ANALYTICITY PROPERTIES AND ESTIMATES OF RESOLVENT KERNELS NEAR THRESHOLDS

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Abstract- Resolvent estimates are derived for the family of ordinary differential operators

$$\left\{-c^2(y)\left[\rho(y)\,\frac{d}{dy}\left(\frac{1}{\rho(y)}\,\frac{d}{dy}\right)-p^2\right]\right\},\quad p\in[0,\infty),\quad y\in\mathbb{R}.$$

It is assumed that $c(y)=c_\pm>0, \rho(y)=\rho_\pm$ for $\pm y>y_c$, and the kernels are studied in neighborhoods of the points $\{c_\pm^2\,p^2\}$, uniformly in compact intervals of p. This family arises in the direct integral decomposition of the acoustic propagator in layered media, $-c^2(y)\,\rho(y)\,\nabla_{x,y}\left(\frac{1}{\rho(y)}\,\nabla_{x,y}\right)$, $x\in\mathbb{R}^{\Bbbk}$, and the results imply "low energy" estimates for this operator, as well as the validity of the "limiting absorption principle".

1 Introduction

Consider the operator $H = -c^2(y)\,\rho(y)\,\nabla \cdot \left(\frac{1}{\rho(y)}\,\nabla\right)$, where $\nabla = \nabla_{x,y}$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$. This operator is known as the "acoustic propagator" in (unperturbed) stratified media ([4,7,9,19]). In what follows we impose the following assumptions on the real functions $\rho(y), c(y)$ $(y \in \mathbb{R})$:

(1.1) $\rho(y) \in C^1(\mathbb{R})$, $c(y)$ is piecewise continuous with finitely many jump discontinuities, and there exist positive constants ρ_m, ρ_M, c_m, c_M such that

$$0 < \rho_m \leq \rho(y) + |\rho'(y)| \leq \rho_M, \quad 0 < c_m \leq c(y) \leq c_M.$$

(1.2) There exist positive constants y_c, $\rho_\pm$, $c_\pm$ such that $c_+ < c_-$ and

$$\rho(y) = \rho_\pm, \quad c(y) = c_\pm \quad \text{for} \quad \pm\, y > y_c.$$

It is well-known that the operator H can be defined as a self-adjoint operator in the space $L^2(\mathbb{R}^{n+1}, c^{-2}(y)\rho^{-1}(y)dxdy)$. Taking the (partial) Fourier transform with respect to the x-coordinates, we have the unitary equivalence.

$$H \cong \int_{R^n} \oplus H_{|\zeta|}\, d\zeta, \tag{1.3}$$

where

$$H_p = -c^2(y)\left[\rho(y)\,\frac{d}{dy}\left(\frac{1}{\rho(y)}\,\frac{d}{dy}\right) - p^2\right], 0 \leq p < \infty. \tag{1.4}$$

For every p, the operator H_p is self-adjoint in $L^2(\mathbb{R}, c^{-2}(y)\,\rho^{-1}(y)\,dy)$ and its essential spectrum $[c_+^2\,p^2, \infty)$ is absolutely continuous (in view of (1.1)-(1.2)). The spectral multiplicity of H_p jumps at $c_-^2 p^2$ from 1 to 2.

It is the purpose of this paper to study the behavior of the resolvent kernels associated with the family $\{H_p\}$, specifically near the points $\{c_\pm^2\,p^2\}$. We obtain estimates on the kernels and their derivatives (in terms of the spectral parameter), uniformly in p in compact intervals of $[0, \infty)$. In particular, these estimates hold at "thresholds" (see Remark 2.6).

Such estimates constitute the main technical tools needed in the study of the resolvent of H near the continuous spectrum, and in particular the "limiting absorption principle" ([4]) and "low energy estimates".

To illustrate the situation at hand we take the Laplacian $-\Delta$ in $\mathbb{R}^{n+1}$ and let T be its self-adjoint realization in $L^2(\mathbb{R}^{n+1})$. Then, parallel to (1.3) -(1.4), we have

$$T \cong \int_{R^n} \oplus T_{|\zeta|} d\zeta, \tag{1.5}$$

$$T_p = -\frac{d^2}{dy^2} + p^2, \qquad 0 \leq p < \infty. \tag{1.6}$$

For $z \in \mathbb{C}, \operatorname{Im} z \neq 0$, we denote the resolvents by $M(z) = (T - z)^{-1}, M_p(z) = (T_p - z)^{-1}$, so that

$$M(z) \cong \int_{R^n} \oplus M_{|\zeta|}(z)\, d\zeta. \tag{1.7}$$

The kernel (Green's function) $L_p(\xi, \eta; z)$ associated with $M_p(z)$ is given by

$$L_p(\xi, \eta; z) = (2i)^{-1}(z - p^2)^{-1/2} \exp[i|\xi - \eta|(z - p^2)^{1/2}], \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, \tag{1.8}$$

where $\operatorname{Im}(z - p^2)^{1/2} > 0$. Various estimates and properties of $-\Delta$ can be derived via (1.7)-(1.8). In particular, this applies to estimates near $z = 0$, the "low energy estimates". We shall derive such estimates in the Appendix, as a motivation for our treatment of H in terms of (1.3).

As we show in this work, the kernels associated with $(H_p - z)^{-1}$ share some of the main properties of the kernels L_p, thus permitting analogous results for H.

We note that in particular we give here new proofs to Propositions A.2, A.4 in [4] and a correction to an error (pointed out to us by Professor J. Ralston) in the proof of Proposition A.1 (b) there (see Remark 2.3 for more details).

Decompositions of the type (1.3)-(1.4) are also encountered in a variety of examples in mathematical physics, such as acoustic or electromagnetic propagators [2, 7, 9, 15, 17] and linear elasticity theory [6, 11, 16]. It is expected that the techniques developed here will serve in the spectral study of those cases as well.

Our method of proof is based on a study of the analyticity properties of the kernels in terms of the variable $k = (zc_+^{-2} - p^2)^{1/2}$ (see next section for details). This is closely related to the work of Cohen and Kappeler [8]. In particular, their treatment of the Wronskian of Jost functions (Proposition 2.4 in [8]) corresponds to our conclusion (2.12). However, since we are dealing with families of operators, we need to study the way in which the zeros of the

Wronskian "move around" (see (2.10)) and get uniform estimates on a family of kernels. Furthermore, in order to get estimates on the z-derivatives of the kernels, we need to study analytic extensions of the kernels to sectors in the lower ($k-$)half-plane. We refer the reader to [1, 18] for works concerning such extensions.

