SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME LINEAR AND NONLINEAR FREDHOLM OPERATORS WITH THE LOGARITHMIC LAPLACIAN

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The article is dedicated to the memory of Vladimir E. Zakharov.

Abstract: We study the solvability of certain linear and nonlinear nonhomogeneous equations in one dimension involving the logarithmic Laplacian and the transport term. In the linear case we show that the convergence in $L^2(\mathbb{R})$ of their right sides yields the existence and the convergence in $L^2(\mathbb{R})$ of the solutions. We generalize the results obtained in the earlier article [18] in the non-Fredholm case without the drift. In the nonlinear part of the work we demonstrate that, under the reasonable technical assumptions, the convergence in $L^1(\mathbb{R})$ of the integral kernels implies the existence and the convergence in $L^2(\mathbb{R})$ of the solutions.

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1. Introduction

Let us consider the equation

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and $V(x)$ is a function converging to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.1) contains the origin. As a consequence, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. Let us recall that elliptic equations with non-Fredholm operators, both linear and nonlinear were studied thoroughly in recent years (see [16], [17], [18], [28], [29], [30], [31], [32], [33], [34], also [5]) along with their potential applications to the theory of reaction-diffusion problems (see [11], [12]). Fredholm structures, topological invariants and their application were covered in [13]. The article [14] is devoted to the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was investigated in [15]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in the space of three dimensions was discussed in [20]. The works [21] and [27] are important for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were treated in [22]. In the article [32] the authors study the Laplace operator with drift from the point of view of the non-Fredholm operators. Standing lattice solitons in the discrete NLS equation with saturation were investigated in [1]. In the particular case of $a = 0$, our operator A mentioned above satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [7], [8], [5]). But the situation when $a \neq 0$ is considerably different and the approach developed in these works cannot be used.

One of the significant issues concerning the problems with non-Fredholm operators is their solvability. Let us address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. We designate by u_n a sequence of functions from $H^2(\mathbb{R}^d)$, such that

$$Au_n = f_n, \; n \in \mathbb{N}.$$

Since the operator A fails to satisfy the Fredholm property, the sequence u_n may be divergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [28]). If such sequence tends to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in certain weaker sense. In this case, the solution in the sense of sequences may not imply the existence of the usual solution. In the works which may include the non-Fredholm operators, we determine the sufficient conditions of equivalence of the solutions in the sense of sequences and the usual solutions. In the other words, we determine the conditions on the sequences f_n under which the corresponding sequences u_n are strongly convergent. The solvability in the sense of sequences for

the problems involving the Schrödinger type non-Fredholm operators was covered in [17], [30], [34]. The current article is our attempt to generalize such results by considering the solvability of the linear and nonlinear problems involving in their left sides the logarithmic Laplacian in one dimension, which can be defined via the spectral calculus along with the transport term.
First we consider the problem

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R} \tag{1.2}$$

with a square integrable right side. Here $a, b \in \mathbb{R}, \ b \neq 0$ are the constants. The logarithmic Laplacian $\ln(-\Delta)$ is the operator with Fourier symbol $2\ln|p|$. It appears as formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of fractional Laplacians at $s = 0$. The operator $(-\Delta)^s$ is actively used, for instance in the studies of the anomalous diffusion problems (see e.g. [34] and the references therein). Spectral properties of the logarithmic Laplacian in an open set of finite measure with Dirichlet boundary conditions were covered in [26] (see also [10]). The studies of $\ln(-\Delta)$ are important for the understanding of the asymptotic spectral properties of the family of fractional Laplacians in the limit $s \to 0^+$. In [24] it has been established that this operator allows to characterize the s-dependence of solution to fractional Poisson equations for the full range of exponents $s \in (0, 1)$. A direct method of moving planes for logarithmic Schrödinger operator was discussed in [35]. The article [36] is devoted to the symmetry of positive solutions for Lane-Emden systems involving the Logarithmic Laplacian. The equation analogous to (1.2) but without the transport term was treated in [18] in the context of the solvability in the sense of sequences. The solvability of certain linear nonhomogeneous equations containing the logarithm of the sum of the two Schrödinger operators in higher dimensions was discussed in [19]. The non self-adjoint operator involved in the left side of problem (1.2) is given by

$$L_{a,b} := \frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big) - b\frac{d}{dx} - a, \quad a, b \in \mathbb{R}, \quad b \neq 0. \tag{1.3}$$

It is considered on $L^2(\mathbb{R})$. By virtue of the standard Fourier transform, it can be trivially obtained that the essential spectrum of (1.3) is given by

$$\lambda_{a,b}(p) = \ln\Big(\frac{|p|}{e^a}\Big) - ibp, \quad a, b \in \mathbb{R}, \quad b \neq 0. \tag{1.4}$$

Evidently, the lower bound

$$|\lambda_{a,b}(p)| = \sqrt{\ln^2\Big(\frac{|p|}{e^a}\Big) + b^2p^2} \geq C_{a,b} > 0, \quad p \in \mathbb{R} \tag{1.5}$$

holds. Here $C_{a,b}$ is a constant. Therefore, as distinct from the case without the drift term discussed in [18], our operator (1.3) satisfies the Fredholm property.

