mathcal{Q}_n$ if and only if writing its sin–Fourier expansion

$$h(x) = \sum \hat{h}_l \sin lx$$

one has that $\hat{h}_l \neq 0$ implies $l = nk$ for some integer $k \geq 1$. Now denote $g = V_n^2 h$, extend g, V_n and h to periodic functions on $[-\pi, \pi]$ by skew-symmetry, and take their Fourier expansion on the basis of the exponential:

$$h(x) = \sum_l \tilde{h}_l \,\mathrm{e}^{\mathrm{i}lx} \ , \quad V_n(x) = \sum_l \tilde{V}_{n,l} \,\mathrm{e}^{\mathrm{i}lx} \ , \quad g(x) = \sum_l \tilde{g}_l \,\mathrm{e}^{\mathrm{i}lx} \ ;$$

since the relation between these coefficients and those on the sin-Fourier basis is $\tilde{h}_l = \frac{\hat{h}_l}{2\mathrm{i}} = \tilde{h}_{-l}$, it follows that $\tilde{h}_l \neq 0$, $\tilde{V}_{n,l} \neq 0$ only if $l = nk$ for some integer $k \geq 1$.

From the relation

$$\tilde{g}_m = \sum_{l+j+i=m} \tilde{h}_l \tilde{V}_{n,j} \tilde{V}_{n,i}$$

and the fact that for non vanishing term one has $l = nk_1$, $j = nk_2$, $i = nk_3$, $\tilde{g}_m$ is different from zero only if $m = n(k_1 + k_2 + k_3)$; thus also m is an integer multiple of n, and therefore $g \in \mathcal{Q}_n$. ☐

Proof of theorem 6.1 We apply theorem 5.3. We have to prove that

$$\big(\, Y \in \mathcal{Q}_n \cap T_{V_n} E^{-1}(1) \ , \quad \langle B_{V_n} X_1, Y \rangle_{L^2} = 0 \quad \forall X_1 \in \mathcal{Q}_n \cap T_{V_n} E^{-1}(1) \,\big) \Longrightarrow Y = 0 \quad (6.5)$$

Take $X_2 \in \mathcal{Q}_n^{\perp} \cap T_{V_n} E^{-1}(1)$ and remark that $T_{V_n} E^{-1}(1) = (\mathcal{Q}_n \cap T_{V_n} E^{-1}(1)) \oplus (\mathcal{Q}_n^{\perp} \cap T_{V_n} E^{-1}(1))$. By $B_{V_n} X_2 \in \mathcal{Q}_n^{\perp}$ one has

$$\langle B_{V_n} X_1, Y \rangle_{L^2} = \langle B_{V_n} (X_1 + X_2), Y \rangle_{L^2}$$

so condition (6.5) is implied by

$$\big(\, Y \in T_{V_n} E^{-1}(1) \ , \quad \langle B_{V_n} X, Y \rangle_{L^2} = 0 \quad \forall X \in T_{V_n} E^{-1}(1) \,\big) \Longrightarrow Y = 0$$

which is nondegeneracy of V_n as a critical point of $\mathcal{F}^{(0)}$, and this is ensured by proposition 6.4. ☐

References

[1] M. Abramowitz, I.A. Stegun: Handbook of Mathematical Functions. Dover, (New York, 1965).

[2] D. Bambusi: *On Lyapunov center theorem for some nonlinear PDE's: a simple proof.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., to appear. At present available in http://www.mat.unimi.it/~bambusi/elenco.html

[3] J. Bourgain: *Construction of periodic solutions of nonlinear wave equations in higher dimension.* GAFA **5**, 629-639 (1995)

[4] J. Bourgain: Nonlinear Schrödinger equations. In "Hyperbolic equations and frequency interactions"; L. Caffarelli, E. Weinan editors. IAS/Park City mathematics series 5. American Mathematical Society (Providence, Rhode Island 1999).

[5] W. Craig, C.E. Wayne: *Newton's Method and Periodic Solutions of Nonlinear Wave Equations* Comm. Pure and Appl. Math. **46** 1409-1501 (1993).

[6] W.Craig; C.E.Wayne: *Nonlinear waves and the* $1:1:2$ *resonance*, in Singular limits of dispersive waves (Lyon, 1991), 297–313, NATO Adv. Sci. Inst. Ser. B Phys., 320, Plenum, New York, 1994.

[7] R. de la Llave: *Variational methods for quasi–periodic solutions of partial differential equations.* Preprint 1999 available in mparc (http://www.ma.utexas.edu/mp_arc/), $n^0$00-56.

[8] G.F. Dell'Antonio: *Fine Tuning of Resonances and Periodic Solutions of Hamiltonian Systems Near Equilibrium* Commun. Math. Phys. **120** 529-546 (1989).

[9] S.B. Kuksin: Nearly Integrable Infinite-Dimensional Hamiltonian Systems, Lect. Notes Math. 1556 (Springer 1994)

[10] S.B. Kuksin, J. Pöschel: *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation* Ann. of Math. **142** 149–179 (1995).

[11] B.V.Lidskij, E.I.Shulman: *Periodic solutions of the equation* $u_{tt} - u_{xx} + u^3 = 0$, Funct. Anal. Appl. **22** 332–333 (1988).

[12] S. Paleari, D. Bambusi, S. Cacciatori: *Exponential stability in a nonlinear string equation.* Quaderno $n^0$6/2000 Dipartimento di Matematica "F.Enriques", Università di Milano (2000). Also available at http://www.mat.unimi.it/~bambusi/elenco.html

[13] J. Pöschel: *A KAM–Theorem for some Partial Differential Equations* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **23**, 119–148 (1996).

[14] W.M. Schmidt: Diophantine Approximation. Lect. Notes Math. 785, Springer Verlag 1980.