To formulate our results, fix $L > 0, a > 0$ and $0 < \beta < \pi$. For every $p \in [0, L]$ set

(1.9)

$$I_p = (c_+^2p^2, c_+^2p^2+a), \quad \Omega_p^+ = \{z \in \mathbb{C}; z - c_+^2p^2 = re^{i\theta}, \quad 0 < r < a, \quad 0 < \theta < \beta\}.$$

Let $\Phi_p(\xi, \eta; z)$ be the resolvent kernel associated with $(H_p - z)^{-1}$.

Theorem A *There exists a constant $C > 0$, depending only on L, a, β, such that*

$$|\Phi_p(\xi, \eta; z)| \leq C[|z - c_+^2p^2|^{-1/2} + |z - c_-^2p^2|^{-1/2}], \tag{1.10}$$

for $z \in \Omega_p^+, \quad p \in [0, L], \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}$. Furthermore, $\Phi_p(\xi, \eta; z)$ can be extended continuously to $\mathbb{R} \times \mathbb{R} \times \mathbb{I}_p$, and the estimate (1.10) is valid for $z \in I_p \backslash \{c_-^2p^2\}$.

Remark 1.1 Remark that $\Phi_p(\xi, \eta; z)$ is continuous at $z = c_-^2p^2$ $(p > 0)$, but (1.10) expresses the fact that

$$\sup_{(\xi,\eta)\in R\times R} |\Phi_p(\xi, \eta; z)|$$

may blow-up as z approaches $c_-^2p^2$.

It follows from (1.10), (1.8), that the "blow-up" rate for the kernel Φ_p at the bottom of the essential spectrum is no worse than that of L_p, associated with $-\Delta$. In particular, repeating verbatim the proofs of Lemmas A.1, A.2 in the Appendix, we obtain a "low energy estimate" for the resolvent $(H - z)^{-1}$. In its formulation, we use the weighted-L^2 norm introduced in the Appendix, and also the "limiting absorption principle", which allows the extension of $(H - z)^{-1}$ to positive z (continuously in the same operator space).

Corollary *Fix $0 < \tau < \pi$ and let $\Lambda_\tau = \{z; 0 \leq \arg z \leq \tau\}$. Then,*

(a) *For any $s > 1$, $n \geq 1$,*

$$||(H - z)^{-1}||_{B(L^{2,s}(R^{n+1}),L^{2,-s}(R^{n+1}))} = \mathcal{O}(\log|z|) \text{ as } z \to 0, z \in \Lambda_\tau. \tag{1.11}$$

(b) *For any* $s > \frac{3}{2}$, $n \geq 3$,

$$||(H-z)^{-1}||_{B(L^{2,s}(R^{n+1}),L^{2,-s}(R^{n+1}))} = \mathcal{O}(\infty) \text{ as } \ddagger \to \prime, \ddagger \in *_T. \tag{1.12}$$

We shall now state a theorem concerning the differentiability of $\Phi_p(\xi, \eta; z)$ with respect to real z embedded in the continuous spectrum. Recall [4] that such a result implies the "limiting absorption principle" as well as the Hölder continuity of the limiting values of the resolvent.

Theorem B *Fix* $a > 0$, $0 < \ell < L$, *and let* I_p *be as in (1.9) with* $p \in [l, L]$. *Then there exists a constant* $C = C(\ell, L, a) > 0$ *such that, for* $\lambda \in I_p \backslash \{c_-^2 p^2\}$,

$$\left| \frac{\partial}{\partial \lambda} \Phi_p(\xi, \eta; \lambda) \right| \leq C \left[|\lambda - c_+^2 p^2|^{-3/2} + |\lambda - c_-^2 p^2|^{-3/2} \right] (1 + |\xi| + |\eta|).$$

The proofs of Theorems A, B are given in the next section.

Remark 1.2 Results similar to those of Theorems A, B can be obtained for an operator of the form $-\frac{d^2}{dy^2} + V(y)$, where $V(y)$ is constant for large $|y|$. See also Remark 2.9.

2 Resolvent Kernels and proofs of Theorems A, B

Given $p \in [0, \infty)$, $z \in \mathbb{C}^+ \equiv \{\precapprox; \mathrm{Im} \precapprox > \nvdash\}$, we are interested in estimating the kernels Φ_p associated with $(H_p - z)^{-1}$. However, it will be more convenient to replace z by another parameter k, defined by

$$k = (z c_+^{-2} - p^2)^{1/2}. \tag{2.1}$$

We shall always take $\mathrm{Im}\, \zeta^{1/2} \geq 0$ for $\zeta \in \mathbb{C}$.

The Jost functions φ, ψ, which are solutions of

$$(H_p - z)\, u = 0, \tag{2.2}$$

are given by their asymptotic behavior for large $|y|$,

$$\begin{aligned} \varphi(y; k, p) &= \exp(iky), \quad y > y_c, \\ \psi(y; k, p) &= \exp[-i((k^2 + p^2)\, c_+^2 c_-^{-2} - p^2)^{1/2} y], \quad y < -y_c. \end{aligned} \tag{2.3}$$

Note that the exponent of ψ vanishes at $k = \pm\tilde{k}$, where

$$\tilde{k} = \tilde{k}(p) = (c_-^2 c_+^{-2} - 1)^{1/2} p, \quad \tilde{k} \geq 0, \tag{2.4}$$

so that

$$\psi(y; k, p) = \exp[-ic_+ c_-^{-1}(k^2 - \tilde{k}^2)^{1/2} y], \quad y < -y_c. \tag{2.5}$$

Fix $L > 0$, $a > 0$. For any $p \in [0, L]$ take I_p, Ω_p^+ as in (1.9). Note that $z \in \Omega_p^+$ implies $k \in \mathbb{C}^+ \cap \{\mathrm{Re}\,\urcorner > \not\vdash\}$. However, using the asymptotic behavior (2.3) for φ, ψ, it is obvious that both functions can be extended to $k \in \overline{\mathbb{C}^+}$, so that they solve (2.2) with $z = c_+^2(k^2 + p^2)$. In view of standard theorems about solutions of ordinary differential equations we can summarize the properties of φ, ψ and their y-derivatives φ', ψ' as follows.

Claim 2.1 *The functions $\varphi(y; k, p)$, $\psi(y; k, p)$, $\varphi'(y; k, p)$, $\psi'(y; k, p)$ are continuous in $\mathbb{R} \times \overline{\mathbb{C}^+} \times [\not\vdash, \mathbb{L}]$ and analytic in $k \in \mathbb{C}^+$ for any fixed $(y, p) \in \mathbb{R} \times [\not\vdash, \mathbb{L}]$.*

(Remark that for the continuity of ψ for real $k < -\tilde{k}$ one takes in (2.3), (2.5) the root satisfying $\arg[(k^2 + p^2)\, c_+^2 c_-^{-2} - p^2]^{1/2} = \arg[(k^2 - \tilde{k}^2)^{1/2}] = \pi$).