We write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where $a, b \in \mathbb{R},\ b \neq 0$ are the constants. Let us assume that the right sides of (1.6) are square integrable on the real line and converge to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. Our first statment deals with the solvability of problem (1.2).

Proposition 1.1. *Let the constants $a, b \in \mathbb{R},\ b \neq 0$ and $f(x) \in L^2(\mathbb{R})$. Then equation (1.2) admits a unique solution $u(x) \in L^2(\mathbb{R})$.*

Note that as distinct from the analogous situation without the transport term considered in [18], the argument of the proposition above does not rely on the orthogonality conditions. Our second statement is devoted to the issue of the solvability in the sense of sequences for our equation.

Proposition 1.2. *Let $m \in \mathbb{N}$, the constants $a, b \in \mathbb{R},\ b \neq 0$, the functions $f_m(x) \in L^2(\mathbb{R})$, such that $f_m(x) \to f(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. Then problems (1.2) and (1.6) possess unique solutions $u(x) \in L^2(\mathbb{R})$ and $u_m(x) \in L^2(\mathbb{R})$ respectively, such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$.*

Throughout the article we use the hat symbol to denote the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ipx}dx, \quad p \in \mathbb{R}. \tag{1.7}$$

Clearly, the upper bound

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\|f(x)\|_{L^1(\mathbb{R})} \tag{1.8}$$

is valid. The second part of our article is devoted to the studies of the nonlinear equation

$$\Big[-\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u + b\frac{du}{dx} + au + \int_{-\infty}^{\infty} G(x-y)F(u(y), y)dy = 0, \quad x \in \mathbb{R} \tag{1.9}$$

with the constants $a, b \in \mathbb{R},\ b \neq 0$. In the Population Dynamics the integro-differential problems are used to describe the biological systems with the nonlocal consumption of resources and the intra-specific competition (see e.g. [4], [6], [23]). The solvability of the equation analogical to (1.9) but with a standard Laplacian in the diffusion term was discussed in [16]. Similarly to [16], we impose the following regularity conditions on the nonlinear part of problem (1.9).

Assumption 1.3. *Function $F(u,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is satisfying the Caratheodory condition (see [25]), so that*

$$|F(u,x)|\le k|u|+h(x) \quad for \quad u\in\mathbb{R}, \quad x\in\mathbb{R} \tag{1.10}$$

with a constant $k>0$ and $h(x):\mathbb{R}\to\mathbb{R}^+,\ h(x)\in L^2(\mathbb{R})$. Moreover, it is a Lipschitz continuous function, so that

$$|F(u_1,x)-F(u_2,x)|\le l|u_1-u_2| \quad for \quad any \quad u_{1,2}\in\mathbb{R}, \quad x\in\mathbb{R} \tag{1.11}$$

with a constant $l>0$.

The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was considered in [9]. The nonlinear function there was allowed to have a sublinear growth. In order to demonstrate the existence of solutions of problem (1.9), we will use the auxiliary equation

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u-b\frac{du}{dx}-au=\int_{-\infty}^{\infty}G(x-y)F(v(y),y)dy, \quad x\in\mathbb{R}, \tag{1.12}$$

where $a,b\in\mathbb{R},\ b\neq 0$ are the constants. We manage to establish that under the reasonable technical assumptions problem (1.12) defines a map $T_{a,b}:L^2(\mathbb{R})\to L^2(\mathbb{R})$, which is a strict contraction.

Theorem 1.4. *Let Assumption 1.3. hold, the function $G(x):\mathbb{R}\to\mathbb{R}$, so that $G(x)\in L^1(\mathbb{R})$ and $\dfrac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}l<1$.*
Then the map $T_{a,b}v=u$ on $L^2(\mathbb{R})$ defined by equation (1.12) has a unique fixed point $v_{a,b}$, which is the only solution of of problem (1.9) in $L^2(\mathbb{R})$.
This fixed point does not vanish identically on the real line provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)}(p)\cap supp\widehat{G}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.