For any two functions $\tau(y)$, $\theta(y)$ we define the Wronskian $[\tau, \theta]$ by

$$[\tau, \theta] = \rho^{-1}(y)\,(\tau'\theta - \tau\theta'). \tag{2.6}$$

Recall that $[\tau, \theta]$ is independent of y if τ, θ are solutions of (2.2). In particular, by evaluating the following Wronskians for $\pm y > y_c$, taking (2.3) into account, we get, for $k \in \mathbb{R}$,

$$\begin{aligned} [\varphi, \bar{\varphi}] &= 2i\,\rho_+^{-1} k, \\ [\psi, \bar{\psi}] &= \begin{cases} 0, & |k| \leq \tilde{k}, \\ -2i\,\rho_-^{-1} c_+ c_-^{-1} (k^2 - \tilde{k}^2)^{1/2} \cdot \operatorname{sgn} k, & |k| > \tilde{k}. \end{cases} \end{aligned} \tag{2.7}$$

(Note that $\psi = \bar{\psi}$ for $|k| \leq \tilde{k}$).

The resolvent kernel (Green's function), $\Phi_p(\xi, \eta; k)$, $k \in \mathbb{C}^+$, is given by [12],

$$\Phi_p(\xi, \eta; k) = \begin{cases} \dfrac{\varphi(\xi; k, p)\,\psi(\eta; k, p)}{[\varphi, \psi]} & \xi \geq \eta, \\[2ex] \dfrac{\varphi(\eta; k, p)\,\psi(\xi; k, p)}{[\varphi, \psi]} & \xi \leq \eta. \end{cases} \tag{2.8}$$

In view of Claim 2.1, the kernel Φ_p is analytic in $k \in \mathbb{C}^+$ and can be extended continuously to $k \in \overline{\mathbb{C}^+}$ except for zeros of $[\varphi, \psi]$. It follows from our next claim that in fact $[\varphi, \psi]$ does not vanish for $k \in \mathbb{R} \backslash \{\nvdash\}$.

Claim 2.2 *(a) The Jost functions* φ, ψ *are linearly independent (hence* $[\varphi, \psi] \neq 0$*) for every* $(k, p) \in \overline{\mathbb{C}^+} \times [0, L]$, $\operatorname{Re} k \neq 0$.
(b) Let $\Gamma \subseteq \overline{\mathbb{C}^+} \backslash \{0\}$ *be compact and such that* $\pm \tilde{k}(p) \notin \Gamma$ *for* $p \in [0, L]$. *Then,*

$$\sup_{\substack{-\infty < \eta \leq \xi < \infty \\ k \in \Gamma, p \in [0, L]}} |\varphi(\xi; k, p)\, \psi(\eta; k, p)| < \infty. \tag{2.9}$$

Proof. (a) Observe that if $\operatorname{Im} k > 0$, $\operatorname{Re} k \neq 0$, then $z = (k^2 + p^2)\, c_+^2$ is non-real. Thus, if $\varphi = \alpha \psi$, $\alpha \in \mathbb{C}$, then in view of (2.3) the function φ is an L^2-eigenfunction of H_p, associated with a non-real eigenvalue z. Thus, it remains to consider $k \in \mathbb{R} \backslash \{\nvdash\}$. But, if $\varphi = \alpha \psi$, $\alpha \in \mathbb{C}$, then $[\varphi, \bar{\varphi}] = |\alpha|^2 [\psi, \bar{\psi}]$, which, in view of (2.7), is a contradiction for all $k \in \mathbb{R} \backslash \{\nvdash\}$.

(b) In view of (2.3) the boundedness claim in (2.9) is clear if $\eta \leq y_c$, $\xi \geq -y_c$. It thus suffices to look at the cases $y_c \leq \eta \leq \xi$ or $\eta \leq \xi \leq -y_c$. In the first case, let $\varphi_1(y; k, p)$ be the solution of

(2.2) such that $\varphi_1(y; k, p) = \exp(-iky)$ for $y > y_c$. Then $\psi(\eta; k, p) = \alpha(k, p)\, \varphi(\eta; k, p) + \beta(k, p)\, \varphi_1(\eta; k, p)$ where α, β are continuous for $(k, p) \in \Gamma \times [0, L]$ (since φ and φ_1 are linearly independent for such (k, p)) and the boundedness of $\psi(\eta; k, p)\, \varphi(\xi; k, p)$, $\xi \geq \eta$, follows from (2.3). The second case follows in exactly the same way, since $k^2 - \tilde{k}^2$ does not vanish for $(k, p) \in \Gamma \times [0, L]$.

$\square$

Remark 2.3 In Claim 2.2 (a) we correct the statement of Proposition A.1 (b) in [4], where the linear independence of φ, ψ was stated also for $k = 0$ (i.e., $z = c_+^2 p^2$). However, for this value the Jost functions may be linearly dependent, corresponding to a solution of $(H_p - c_+^2 p^2)\, u = 0$ which is constant for $y > y_c$ and decays exponentially as $y \to -\infty$. This is the case labeled as "exceptional" in [8], in contrast to the "generic" case, where $[\varphi, \psi] \neq 0$ at $c_+^2 p^2$. Since the proof of Proposition A.2 in [4] relies on Proposition A.1 (b), we need to modify it, which we do below in the proof of Theorem A. We are grateful to J. Ralston for bringing this error to our attention.

It follows from Claim 2.1 that the Wronskian $[\varphi, \psi]\,(k, p)$ is continuous in $\overline{\mathbb{C}^+} \times [0, L]$ and analytic in $k \in \mathbb{C}^+$, for every fixed $p \in [0, L]$. By Claim 2.2 (a) it does not vanish if $\operatorname{Re} k \neq 0$. Its zeros on the positive imaginary k-axis correspond to the L^2-eigenvalues of H_p below the threshold $c_+^2 p^2$. In fact, we can obtain more information by noting the following claim, which extends the analyticity (in k) domain of $[\varphi, \psi]$. Its proof is immediate, in view of (2.3) and standard theorems concerning ordinary differential equations.

Claim 2.4 *Fix $0 < p \leq L$. Then the Jost functions $\varphi(\eta; k, p)$, $\psi(y; k, p)$ and their y-derivatives φ', ψ', can be extended analytically to the disk $|k| < \frac{1}{2}\tilde{k}(p)$. If $p = 0$ these functions can be extended analytically to all of $\mathbb{C}$.*

In particular, the Wronskian $[\varphi, \psi](k, p)$ is analytic in k in $\mathbb{C}^+ \cup \{|\daleth| < \frac{\nVdash}{\nvDash}\hbar(\imath)\}$, for any fixed $p \in [0, L]$ (in $\mathbb{C}$ for $p = 0$).

We infer immediately from this claim that there are only finitely many zeros (including multiplicity) of $[\varphi, \psi](k, p)$ for $\operatorname{Im} k > 0$, for any fixed $p \in [0, L]$. Note that $[\varphi, \psi](i\zeta, p) = 0$, $\zeta > 0$, implies that $\lambda = c_+^2(p^2 - \zeta^2)$ is an eigenvalue of H_p, and since $H_p \geq 0$ we have $\zeta \leq p$. Let, for $p \in [0, L]$, $\{i\zeta_j(p)\}_{j=1}^{J(p)}$ be the finitely many zeros of $[\varphi, \psi](\cdot, p)$ in $\mathbb{C}^+$ (including multiplicity. Actually, following the argument of Lemma 1.3 in [8], we see that they are all simple, but we shall not need this fact). We arrange these zeros such that

$$p \geq \zeta_1(p) \geq \ldots \geq \zeta_J(p) > 0, \quad J = J(p). \tag{2.10}$$

The continuity of $[\varphi, \psi](k, p)$ in p and standard analyticity arguments imply that the integer function $J(p)$ is constant in intervals where $[\varphi, \psi](0, p) \neq 0$. However, the following claim shows that this is the case except possibly for a finite number of values.

Claim 2.5 *There are at most finitely many points $0 \leq p_1 < p_2 < \ldots < p_m \leq L$ such that $[\varphi, \psi](0, p_j) = 0$, $j = 1, \ldots, m$.*

Proof. As is seen from (1.4) and (2.3) the function $[\varphi, \psi](0, p)$ can be extended analytically to $p \in \mathbb{C}$.

□

Remark 2.6 The values $\{c_+^2 p_j^2\}$ are called the "thresholds" of H. Generically, as in Remark 2.3, they correspond to a "semi-bound" state and mark the "birth" of a new branch of proper eigenvalues (i.e., $\zeta_{J(p)}(p) \to 0$ as $p \to p_j + 0$). We refer to [4] for details.

¿From Claim 2.5 and the discussion preceding it we conclude that there ¿exists $M = M(L) > 0$ such that

$$J(p) \leq M, \qquad p \in [0, L]. \tag{2.11}$$

We now turn to the study of $[\varphi,\psi](k,p)$ for real k near $k=0$. Our next claim will show, in particular, that $k=0$ is at most a simple zero for $p=p_j$, $j=1,\dots,m$. For a single operator, it follows from Proposition 2.4 in [8].

Claim 2.7 *Fix $\delta>0$. Then there exists a positive constant $\gamma>0$ such that*

$$|[\varphi,\psi](k,p)|\geq\gamma|k|,\quad k\in[-\delta,\delta],p\in[0,L]. \tag{2.12}$$

Proof. As in the proof of Claim 2.2 (b), we write for $k\neq 0$,

$$\psi(y;k,p)=\alpha(k,p)\,\varphi(y;k,p)+\beta(k,p)\,\varphi_1(y;k,p), \tag{2.13}$$

so that by (2.7),

$$[\varphi,\psi](k,p)=2i\rho_+^{-1}k\beta(k,p). \tag{2.14}$$

On the other hand, using (2.7) once again to evaluate $[\psi,\bar{\psi}]$ by (2.13),

$$-2i\rho_-^{-1}c_+c_-^{-1}(k^2-\tilde{k}^2)\cdot(1-\chi(k))\operatorname{sgn}k=2ik\rho_+^{-1}[|\alpha|^2-|\beta|^2] \tag{2.15}$$

($\chi(k)=1$ for $k\in[-\tilde{k},\tilde{k}]$, $=0$ otherwise).

We conclude that

$$|\alpha(k,p)|\leq|\beta(k,p)|\qquad(\alpha(k,p)=\overline{\beta(k,p)}\quad\text{for}\quad k\in(-\tilde{k},\tilde{k})). \tag{2.16}$$

Suppose that there exist sequences $\{k_n\}\subseteq[-\delta,\delta]\backslash\{0\}$, $\{p_n\}\subseteq[0,L]$ such that $\beta(k_n,p_n)\to 0$. Necessarily $\alpha(k_n,p_n)\to 0$. Without loss of generality assume $k_n\to K\in[-\delta,\delta]$, $p_n\to q\in[0,L]$. By continuity we infer from (2.13) that $\psi(y;K,q)\equiv 0$, a contradiction to (2.3). Thus,

$$\inf|\beta(k,p)|>0\quad\text{in}\quad([-\delta,\delta]\backslash\{0\})\times[0,L], \tag{2.17}$$

which proves (2.12) in view of (2.14).

□

We may now conclude the proof of Theorem A.

Proof of Theorem A. Take $\delta\geq 2\,(L+\tilde{k}(L))$, and let $\Gamma_\delta=\overline{\mathbb{C}^+}\cap\{|k|\leq\delta\}$. In what follows we use $C>0$ to denote positive constants not depending on k, p.

In view of (2.9) we have, for $k\in\overline{\mathbb{C}^+}$, $|k|=\delta$,

$$|\varphi(\xi;k,p)\,\psi(\eta;k,p)|\leq C,\qquad p\in[0,L],\quad -\infty<\eta\leq\xi<\infty, \tag{2.18}$$

and, by (2.10) and Claim 2.2 (a), $[\varphi, \psi]\,(k,p) \neq 0$ for $k \in \overline{\mathbb{C}^+}$, $|k| = \delta$, $p \in [0, L]$. The expression (2.8) now yields, on the circular part of $\partial\Gamma_\delta$,

$$|\Phi_p(\xi, \eta; k)| \leq C, \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, \quad p \in [0, L], \quad k \in \overline{\partial\Gamma_\delta \cap \mathbb{C}^+}. \tag{2.19}$$

Next, we estimate $\Phi_p(\xi, \eta; k)$ for k in the real part of $\partial\Gamma_\delta$. This means that we need again to estimate products of the type $\varphi(\xi; k, p)\,\psi(\eta; k, p)$, $\eta \leq \xi$, this time for real $k \in [-\delta, \delta]$. Note first that, by (2.3) and Claim 2.1, we have the individual uniform boundedness,

$$|\varphi(\xi; k, p)| \leq C, \quad |\psi(\eta; k, p)| \leq C, \quad k \in [-\delta, \delta], \quad \xi \geq -y_c, \quad \eta \leq y_c. \tag{2.20}$$