Related to problem (1.9), we study the sequence of the approximate equations for $m\in\mathbb{N}$, namely

$$\Big[-\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u^{(m)}+$$

$$+b\frac{du^{(m)}}{dx}+au^{(m)}+\int_{-\infty}^{\infty}G_m(x-y)F(u^{(m)}(y),y)dy=0, \tag{1.13}$$

where $x\in\mathbb{R}$ and $a,b\in\mathbb{R},\ b\neq 0$ are the constants. The sequence of kernels $\{G_m(x)\}_{m=1}^{\infty}$ tends to $G(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$. Let us show that, under the appropriate technical conditions, each problem (1.13) possesses a unique solution $u^{(m)}(x)\in L^2(\mathbb{R})$, limiting equation (1.9) admits a unique solution $u(x)\in L^2(\mathbb{R})$, and $u^{(m)}(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. The importance of Theorem 1.5 below is the continuous dependence of the solution with respect to the integral kernel.

Theorem 1.5. *Let Assumption 1.3 hold, $m \in \mathbb{N}$, the functions $G_m(x) : \mathbb{R} \to \mathbb{R}$ are such that $G_m(x) \in L^1(\mathbb{R})$ and $G_m(x) \to G(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. Moreover, we suppose that*

$$\frac{\|G_m(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} l \leq 1 - \varepsilon \tag{1.14}$$

is valid for each $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$.
Then each equation (1.13) admits a unique solution $u^{(m)}(x) \in L^2(\mathbb{R})$, limiting problem (1.9) has a unique solution $u(x) \in L^2(\mathbb{R})$, and $u^{(m)}(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$.
The unique solution $u^{(m)}(x)$ of each equation (1.13) is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)}(p) \cap supp\widehat{G_m}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Similarly, the unique solution $u(x)$ of limiting problem (1.9) does not vanish identically if $supp\widehat{F(0,x)}(p) \cap supp\widehat{G}(p)$ is a set of nonzero Lebesgue measure on the real line.

2. Proofs of the main results

Proof of Proposition 1.1. To establish the uniqueness of solutions for our problem, we suppose that (1.2) admits two solutions $u_1(x),\ u_2(x) \in L^2(\mathbb{R})$. Evidently, their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R})$ as well and it satisfies the homogeneous equation

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]w - b\frac{dw}{dx} - aw = 0. \tag{2.1}$$

Because the operator $L_{a,b}$ on $L^2(\mathbb{R})$ given by (1.3) has only the essential spectrum and no nontrivial zero modes (see (1.4) and (1.5)), the function $w(x)$ vanishes a.e. in $\mathbb{R}$.
Let us apply the standard Fourier transform (1.7) to both sides of equation (1.2). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big) - ibp}, \quad p \in \mathbb{R}. \tag{2.2}$$

By means of (1.5), we have

$$|\widehat{u}(p)| \leq \frac{|\widehat{f}(p)|}{C_{a,b}} \in L^2(\mathbb{R})$$

due to our assumption, such that $u(x) \in L^2(\mathbb{R})$. ∎

We proceed to addressing the issue of the solvability in the sense of sequences for our linear problem.

Proof of Proposition 1.2. By virtue of Proposition 1.1 above, each equation (1.6) has a unique solution $u_m(x) \in L^2(\mathbb{R}),\ m \in \mathbb{N}$ and limiting problem (1.2) possesses a unique solution $u(x) \in L^2(\mathbb{R})$.
We apply the standard Fourier transform (1.7) to both sides of equations (1.6). This gives us for $m \in \mathbb{N}$ that

$$\widehat{u_m}(p) = \frac{\widehat{f_m}(p)}{\ln\left(\frac{|p|}{e^a}\right) - ibp}, \quad p \in \mathbb{R}. \tag{2.3}$$

By means of (2.3) along with (2.2), we arrive at

$$\widehat{u_m}(p) - \widehat{u}(p) = \frac{\widehat{f_m}(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right) - ibp}. \tag{2.4}$$

Using (1.5), we derive

$$|\widehat{u_m}(p) - \widehat{u}(p)| \le \frac{|\widehat{f_m}(p) - \widehat{f}(p)|}{C_{a,b}},$$

such that

$$\|u_m(x) - u(x)\|_{L^2(\mathbb{R})} \le \frac{\|f_m(x) - f(x)\|_{L^2(\mathbb{R})}}{C_{a,b}} \to 0, \quad m \to \infty$$

as assumed. Therefore, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. ∎

Let us turn our attention to the solvability of the nonlinear problem.