Thus, it suffices to treat the cases (a) $y_c \leq \eta \leq \xi$ and (b) $\eta \leq \xi \leq -y_c$. In case (a) we use (2.13) to write

$$\varphi(\xi; k, p)\,\psi(\eta; k, p) = [\alpha(k, p)\,\varphi(\eta; k, p) + \beta(k, p)\,\varphi_1(\eta; k, p)]\,\varphi(\xi; k, p),$$

and in conjunction with (2.14), for $k \neq 0$,

$$\frac{\varphi(\xi; k, p)\,\psi(\eta; k, p)}{[\varphi, \psi]\,(k, p)} = -\frac{i}{2}\,\rho_+ k^{-1} \left[\frac{\alpha(k, p)}{\beta(k, p)}\,\varphi(\eta; k, p) + \varphi_1(\eta; k, p)\right]\,\varphi(\xi; k, p)$$

so that (2.16) and (2.3) yield

$$|k\,\Phi_p(\xi, \eta; k)| \leq C, \qquad k \in [-\delta, \delta]\backslash\{0\}, \qquad y_c \leq \eta \leq \xi, \qquad p \in [0, L]. \tag{2.21}$$

In case (b) we first define (analogously to φ_1) a second solution ψ_1 to (2.2), such that, for $y < -y_c$,

$$\psi_1(y; k, p) = \exp[i((k^2 + p^2)\,c_+^2 c_-^{-2} - p^2)^{1/2} y] = \exp[i c_+ c_-^{-1} (k^2 - \tilde{k}^2)^{1/2} y], \tag{2.22}$$

(Recall that $\arg(k^2 - \tilde{k}^2)^{1/2} = \pi$ if $k < -\tilde{k}$).

The solutions ψ, ψ_1 are linearly independent if $k \neq \pm\tilde{k}$, but coincide if $k = \pm\tilde{k}$ (and are identically 1 for $y < -y_c$). However, the pair $\{\psi, (k^2 - \tilde{k}^2)^{-1/2}(\psi - \psi_1)\}$ is linearly independent if $k \neq \pm\tilde{k}$ and remains linearly independent when extended continuously to $k = \pm\tilde{k}$. We may therefore write,

$$\varphi(y; k, p) = \theta(k, p)\,\psi(y; k, p) + \tau(k, p)\,\frac{(\psi - \psi_1)\,(y; k, p)}{(k^2 - \tilde{k}^2)^{1/2}} \tag{2.23}$$

(with suitable modification at $k = \pm\tilde{k}$), with $\theta(k, p)$, $\tau(k, p)$ continuous in $[-\delta, \delta] \times [0, L]$. We obtain readily,

$$[\varphi, \psi]\,(k, p) = -2i c_+ c_-^{-1} \rho_-^{-1}\,\tau(k, p). \tag{2.24}$$

We write by (2.8), for $\eta \le \xi \le -y_c$, $k \ne 0$,

$$\Phi_p(\xi,\eta;k) = \frac{\psi(\eta;k,p)\left[\theta(k,p)\psi(\xi;k,p) + \tau(k,p)\,\dfrac{(\psi-\psi_1)(\xi;k,p)}{(k^2-\tilde{k}^2)^{1/2}}\right]}{[\varphi,\psi]\,(k,p)}$$

so that in view of (2.12), (2.24) and the uniform boundedness of $\psi(\eta;k,p)\,\psi(\xi;k,p)$, $\eta \le \xi \le -y_c$, $(k,p) \in [-\delta,\delta] \times [0,L]$, we get

$$\begin{aligned} &\left|[\varphi,\psi](k,p)\,\left[\Phi_p(\xi,\eta;k) - \tfrac{i}{2}\,c_+^{-1}c_-\rho_-\,\tfrac{\psi(\eta;k,p)\,(\psi-\psi_1)\,(\xi;k,p)}{(k^2-\tilde{k}^2)^{1/2}}\right]\right| \le C, \\ &\eta \le \xi \le -y_c, \quad (k,p) \in ([-\delta,\delta]\backslash\{0,\pm\tilde{k}\}) \times [0,L]. \end{aligned} \tag{2.25}$$

Note that by Claims 2.4 and 2.7 the function $k\,\Phi_p(\xi,\eta;k)$, for fixed ξ,η,p, is analytic near $k=0$, so that the estimates (2.21), (2.25) are valid also at $k=0$. Furthermore, the function

$$F_p(\xi,\eta;k) := \prod_{j=1}^{J(p)} \frac{k-i\zeta_j(p)}{k+i\zeta_j(p)}\,k\,\Phi_p(\xi,\eta;k) \tag{2.26}$$

(see (2.10) for $\zeta_j(p)$) is analytic in $\overset{\circ}{\Gamma}_\delta$ and continuous in Γ_δ. It therefore attains its maximum (absolute) value on $\partial\Gamma_\delta$. Take $0 < \varepsilon < \frac{\pi}{2}\cdot$ Due to (2.11) the product $\Lambda = \displaystyle\prod_{j=1}^{J(p)} \frac{k-i\zeta_j(p)}{k+i\zeta_j(p)}$ is uniformly bounded on the line segments $\{\arg\, k = \varepsilon\} \cap \Gamma_\delta$ and $\{\arg\, k = 0\} \cap \Gamma_\delta$, for all $p \in [0,L]$. The same is clearly true for Λ^{-1}. Noting the boundary estimates (2.19), (2.20) and (2.21), we get, in the sector $\{0 \le \arg\, k \le \varepsilon\} \cap \Gamma_\delta$,

$$|k\,\Phi_p(\xi,\eta;k)| \le C, \tag{2.27}$$

$(\xi,\eta) \in \mathbb{R} \times \mathbb{R}$, $\max(\xi,\eta) > -y_c$, $p \in [0,L]$, $0 \le \arg\, k \le \varepsilon$, $k \ne \tilde{k}(p)$ and $|k| \le \delta$, where $C = C(L,\varepsilon)$. In the case $\eta \le \xi \le -y_c$, we repeat the argument above, replacing in (2.26) the function $k\,\Phi_p$ by (see (2.25)),

$$[\varphi,\psi]\,(k,p)\,\left[\Phi_p(\xi,\eta;k) - \frac{i}{2}\,c_+^{-1}c_-\rho_-\,\frac{\psi(\eta;k,p)\cdot(\psi-\psi_1)\,(\xi;k,p)}{(k^2-\tilde{k}^2)^{1/2}}\right]\cdot$$

We conclude that this function is uniformly bounded in $\Gamma_\delta \cap \{0 \le \arg\, k \le \varepsilon\}$, so that the uniform boundedness of $\psi(\eta;k,p)\,\psi_1(\xi;k,p)$, $\eta \le \xi \le -y_c$, yields now

$$|\Phi_p(\xi,\eta;k)| \le C\left(\frac{1}{|k|} + \frac{1}{|k^2-\tilde{k}^2|^{1/2}}\right), \tag{2.28}$$

$\eta \leq \xi \leq -y_c$, $0 \leq \arg\, k \leq \varepsilon$, $0 < |k| \leq \delta$, $k \neq \tilde{k}$, $p \in [0, L]$.

We can now conclude easily the proof of Theorem A. Indeed, if $z \in \Omega_p^+$, $p \in [0, L]$, then the associated k (by (2.1)) satisfies $0 < \arg\, k < \varepsilon$, for $\varepsilon = \frac{\beta}{2}\cdot$

Since $|k| = |zc_+^{-2} - p^2|^{1/2}$ and $|k^2 - \tilde{k}^2|^{1/2} = c_+^{-1} c_- |zc_-^{-2} - p^2|^{1/2}$, the estimate (1.10) follows from (2.27), (2.28), which also imply the same estimates for the extension to I_p.

□

Remark 2.8 Observe that the fact that we are dealing with a family of operators H_p, $p \in [0, L]$, complicates the proof. Indeed, in our estimates we had to account for the various positions of $\tilde{k} = \tilde{k}(p)$ (approaching 0 as $p \to 0$) and $i\zeta_j(p)$, the zeros of the Wronskian $[\varphi, \psi]\,(k, p)$. The latter could also vanish as p approaches a threshold.

As noted in Remark 1.1, the kernel $\Phi_p(\xi, \eta; k)$ is continuous at $k = \tilde{k}$ if $p > 0$ (Claim 2.2 (a)), but $\sup\limits_{(\xi,\eta) \in R \times R} |\Phi_p(\xi, \eta; k)|$ may blow-up as k approaches $\tilde{k}(p)$.

Remark 2.9 The method used here can clearly be used in the study of resolvent kernels for an operator of the form $-\frac{d^2}{dy^2} + V(y)$, where $V(y) = V_\pm$ for $\pm y > y_c$.

The case $V_+ = V_-$ corresponds to $p = 0$ while $V_- > V_+$ corresponds to $p > 0$. Similar estimates are obtained for the kernel in the neighborhood of $V_\pm$.

Proof of Theorem B. By Claim 2.4 we have, for $0 < \delta < \frac{1}{2}\tilde{k}(\ell)$, that $\varphi(y; k, p)$, $\psi(y; k, p)$ and their y-derivatives are analytic (in k) in the disk $B_\delta = \{|k| < \delta\}$. In particular, the derivative $\frac{\partial}{\partial k}\,[\varphi, \psi]\,(k, p)$ is bounded for $(k, p) \in B_\delta \times [\ell, L]$. This implies, in view of (2.12), that for some $0 < \varepsilon < \frac{\pi}{2}$,

$$(2.29) \qquad |[\varphi, \psi]\,(k, p)| \geq \gamma |k|, \quad k \in B_\delta^\varepsilon := B_\delta \cap \{|\arg\, k| < \varepsilon\}, \quad p \in [\ell, L],$$

where $\gamma = \gamma(\varepsilon, \delta, \ell, L) > 0$. It follows ((2.8)) that $\Phi_p(\xi, \eta; k)$ can be extended analytically to $B_\delta^\varepsilon \backslash \{0\}$. Remark that for $\arg\, k < 0$ it is no longer the resolvent kernel associated with $H_p - z$, $z = (k^2 + p^2)\, c_+^2$, but just its analytic continuation (from $\mathbb{C}^+ \cap \mathbb{B}_\delta^\varepsilon$). However, the equality (2.8) remains valid throughout B_δ^ε, and φ, ψ are solutions of (2.2), satisfying the asymptotic behavior (2.3) in $k \in B_\delta^\varepsilon$. We also note that, as before, the exponent of ψ,

$$(2.30) \qquad \mathrm{Im}((k^2 + p^2)\, c_+^2 c_-^{-2} - p^2)^{1/2} > 0 \quad \text{for} \quad p \in [\ell, L], k \in B_\delta^\varepsilon,$$

and ψ decays exponentially as $y \to -\infty$, uniformly in $k \in B^\varepsilon_\delta$. We have the representation (2.13) for $k \in B^\varepsilon_\delta \backslash \{0\}$, $p \in [\ell, L]$, hence also (2.14). By (2.29), (2.14),

$$|\beta(k,p)| \geq \frac{\gamma}{2}\,\rho_+ = \gamma_1. \tag{2.31}$$

Evaluating (2.13) at $y = y_c$, we get for $k \neq 0$,

$\beta(k,p)^{-1}\psi(y_c;k,p) = \frac{\alpha(k,p)}{\beta(k,p)}\,\varphi(y_c;k,p) + \varphi_1(y_c;k,p), and in view of (2.31) and — \varphi(y_c;k,p)| = |\varphi_1(y_c;k,p)| = 1$, we conclude

$$\sup\left\{\left|\frac{\alpha(k,p)}{\beta(k,p)}\right|; \quad k \in B^\varepsilon_\delta \backslash \{0\}, \quad p \in [\ell, L]\right\} < \infty. \tag{2.32}$$

We can now establish the key estimate
(2.33)
$|k\,\Phi_p(\xi,\eta;k)| \leq C\,\exp[(-\mathrm{Im}\,k)^+(\xi^+ + \eta^+)], (\xi,\eta) \in \mathbb{R}\times\mathbb{R}, (\daleth, \imath) \in \mathbb{B}^\varepsilon_\delta \times [\ell, \mathbb{L}]$

where $b^+ = \max(b, 0)$. Indeed, for $\mathrm{Im}\,k \geq 0$ we have the stronger estimate (2.27)-(2.28) (note that $|k^2 - \tilde{k}^2| > \frac{1}{2}\,\tilde{k}(\ell)^2$). In general, using (2.13), (2.14), (2.8), we get, for $k \neq 0$,

$$k\,\Phi_p(\xi,\eta;k) = \frac{\rho_+}{2i}\,\varphi(\xi;k,p)\,\left[\frac{\alpha(k,p)}{\beta(k,p)}\,\varphi(\eta;k,p) + \varphi_1(\eta;k,p)\right], \xi \geq \eta, \tag{2.34}$$

so that (2.32) and the asymptotic behavior (2.3) yield (2.33) in the case $-y_c \leq \eta \leq \xi$. Obviously, (2.33) is valid if $\eta < -y_c < \xi$. Finally, if $\eta \leq \xi \leq -y_c$ the estimate (2.33) follows from (2.29) and the uniform exponential decay of ψ (as $y \to -\infty$).