Proof of Theorem 1.4. First we suppose that for a certain $v(x) \in L^2(\mathbb{R})$ there exist two solutions $u_{1,2}(x) \in L^2(\mathbb{R})$ of equation (1.12). Obviously, the difference function $w(x) = u_1(x) - u_2(x) \in L^2(\mathbb{R})$ solves (2.1). Since the operator $L_{a,b}$ on $L^2(\mathbb{R})$ defined in (1.3) does not possess any nontrivial zero modes as discussed above, the function $w(x)$ is trivial on the real line.
Let us choose an arbitrary $v(x) \in L^2(\mathbb{R})$ and apply the standard Fourier transform (1.7) to both sides of problem (1.12). We obtain

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{\varphi}(p)}{\ln\left(\frac{|p|}{e^a}\right) - ibp}, \quad p \in \mathbb{R}. \tag{2.5}$$

Here $\widehat{\varphi}(p)$ stands for the Fourier image of $F(v(x), x)$. By means of estimates (1.5) and (1.8), we derive

$$|\widehat{u}(p)| \le \frac{\|G(x)\|_{L^1(\mathbb{R})}|\widehat{\varphi}(p)|}{C_{a,b}},$$

such that

$$\|u(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} \|F(v(x),x)\|_{L^2(\mathbb{R})}. \tag{2.6}$$

Let us recall inequality (1.10) of Assumption 1.3. Hence, $F(v(x),x)$ is square integrable on the real line for $v(x) \in L^2(\mathbb{R})$. Thus, for an arbitrarily chosen $v(x) \in L^2(\mathbb{R})$ there exists a unique solution $u(x) \in L^2(\mathbb{R})$ of equation (1.12), such that its Fourier image is given by (2.5). Therefore, the map $T_{a,b} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is well defined.

This allows us to choose the arbitrary functions $v_{1,2}(x) \in L^2(\mathbb{R})$, such that their images $u_{1,2} := T_{a,b}v_{1,2} \in L^2(\mathbb{R})$. Clearly, (1.12) yields

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_1 - b\frac{du_1}{dx} - au_1 = \int_{-\infty}^{\infty} G(x-y)F(v_1(y),y)dy, \quad x \in \mathbb{R}, \tag{2.7}$$

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_2 - b\frac{du_2}{dx} - au_2 = \int_{-\infty}^{\infty} G(x-y)F(v_2(y),y)dy, \quad x \in \mathbb{R}, \tag{2.8}$$

where $a, b \in \mathbb{R},\ b \neq 0$ are the constants. We apply the standard Fourier transform (1.7) to both sides of the equations of system (2.7), (2.8) and arrive at

$$\widehat{u_1}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{\varphi_1}(p)}{\ln\Big(\frac{|p|}{e^a}\Big) - ibp}, \quad \widehat{u_2}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{\varphi_2}(p)}{\ln\Big(\frac{|p|}{e^a}\Big) - ibp}, \quad p \in \mathbb{R}. \tag{2.9}$$

Here $\widehat{\varphi_{1,2}}(p)$ designate the Fourier images of $F(v_{1,2}(x),x)$. Let us use (2.9) along with (1.8) and (1.5) to derive the upper bound

$$|\widehat{u_1}(p) - \widehat{u_2}(p)| \leq \frac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}|\widehat{\varphi_1}(p) - \widehat{\varphi_2}(p)|,$$

so that

$$\|u_1(x) - u_2(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F(v_1(x),x) - F(v_2(x),x)\|_{L^2(\mathbb{R})}.$$

We recall condition (1.11) of Assumption 1.3 above. Hence,

$$\|F(v_1(x),x) - F(v_2(x),x)\|_{L^2(\mathbb{R})} \leq l\|v_1(x) - v_2(x)\|_{L^2(\mathbb{R})},$$

such that

$$\|T_{a,b}v_1(x) - T_{a,b}v_2(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} l \|v_1(x) - v_2(x)\|_{L^2(\mathbb{R})}. \tag{2.10}$$

The constant in the right side of (2.10) is less than one as we assume. Therefore, by means of the Fixed Point Theorem, there exists a unique function $v_{a,b} \in L^2(\mathbb{R})$, so that $T_{a,b}v_{a,b} = v_{a,b}$. This is the only solution of equation (1.9) in $L^2(\mathbb{R})$. Suppose

$v_{a,b}$ vanishes identically in $\mathbb{R}$. This will contradict to the given condition that the intersection of the supports of the Fourier images $supp\widehat{F(0,x)}(p) \cap supp\widehat{G}(p)$ is a set of nonzero Lebesgue measure on the real line. ∎

We conclude the article by discussing the issue of the solvability in the sense of sequences for our nonlinear equation.