Turning back to the proof of the theorem, we take $k_0 \in (0, \frac{\delta}{2})$ and let $\Lambda_0 \subseteq B^\varepsilon_\delta$ be the circle centered at k_0 with radius equal to $(2 + \xi^+ + \eta^+)^{-1}k_0 \sin\varepsilon$. By (2.33) the analytic function $k\,\Phi_p(\xi,\eta;k)$ $((\xi,\eta) \in \mathbb{R}\times\mathbb{R})$ is bounded on Λ_0 with a bound which is independent of (ξ,η), so that the Cauchy formula implies

$$\begin{aligned} &\left|\tfrac{\partial}{\partial k}\,(k\,\Phi_p(\xi,\eta;k))_{k=k_0}\right| \leq C\,k_0^{-1}(1 + \xi^+ + \eta^+), \\ &(\xi,\eta) \in \mathbb{R}\times\mathbb{R}, \quad \imath \in [\ell, \mathbb{L}], \quad \daleth_{\nvdash} \in (\nvdash, \tfrac{\delta}{\nvDash}). \end{aligned} \tag{2.35}$$

Changing back to $\lambda = (k^2 + p^2)\,c_+^2 \in I_p$, $\lambda < c_-^2 p^2 - \delta$, and noting (2.33), we obtain (1.13) away from $c_-^2 p^2$.

To prove (1.13) for λ near $c_-^2 p^2$, we need to study $\frac{\partial}{\partial k}\Phi_p(\xi,\eta;k)$ near $k = \tilde{k}(p)$. We observe first that while φ, φ' can be extended analytically to a full neighborhood of $\tilde{k}(p)$, the functions ψ, ψ' and ψ_1, ψ_1' (see (2.22)) can be extended analytically to sectors

$$B^{\varepsilon}_{\delta\pm} = \{k; |k-\tilde{k}(p)| < \delta, \quad -\varepsilon < \arg[\pm(k-\tilde{k})] < \varepsilon\}, \tag{2.36}$$

for δ, $\varepsilon > 0$ sufficiently small (uniformly in $p \in [\ell, L]$). In order to maintain continuity we need to take now $\mathrm{Im}(k \pm \tilde{k})^{1/2} < 0$ for $k \in B^{\varepsilon}_{\delta+}$, $\mathrm{Im}\,k < 0$.

Using the analytic extensions, we can now proceed as in the study above near $k = 0$. First, by (2.12), $[\varphi, \psi]\,(k,p)$ is bounded away from zero in $B^{\varepsilon}_{\delta\pm} \times [\ell, L]$. Then, using (2.34) for $\xi \geq \eta \geq -y_c$ and the expression following (2.24) for $\eta \leq \xi \leq y_c$ we obtain

$$\begin{aligned} |\Phi_p(\xi,\eta;k)| \leq \ & C\{\exp[(-\mathrm{Im}\,k)^+(1+\xi^+ +\eta^+)] \\ & +|k^2-\tilde{k}^2|^{-1/2}\exp[(-\mathrm{Im}(k^2-\tilde{k}^2)^{1/2})^+(1+\xi^- +\eta^-)]\} \end{aligned} \tag{2.37}$$

$(k,p) \in (B^{\varepsilon}_{\delta\pm}\backslash\{\tilde{k}\}) \times [\ell, L]$, $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}$ and we denoted $b^- = -\min(b,0)$.

We now proceed as in the argument leading up to (2.35) with suitable modifications. So, taking $k_0 \in (\tilde{k}, \tilde{k}+\frac{\delta}{2})$ (the case $k_0 \in (\tilde{k}-\frac{\delta}{2}, \tilde{k})$ is treated similarly), let Λ_0 be the circle centered at k_0, having radius $(2+|\xi|+|\eta|)^{-1}(k_0-\tilde{k})\sin\varepsilon$, so that $\Lambda_0 \subset B^{\varepsilon}_{\delta\pm}$. In order to estimate $\mathrm{Im}(k^2-\tilde{k}^2)^{1/2}$ we note that $2\mathrm{Im}(k^2-\tilde{k}^2)^{1/2} = \mathrm{Im}(k^2-\tilde{k}^2)(\mathrm{Re}(k^2-\tilde{k}^2)^{1/2})^{-1}$, the numerator of which is less than $C|k-k_0|$. The denominator is greater than $|\mathrm{Re}(k^2-\tilde{k}^2)|^{1/2}$ and this last one greater than $C|\mathrm{Re}(k-\tilde{k})\mathrm{Re}(k+\tilde{k}) - \mathrm{Im}(k-\tilde{k})\mathrm{Im}(k+\tilde{k})|^{1/2}$. For $k \in \Lambda_0$ (ε is small enough and depends on $\tilde{k}(l)$), we have

$$|\mathrm{Im}(k^2-\tilde{k}^2)^{1/2}| \leq C|k-\tilde{k}|^{-1/2}|k-k_0| \leq C(k_0-\tilde{k})^{1/2}(2+|\xi|+|\eta|)^{-1}\mathrm{sin}\varepsilon, \tag{2.38}$$

where $C > 0$ depends only on l, L, δ. We can now use (2.38) in (2.37) to conclude that the analytic function $(k^2-\tilde{k}^2)^{1/2}\Phi_p(\xi,\eta;k)$ is bounded on Λ_0, with a bound independent of $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}$, so that the Cauchy formula yields,

$$\left|\frac{\partial}{\partial k}[(k^2-\tilde{k}^2)^{1/2}\Phi_p(\xi,\eta;k)]_{k=k_0}\right| \leq C(k_0-\tilde{k})^{-1}\frac{(1+|\xi|+|\eta|)}{\mathrm{sin}\varepsilon}. \tag{2.39}$$

Observe that $\lambda - c_-^2 p^2 = c_+^2(k^2-\tilde{k}^2)$, so that $\frac{\partial\lambda}{\partial k}$ and $\frac{\partial k}{\partial\lambda}$ are uniformly bounded for $p \in [l, L]$ and $k \in (\tilde{k}-\frac{\delta}{2}, \tilde{k}+\frac{\delta}{2})$. Replacing the k-derivative in (2.39) by λ-derivative we obtain (1.13). Observe that outside neighborhoods of $c_\pm^2 p^2$ the theorem is obvious.

□

Appendix

In this appendix we show how the decomposition (1.3)-(1.7), combined with estimates on the kernel L_p, leads to estimates on the behavior of the resolvent of the Laplacian $-\Delta$ near the lower endpoint of its spectrum, $\lambda = 0$. For this purpose, we shall use only the global bound of L_p.

For $\sigma \in \mathbb{R}$ we denote by $L^{2,\sigma}(\mathbb{R}^{\ltimes})$ the weighted$-L^2$ space normed by

$$||f||^2_{L^{2,\sigma}(R^n)} = \int_{R^n} (1+|x|^2)^\sigma |f(x)|^2 dx.$$

Since $|L_p(\xi,\eta;z)| \le C|z-p^2|^{-1/2}$, we get

$$||M_p(z)f||_{L^{2,-s}(R)} \le C_s|z-p^2|^{-1/2}||f||_{L^{2,s}(R)}, \quad s > \frac{1}{2}. \tag{A.1}$$

We shall now use this estimate to establish an estimate for $M(z)$ near $z = 0$. So, using the notation in the Introduction, let $f(x,y), g(x,y) \in C_0^\infty(\mathbb{R}^{\ltimes+\nVdash})$ and let $\hat{f}(\theta,y)$, $\hat{g}(\theta,y)$ be their partial Fourier transforms with respect to $x \in \mathbb{R}^{\ltimes}$. Then, by (1.7),

$$(M(z)f,g) = \int_0^\infty \int_{\Sigma_p} (M_p(z)\,\hat{f}(\theta,\cdot),\hat{g}(\theta,\cdot))\,dS_p\,dp, \tag{A.2}$$

where $\Sigma_p = \{\theta \in \mathbb{R}^{\ltimes};|\theta| = \imath\}$ and dS_p is the Lebesgue surface measure.