Proof of Theorem 1.5. By virtue of the result of Theorem 1.4 above, under the stated assumptions each equation (1.13) admits a unique solution $u^{(m)}(x) \in L^2(\mathbb{R}),\ m \in \mathbb{N}$.
We have $\|G_m(x)\|_{L^1(\mathbb{R})} \to \|G(x)\|_{L^1(\mathbb{R})}$ as $m \to \infty$ via the standard triangle inequality. From (1.14) we easily deduce that

$$\frac{\|G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} l \leq 1 - \varepsilon$$

via the trivial limiting argument. Hence, by means of Theorem 1.4, limiting problem (1.9) possesses a unique solution $u(x) \in L^2(\mathbb{R})$ as well.
Let us apply the standard Fourier transform (1.7) to both sides of equations (1.9) and (1.13). Thus,

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right) - ibp}, \quad \widehat{u^{(m)}}(p) = \sqrt{2\pi}\frac{\widehat{G_m}(p)\widehat{f^{(m)}}(p)}{\ln\left(\frac{|p|}{e^a}\right) - ibp}, \quad p \in \mathbb{R}. \tag{2.11}$$

Here $\widehat{f}(p)$ and $\widehat{f^{(m)}}(p)$ denote the Fourier images of $F(u(x),x)$ and $F(u^{(m)}(x),x)$ respectively. From (2.11), we easily deduce that

$$|\widehat{u^{(m)}}(p) - \widehat{u}(p)| \leq$$

$$\leq \sqrt{2\pi}\frac{|\widehat{G_m}(p)|}{\sqrt{\ln^2\left(\frac{|p|}{e^a}\right) + b^2p^2}}|\widehat{f^{(m)}}(p) - \widehat{f}(p)| + \sqrt{2\pi}\frac{|\widehat{G_m}(p) - \widehat{G}(p)|}{\sqrt{\ln^2\left(\frac{|p|}{e^a}\right) + b^2p^2}}|\widehat{f}(p)|.$$

Let us use inequalities (1.5) and (1.8) to derive that

$$|\widehat{u^{(m)}}(p) - \widehat{u}(p)| \leq \frac{\|G_m(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}|\widehat{f^{(m)}}(p) - \widehat{f}(p)| + \frac{\|G_m(x) - G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}|\widehat{f}(p)|,$$

so that

$$\|u^{(m)}(x) - u(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G_m(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F(u^{(m)}(x),x) - F(u(x),x)\|_{L^2(\mathbb{R})} +$$

$$+\frac{\|G_m(x) - G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F(u(x),x)\|_{L^2(\mathbb{R})}.$$

We recall bound (1.11) of Assumption 1.3. above. Hence,

$$\|F(u^{(m)}(x),x)-F(u(x),x)\|_{L^2(\mathbb{R})}\leq l\|u^{(m)}(x)-u(x)\|_{L^2(\mathbb{R})}. \tag{2.12}$$

This enables us to derive the estimate

$$\left[1-\frac{\|G_m(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}l\right]\|u^{(m)}(x)-u(x)\|_{L^2(\mathbb{R})}\leq$$

$$\leq\frac{\|G_m(x)-G(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F(u(x),x)\|_{L^2(\mathbb{R})}.$$

Using (1.14), we arrive at

$$\|u^{(m)}(x)-u(x)\|_{L^2(\mathbb{R})}\leq\frac{\|G_m(x)-G(x)\|_{L^1(\mathbb{R})}}{\varepsilon C_{a,b}}\|F(u(x),x)\|_{L^2(\mathbb{R})}.$$

Let us recall upper bound (1.10) of Assumption 1.3. Thus, $F(u(x),x)$ is square integrable on the real line for $u(x)\in L^2(\mathbb{R})$. Therefore, under the given conditions

$$u^{(m)}(x)\to u(x),\quad m\to\infty \tag{2.13}$$

in $L^2(\mathbb{R})$. If we suppose that $u^{(m)}(x)$ vanishes identically in $\mathbb{R}$, we will obtain the contradiction to the stated assumption that the intersection of the supports of the Fourier transforms $supp\widehat{F(0,x)}(p)\cap supp\widehat{G_m}(p)$ is a set of nonzero Lebesgue measure on the real line. The similar argument is valid for the solution $u(x)$ of limiting equation (1.9). ■

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Declarations

Data availability statement

All data generated or analyzed during this study are included in this published article.

Conflict of interest

The authors declare that they have no competing interests as defined by Taylor and Francis, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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