Now fix $\beta > 1$ and let $|z| < \beta^2 - 1$. The family $\{M_p(z), p > \beta\}$ is uniformly bounded in $L^2(\mathbb{R})$, so that

$$\left|\int_\beta^\infty \int_{\Sigma_p} (M_p(z)\,\hat{f}(\theta,\cdot),\hat{g}(\theta,\cdot))\,dS_p\,dp\right| \le C||f||_{L^2(R^{n+1})}||g||_{L^2(R^{n+1})}. \tag{A.3}$$

Thus, in (A.2), it suffices to estimate the right-hand side with $\int_0^\infty$ replaced by $\int_0^\beta$.

Let $H^s(\mathbb{R}^{\ltimes})$ be the Sobolev space of order $s \in \mathbb{R}$, so that $||\hat{u}||_{H^s(R^n)} = ||u||_{L^{2,s}(R^n)}$. We recall the classical trace lemma

$$||\hat{u}||_{L^2(\Sigma_p)} \le C_{s,n}||\hat{u}||_{H^s(R^n)}, \quad s > \frac{1}{2}, \tag{A.4}$$

where $C_{s,n} > 0$ is independent of $p \in (0,\infty)$.

Combining (A.1)-(A.4) we obtain, for $|z| < \beta^2 - 1$,

$$(A.5)\qquad \begin{aligned} |(M(z)\, f, g)| \leq\ & C_{s,n,\beta}\left[||f||_{L^2(R^{n+1})}||g||_{L^2(R^{n+1})}\right. \\ & \left. + \int_0^\beta |z-p^2|^{-1/2}||\hat{f}||_{L^2(\Sigma_p)\otimes L^{2,s}(R)}\,||\hat{g}||_{L^2(\Sigma_p)\otimes L^{2,s}(R)}\,dp\right] \end{aligned}$$

$$\leq C_{s,n,p}\left[||f||_{L^2(R^{n+1})}||g||_{L^2(R^{n+1})} + \int_0^\beta |z-p^2|^{-1/2}dp\,||f||_{L^{2,s}(R^n)\otimes L^{2,s}(R)}\,||g||_{L^{2,s}(R^n)\otimes L^{2,s}(R)}\right]$$

We can simplify (A.5) by noting that the norm of $L^{2,2s}(\mathbb{R}^{n+1})$ is stronger than that of $L^{2,s}(\mathbb{R}^n)\otimes L^{2,s}(\mathbb{R})$, so that, for $|z| < \beta^2 - 1$, $s > \frac{1}{2}$,

$$\text{(A.6)}\qquad ||M(z)||_{B(L^{2,2s}(R^{n+1}),L^{2,-2s}(R^{n+1}))} \leq C_{s,n,\beta}\left[1 + \int_0^\beta |z-p^2|^{-1/2}dp\right].$$

($B(X, Y)$ is the space of bounded linear operators from X to Y, equipped with the operator norm).

The estimate (A.6) holds for $z \in \mathbb{C}^+$. However, we know that the operator-valued function $M(z)$ can be extended continuously, in the same operator space, to positive z (the "limiting absorption principle"). In fact, this can be proved using the above arguments ([4]). Retaining the notation $M(z)$ for the extended function, we have the validity of (A.6) for $|z| < \beta^2 - 1$, $0 \leq \arg\, z \leq \tau$, where $\beta > 1$ and $0 < \tau < \pi$ are fixed. (In the absence of negative eigenvalues, we could actually take here $\tau = \pi$. However, this is not true in the case of Hypothesis (1.3), where H_p possesses, in general, negative eigenvalues).

Inspecting the behavior of the integral in (A.6) as z approaches 0, $0 \leq \arg\, z \leq \tau$, we easily obtain the following "low energy estimate" for the resolvent $M(z) = (-\Delta - z)^{-1}$.

Lemma A.1 For $0 \leq \arg\, z \leq \tau < \pi$, $s > 1$, $n \geq 1$,

$$\text{(A.7)}\qquad ||M(z)||_{B(L^{2,s}(R^{n+1}),L^{2,-s}(R^{n+1}))} = \mathcal{O}(\log|z|), \quad z \to 0.$$

Certainly (A.7) is not "optimal", at least for $n \geq 2$ (see [3, 5, 13, 14]). However, the simplicity of its derivation makes it easy to generalize to cases such as those mentioned in the Introduction. Also, if $n \geq 3$ we can improve the trace estimate (A.4) to read ([5])

$$\text{(A.8)}\qquad ||\hat{u}||_{L^2(\Sigma_p)} \leq C_{s,n}\, p^{s-1/2}||\hat{u}||_{H^s(R^n)}, \quad \frac{1}{2} < s \leq 1, \quad n \geq 3.$$

Using this in (A.5) we obtain, instead of (A.6),

(A.9)

$$||M(z)||_{B(L^{2,1}(R^n)\otimes L^{2,s}(R),L^{2,-1}(R^n)\otimes L^{2,-s}(R))} \leq C_{s,n,\beta}\left[1 + \int_0^\beta |z-p^2|^{-1/2}p\,dp\right],$$

where $s > \frac{1}{2}$, $n \geq 3$. Replacing $L^{2,1}(\mathbb{R}^{\ltimes}) \otimes \mathbb{L}^{\nvDash,\sim}(\mathbb{R})$ by the stronger norm of $L^{2,1+s}(\mathbb{R}^{\ltimes+\nvDash})$, we can modify Lemma A.1 as follows.

Lemma A.2 For $0 \leq \arg z \leq \tau < \pi$, $s > \frac{3}{2}$, $n \geq 3$,

$$||M(z)||_{B(L^{2,s}(R^{n+1}),L^{2,-s}(R^{n+1}))} = \mathcal{O}(\infty) \quad \text{as} \quad \ddagger \to / \tag{A.10}$$

Acknowledgements

We thank Professor J. Ralston for pointing out to us an error in our previous work, which led to the present paper.

M. Ben-Artzi and Y. Dermenjian were supported by an Arc-en-Ciel grant and they also thank the Hebrew University of Jerusalem and Université de Provence for their hospitality during mutual visits.

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